Manipulating flavour models with invariants

Nucl. Phys. B928 (2018), 535–554

Wolfgang Gregor Hollik and Ulises Jésus Saldaña-Salazar

DESY Hamburg Theory

March 19 2018 | DPG Spring Meeting Würzburg

What is allowed to model flavour

- massless Standard Model: $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$

$$\mathrm{U}(3)_Q \times \mathrm{U}(3)_u \times \mathrm{U}(3)_d \times \mathrm{U}(3)_L \times \mathrm{U}(3)_\ell$$

- 3 generations (why?)
- gauge couplings U(3)-invariant for complex triplets $\mathbf{3}, \bar{\mathbf{3}}$
- $\overline{\psi}_i \not{D} \psi_i$ for $i = 1, \ldots, 3$ generations
- broken by Yukawa couplings (mix gauge representations)

$$\overline{\psi}^L_i \, Y_{ij} \, \psi^R_j$$

- $\mathrm{U}(3)^2$ freedom of rotating Yukawas: $\mathrm{U}(3)^5$ not all independent

$$\overline{\psi}^L_{i'} \, {U^L_{i'i}}^* \, Y_{ij} \, U^R_{jj'} \, \psi^R_{j'} \quad \xrightarrow{\text{diagonalization}} \quad \overline{\psi}^L_i \, m_{ii} \, \psi^R_i$$

Quark mass matrices

$$\boldsymbol{M}_u = v\,\boldsymbol{Y}_u,$$
$$\boldsymbol{M}_d = v\,\boldsymbol{Y}_d.$$

Quark mass matrices

$$M_u = v\, Y_u,$$
$$M_d = v\, Y_d.$$

Masses: diagonalize mass matrices (eigenvalues / singular values)

Singular Value Decomposition: $Y \to L^\dagger \Sigma R$: $\quad \Sigma = L Y R^\dagger$

Quark mass matrices

$$M_u = v\, Y_u,$$
$$M_d = v\, Y_d.$$

Masses: diagonalize mass matrices (eigenvalues / singular values)

Singular Value Decomposition: $\boldsymbol{Y} \to \boldsymbol{L}^\dagger \boldsymbol{\Sigma} \boldsymbol{R}$: $\quad \boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{Y}\boldsymbol{R}^\dagger$

Rotate fields in flavour space:

$$Q_L \to \boldsymbol{L}_Q\, Q_L,$$
$$u_R \to \boldsymbol{R}_u\, u_R,$$
$$d_R \to \boldsymbol{R}_d\, d_R.$$

Quark mass matrices

$$M_u = v\, Y_u,$$
$$M_d = v\, Y_d.$$

Masses: diagonalize mass matrices (eigenvalues / singular values)

Singular Value Decomposition: $Y \to L^\dagger \Sigma R$: $\quad \Sigma = LYR^\dagger$

Rotate fields in flavour space:

$$Q_L \to L_Q\, Q_L,$$
$$u_R \to R_u\, u_R,$$
$$d_R \to R_d\, d_R.$$

$$\mathcal{L}_{\mathrm{Y}}^{q} = \bar{Q}_L\, Y_d\, \Phi\, d_R + \bar{Q}_L\, Y_u\, \tilde{\Phi}\, u_{R,k} + \text{ h. c.}$$

Quark mass matrices

$$M_u = v\, Y_u,$$
$$M_d = v\, Y_d.$$

Masses: diagonalize mass matrices (eigenvalues / singular values)

Singular Value Decomposition: $Y \to L^\dagger \Sigma R$: $\quad \Sigma = L Y R^\dagger$

Rotate fields in flavour space:

$$Q_L \to L_Q\, Q_L,$$
$$u_R \to R_u\, u_R,$$
$$d_R \to R_d\, d_R.$$

$$\mathcal{L}_{\mathrm{Y}}^q = \bar{Q}_L\, L_Q^\dagger \Sigma_d\, \Phi\, R_d\, d_R + \bar{Q}_L\, L_Q^\dagger \Sigma_u\, \tilde{\Phi}\, R_u\, u_{R,k} + \text{ h. c.}$$

Mass matrices: arbitrary 3×3 *complex* matrices

$$\boldsymbol{M} = \frac{v}{\sqrt{2}} \begin{pmatrix} |y_{11}|e^{i\delta_{11}} & |y_{12}|e^{i\delta_{12}} & |y_{13}|e^{i\delta_{13}} \\ |y_{21}|e^{i\delta_{21}} & |y_{22}|e^{i\delta_{22}} & |y_{23}|e^{i\delta_{23}} \\ |y_{31}|e^{i\delta_{31}} & |y_{32}|e^{i\delta_{32}} & |y_{33}|e^{i\delta_{33}} \end{pmatrix}$$

Matrix invariants

- do not change for different bases
- relate matrix elements with their singular values (i. e. masses)

$$\xi = \frac{1}{2}\left[\mathrm{Tr}\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right]^2 - \mathrm{Tr}\left[\left(\boldsymbol{M}\boldsymbol{M}^\dagger\right)^2\right]\right] = m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2\,,$$

$$D = \det\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right] = m_1^2 m_2^2 m_3^2\,,$$

$$R^2 = \mathrm{Tr}\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right] = m_1^2 + m_2^2 + m_3^2\,.$$

Mass matrices: arbitrary 3×3 *complex* matrices

$$\boldsymbol{M} = \frac{v}{\sqrt{2}} \begin{pmatrix} |y_{11}|e^{i\delta_{11}} & |y_{12}|e^{i\delta_{12}} & |y_{13}|e^{i\delta_{13}} \\ |y_{21}|e^{i\delta_{21}} & |y_{22}|e^{i\delta_{22}} & |y_{23}|e^{i\delta_{23}} \\ |y_{31}|e^{i\delta_{31}} & |y_{32}|e^{i\delta_{32}} & |y_{33}|e^{i\delta_{33}} \end{pmatrix}$$

Matrix invariants

- do not change for different bases
- relate matrix elements with their singular values (i. e. masses)

$$\xi = \frac{1}{2}\left[\mathrm{Tr}\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right]^2 - \mathrm{Tr}\left[\left(\boldsymbol{M}\boldsymbol{M}^\dagger\right)^2\right]\right] = m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2 \,,$$

$$D = \det\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right] = m_1^2 m_2^2 m_3^2 \,,$$

$$R^2 = \mathrm{Tr}\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right] = m_1^2 + m_2^2 + m_3^2 \,.$$

Frobenius norm

$$||\boldsymbol{M}||_F^2 = \sum_{i,j} |m_{ij}|^2 = \mathrm{Tr}\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right] = R^2$$

Consider a *real* 3×3 matrix

$$\widetilde{\boldsymbol{M}} = \begin{pmatrix} \widetilde{m}_{11} & \widetilde{m}_{12} & \widetilde{m}_{13} \\ \widetilde{m}_{21} & \widetilde{m}_{22} & \widetilde{m}_{23} \\ \widetilde{m}_{31} & \widetilde{m}_{32} & \widetilde{m}_{33} \end{pmatrix}, \qquad \text{with}$$

$$\begin{aligned}
\widetilde{m}_{11} &= R \sin\chi \, \sin\phi_1 \sin\phi_2 \sin\phi_3 \sin\phi_4 \sin\phi_5 \sin\phi_6 \sin\phi_7, \\
\widetilde{m}_{12} &= R \sin\chi \, \sin\phi_1 \sin\phi_2 \sin\phi_3 \sin\phi_4 \sin\phi_5 \sin\phi_6 \cos\phi_7, \\
\widetilde{m}_{13} &= R \sin\chi \, \sin\phi_1 \sin\phi_2 \sin\phi_3 \sin\phi_4 \sin\phi_5 \cos\phi_6, \\
\widetilde{m}_{21} &= R \sin\chi \, \sin\phi_1 \sin\phi_2 \sin\phi_3 \sin\phi_4 \cos\phi_5, \\
\widetilde{m}_{22} &= R \sin\chi \, \sin\phi_1 \sin\phi_2 \sin\phi_3 \cos\phi_4, \\
\widetilde{m}_{23} &= R \sin\chi \, \sin\phi_1 \sin\phi_2 \cos\phi_3, \\
\widetilde{m}_{31} &= R \sin\chi \, \sin\phi_1 \cos\phi_2, \\
\widetilde{m}_{32} &= R \sin\chi \, \cos\phi_1, \\
\widetilde{m}_{33} &= R \cos\chi \,.
\end{aligned}$$

The angles are $\phi_i \in [0, 2\pi)$, $i = 1, \ldots, 7$, and $\chi \in [0, \pi]$.

Personal bias: define $\tilde{m}_{33}$ distinguished direction

$$\widetilde{\boldsymbol{M}} = R \begin{pmatrix} \sin\chi\left(\prod_{i=1}^{6}\sin\phi_i\right)\sin\phi_7 & \sin\chi\left(\prod_{i=1}^{6}\sin\phi_i\right)\cos\phi_7 & \sin\chi\left(\prod_{i=1}^{5}\sin\phi_i\right)\cos\phi_6 \\ \sin\chi\left(\prod_{i=1}^{4}\sin\phi_i\right)\cos\phi_5 & \sin\chi\left(\prod_{i=1}^{3}\sin\phi_i\right)\cos\phi_4 & \sin\chi\left(\prod_{i=1}^{2}\sin\phi_i\right)\cos\phi_3 \\ \sin\chi\sin\phi_1\cos\phi_2 & \sin\chi\cos\phi_1 & \cos\chi \end{pmatrix}$$

9-dimensional vector

$$\overrightarrow{\mathfrak{m}} = (\widetilde{m}_{11}, \widetilde{m}_{12}, \widetilde{m}_{13}, \widetilde{m}_{21}, \widetilde{m}_{22}, \widetilde{m}_{23}, \widetilde{m}_{31}, \widetilde{m}_{32}, \widetilde{m}_{33})^T$$

"flavor space" expansion

A new type of alignment

Personal bias: define $\tilde{m}_{33}$ distinguished direction

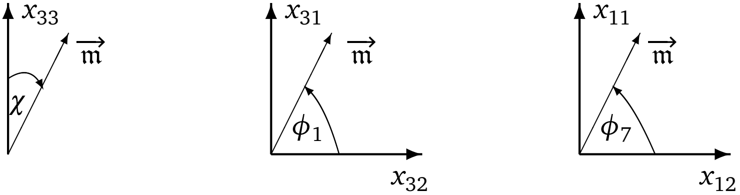
$$\widetilde{\boldsymbol{M}} = R \begin{pmatrix} \sin\chi\left(\prod_{i=1}^{6}\sin\phi_i\right)\sin\phi_7 & \sin\chi\left(\prod_{i=1}^{6}\sin\phi_i\right)\cos\phi_7 & \sin\chi\left(\prod_{i=1}^{5}\sin\phi_i\right)\cos\phi_6 \\ \sin\chi\left(\prod_{i=1}^{4}\sin\phi_i\right)\cos\phi_5 & \sin\chi\left(\prod_{i=1}^{3}\sin\phi_i\right)\cos\phi_4 & \sin\chi\left(\prod_{i=1}^{2}\sin\phi_i\right)\cos\phi_3 \\ \sin\chi\sin\phi_1\cos\phi_2 & \sin\chi\cos\phi_1 & \cos\chi \end{pmatrix}$$

9-dimensional vector

$$\overrightarrow{\mathfrak{m}} = (\widetilde{m}_{11}, \widetilde{m}_{12}, \widetilde{m}_{13}, \widetilde{m}_{21}, \widetilde{m}_{22}, \widetilde{m}_{23}, \widetilde{m}_{31}, \widetilde{m}_{32}, \widetilde{m}_{33})^T$$

"flavor space" expansion

$$-\mathcal{L} = \sum_{i,j=1}^{3} \overline{\psi}_{L,i}\widetilde{m}_{ij}\psi_{R,j} \equiv \sum_{i,j=1}^{3} \widetilde{m}_{ij}\hat{x}_{ij}$$



"Nearest Neighbour Interaction"

$$|\boldsymbol{M}| = \begin{pmatrix} 0 & A & 0 \\ A' & 0 & B \\ 0 & B' & C \end{pmatrix}$$

with $\quad \phi_{2,4,6} = \frac{\pi}{2} \quad$ and $\quad \phi_7 = 0$

$$\begin{aligned} A &= R \sin\chi \sin\phi_1 \sin\phi_3 \sin\phi_5 \\ A' &= R \sin\chi \sin\phi_1 \sin\phi_3 \cos\phi_5 \\ B &= R \sin\chi \sin\phi_1 \cos\phi_3 \\ B' &= R \sin\chi \cos\phi_1 \\ C &= R \cos\chi \end{aligned}$$

$$\tan\phi_5 = \frac{A}{A'}$$

$$\tan\phi_3 = \sqrt{1 + \left(\frac{A}{A'}\right)^2}$$

$$\tan\phi_1 = \sqrt{1 + \left(1 + \left(\frac{A}{A'}\right)^2\right)\left(\frac{A^2}{A'B}\right)^2 \frac{B}{B'}}$$

Parameter Counting

- $\mathrm{U}(n)^3 \quad \hookrightarrow \quad [3n(n+1)-2]/2$ arbitrary phases
- $n=3$: 17 free phases
- reducing phases from texture zeros: $17-8=9$ unphysical
- in total 10 phases in the mass matrix: 1 independent!

define $\gamma = \delta_{21}^{(b)} + \delta_{33}^{(b)} - \delta_{31}^{(b)} - \delta_{23}^{(b)}$

$$\boldsymbol{M}_a = \begin{pmatrix} 0 & A_a & 0 \\ A'_a & 0 & B_a \\ 0 & B'_a & C_a \end{pmatrix}, \qquad \boldsymbol{M}_b = \begin{pmatrix} 0 & A_b e^{i\gamma} & 0 \\ A_b e^{-i\gamma} & 0 & B_b \\ 0 & B'_b & C_b \end{pmatrix}$$

- 10 parameters in $\boldsymbol{M}_a$ and $\boldsymbol{M}_b$
- 6 masses, 3 mixing angles, one CP-phase: “10 observables”
- weak basis invariant statement [see Branco et al.]

Majorana neutrinos

full ignorance about high-scale model: [Weinberg 1979]

$$\mathcal{L}_5 = \frac{1}{2}\frac{c_{\alpha\beta}}{\Lambda_{\mathrm{NP}}}\left(\bar{L}^c_{L\alpha}\widetilde{H}^*\right)\left(\widetilde{H}^\dagger L_{L\beta}\right) + \mathrm{h.\,c.}$$

complex, symmetric mass matrix

$$\boldsymbol{M}^\nu = \begin{pmatrix} \widetilde{m}^\nu_{11} e^{i\varphi^\nu_{11}} & \frac{1}{\sqrt{2}}\widetilde{m}^\nu_{12} e^{i\varphi^\nu_{12}} & \frac{1}{\sqrt{2}}\widetilde{m}^\nu_{13} e^{i\varphi^\nu_{13}} \\ \frac{1}{\sqrt{2}}\widetilde{m}^\nu_{12} e^{i\varphi^\nu_{12}} & \widetilde{m}^\nu_{22} e^{i\varphi^\nu_{22}} & \frac{1}{\sqrt{2}}\widetilde{m}^\nu_{23} e^{i\varphi^\nu_{23}} \\ \frac{1}{\sqrt{2}}\widetilde{m}^\nu_{13} e^{i\varphi^\nu_{13}} & \frac{1}{\sqrt{2}}\widetilde{m}^\nu_{23} e^{i\varphi^\nu_{23}} & \widetilde{m}^\nu_{33} e^{i\varphi_{33}} \end{pmatrix}$$

$$\widetilde{m}^\nu_{11} = R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \sin\omega^\nu_3 \sin\omega^\nu_4$$

$$\widetilde{m}^\nu_{12} = R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \sin\omega^\nu_3 \cos\omega^\nu_4$$

$$\widetilde{m}^\nu_{13} = R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \cos\omega^\nu_3$$

$$\widetilde{m}^\nu_{22} = R^\nu \sin\chi^\nu \sin\omega^\nu_1 \cos\omega^\nu_2$$

$$\widetilde{m}^\nu_{23} = R^\nu \sin\chi^\nu \cos\omega^\nu_1 \qquad \widetilde{m}^\nu_{33} = R^\nu \cos\chi^\nu$$

The Altarelli–Feruglio model

special choice of matrix elements motivated by discrete symmetry

$$|M_\nu| = \begin{pmatrix} 0 & a^\nu & a^\nu \\ a^\nu & -2a^\nu & b^\nu \\ a^\nu & b^\nu & -2a^\nu \end{pmatrix}$$

with $m^\nu_{12} = m^\nu_{13}$, $m^\nu_{22} = m^\nu_{33}$, and $m^\nu_{22} = -2m^\nu_{12}$

$$\begin{aligned}
\widetilde{m}^\nu_{11} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \cos\omega^\nu_3 \\
\widetilde{m}^\nu_{12} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \sin\omega^\nu_3 \sin\omega^\nu_4 \\
\widetilde{m}^\nu_{13} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \sin\omega^\nu_3 \cos\omega^\nu_4 \\
\widetilde{m}^\nu_{22} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \cos\omega^\nu_2 \\
\widetilde{m}^\nu_{23} &= R^\nu \sin\chi^\nu \cos\omega^\nu_1 \\
\widetilde{m}^\nu_{33} &= R^\nu \cos\chi^\nu
\end{aligned}$$

The Altarelli–Feruglio model

special choice of matrix elements motivated by discrete symmetry

$$|M_\nu| = \begin{pmatrix} 0 & a^\nu & a^\nu \\ a^\nu & -2a^\nu & b^\nu \\ a^\nu & b^\nu & -2a^\nu \end{pmatrix}$$

with $m^\nu_{12} = m^\nu_{13}$, $m^\nu_{22} = m^\nu_{33}$, and $m^\nu_{22} = -2m^\nu_{12}$

$$\begin{aligned}
\widetilde{m}^\nu_{11} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \cos\omega^\nu_3 \\
\widetilde{m}^\nu_{12} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \sin\omega^\nu_3 \sin\omega^\nu_4 \\
\widetilde{m}^\nu_{13} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \sin\omega^\nu_3 \cos\omega^\nu_4 \\
\widetilde{m}^\nu_{22} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \cos\omega^\nu_2 \\
\widetilde{m}^\nu_{23} &= R^\nu \sin\chi^\nu \cos\omega^\nu_1 \\
\widetilde{m}^\nu_{33} &= R^\nu \cos\chi^\nu
\end{aligned}$$

1. $\omega^\nu_4 = \frac{\pi}{4}$

The Altarelli–Feruglio model

special choice of matrix elements motivated by discrete symmetry

$$|M_\nu| = \begin{pmatrix} 0 & a^\nu & a^\nu \\ a^\nu & -2a^\nu & b^\nu \\ a^\nu & b^\nu & -2a^\nu \end{pmatrix}$$

with $m^\nu_{12} = m^\nu_{13}$, $m^\nu_{22} = m^\nu_{33}$, and $m^\nu_{22} = -2m^\nu_{12}$

$$\begin{aligned}
\widetilde{m}^\nu_{11} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \cos\omega^\nu_3 \\
\widetilde{m}^\nu_{12} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \sin\omega^\nu_3/\sqrt{2} \\
\widetilde{m}^\nu_{13} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2 \sin\omega^\nu_3/\sqrt{2} \\
\widetilde{m}^\nu_{22} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \cos\omega^\nu_2 \\
\widetilde{m}^\nu_{23} &= R^\nu \sin\chi^\nu \cos\omega^\nu_1 \\
\widetilde{m}^\nu_{33} &= R^\nu \cos\chi^\nu
\end{aligned}$$

1. $\omega^\nu_4 = \frac{\pi}{4}$
2. $\omega^\nu_3 = \frac{3\pi}{2}$

The Altarelli–Feruglio model

special choice of matrix elements motivated by discrete symmetry

$$|M_\nu| = \begin{pmatrix} 0 & a^\nu & a^\nu \\ a^\nu & -2a^\nu & b^\nu \\ a^\nu & b^\nu & -2a^\nu \end{pmatrix}$$

with $m^\nu_{12} = m^\nu_{13}$, $m^\nu_{22} = m^\nu_{33}$, and $m^\nu_{22} = -2m^\nu_{12}$

$$\begin{aligned}
\widetilde{m}^\nu_{11} &= 0 \\
\widetilde{m}^\nu_{12} &= -R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2/\sqrt{2} \\
\widetilde{m}^\nu_{13} &= -R^\nu \sin\chi^\nu \sin\omega^\nu_1 \sin\omega^\nu_2/\sqrt{2} \\
\widetilde{m}^\nu_{22} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1 \cos\omega^\nu_2 \\
\widetilde{m}^\nu_{23} &= R^\nu \sin\chi^\nu \cos\omega^\nu_1 \\
\widetilde{m}^\nu_{33} &= R^\nu \cos\chi^\nu
\end{aligned}$$

1. $\omega^\nu_4 = \frac{\pi}{4}$
2. $\omega^\nu_3 = \frac{3\pi}{2}$
3. $\omega^\nu_2 = \frac{5\pi}{4}$

The Altarelli–Feruglio model

special choice of matrix elements motivated by discrete symmetry

$$|\boldsymbol{M}_\nu| = \begin{pmatrix} 0 & a^\nu & a^\nu \\ a^\nu & -2a^\nu & b^\nu \\ a^\nu & b^\nu & -2a^\nu \end{pmatrix}$$

with $m^\nu_{12} = m^\nu_{13}$, $m^\nu_{22} = m^\nu_{33}$, and $m^\nu_{22} = -2m^\nu_{12}$

$$\begin{aligned}
\widetilde{m}^\nu_{11} &= 0 \\
\widetilde{m}^\nu_{12} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1/2 \\
\widetilde{m}^\nu_{13} &= R^\nu \sin\chi^\nu \sin\omega^\nu_1/2 \\
\widetilde{m}^\nu_{22} &= -R^\nu \sin\chi^\nu \sin\omega^\nu_1/\sqrt{2} \\
\widetilde{m}^\nu_{23} &= R^\nu \sin\chi^\nu \cos\omega^\nu_1 \\
\widetilde{m}^\nu_{33} &= R^\nu \cos\chi^\nu
\end{aligned}$$

1. $\omega^\nu_4 = \frac{\pi}{4}$
2. $\omega^\nu_3 = \frac{3\pi}{2}$
3. $\omega^\nu_2 = \frac{5\pi}{4}$

The Altarelli–Feruglio model

special choice of matrix elements motivated by discrete symmetry

$$|M_\nu| = \begin{pmatrix} 0 & a^\nu & a^\nu \\ a^\nu & -2a^\nu & b^\nu \\ a^\nu & b^\nu & -2a^\nu \end{pmatrix}$$

with $m_{12}^\nu = m_{13}^\nu$, $m_{22}^\nu = m_{33}^\nu$, and $m_{22}^\nu = -2m_{12}^\nu$

$$\begin{aligned}
\widetilde{m}_{11}^\nu &= 0 \\
\widetilde{m}_{12}^\nu &= R^\nu \sin\chi^\nu \sin\omega_1^\nu/2 \\
\widetilde{m}_{13}^\nu &= R^\nu \sin\chi^\nu \sin\omega_1^\nu/2 \\
\widetilde{m}_{22}^\nu &= -R^\nu \sin\chi^\nu \sin\omega_1^\nu/\sqrt{2} \\
\widetilde{m}_{23}^\nu &= R^\nu \sin\chi^\nu \cos\omega_1^\nu \\
\widetilde{m}_{33}^\nu &= R^\nu \cos\chi^\nu
\end{aligned}$$

1. $\omega_4^\nu = \frac{\pi}{4}$
2. $\omega_3^\nu = \frac{3\pi}{2}$
3. $\omega_2^\nu = \frac{5\pi}{4}$

Note, that 1-1 element vanishes!

Define

$$a^{\nu} = R^{\nu} \sin\chi^{\nu} \sin\omega_1^{\nu}/(2\sqrt{2})$$
$$b^{\nu} = R^{\nu} \sin\chi^{\nu} \cos\omega_1^{\nu}/\sqrt{2}$$

with $\tan\chi^{\nu} \sin\omega_1^{\nu} = -\sqrt{2}$. Leads to close-to-TBM mixing matrix.

Define

$$a^\nu = R^\nu \sin\chi^\nu \sin\omega_1^\nu/(2\sqrt{2})$$
$$b^\nu = R^\nu \sin\chi^\nu \cos\omega_1^\nu/\sqrt{2}$$

with $\tan\chi^\nu \sin\omega_1^\nu = -\sqrt{2}$. Leads to close-to-TBM mixing matrix.

- Fit a^ν, b^ν via Δm_{21}^2 and Δm_{31}^2.
- Vanishing $\tilde{m}_{11}^\nu$ element (in contrast to original AF).
- $\theta_{13}^\nu = 0$ by definition.

Disturb the model

Deviate with a small perturbation $\omega_4^\nu = \frac{\pi}{4} + \varepsilon$ such that

$$|M_\nu| = \begin{pmatrix} 0 & a^\nu + \delta^\nu & a^\nu - \delta^\nu \\ a^\nu + \delta^\nu & -2a^\nu & b^\nu \\ a^\nu - \delta^\nu & b^\nu & -2a^\nu \end{pmatrix} + \mathcal{O}(\varepsilon^2)$$

with $\delta^\nu = a^\nu \varepsilon$.

- Accommodate for CP violating phase: $\omega_4^\nu = \frac{\pi}{4} + \mathrm{i}\,\varepsilon$
- $\delta^\nu = 0.005\,\mathrm{i}$ gives $\sin\theta_{13}^\nu \approx 0.15$.

Fitting $\Delta m_{21}^2 = 7.40 \times 10^{-5}\,\mathrm{eV}^2$ and $\Delta m_{31}^2 = 2.494 \times 10^{-3}\,\mathrm{eV}^2$ [nu-fit.org], we find

$$a^\nu = 0.0126\,\mathrm{eV}, \qquad \text{and} \qquad b^\nu = 0.0263\,\mathrm{eV}$$

corresponding to

$$m_3^\nu = 0.0526\,\mathrm{eV}, \quad m_2^\nu = 0.0187\,\mathrm{eV} \quad \text{and} \quad m_1^\nu = 0.0166\,\mathrm{eV}$$

and

$$|U_{\mathrm{PMNS}}| = \begin{pmatrix} 0.696 & 0.702 & 0.150 \\ 0.398 & 0.551 & 0.733 \\ 0.598 & 0.451 & 0.663 \end{pmatrix}.$$

(Deviations from 3σ regime.)

Hierarchical matrix elements from *mis*alignment

All small angles: $\varepsilon \equiv \chi \sim \phi_k \ll 1$ gives

$$|M| \sim R \begin{pmatrix} \varepsilon^8 & \varepsilon^7 & \varepsilon^6 \\ \varepsilon^5 & \varepsilon^4 & \varepsilon^3 \\ \varepsilon^2 & \varepsilon & 1 \end{pmatrix} .$$

- A bigger variety possible: not all angles have to be the same.
- Treat up- and down-type masses differently, e. g. $\chi^d \to \chi^d - \frac{\pi}{2}$.
- Higher powers of ε possible.
- Smaller powers sufficient (i. e. only two small angles)

Hierarchical matrix elements from *mis*alignment

All small angles: $\varepsilon \equiv \chi \sim \phi_k \ll 1$ gives

$$|M| \sim R \begin{pmatrix} \varepsilon^8 & \varepsilon^7 & \varepsilon^6 \\ \varepsilon^5 & \varepsilon^4 & \varepsilon^3 \\ \varepsilon^2 & \varepsilon & 1 \end{pmatrix} .$$

- A bigger variety possible: not all angles have to be the same.
- Treat up- and down-type masses differently, e. g. $\chi^d \to \chi^d - \frac{\pi}{2}$.
- Higher powers of ε possible.
- Smaller powers sufficient (i. e. only two small angles)

$$|M| \sim R \begin{pmatrix} \varepsilon^2 & \varepsilon^2 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon^2 & \varepsilon \\ \varepsilon & \varepsilon & 1 \end{pmatrix} \quad \Rightarrow \quad |MM^\dagger| \sim R^2 \begin{pmatrix} \varepsilon^4 & \varepsilon^3 & \varepsilon^2 \\ \varepsilon^3 & \varepsilon^2 & \varepsilon \\ \varepsilon^2 & \varepsilon & 1+\varepsilon^2 \end{pmatrix}$$

- There is a fundamental thirst of reducing arbitrary parameters in the Standard Model.
- The flavour sector has too many.
- Relating observable parameters with the free parameters.
- Omitting unphysical ones.
- Exploiting invariant statements.

The spherical mass matrix interpretation

Frobenius norm defines surface of a hypersphere:
$R^2 = ||\boldsymbol{M}||_F^2 = \sum_{i,j} |m_{ij}|^2 = \mathrm{Tr}\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right]$

$$\widetilde{\boldsymbol{M}}/R = \begin{pmatrix} \sin\chi\left(\prod_{i=1}^{6}\sin\phi_i\right)\sin\phi_7 & \sin\chi\left(\prod_{i=1}^{6}\sin\phi_i\right)\cos\phi_7 & \sin\chi\left(\prod_{i=1}^{5}\sin\phi_i\right)\cos\phi_6 \\ \sin\chi\left(\prod_{i=1}^{4}\sin\phi_i\right)\cos\phi_5 & \sin\chi\left(\prod_{i=1}^{3}\sin\phi_i\right)\cos\phi_4 & \sin\chi\left(\prod_{i=1}^{2}\sin\phi_i\right)\cos\phi_3 \\ \sin\chi\sin\phi_1\cos\phi_2 & \sin\chi\cos\phi_1 & \cos\chi \end{pmatrix}$$