To Higgs or not to Higgs

parameter constraints in SUSY models

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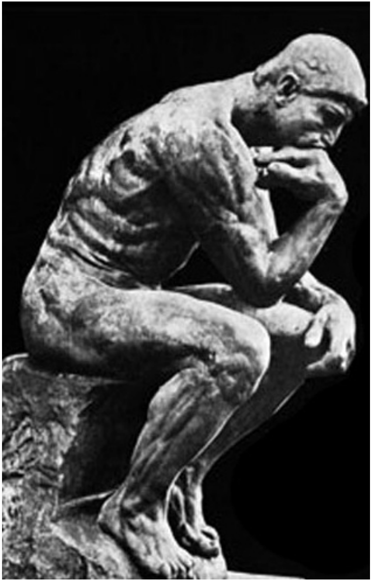

Open questions

- What is the origin of space-time?
- Which is the fundamental symmetry of nature?
- How does spontaneous symmetry breaking work?
- Why do we observe three copies of matter states?
- ...

Possible answer

- Nature is supersymmetric!

More than a symmetry

Latin: *superus*—being above/beyond, part of the Olympus, celestial

Haag–Łopuszánski–Sohnius theorem

- evades no-go theorem by Coleman and Mandula
  ["*All Possible Symmetries of the S Matrix*"; Coleman, Mandula 1967]
- only direct products of any internal symmetries and Poincaré group allowed (commutator relations)
- supersymmetry is an exception (anticommutator relations)
  [Haag, Łopuszánski and Sohnius 1975]

a SUSY algebra obeys a *pseudo* Lie algebra

$$\{Q_\alpha^N, \bar{Q}_\beta^M\} = 2\gamma^\mu_{\alpha\beta} P_\mu \delta^{NM},$$

where N, M count the number of SUSY generators, α, β are (Dirac) spinor indices, μ a Lorentz index, P_μ the 4-momentum and $Q, \bar{Q}$ generators of a supersymmetry

Supersymmetry is there

- more than a solution to the hierarchy problem
- more than simply stabilizing the Higgs mass
- less than "doubling" the number of states

[https://blogphysica.wordpress.com]

How SUSY acts

$$Q|\text{scalar}\rangle = |\text{fermion}\rangle$$
$$\bar{Q}|\text{fermion}\rangle = |\text{scalar}\rangle$$
$$QQ|\text{scalar}\rangle = 0$$

but

$$Q|\text{fermion}'\rangle = |\text{vector}\rangle$$

at least two different kind of multiplets

The (left) chiral Supermultiplet

$\Phi = \{\phi, \xi, F\}$ with an auxiliary (scalar) field F

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\xi(y) + \theta\theta F(y),$$

for superspace coordinates $y^\mu = x^\mu - \mathrm{i}\,\theta\sigma^\mu\bar{\theta}$ and Grassmann numbers θ (two-component spinors)

How SUSY acts

$$Q|\text{scalar}\rangle = |\text{fermion}\rangle$$
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$$QQ|\text{scalar}\rangle = 0$$

but

$$Q|\text{fermion}'\rangle = |\text{vector}\rangle$$

at least two different kind of multiplets

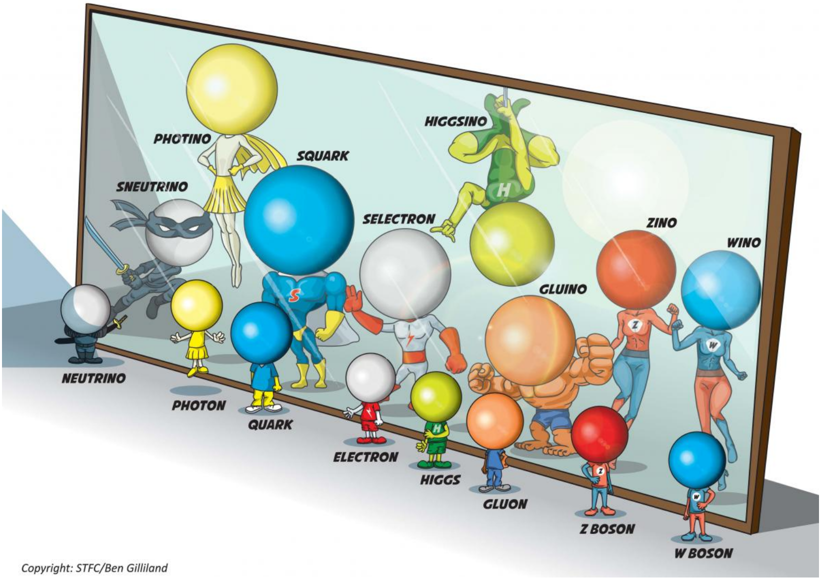
The vector Supermultiplet

$V = \{A_\mu, \lambda, D\}$ with an auxiliary (scalar) field D

$$\Phi(x, \theta, \bar{\theta}) = \theta\sigma^\mu\bar{\theta}A_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \theta\lambda(x)\bar{\theta}\bar{\theta} + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x),$$

in a special *supergauge* (Wess–Zumino gauge) and where $\theta_\alpha^2 = 0$, but $\theta\theta = \theta^\alpha\theta_\alpha \neq 0$

SUSY heroes: The Minimal Supersymmetric Standard Model



Chiral Superfields of the MSSM

	spin 0	spin 1/2	$\mathrm{SU}(3)_c, \mathrm{SU}(2)_L, \mathrm{U}(1)_Y$
Q	$\begin{pmatrix}\tilde{u}_L, \tilde{d}_L\end{pmatrix}$	(u_L, d_L)	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$
$\bar{U}$	$\tilde{u}_R^*$	u_R^c	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})$
$\bar{D}$	$\tilde{d}_R^*$	d_R^c	$(\bar{\mathbf{3}}, \mathbf{1}, \frac{1}{3})$
L	$(\tilde{\nu}_L, \tilde{e}_L)$	(ν_L, e_L)	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$
$\bar{E}$	$\tilde{e}_R^*$	e_R^c	$(\bar{\mathbf{1}}, \mathbf{1}, 1)$
H_u	(h_u^+, h_u^0)	$\begin{pmatrix}\tilde{h}_u^+, \tilde{h}_u^0\end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, \frac{1}{2})$
H_d	(h_d^0, h_d^-)	$\begin{pmatrix}\tilde{h}_d^0, \tilde{h}_d^-\end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, \frac{1}{2})$

Gauge Supermultiplets of the MSSM

spin 1/2	spin 1	$\mathrm{SU}(3)_c, \mathrm{SU}(2)_L, \mathrm{U}(1)_Y$
$\tilde{g}$	g	$(\mathbf{8}, \mathbf{1}, 0)$
$\tilde{W}^{\pm}, \tilde{W}^0$	$W_\mu^{\pm}, W_\mu^0$	$(\mathbf{1}, \mathbf{3}, 0)$
$\tilde{B}$	B_μ	$(\mathbf{1}, \mathbf{1}, 0)$

The advantages of SUSY and the MSSM

Electroweak symmetry breaking is automatic

- Higgs potential determined by theory, not put by hand
- Higgs mass calculated, not an input
- potential stable up to high scales, quartics $\sim g_1^2 + g_2^2$

There are, however, problems

- (tree level) Higgs mass bounded by M_Z
- need (large) loop corrections to shift ${m_h}^0 \to 125\,\mathrm{GeV}$
- Supersymmetry apparently not exact in nature @ EW scale

Higgs potential of 2HDM type II

$$
\begin{aligned}
V = {} & m_{11}^2\ H_d^\dagger H_d + m_{22}^2\ H_u^\dagger H_u + \left(m_{12}^2\ H_u \cdot H_d + \text{h.c.}\right) \\
& + \frac{\lambda_1}{2}\left(H_d^\dagger H_d\right)^2 + \frac{\lambda_2}{2}\left(H_u^\dagger H_u\right)^2 \\
& + \lambda_3\left(H_u^\dagger H_u\right)\left(H_d^\dagger H_d\right) + \lambda_4\left(H_u^\dagger H_d\right)\left(H_d^\dagger H_u\right) + \{\lambda_5, \lambda_6, \lambda_7\}
\end{aligned}
$$

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$$m_{11}^2 = |\mu|^2 + m_{H_d}^2, \qquad \lambda_1 = \lambda_2 = -\lambda_3 = \frac{g_1^2 + g_2^2}{4},$$

$$m_{22}^2 = |\mu|^2 + m_{H_u}^2, \qquad \lambda_4 = \frac{g^2}{2},$$

$$m_{12}^2 = B_\mu, \qquad \lambda_5 = \lambda_6 = \lambda_7 = 0.$$

The desired vacuum of the MSSM

Desired = constructed

$$V = (m_{H_u}^2 + |\mu|^2)|H_u|^2 + (m_{H_d}^2 + |\mu|^2)|H_d|^2 - 2\,\mathrm{Re}\,(B_\mu H_u \cdot H_d)$$
$$+ \frac{g_1^2 + g_2^2}{8} \left(|H_u|^2 - |H_d|^2\right)^2 + \frac{g_2^2}{2}|H_d^\dagger H_u|^2$$

with

$$\langle H_u \rangle = \begin{pmatrix} 0 \\ v_u \end{pmatrix}, \langle H_d \rangle = \begin{pmatrix} v_d \\ 0 \end{pmatrix},$$

and $v_u^2 + v_d^2 = (174\,\mathrm{GeV})^2$, $v_u/v_d = \tan\beta$.

How to?

$$\left.\frac{\partial V}{\partial h_u^0}\right|_{h_{u,d}\to v_{u,d}} = 2(\boldsymbol{m_{H_u}^2} + |\mu|^2)v_u - 2\,\mathrm{Re}\,B_\mu v_d + \tfrac{g_1^2+g_2^2}{4}(v_u^2 - v_d^2)v_u$$
$$\left.\frac{\partial V}{\partial h_d^0}\right|_{h_{u,d}\to v_{u,d}} = 2(\boldsymbol{m_{H_d}^2} + |\mu|^2)v_d - 2\,\mathrm{Re}\,B_\mu v_u - \tfrac{g_1^2+g_2^2}{4}(v_u^2 - v_d^2)v_d$$

The Minimal Supersymmetric Standard Model: A multi-scalar theory

$$V = V_F + V_D + V_{\text{soft}}$$

with

$$V_F = \sum_i \left| \frac{\partial \mathcal{W}}{\partial \phi_i} \right|^2,$$

$$V_D = \frac{1}{2} \sum_a g_a^2 \left(\sum_i \phi_i^\dagger T^a \phi_i \right)^2$$

$$V_{\text{soft}} = \sum_i m_{\phi_i}^2 |\phi_i|^2 + \sum_{ijk} A_{ik}^{(j)} \phi_i^\dagger \phi_j \phi_k$$

$$\leftrightarrow$$

The Standard Model: A single scalar theory

$$V_{\text{SM}} = -\mu^2 H^\dagger H + \frac{\lambda}{4} \left(H^\dagger H \right)^2$$

The field content of the MSSM...

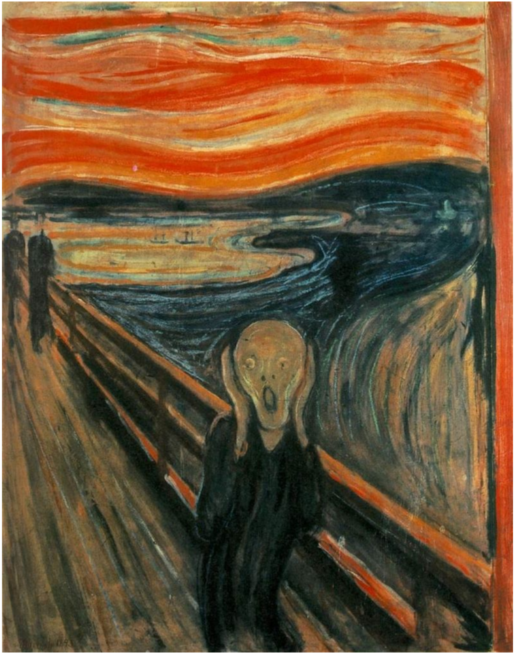
$$\mathcal{W} = \mu\, H_d \cdot H_u + Y^u_{ij}\, H_u \cdot Q_i \bar{T}_j - Y^d_{ij}\, H_d \cdot Q_i \bar{B}_j - Y^\ell_{ij}\, H_d \cdot L_i \bar{E}_j,$$

$i, j = 1, \ldots, 3$ generations; $H_d \cdot H_u = h_d^- h_u^+ - h_d^0 h_u^0$.

Soft SUSY breaking terms for the scalars...

$$\begin{aligned}\mathcal{L}_{\text{soft}} = {} & m^2_{H_d}\, |h_d|^2 + m^2_{H_u}\, |h_u|^2 \\ & + \left(\tilde{m}^2_Q\right)_{ij}\, \tilde{Q}^*_i \tilde{Q}_j + \left(\tilde{m}^2_u\right)_{ij}\, \tilde{u}^*_i \tilde{u}_j + \left(\tilde{m}^2_d\right)_{ij}\, \tilde{d}^*_i \tilde{d}_j \\ & + \left(\tilde{m}^2_L\right)_{ij}\, \tilde{L}^*_i \tilde{L}_j + \left(\tilde{m}^2_e\right)_{ij}\, \tilde{e}^*_i \tilde{e}_j \\ & + A^u_{ij}\, h_u \cdot \tilde{Q}\, \tilde{u}^* - A^d_{ij}\, h_d \cdot \tilde{Q}\, \tilde{d}^* \\ & - A^\ell_{ij}\, h_d \cdot \tilde{Q}\, \tilde{e}^*\end{aligned}$$

- about 100 new parameters
- in the scalar sector only!



[commons.wikimedia.org]

A multi-scalar theory

- 2 Higgs doublets
- 2×6 scalar quarks, $6 + 3$ scalar leptons
- 12 colored and 18 + 2 charged directions
- charged Higgs directions “safe” [Casas et al. 1996]
- SM Higgs potential: $\mathrm{SO}(4)$ symmetry

The hazard

- impossible to minimize directly, analytically
- colored directions sensitive to all kinds of SUSY breaking
- spontaneous breaking of color charge: $\langle \tilde{q} \rangle \neq 0$

The true vacuum

- effective potential: average energy density
- global minimum: true ground state of the theory

The third generation MSSM

$$\mathcal{W} = \mu\, H_d \cdot H_u + y_t\, H_u \cdot Q\bar{T} - y_b\, H_d \cdot Q\bar{B}$$

- large couplings to Higgs doublets (y_t and y_b comparably large)
- large stop contribution (X_t, A_t) to light Higgs mass needed
- $\tan\beta$ resummation for m_b influences y_b

Properties of the (effective) scalar potential

- no UFB directions (due to quantum corrections)
- D-terms: (comparably) large contributions ϕ^4
- "dangerous" directions: small quadrilinears + large trilinears

Analytic constraints

- define certain directions in field space: great simplification
- e.g. D-terms absent: $|\tilde{Q}_L| = |\tilde{t}_R| = |h_2|$ (possibly miss sth.)

$$
\begin{aligned}
V_{\tilde{q},h} = {} & \tilde{t}_L{}^* \left(\tilde{m}_L^2 + |y_t h_u|^2\right) \tilde{t}_L + \tilde{t}_R{}^* \left(\tilde{m}_t^2 + |y_t h_u|^2\right) \tilde{t}_R \\
& + \tilde{b}_L{}^* \left(\tilde{m}_L^2 + |y_b h_d|^2\right) \tilde{b}_L + \tilde{b}_R{}^* \left(\tilde{m}_b^2 + |y_b h_d|^2\right) \tilde{b}_R \\
& - \left[\tilde{t}_L{}^* \left(\mu^* y_t\, h_d{}^* - A_t h_u\right) \tilde{t}_R + \text{h.c.}\right] \\
& - \left[\tilde{b}_L{}^* \left(\mu^* y_b\, h_u{}^* - A_b h_d\right) \tilde{b}_R + \text{h.c.}\right] \\
& + |y_t|^2 |\tilde{t}_L|^2 |\tilde{t}_R|^2 + |y_b|^2 |\tilde{b}_L|^2 |\tilde{b}_R|^2 \\
& + \frac{g_1^2}{8} \left(|h_u|^2 - |h_d|^2 + \frac{1}{3}|\tilde{b}_L|^2 + \frac{2}{3}|\tilde{b}_R|^2 + \frac{1}{3}|\tilde{t}_L|^2 - \frac{4}{3}|\tilde{t}_R|^2\right)^2 \\
& + \frac{g_2^2}{8} \left(|h_u|^2 - |h_d|^2 + |\tilde{b}_L|^2 - |\tilde{t}_L|^2\right)^2 \\
& + \frac{g_3^2}{8} \left(|\tilde{t}_L|^2 - |\tilde{t}_R|^2 + |\tilde{b}_L|^2 - |\tilde{b}_R|^2\right)^2 \\
& + (m_{h_u}^2 + |\mu|^2)|h_u|^2 + (m_{h_d}^2 + |\mu|^2)|h_d|^2 - 2\,\text{Re}(B_\mu\, h_d h_u).
\end{aligned}
$$

The tree-level scalar potential

$$
\begin{aligned}
V_{\tilde{q},h} = {} & \tilde{t}_L{}^* \left(\tilde{m}_L^2 + |y_t h_u|^2\right) \tilde{t}_L + \tilde{t}_R{}^* \left(\tilde{m}_t^2 + |y_t h_u|^2\right) \tilde{t}_R \\
& + \tilde{b}_L{}^* \left(\tilde{m}_L^2 + |y_b h_d|^2\right) \tilde{b}_L + \tilde{b}_R{}^* \left(\tilde{m}_b^2 + |y_b h_d|^2\right) \tilde{b}_R \\
& - \left[\tilde{t}_L{}^* \left(\mu^* y_t \, h_d{}^* - A_t h_u\right) \tilde{t}_R + \text{h.c.}\right] \\
& - \left[\tilde{b}_L{}^* \left(\mu^* y_b \, h_u{}^* - A_b h_d\right) \tilde{b}_R + \text{h.c.}\right] \\
& + |y_t|^2 |\tilde{t}_L|^2 |\tilde{t}_R|^2 + |y_b|^2 |\tilde{b}_L|^2 |\tilde{b}_R|^2 \\
& + \frac{g_1^2}{8} \left(|h_u|^2 - |h_d|^2 + \frac{1}{3}|\tilde{b}_L|^2 + \frac{2}{3}|\tilde{b}_R|^2 + \frac{1}{3}|\tilde{t}_L|^2 - \frac{4}{3}|\tilde{t}_R|^2\right)^2 \\
& + \frac{g_2^2}{8} \left(|h_u|^2 - |h_d|^2 + |\tilde{b}_L|^2 - |\tilde{t}_L|^2\right)^2 \\
& + \frac{g_3^2}{8} \left(|\tilde{t}_L|^2 - |\tilde{t}_R|^2 + |\tilde{b}_L|^2 - |\tilde{b}_R|^2\right)^2 \\
& + (m_{h_u}^2 + |\mu|^2)|h_u|^2 + (m_{h_d}^2 + |\mu|^2)|h_d|^2 - 2\,\mathrm{Re}(B_\mu \, h_d h_u).
\end{aligned}
$$

$|\tilde{t}_L| = |\tilde{t}_R| = |\tilde{t}|$, $|\tilde{b}_L| = |\tilde{b}_R| = |\tilde{b}|$

$$
\begin{aligned}
V_{\tilde{q},h} = \; & \tilde{t}^{\,*}\left(\tilde{m}_L^2 + |y_t h_u|^2\right)\tilde{t} \;+ \tilde{t}^{\,*}\left(\tilde{m}_t^2 + |y_t h_u|^2\right)\tilde{t} \\
& + \tilde{b}^{\,*}\left(\tilde{m}_L^2 + |y_b h_d|^2\right)\tilde{b} \;+ \tilde{b}^{\,*}\left(\tilde{m}_b^2 + |y_b h_d|^2\right)\tilde{b} \\
& - \left[\tilde{t}^{\,*}\left(\mu^* y_t\, h_d{}^* - A_t h_u\right)\tilde{t} \;+ \text{h.c.}\right] \\
& - \left[\tilde{b}^{\,*}\left(\mu^* y_b\, h_u{}^* - A_b h_d\right)\tilde{b} \;+ \text{h.c.}\right] \\
& + |y_t|^2|\tilde{t}|^2|\tilde{t}|^2 + |y_b|^2|\tilde{b}|^2|\tilde{b}|^2 \\
& + \frac{g_1^2}{8}\left(|h_u|^2 - |h_d|^2 + |\tilde{b}|^2 - |\tilde{t}|^2\right)^2 \\
& + \frac{g_2^2}{8}\left(|h_u|^2 - |h_d|^2 + |\tilde{b}|^2 - |\tilde{t}|^2\right)^2 \\
& + (m_{h_u}^2 + |\mu|^2)|h_u|^2 + (m_{h_d}^2 + |\mu|^2)|h_d|^2 - 2\,\mathrm{Re}(B_\mu\, h_d h_u).
\end{aligned}
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$|\tilde{t}_L| = |\tilde{t}_R| = |\tilde{t}|$, $|\tilde{b}_L| = |\tilde{b}_R| = |\tilde{b}|$

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\begin{aligned}
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& + \tilde{b}\ ^* \left(\tilde{m}_L^2 + |y_b h_d|^2\right) \tilde{b}\ \ + \tilde{b}\ ^* \left(\tilde{m}_b^2 + |y_b h_d|^2\right) \tilde{b} \\
& - \left[\tilde{t}\ ^* (\mu^* y_t\ h_d{}^* - A_t h_u)\, \tilde{t}\ \ + \mathrm{h.c.}\right] \\
& - \left[\tilde{b}\ ^* (\mu^* y_b\ h_u{}^* - A_b h_d)\, \tilde{b}\ \ + \mathrm{h.c.}\right] \\
& + |y_t|^2 |\tilde{t}\ |^2 |\tilde{t}\ |^2 + |y_b|^2 |\tilde{b}\ |^2 |\tilde{b}\ |^2 \\
& + \frac{g_1^2}{8} \left(|h_u|^2 - |h_d|^2 + \ |\tilde{b}\ |^2 \qquad - \ |\tilde{t}\ |^2 \qquad \right)^2 \\
& + \frac{g_2^2}{8} \left(|h_u|^2 - |h_d|^2 + |\tilde{b}\ |^2 - |\tilde{t}\ |^2\right)^2 \\
& \\
& + (m_{h_u}^2 + |\mu|^2)|h_u|^2 + (m_{h_d}^2 + |\mu|^2)|h_d|^2 - 2\,\mathrm{Re}(B_\mu\ h_d h_u).
\end{aligned}
$$

$|\tilde{t}_L| = |\tilde{t}_R| = |\tilde{t}|,\ |\tilde{b}_L| = |\tilde{b}_R| = |\tilde{b}|\ ;\ |\tilde{b}| = |h_d| = |\phi_1|,\ |\tilde{t}| = |h_u| = |\phi_2|$

$$
\begin{aligned}
V_{\tilde{q},h} = {} & \phi_2{}^* \left(\tilde{m}_L^2 + |y_t \phi_2|^2\right) \phi_2 + \phi_2{}^* \left(\tilde{m}_t^2 + |y_t \phi_2|^2\right) \phi_2 \\
& + \phi_1{}^* \left(\tilde{m}_L^2 + |y_b \phi_1|^2\right) \phi_1 + \phi_1{}^* \left(\tilde{m}_b^2 + |y_b \phi_1|^2\right) \phi_1 \\
& - \left[\phi_2{}^* \left(\mu^* y_t \ \phi_1{}^* - A_t \phi_2\right) \phi_2 + \text{h.c.}\right] \\
& - \left[\phi_1{}^* \left(\mu^* y_b \ \phi_2{}^* - A_b \phi_1\right) \phi_1 + \text{h.c.}\right] \\
& + |y_t|^2 |\phi_2|^2 |\phi_2|^2 + |y_b|^2 |\phi_1|^2 |\phi_1|^2
\end{aligned}
$$

$$
+ (m_{h_u}^2 + |\mu|^2)|\phi_2|^2 + (m_{h_d}^2 + |\mu|^2)|\phi_1|^2 - 2\,\text{Re}(B_\mu \ \phi_1 \phi_2).
$$

$|\tilde{t}_L| = |\tilde{t}_R| = |\tilde{t}|$, $|\tilde{b}_L| = |\tilde{b}_R| = |\tilde{b}|$; $|\tilde{b}| = |h_d| = |\phi_1|$, $|\tilde{t}| = |h_u| = |\phi_2|$

$$V_{\tilde{q},h} = \phi_2{}^* \left(\tilde{m}_L^2 + |y_t \phi_2|^2\right) \phi_2 + \phi_2{}^* \left(\tilde{m}_t^2 + |y_t \phi_2|^2\right) \phi_2$$

$$- \left[\phi_2{}^* \left(\qquad\qquad - A_t \phi_2\right) \phi_2 + \text{h.c.}\right]$$

$$+ |y_t|^2 |\phi_2|^2 |\phi_2|^2$$

$$+ (m_{h_u}^2 + |\mu|^2)|\phi_2|^2 \qquad\qquad\qquad\qquad .$$

$|\tilde{t}_L| = |\tilde{t}_R| = |\tilde{t}|$, $|\tilde{b}_L| = |\tilde{b}_R| = |\tilde{b}|$; $\cancel{|\tilde{b}| = |h_d| = |\phi_1|}$, $|\tilde{t}| = |h_u| = |\phi_2|$

Minimize the potential

$$V(\phi) = m^2\phi^2 - A\phi^3 + \lambda\phi^4,$$

with $m^2 = m_{h_2}^2 + |\mu|^2 + \tilde{m}_L^2 + \tilde{m}_t^2$, $A = -A_t$ and $\lambda = 3y_t^2$.

Mathematics for the kindergarden

Minimize the potential

$$V(\phi) = m^2\phi^2 - A\phi^3 + \lambda\phi^4,$$

with $m^2 = m_{h_2}^2 + |\mu|^2 + \tilde{m}_L^2 + \tilde{m}_t^2$, $A = -A_t$ and $\lambda = 3y_t^2$.

Answer:

$$\phi_0 = 0, \qquad \phi_\pm = \frac{3A \pm \sqrt{9A^2 32\lambda\, m^2}}{8\lambda}.$$

Condition to be safe from non-standard (i.e. non-trivial) minima:

$$V(\phi_\pm) > 0 \quad \hookrightarrow \quad m^2 > \frac{A^2}{4\lambda}$$

Minimize the potential

$$V(\phi) = m^2\phi^2 - A\phi^3 + \lambda\phi^4,$$

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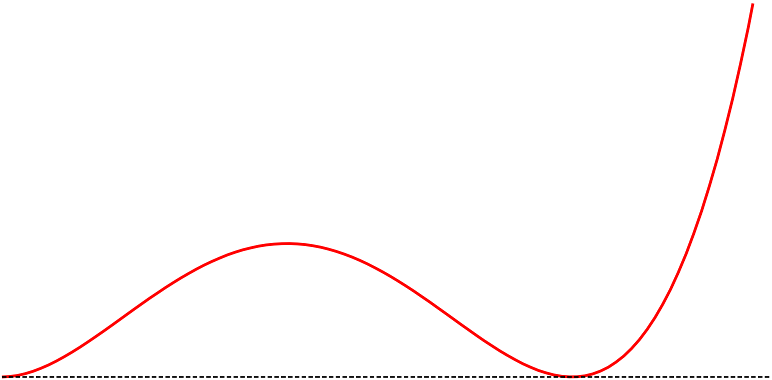
$$V(\phi_\pm) > 0 \quad \hookrightarrow \quad m^2 > \frac{A^2}{4\lambda}$$

Well-known constraints [Gunion, Haber, Sher '88]

$$|A_t|^2 < 3y_t^2\left(m_{h_2}^2 + |\mu|^2 + \tilde{m}_L^2 + \tilde{m}_t^2\right)$$
$$|A_b|^2 < 3y_b^2\left(m_{h_1}^2 + |\mu|^2 + \tilde{m}_L^2 + \tilde{m}_b^2\right)$$

for the limiting cases $|\tilde{t}_L| = |\tilde{t}_R| = |h_2|$ and $|\tilde{b}_L| = |\tilde{b}_R| = |h_1|$!

$$A^2 = 4\lambda m^2$$



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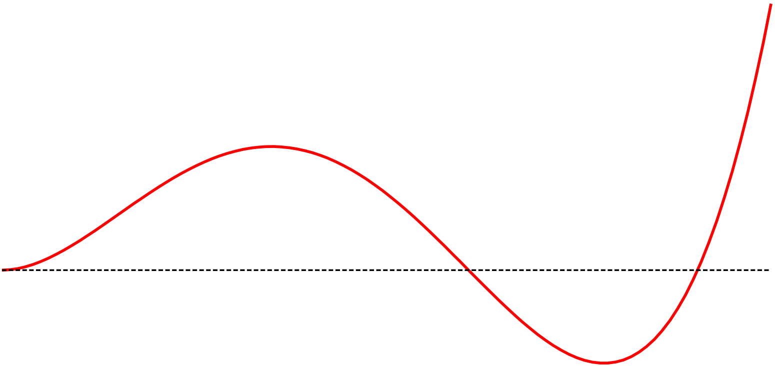

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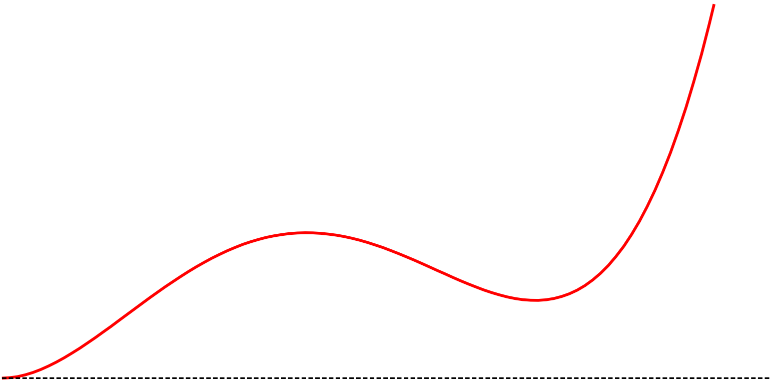

Problem already known for a while

- problem noticed [Frere, Jones, Raby '83]
- "A-parameter bounds" [Gunion, Haber, Sher '87]
- classification of all dangerous directions [Casas, Lleyda, Muñoz '96]
- including flavor violation [Casas and Dimopuolos '96]

Stability $\neq$ no Instability $\Rightarrow$ Metastability

Vacuum tunneling [Kusenko, Langacker '96; Blinov, Morissey '13]

The tool

VeVacious [Camargo-Molina, O'Leary, Porod, Staub '13]

- finds all (?) tree-level minima
- minimizes scalar potential in the vicinity at one loop
- calculates bounce action / tunneling times [CosmoTransitions]

What has changed since the mid 90s?

1. We have discovered the Higgs!
2. No sign of SUSY so far...
3. $\boldsymbol{m_h = 125\,\mathrm{GeV}}$
   (all SUSY literature during LEP era expected it to be $\lesssim 100\,\mathrm{GeV}$)
4. Consequently: large radiative corrections!
5. large stop mixing needed? heavy SUSY spectrum?
   (or hidden in some hardly accessible valley)
6. approach today:
   - less focused on unified models
   - still certain scenarios
   - $\tan\beta$ resummation for bottom quark mass (large $\tan\beta$)
   - low $\tan\beta$ favored (for $M_A \lesssim 800\,\mathrm{GeV}$, direct search $A \to \tau\tau$)

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Semi-analytical bounds/exclusions important for fast processing!

My "pMSSM"

- no unification (more than m_0, $m_{1/2}$, A_0, $\tan\beta$ and $\mathrm{sign}\,\mu$)
- free parameters: (although similar choice as in CMSSM)
  - $\tilde{m}_L^2 = \tilde{m}_t^2 = \tilde{m}_b^2 = M_{\mathsf{SUSY}}^2$
  - μ, $\tan\beta$
  - A_t, A_b (not necessarily equal)
- $m_{h_{1,2}}^2$ determined from ew breaking, B_μ related to M_A
- no RG running needed = parameters taken at the SUSY scale

Why no RG-improvement?

- SUSY scale parameters; limits on this parameters
- destabilization of ew vacuum *around* SUSY scale
- no Planck scale vevs! (maybe there are... in addition)
- in the spirit of the pMSSM as phenomenologically as possible
- only small RG modifications, qualitative features unchanged

Less constraints, more parameters, more fields, more vevs...

Restriction to certain directions too restrictive!

- give up $|\tilde{q}_L| = |\tilde{t}_R| = |h_2|$
- allow for $h_1 \neq 0$ and $\tilde{b} \neq 0$
- "break" $\tilde{q}_L \to \tilde{t}_L + \tilde{b}_L$
- back to full scalar potential!

Simplify your life

- $h_2 = \phi$
- $|\tilde{t}_L| = |\tilde{t}_R| = |\tilde{t}| = \alpha|\phi|$
- $h_1 = \eta\phi$
- $|\tilde{b}_L| = |\tilde{b}_R| = |\tilde{b}| = \beta|\phi|$
- all fields and parameters real, $\alpha, \beta > 0$, $\eta \in \mathbb{R}$
- $\mathrm{SU}(3)_c$-flatness: $\tilde{t}_L = \tilde{t}_R$ and $\tilde{b}_L = \tilde{b}_R$

$$h_2 = \phi, \qquad |\tilde{t}| = \alpha|\phi|, \qquad h_1 = \eta\phi, \qquad |\tilde{b}| = \beta|\phi|$$

$$\begin{aligned}
V_\phi =& \Big(m_{h_2}^2 + \eta^2 m_{h_1}^2 + (1+\eta^2)\mu^2 - 2B_\mu\eta \\
&\qquad + (\alpha^2+\beta^2)\tilde{m}_L^2 + \alpha^2\tilde{m}_t^2 + \beta^2\tilde{m}_b^2\Big)\phi^2 \\
&- 2\left(\alpha^2(\mu y_t\eta - A_t) + \beta^2(\mu y_t - \eta A_b)\right)\phi^3 + (\alpha^2 y_t^2 + \beta^4 y_b^2)\phi^4 \\
&+ \left(\frac{g_1^2+g_2^2}{8}(1-\eta^2+\beta^2-\alpha^2)^2 + 2\alpha^2 y_t^2 + 2\beta^2 y_b^2\right)\phi^4 \\
\equiv& M^2(\eta,\alpha,\beta)\phi^2 - \mathcal{A}(\eta,\alpha,\beta)\phi^3 + \lambda(\eta,\alpha,\beta)\phi^4,
\end{aligned}$$

with

$$\begin{aligned}
M^2 =& \; m_{h_2}^2 + \eta^2 m_{h_1}^2 + (1+\eta^2)\mu^2 - 2B_\mu\eta \\
&+ (\alpha^2+\beta^2)\tilde{m}_L^2 + \alpha^2\tilde{m}_t^2 + \beta^2\tilde{m}_b^2\,, \\
\mathcal{A} =& \; 2\alpha^2\eta\mu y_t - 2\alpha^2 A_t + 2\beta^2\mu y_b - 2\eta\beta^2 A_b\,, \\
\lambda =& \; \frac{g_1^2+g_2^2}{8}(1-\eta^2+\beta^2-\alpha^2)^2 \\
&+ (2+\alpha^2)\alpha^2 y_t^2 + (2\eta^2+\beta^2)\beta^2 y_b^2\,.
\end{aligned}$$

[Gunion, Haber, Sher '88; Casas, Lleyda, Muñoz '96]

The same but different ("A-parameter bounds")

$$\mathcal{A}^2 < 4\lambda M^2$$

$$\downarrow$$

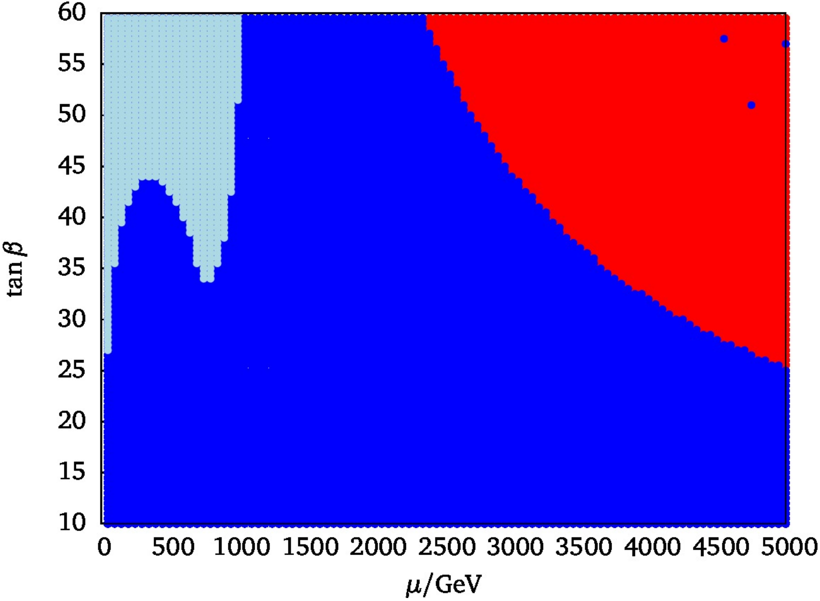
$$4 \min_{\{\eta,\alpha,\beta\}} \lambda(\eta,\alpha,\beta) M^2(\eta,\alpha,\beta) > \max_{\{\eta,\alpha,\beta\}} \left(\mathcal{A}(\eta,\alpha,\beta)\right)^2$$

$h_u = \tilde{b}$, $h_d^0 = 0$ [WGH'15]

$$m_{H_u}^2 + \mu^2 + \tilde{m}_Q^2 + \tilde{m}_b^2 > \frac{(\mu y_b)^2}{y_b^2 + (g_1^2 + g_2^2)/2}$$

$|h_d|^2 = |h_u|^2 + |\tilde{b}|^2$, $\tilde{b} = \alpha h_u$ [WGH'15]

$$m_{11}^2(1+\alpha^2) + m_{22}^2 \pm 2m_{12}^2\sqrt{1+\alpha^2} + \alpha^2(\tilde{m}_Q^2 + \tilde{m}_b^2) > \frac{4\mu^2\alpha^2}{2+3\alpha^2}$$



60
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$\tan\beta$
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3500
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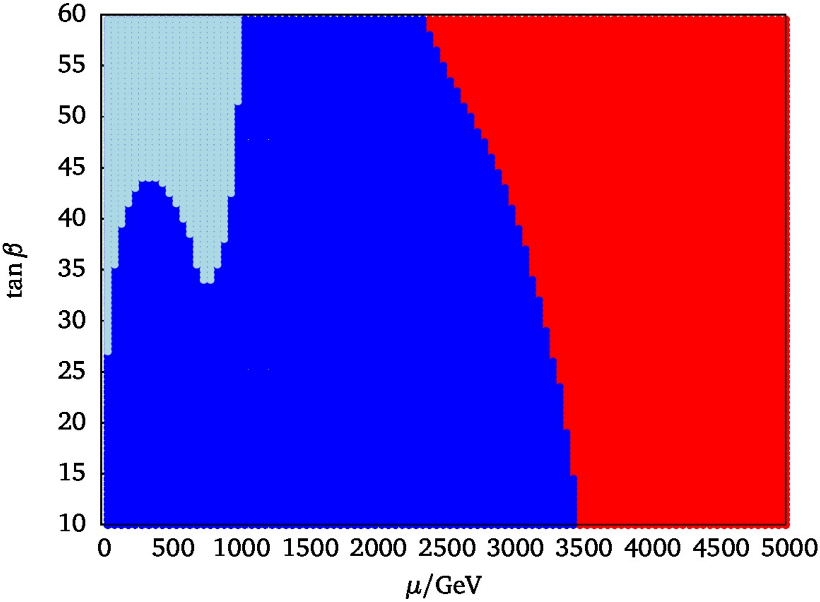
4500
5000
μ/GeV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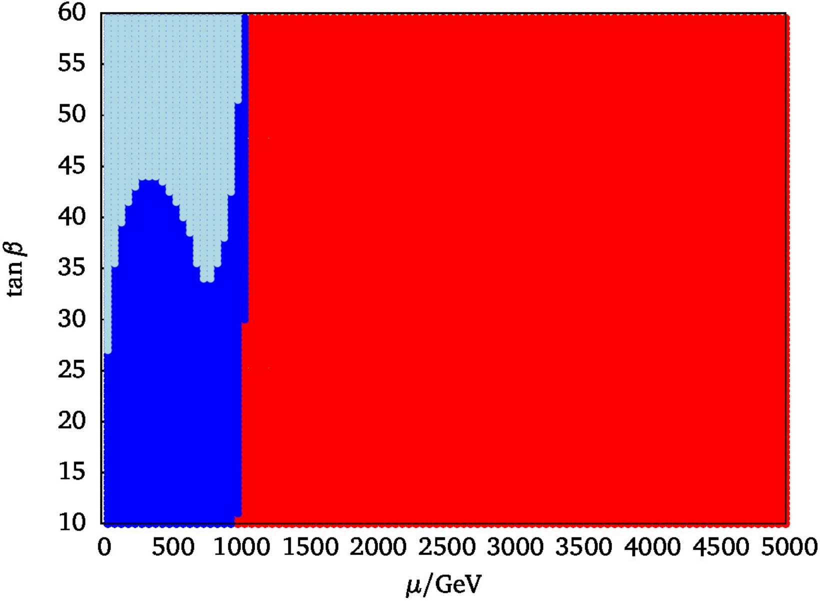
$\tan\beta$
μ/GeV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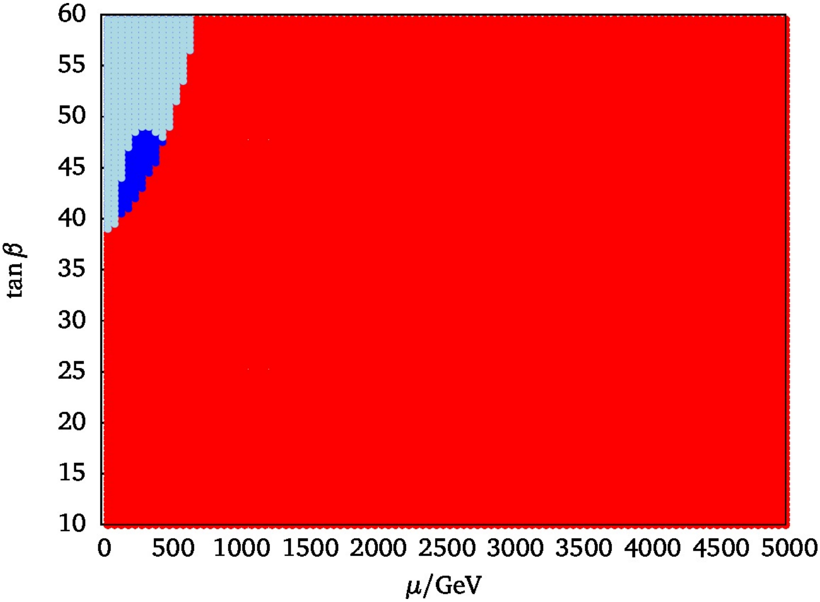
$\tan\beta$
0 500 1000 1500 2000 2500 3000 3500 4000 4500 5000
μ/GeV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20
15
10
$\tan\beta$
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
μ/GeV

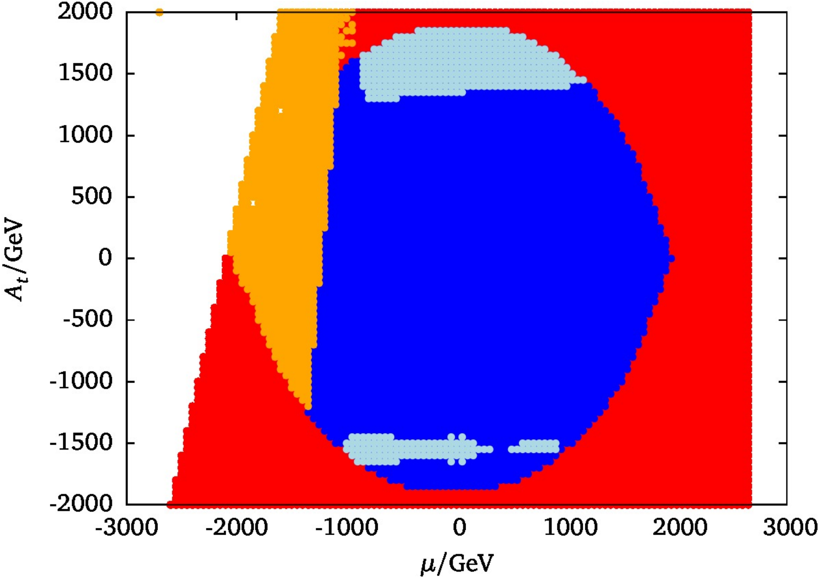
Higgs mass bounded by M_Z

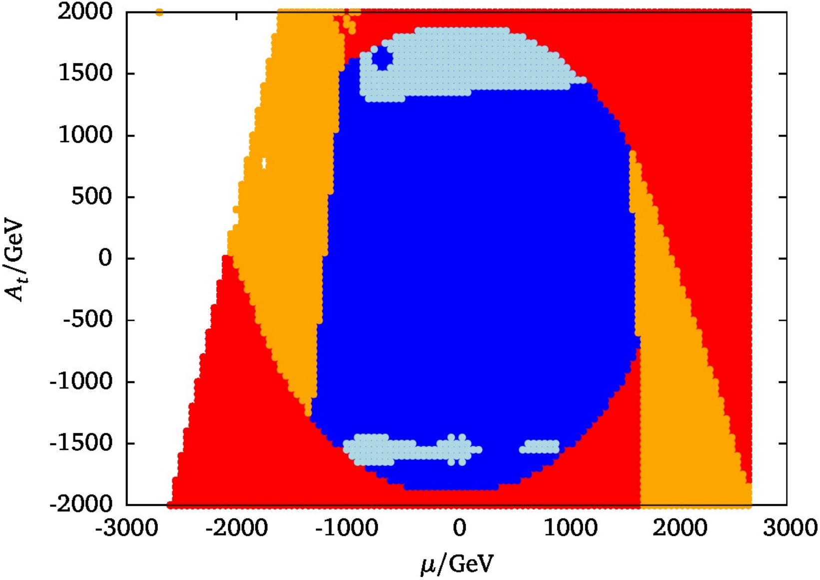
$$m_{h^0} \leq M_Z^2 \cos^2(2\beta)$$

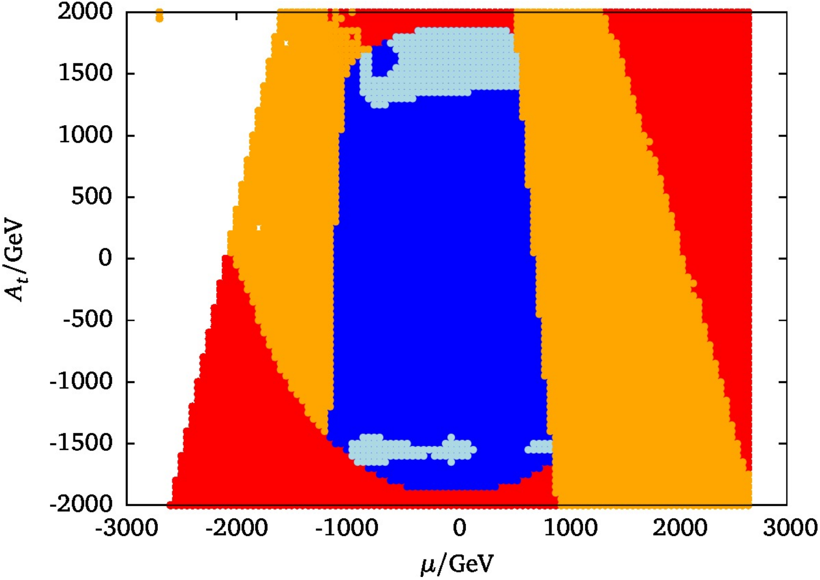
- large loop contribution in the MSSM needed

$$\Delta m_{h^0}^2 = \frac{3v^2}{4\pi^2} y_t^4 \left[\log\left(\frac{M_{\mathsf{SUSY}}^2}{m_t^2}\right) + \frac{X_t^2}{M_{\mathsf{SUSY}}^2}\left(1 - \frac{X_t^2}{12 M_{\mathsf{SUSY}}^2}\right)\right]$$

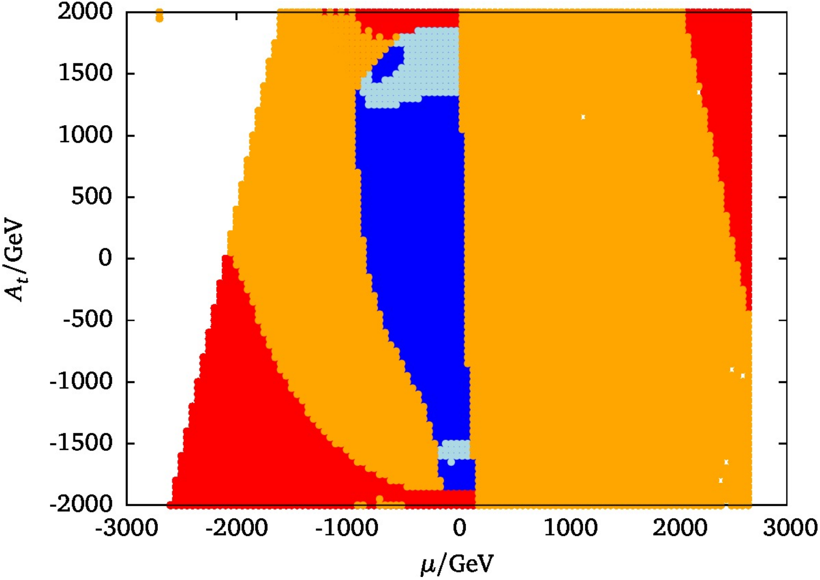
- dominant contribution with y_t, $X_t = (A_t/y_t - \mu \cot\beta)$
- large A_t and/or M_{SUSY} needed


2000
1500
1000
500
0
-500
-1000
-1500
-2000
A_t/GeV
-3000
-2000
-1000
0
1000
2000
3000
μ/GeV

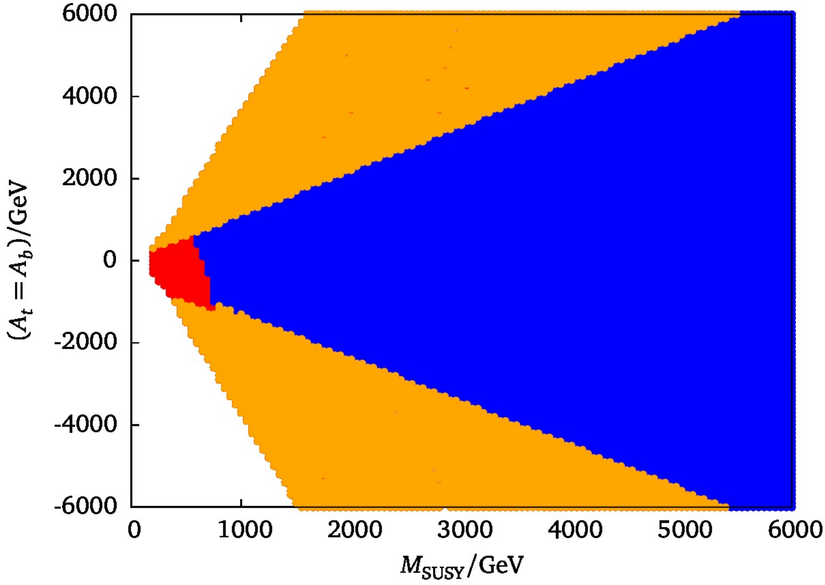

2000
1500
1000
500
0
-500
-1000
-1500
-2000
-3000
-2000
-1000
0
1000
2000
3000
A_t/GeV
μ/GeV


2000
1500
1000
500
0
-500
-1000
-1500
-2000
A_t/GeV
-3000
-2000
-1000
0
1000
2000
3000
μ/GeV

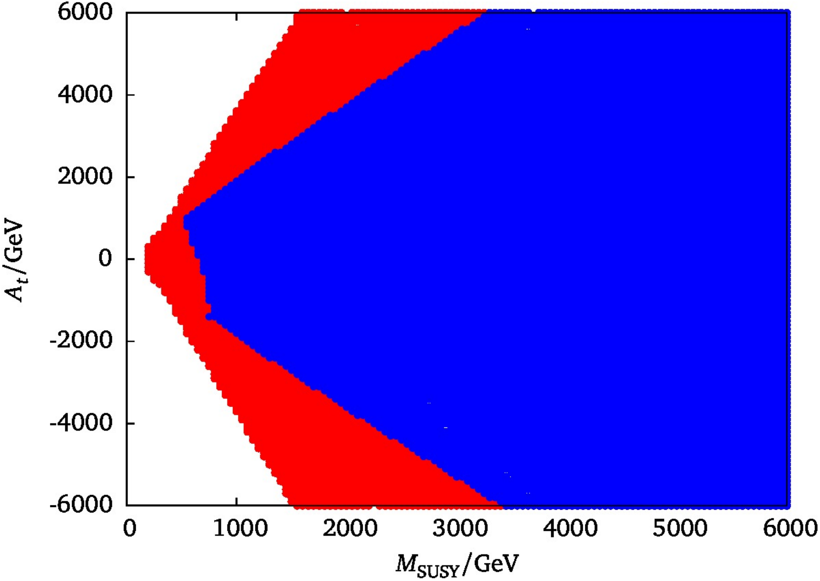
Closing in on the parameter space $A_b = 1500\,\mathrm{GeV}$



Open up parametrs space increasing MSUSY

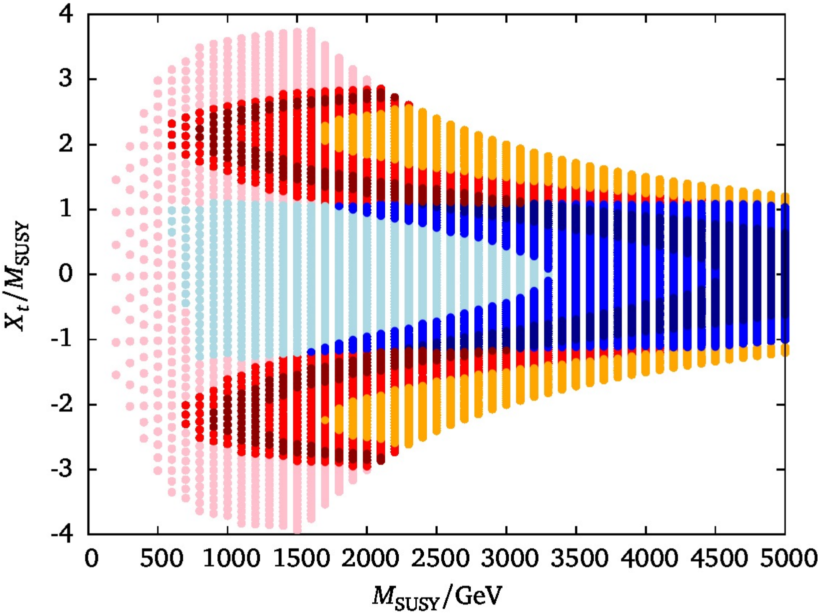


Open up parametrs space increasing MSUSY $A_b = 0$



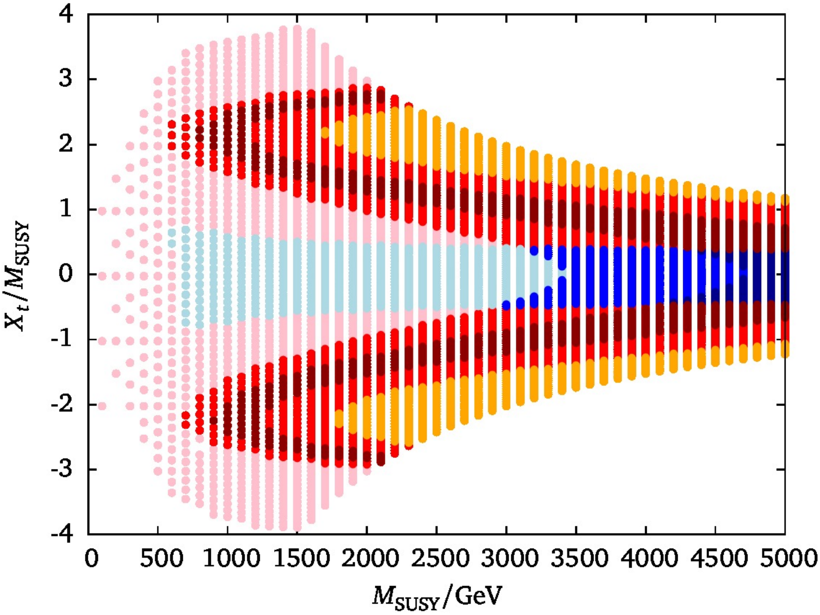
Parameter choice crucial

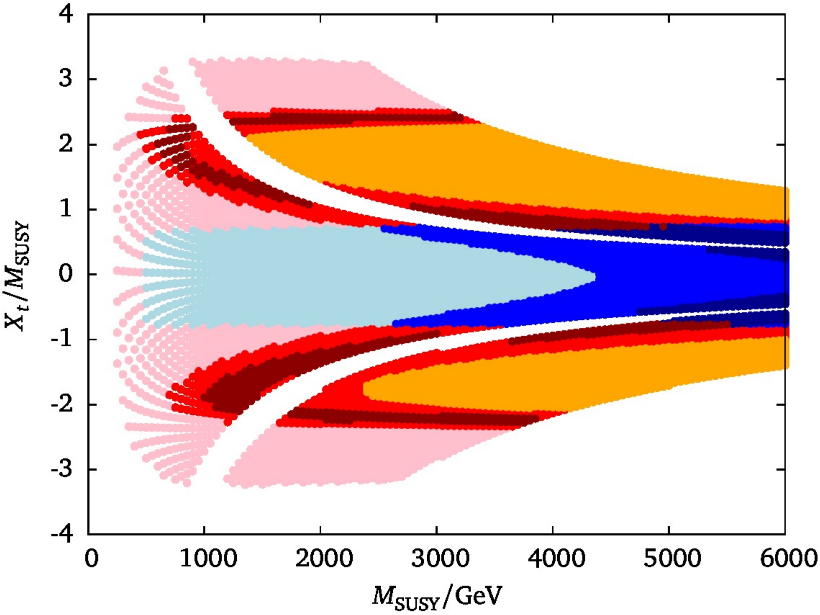
$\mu = 350\,\mathrm{GeV}$



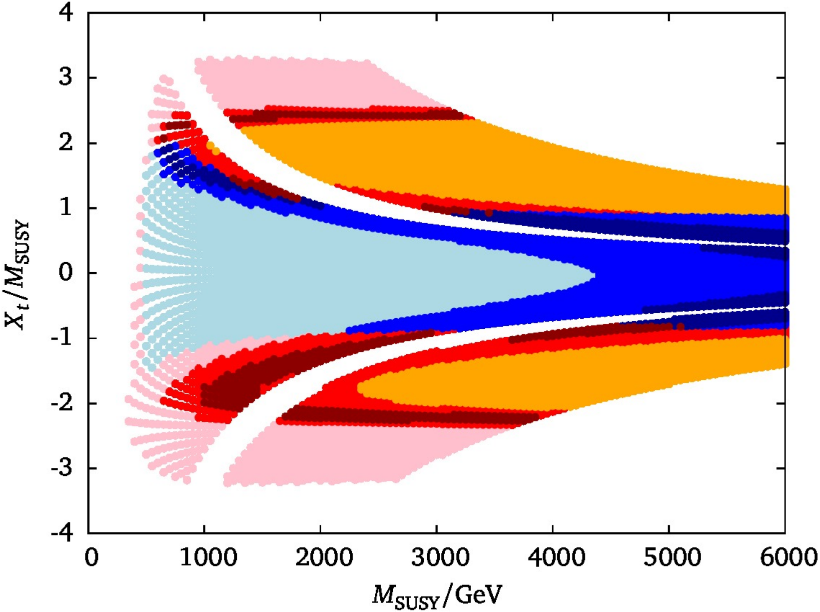
Parameter choice crucial

$\mu = M_{\text{MSUSY}}$




4
3
2
1
0
-1
-2
-3
-4
X_t/M_{SUSY}
0
1000
2000
3000
4000
5000
6000
$M_{\mathrm{SUSY}}/\mathrm{GeV}$

Parameter choice crucial $\mu = -500\,\mathrm{GeV}$



Why next?

- Why not?
- Add a SM singlet superfield.
- Richer phenomenology (Higgs and neutralino sector)

The NMSSM solves the "μ-problem"

$$\mathcal{W}_{\text{MSSM}} = \mu\, H_u \cdot H_d + \text{Yukawa}$$

only dimensionful parameter μ has to be $\sim$ electroweak scale

$$\mathcal{W}_{\text{NMSSM}} \supset \lambda\, S H_u \cdot H_d + \frac{\kappa}{3} S^3$$

dynamical μ-term: $\lambda \langle S \rangle = \mu_{\text{eff}}$

$\mathbb{Z}_3$ symmetry forbids dimensionful couplings (bilinear, tadpole terms)

Trilinear terms in the Higgs sector

$$V_{\text{soft}} = m_{H_u}^2|h_u|^2 + m_{H_d}^2|h_d|^2 + m_S^2|s|^2 + \left(A_\lambda \lambda\, s h_u \cdot h_d + \frac{1}{3}A_\kappa \kappa\, s^3 + \text{h. c.}\right)$$

V_{Higgs} is a *multivariate* polynomial to order 4

$$V_{\text{Higgs}} \subset \{h_u^2, h_d^2, s^2, s h_u h_d, s^2, s^2 h_u h_d, s^2 h_u^2, s^2 h_d^2, h_u^2 h_d^2, h_u^4, h_d^4, s^4\}$$

Not only one minimum! V_{ew} not necessarily the global minimum!

Minimisation conditions are in general misleading!

$$\left.\frac{\partial V}{\partial h_u}\right|_{\text{vev}} = 2m_{H_u}^2 v_u + \dots$$
$$\left.\frac{\partial V}{\partial h_d}\right|_{\text{vev}} = 2m_{H_d}^2 v_d + \dots$$
$$\left.\frac{\partial V}{\partial h_u}\right|_{\text{vev}} = 2m_S^2 v_s + \dots$$

linear equations for soft SUSY breaking masses $m_{H_u}^2$, $m_{H_d}^2$, m_S^2, can be solved uniquely; determine numerical values for those

Simplification: h_u^0, h_d^0, s^0 only (three fields, many vacua), real fields and parameters

$$\begin{aligned} V_{\text{Higgs}} = {} & \boldsymbol{m^2_{H_u}}\ h_u^2 + \boldsymbol{m^2_{H_d}}\ h_d^2 + \boldsymbol{m^2_S}\ s^2 \\ & + \tfrac{2}{3}\kappa A_\kappa\ s^3 + 2\lambda A_\lambda\ s h_u h_d \\ & + \left(\kappa\ s^2 - \lambda\ h_u h_d\right)^2 + \lambda^2\ s^2 \left(h_u^2 + h_d^2\right) \\ & + \tfrac{g_1^2+g_2^2}{8}\left(h_u^2 - h_d^2\right)^2 \end{aligned}$$

However

Solutions for minimization equations with $\langle h_u \rangle \neq v_u$, $\langle h_d \rangle \neq v_d$ and $\langle s \rangle \neq \mu_{\text{eff}}/\lambda$ possible, viable, existing *and* leading to a true vacuum.

Potential value at the minimum to be compared with

$$\begin{aligned} V_{\text{min}}^{\text{des}} = {} & -\tfrac{g_1^2+g_2^2}{8} v^4 \cos^2(2\beta) - \tfrac{\lambda^2}{4} v^4 \sin^2(2\beta) - \tfrac{\kappa^2}{\lambda^4}\mu_{\text{eff}}^4 \\ & - v^2 \mu_{\text{eff}}^2 \left(1 - \tfrac{\kappa^2}{\lambda^2}\sin(2\beta)\right) - \tfrac{1}{3}\frac{\kappa A_\kappa}{\lambda^3}\mu_{\text{eff}}^3 + \tfrac{v^2}{2} A_\lambda \mu_{\text{eff}} \sin(2\beta) \end{aligned}$$

Constraint on A_κ

$$A_\kappa^2 > 9m_S^2$$

$A_\kappa^2 < 8m_S^2$: no $\langle s\rangle \neq 0$. [Derendinger, Savoy '84; Ellwanger et al. '97]

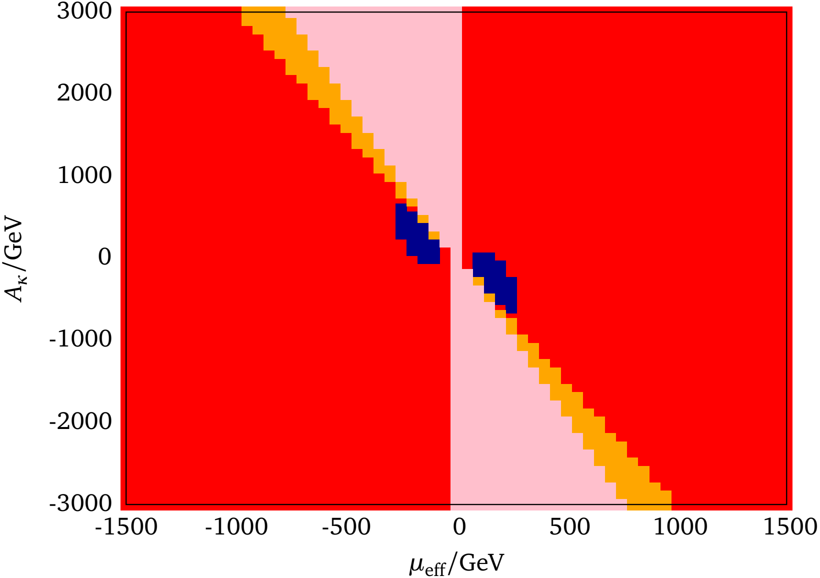
"Tachyonic" Higgs masses

- "problem" of tachyonic masses well known
- one mass eigenvalue of $\mathcal{M}_S^2$, $\mathcal{M}_P^2$ or charged Higgs mass $m_{H^\pm}^2$ negative
- tachyonic mass = negative curvature = alternative vev (!)

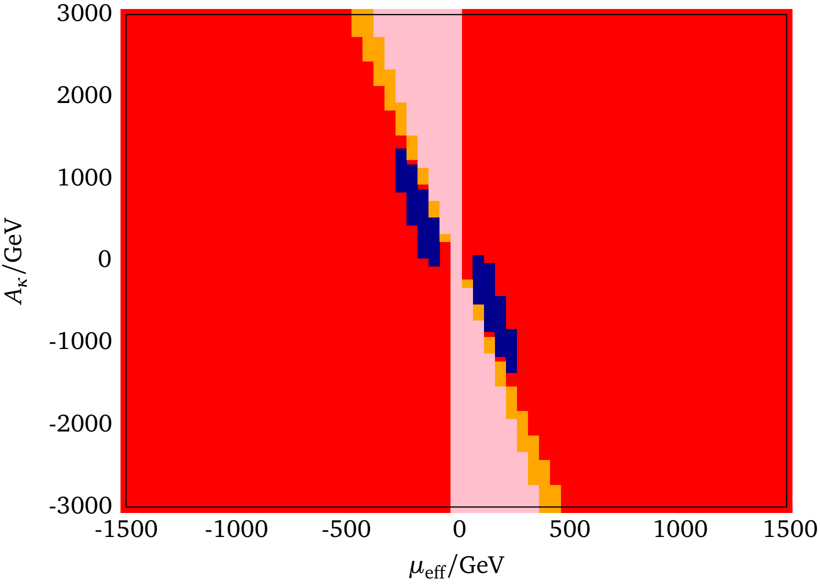
However

Careful analysis shows that tachyonic masses are not enough!

[see e.g. Kanehata, Kobayashi, Konishi, Seto, Shimomura '11]



3000
2000
1000
0
-1000
-2000
-3000
A_κ/GeV
-1500
-1000
-500
0
500
1000
1500
μ_{eff}/GeV



3000
2000
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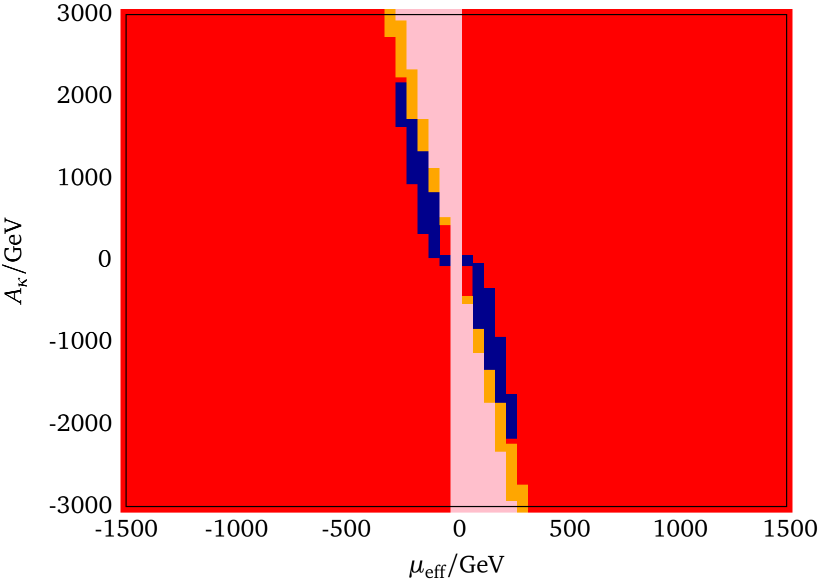
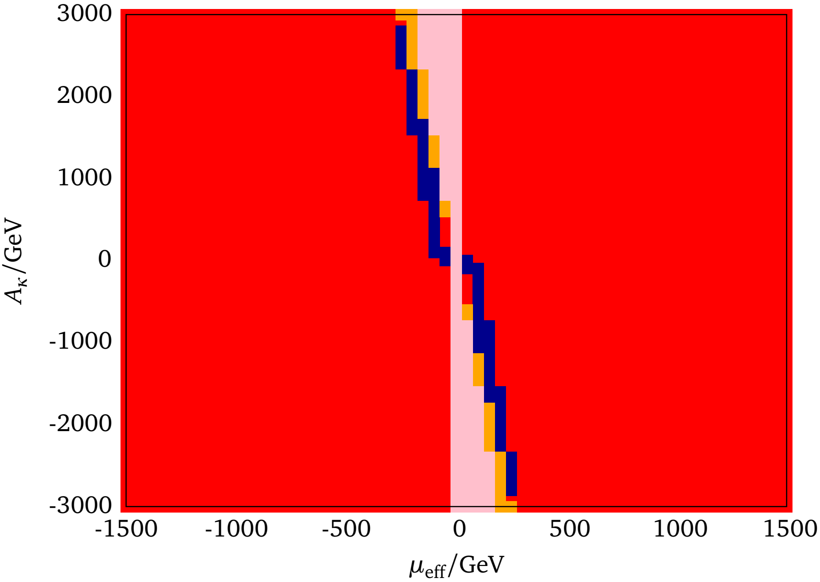
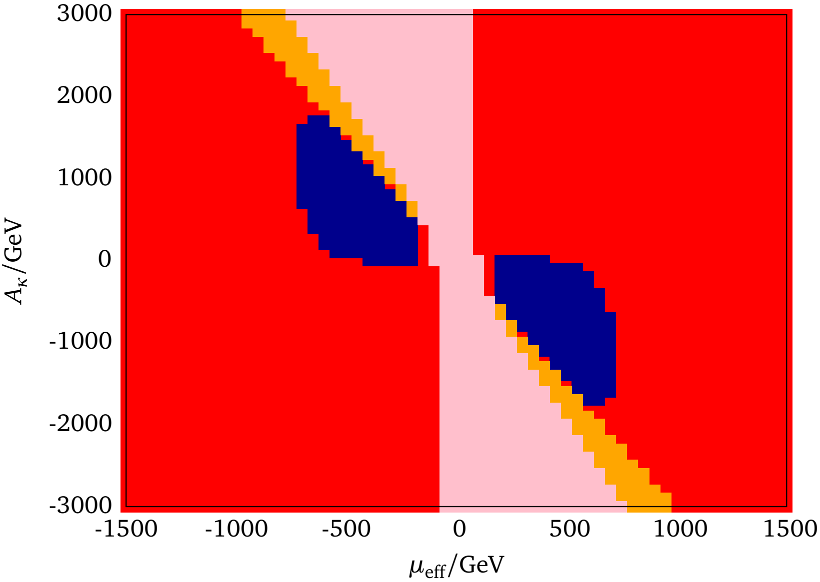
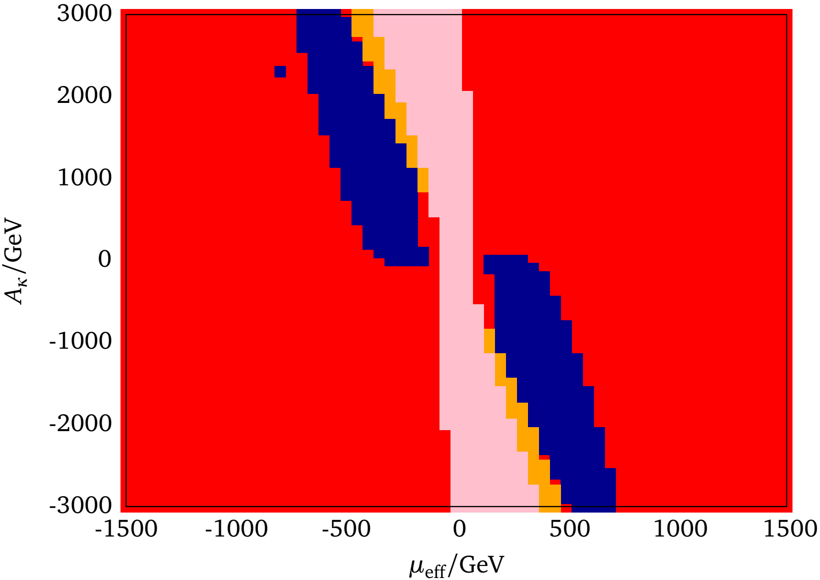
-1000
-2000
-3000
A_κ/GeV
-1500
-1000
-500
0
500
1000
1500
μ_{eff}/GeV


3000
2000
1000
0
-1000
-2000
-3000
A_κ/GeV
-1500
-1000
-500
0
500
1000
1500
$\mu_{\rm eff}$/GeV

Another view: A_κ vs. $\mu_{\rm eff}$ $\lambda = 0.1$ $\kappa = 0.4$, $m_{H^\pm} = 300\,\mathrm{GeV}$




3000
2000
1000
0
-1000
-2000
-3000
A_κ/GeV
-1500
-1000
-500
0
500
1000
1500
$\mu_{\rm eff}/\mathrm{GeV}$


3000
2000
1000
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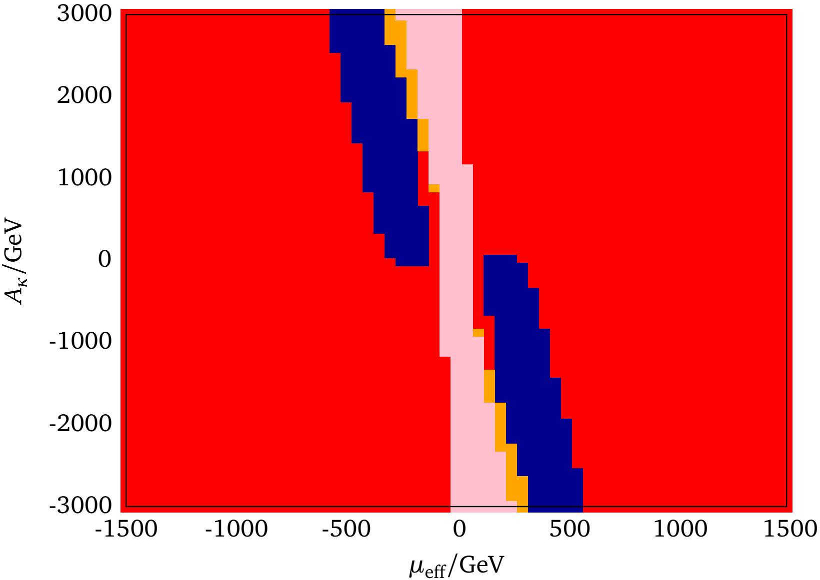
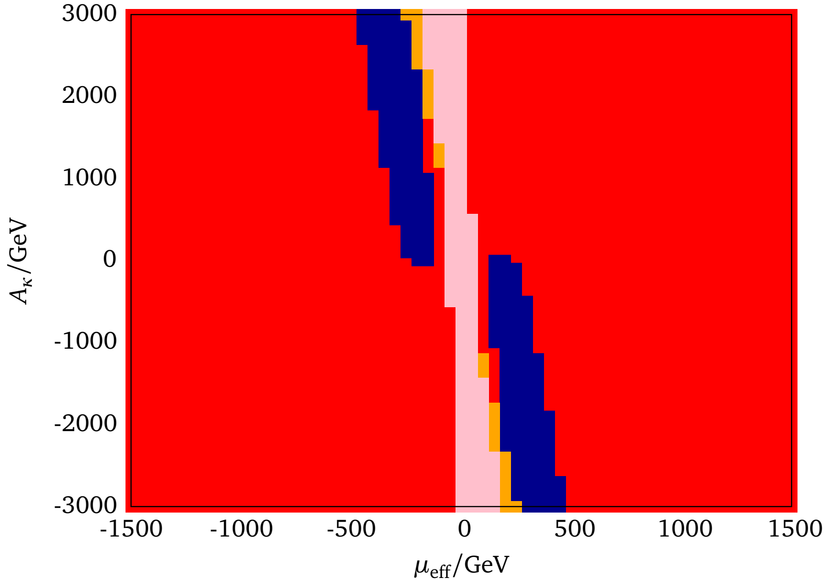
-1000
-2000
-3000
A_κ/GeV
-1500
-1000
-500
0
500
1000
1500
$\mu_{\rm eff}$/GeV



3000
2000
1000
0
-1000
-2000
-3000
A_κ/GeV
-1500
-1000
-500
0
500
1000
1500
μ_{eff}/GeV

Another view: A_κ vs. μ_{eff} $\lambda = 0.1$ $\kappa = 0.4$, $m_{H^\pm} = 800\,\mathrm{GeV}$




Conclusions

The SUSY paradigm and a $125\,\mathrm{GeV}$ Higgs...

- leads to severe constraints in MSSM parameterspace
- theoretical consistency: Higgs vacuum is true vacuum
- vacuum tunneling neglected/ignored

Extending the MSSM

- Next-to-Minimal: add gauge singlet
- changes phenomenology dramatically
- tachyonic Higgses and/or alternative Higgs vevs

The Cosmic Higgs connection [CHICO, canceled :(]

- possibility of incproporating *Higgs inflation* in the (N)MSSM
- Singlet stabilizes inflationary trajectory
- NMSSM + $\mu\, H_d \cdot H_u$, changes phenomenology drastically

local $\mathrm{U}(1)$ $\mathcal{R}$ symmetry

- part of the superconformal $\mathrm{SU}(2,2|1)$
- χ term breaks continous $\mathcal{R}$ *and* discrete $\mathbb{Z}_3$ symmetry
- breaking at dimension 6: $\sim \chi \frac{\lambda^2 h^6}{M_P^2}$

Corrected Superpotential

$$\begin{aligned} \mathcal{W}_{\text{eff}} \to \mathcal{W} e^{X(\Phi)/M_P^2} &= \mathcal{W} + \frac{\langle \mathcal{W}_{\text{hid}} \rangle}{M_P^2} X(\Phi) \\ &\simeq \mathcal{W} + m_{3/2} X(\Phi) \end{aligned}$$

The iNMSSM

$$\mathcal{W}_{\text{eff}} = \lambda\, S H_u \cdot H_d + \frac{\kappa}{3}\, S^3 + \frac{3}{2} \chi m_{3/2} H_u \cdot H_d$$

Cosmo pheno requires $\boldsymbol{\chi \simeq 10^5 \lambda}$

like the NMSSM with an extended effective μ term

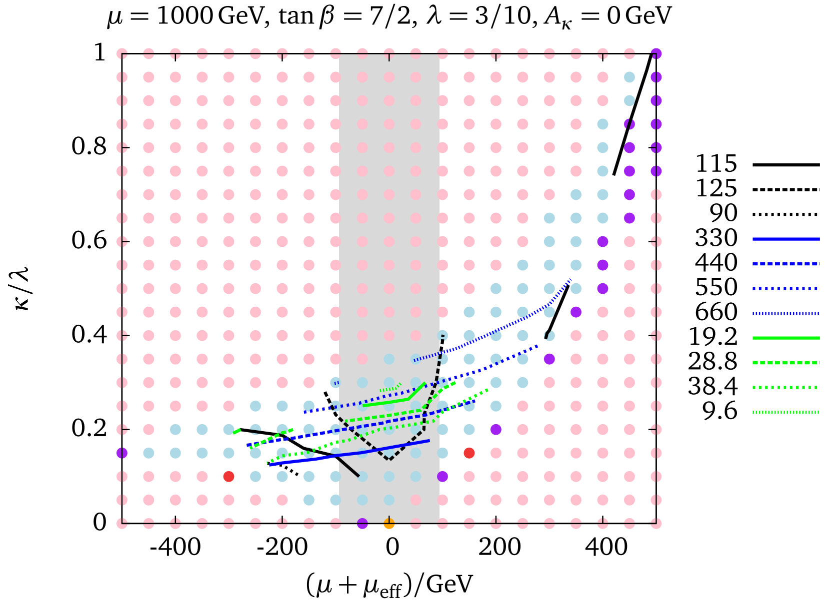
$$\mu'_{\text{eff}} = \lambda\langle S\rangle + \frac{3}{2}\chi m_{3/2} = \mu_{\text{eff}} + \mu$$

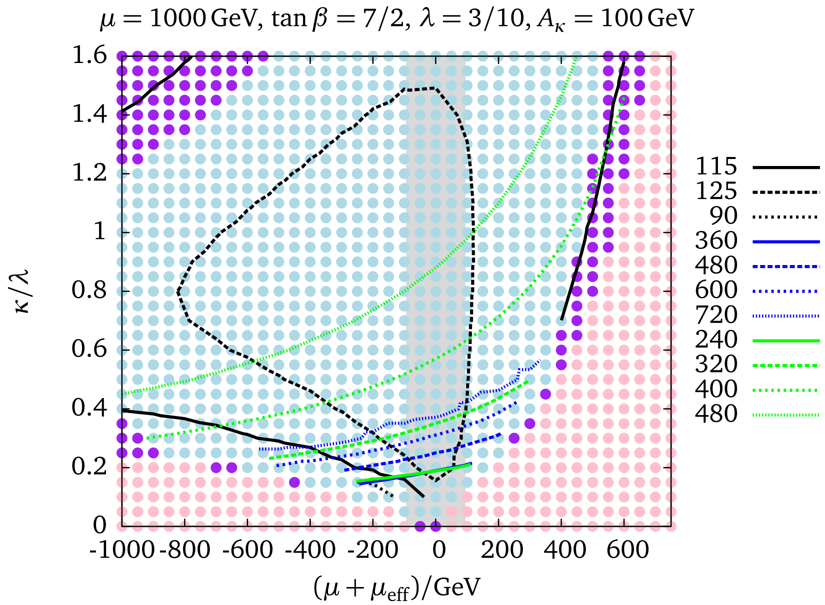
Additional soft SUSY breaking term

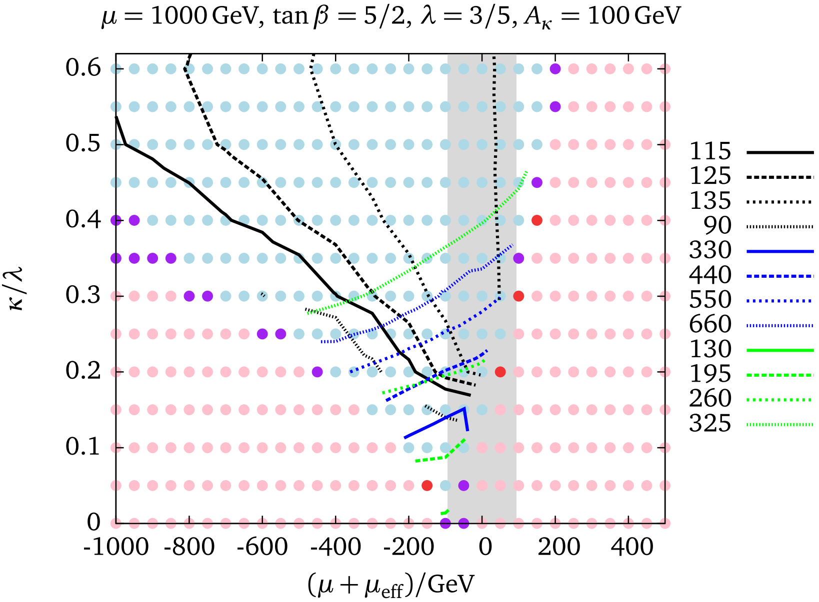
$$V_{\text{soft}} = \lambda A_\lambda\, S H_u \cdot H_d + \frac{1}{3}\kappa A_\kappa\, S^3 + \frac{3}{2} B_\mu \chi m_{3/2} \left(H_u \cdot H_d + \text{h.c.}\right)$$

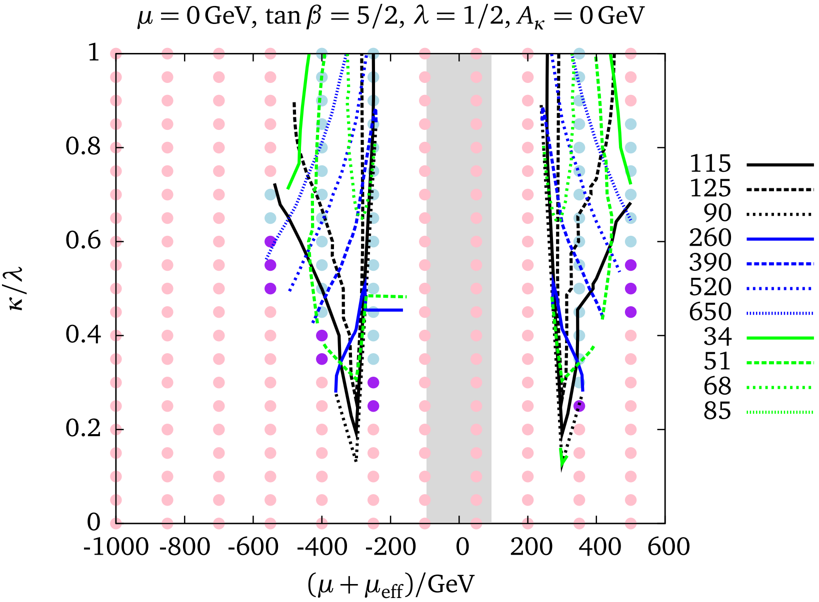
Higgs potential of the iNMSSM

$$\begin{aligned} V = &\left[m_{H_d}^2 + (\mu + \lambda\, S)^2\right] |H_d|^2 + \left[m_{H_u}^2 + (\mu + \lambda\, S)^2\right] |H_u|^2 + m_S^2 S^2 \\ &+ \tfrac{2}{3}\,\kappa\, A_\kappa\, S^3 + \left[\kappa\, S^2 + \lambda\, H_u \cdot H_d\right]^2 + 2\left(B_\mu\, \mu + \lambda\, A_\lambda\, S\right) H_u \cdot H_d \\ &+ \tfrac{g_1^2 + g_2^2}{8}\left(|H_d|^2 - |H_u|^2\right)^2 + \tfrac{g_2^2}{2} |H_d^\dagger\, H_u|^2 \end{aligned}$$


$\mu = 1000\,\mathrm{GeV}$, $\tan\beta = 7/2$, $\lambda = 3/10$, $A_\kappa = 0\,\mathrm{GeV}$
κ/λ
1
0.8
0.6
0.4
0.2
0
-400
-200
0
200
400
$(\mu + \mu_{\mathrm{eff}})/\mathrm{GeV}$
115
125
90
330
440
550
660
19.2
28.8
38.4
9.6


$\mu = 1000\,\mathrm{GeV}$, $\tan\beta = 7/2$, $\lambda = 3/10$, $A_\kappa = 100\,\mathrm{GeV}$
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1000
-800
-600
-400
-200
0
200
400
600
$(\mu + \mu_{\mathrm{eff}})/\mathrm{GeV}$
κ/λ
115
125
90
360
480
600
720
240
320
400
480


$\mu = 1000\,\mathrm{GeV}$, $\tan\beta = 5/2$, $\lambda = 3/5$, $A_\kappa = 100\,\mathrm{GeV}$
0.6
0.5
0.4
0.3
0.2
0.1
0
κ/λ
-1000
-800
-600
-400
-200
0
200
400
$(\mu + \mu_{\mathrm{eff}})/\mathrm{GeV}$
115
125
135
90
330
440
550
660
130
195
260
325


$\mu = 0\,\text{GeV}$, $\tan\beta = 5/2$, $\lambda = 1/2$, $A_\kappa = 0\,\text{GeV}$
κ/λ
$(\mu + \mu_{\text{eff}})/\text{GeV}$
1
0.8
0.6
0.4
0.2
0
-1000
-800
-600
-400
-200
0
200
400
600
115
125
90
260
390
520
650
34
51
68
85

- (heavy) Higgs decays depending on μ and $\mu + \mu_{\text{eff}}$
- Neutralino sector:
  - Higgsinos $\sim (\mu + \mu_{\text{eff}})$
  - Singlino $\sim \frac{\kappa}{\lambda}\mu_{\text{eff}}$
- However, μ can also be small

  [Ferrara, Kallosh, Linde, Marrani, Van Proeyen 2010]
- Work in progress ...

[1] **A new view on vacuum stability in the MSSM**, JHEP **08** (2016) 126, *Wolfgang G. Hollik* [1606.08356]

[2] **Charge and color breaking constraints in the Minimal Supersymmetric Standard Model associated with the bottom Yukawa coupling**, Physics Letters B **752** 7–12 (2016), *Wolfgang G. Hollik* [1508.07201]

[3] **Vacuum stability of the effective Higgs potential in the Minimal Supersymmetric Standard Model**, Physical Review D **90** 035025 (2014) *Markus Bobrowski, Guillaume Chalons, Wolfgang G. Hollik, Ulrich Nierste* [1407.2814]

Work in progress. . .

[1] **Higgs Phenomenology of NMSSM inflation**, with *Stefan Liebler, Gudrid Moortgat-Pick, Sebastian Paßehr, Georg Weiglein*, arXiv:1711.0xxxx

[2] **The Higgs vacuum in the NMSSM**, with *Georg Weiglein, Jonas Wittbrod* arXiv:171x.0yyyy

Backup

Slides

Yukawa coupling not directly proportional to mass (same for y_t)

$$y_b = \frac{m_b}{v_d(1+\Delta_b)}$$

[Hall, Rattazzi, Sarid '94; Carena, Garcia, Nierste, Wagner '99]

$$\Delta_b^{\text{gluino}} = \frac{2\alpha_s}{3\pi}\mu M_{\tilde{G}} \tan\beta C_0(\tilde{m}_{\tilde{b}_1}, \tilde{m}_{\tilde{b}_2}, M_{\tilde{G}}),$$
$$\Delta_b^{\text{higgsino}} = \frac{Y_t^2}{16\pi^2}\mu A_t \tan\beta C_0(\tilde{m}_{\tilde{t}_1}, \tilde{m}_{\tilde{t}_2}, \mu).$$