The Cosmological Constant Problem
and (no) solutions to it

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"Physics thrives on crisis. [. . .] Unfortunately, we have run short on crisis lately."
(Steven Weinberg, 1989 [1])

1 Introduction: the Cosmological Constant

Besides the fact, that we are still running short on severe crises in physics also about 30 years after Weinberg's statement, there is still yet no solution to what is called the "Cosmological Constant problem". Neither is there any clue what the Cosmological Constant (CC) is made of and if the problem is indeed an outstanding problem.

Weinberg defines and proves in his review on the Cosmological Constant [1] a *"no-go" theorem*. This no-go theorem is actually *not* a theorem on the smallness of the CC in the sense of a Vacuum Energy,[1] it is rather a theorem prohibiting any kind of *adjustment mechanisms* that lead effectively to a universe with a flat and static (i. e. Minkowski) space-time metric in the presence of a CC. In other words: with a CC there is no Minkowski universe possible.

An old crisis When Einstein first formulated his field equations for General Relativity, he was neither aware of a possible Cosmological Constant nor of an expanding universe solution to them. Originally, the proposed equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G\, T_{\mu\nu}, \tag{1}$$

using Weinberg's $(-+++)$-metric. The matter content is given by the energy-stress tensor $T_{\mu\nu}$, where the geometry of space-time is encoded in the Ricci tensor $R_{\mu\nu}$ and the scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$ of the metric $g_{\mu\nu}$. The gravitational coupling strength is given by Newton's constant G. Solutions to this set of equations published in 1915 were found to have continuously expanding space-times. Applying this result to the whole universe worried Einstein, who believed (for observational reasons[2]) in a static solution.

[1] Weinberg does not argue the Vacuum Energy to be small, which is rather what we would expect today; instead, he focuses on flat and static solutions for the Einstein equations with a spatially constant set of matter fields corresponding to an isotropically and homogeneously filled universe.

[2] "The most important fact that we draw from experience is that the relative velocities of the stars are very small as compared with the velocity of light." [1]

A static solution, however, was easily obtained by a slight modification of the initial set of equations with a new constant parameter $\lambda > 0$, known as the Cosmological Constant:[3]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda \, g_{\mu\nu} = -8\pi G \, T_{\mu\nu} \,. \tag{2}$$

For a static universe filled with pressureless dust, the mass density can be found to be

$$\rho = \frac{\lambda}{8\pi G} \,. \tag{3}$$

Even the mass and size of the universe can be then determined from the fundamental parameters of the theory. The radius of the S_3 sphere universe is given by

$$r = \frac{1}{\sqrt{8\pi\rho G}} \qquad \text{and the mass} \qquad M = 2\pi^2 r^3 \rho = \frac{\pi}{4} \frac{1}{\sqrt{\lambda} G} \,.$$

This solution was based on a simple and well-motivated assumption: a homogeneous and isotropic universe. However, besides the discovery of the expansion of the universe by Hubble in 1929, which lead Einstein call his Cosmological Constant his biggest folly ("*größte Eselei*"), de Sitter proposed in 1917 another static solution with no matter at all! (One may ask if this is a valid approximation for the universe, although we know that it is mainly empty.) This solution can explain redshift which increases with distance; and although the metric is time-independent, testbodies in it are not at rest. The line element is given by the expression

$$d\tau^2 = \frac{1}{\cosh^2 Hr} \left[dt^2 - dr^2 - H^{-2} \tanh^2 Hr \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right], \tag{4}$$

with the constant $H = \sqrt{\lambda/3}$ and $\rho = p = 0$.

Also de Sitter's solution needs a CC term for the static solution; an expanding universe, however, can live without and this is described by the Friedmann–Lemaître–Robertson–Walker metric

$$d\tau^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right], \tag{5}$$

defining *comoving* coordinates with a cosmic scale factor $R(t)$. Its time-evolution is given by

$$\left[\frac{dR}{dt} \right]^2 = -k + \frac{1}{3} R^2 (8\pi G\rho + \lambda) \,, \tag{6}$$

[3] Weinberg, however, shows in his review [1] that such a solution does not exist (under certain assumptions).

which is an energy-conservation equation. The de Sitter model can be found with $k = 0$ and $\rho = 0$; there are also expanding solutions with $\lambda = 0$ and $\rho > 0$.

Weinberg [1] cites Pais (1982) quoting Einstein in a letter to Weyl 1923 after the discovery of the universe expansion: "*If there is no quasi-static world, then away with the cosmological term!*". Weinberg himself provides now no-go theorem that even states the contrary: the presence of such a cosmological term forbids a quasi-static world!

2 The Problem

A modern formulation of the CC problem is quite different from just being another fine-tuning problem. It is a problem of radiative stability, as outlined in the lecture notes by Padilla [2]. Especially quantum corrections might generate a vacuum energy that is not present at the classical level and every energy density of the vacuum acts as cosmological constant. Because of Lorentz invariance, the equation

$$\langle T_{\mu\nu} \rangle = -\langle \rho \rangle g_{\mu\nu}$$

holds and redefines the cosmological constant as $\lambda_{\text{eff}} = \lambda + 8\pi G \langle \rho \rangle$. Correspondingly, the total vacuum energy is given by $\rho_V = \langle \rho \rangle + \frac{\lambda}{8\pi G} \equiv \frac{\lambda_{\text{eff}}}{8\pi G}$.

Now, we can estimate λ_{eff} from the redshift of an expanding universe,

$$\left[\frac{1}{R}\frac{dR}{dt}\right]_{\text{today}} \equiv H_0 \simeq 50 \ldots 100\,\text{km/s}\,\text{Mpc} \simeq \left(\frac{1}{2} \ldots 1\right) \times 10^{-10}/\text{yr}. \tag{7}$$

The universe is known to be rather flat, so $|k|/R^2_{\text{today}} \lesssim H_0^2$, and thus, in comparison with the critical density,

$$|\rho - \langle \rho \rangle| \lesssim 3H_0^2/8\pi G, \tag{8}$$

the effective CC can be estimated to be $|\lambda_{\text{eff}}| \lesssim H_0^2$, and the total vacuum energy

$$|\rho_V| \lesssim 10^{-29}\text{g/cm}^3 \approx 10^{-47}\,\text{GeV}^4\,. \tag{9}$$

This potentially crude estimate results in a rather small number, taking possible contributions from High Energy physics into account. Actually, any quantum field theory supplies in general a much larger vacuum energy. For any field with mass m, the summation of the zero-point energies for all normal modes (i. e. the one-loop vacuum bubbles) up to a given cut-off $\Lambda \gg m$

results in

$$\langle\rho\rangle = \int_0^\lambda \frac{4\pi k^2 dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2} \simeq \frac{\Lambda^4}{16\pi^2}\,. \tag{10}$$

That means, with a cut-off at the Planck-scale, $\Lambda \simeq 1/\sqrt{8\pi G}$, we estimate

$$\langle\rho\rangle \approx 2^{-10}\pi^{-4}G^{-2} = 2\times 10^{71}\,\mathrm{GeV}^4. \tag{11}$$

Apparently, the two terms of Eqs. (9) and (11) have to cancel to more than 118 digits! It does not help to consider the QCD-scale as probable cut-off, such that $\langle\rho\rangle \sim \Lambda_{\mathrm{QCD}}^4/16\pi^2 \approx 10^{-6}\,\mathrm{GeV}^4$, which still needs a cancellation up to 41 digits. Other estimates have been performed e. g. by Zeldovich, who "for no clear reasons" [1] took $\Lambda = 1\,\mathrm{GeV}$. This leads, of course, to a much smaller vacuum energy if the one-loop terms are canceled by $\lambda/8\pi G$ and the other only contribute as higher-order effect. Thus, for Λ^3 particles of energy Λ per unit volume gives with $\Lambda = 1\,\mathrm{GeV}$

$$\langle\rho\rangle \approx \left(\frac{G\Lambda^2}{\Lambda^{-1}}\right)\Lambda^3 = G\Lambda^6 \approx 10^{-38}\,\mathrm{GeV}.$$

The "*real [...] serious worry*" [1], however, begins when one tries to take spontaneous symmetry breaking in the electroweak sector into account. The scalar field potential

$$V = V_0 - \mu^2\Phi^\dagger\Phi + g\left(\Phi^\dagger\Phi\right)^2 \tag{12}$$

with $\mu^2 > 0$ and $g > 0$ takes a value at its minimum which corresponds to a vacuum energy

$$\langle\rho\rangle = V_{\mathrm{min}} = V_0 - \frac{\mu^4}{4g}\,. \tag{13}$$

If we assume the potential to vanish at the origin, $V(\Phi = 0) = V_0 = 0$, the energy density is found to be

$$\langle\rho\rangle \simeq -g(300\,\mathrm{GeV})^4 \simeq 10^6\,\mathrm{GeV}^4, \tag{14}$$

for $g \approx \alpha^2$, which is still too large by a factor of 10^{53}. However, neither V_0 nor λ must vanish, so a cancellation is still possible.

Things may get more complicated when the thermal history of the universe is taken into account. At early times, temperature effects drive the minimum to be in the symmetric phase, so $\Phi = 0$, because of a positive temperature coefficient $\sim \Phi^\dagger\Phi$. Now, compared with the value of the potential at the minimum today, if this has to be zero because of zero CC today, formerly

$V(\Phi = 0) = V_0$ and thus an enormous CC before the electroweak phase transition. This large early CC can drive inflation and is not necessarily seen to be bad. The issue is nevertheless, why is the CC small *today*.

In summary, we can be assured that there is a CC around because any reasonable theory produces it—and during the cosmological evolution it changed its value. So even if there is some magic cancellation happening today or even if it is just a small number, the cosmological term is there in Einstein's equations. Weinberg argues briefly, before preparing the ground for the proof of this no-go theorem, why there are no constant flat-space solutions to the equations in the presence of such a term.[4]

"*That is, the original symmetry of general covariance, which is always broken by the appearance of any given metric $g_{\mu\nu}$, cannot, without fine-tuning, be broken in such a way as to preserve the subgroup of space-time translations.*" [1]

In mathematics, that means we are looking for solutions of General Relativity with all fields constant over the full space in order to have translational invariance. The field equations are

$$\frac{\partial \mathcal{L}}{\partial \psi_i} = 0\,, \qquad \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = 0\,. \tag{15}$$

There are N generic fields ψ around and the metric is symmetric, so there are $N + 6$ equations for $N + 6$ unknowns. The first set of equations can be easily satisfied; a GL(4) symmetry survives. That means under the transformations

$$g_{\mu\nu} \to A^{\rho}{}_{\mu} A^{\sigma}{}_{\nu}\, g_{\rho\sigma} \qquad \text{and} \qquad \psi_i \to \mathcal{D}_{ij}(A)\, \psi_j\,, \tag{16}$$

the Lagrangian density transforms as

$$\mathcal{L} \to \det(A)\, \mathcal{L}\,. \tag{17}$$

With the ψ fields constant, Eqs. (15) have a unique solution

$$\mathcal{L} = c\sqrt{\det(g)}, \tag{18}$$

with a constant c independent of $g_{\mu\nu}$. The second part of Eqs. (15) can only be satisfied for a vanishing c.

[4]The crucial point is, that there are no static solutions without! Einstein introduced the CC term in order to balance the expansion; the solution, however, is a tuned one.

3 A No-go Theorem

"*No-go theorems are a way of relying on apparently technical assumptions that later turn out to have exceptions of great physical interest.*" [1][5]

Adjustment mechanism We relax the translational invariance condition posed in Eqs. (15) and do not impose the two sets of equations to hold independently but

$$g_{\lambda\nu}\frac{\partial\mathcal{L}(g,\psi)}{\partial g_{\lambda\nu}} = \sum_{n}^{N}\frac{\partial\mathcal{L}(g,\psi)}{\partial\psi_n}f_n(\psi)\,, \tag{19}$$

with some coefficient functions $f_n(\psi)$ and constant fields $g_{\mu\nu}$ and ψ_i. This describes an equilibrium solution in which $g_{\mu\nu}$ and all the fields ψ_i adjust in such a way that they are constant in space-time.

This can be rephrased in a symmetry-condition:

$$\delta g_{\lambda\nu} = 2\varepsilon g_{\lambda\nu}\,, \qquad \delta\psi_n = -\varepsilon f_n(\psi)\,. \tag{20}$$

Once again, we want to have constant fields over space-time, so there is a solution $\psi^{(0)}$, such that

$$\frac{\partial\mathcal{L}}{\partial\psi_n} = 0 \quad \text{at} \quad \psi_n = \psi_n^{(0)}. \tag{21}$$

Apparently, $\partial\mathcal{L}/\partial g_{\mu\nu} = 0$ is then trivially fulfilled, imposing Eq. (19).

However, the solution $\psi^{(0)}$ does not exist ("*without fine-tuning* $\mathcal{L}$" [1]). As a proof, we decompose the set of N fields ψ_n by $N-1$ fields σ_a that do not have to be scalars and one scalar ϕ. For a particular choice of $f_n(\psi)$, we find out of Eq. (20)

$$\delta g_{\lambda\nu} = 2\varepsilon g_{\lambda\nu}\,, \quad \delta\sigma_a = 0\,, \quad \delta\phi = -\varepsilon\,. \tag{22}$$

Under these symmetry transformations, the Lagrangian can only depend on $g_{\lambda\nu}$ and ϕ in the combination $e^{2\phi}g_{\lambda\nu}$. The Lagrangian satisfying $\partial\mathcal{L}/\partial\sigma_a = 0$ thus takes the form

$$\mathcal{L} = e^{4\phi}\sqrt{\det(g)}\,\mathcal{L}_0(\sigma)\,. \tag{23}$$

[5] One of the most famous no-go theorems and its exception is the Coleman–Mandula theorem and Supersymmetry as proposed by Haag, Łopuszański and Sohnius.

The source of the field ϕ is then found to be the trace of the energy-momentum tensor $T_{\mu\nu}$

$$\frac{\partial \mathcal{L}}{\partial \phi} = T^{\mu}{}_{\mu}\sqrt{\det(g)}\,, \quad \text{where} \quad T^{\mu\nu} = g^{\mu\nu}\, e^{4\phi}\, \mathcal{L}_0(\sigma)\,. \tag{24}$$

We now can redefine the metric as $\mathcal{L}$ depends only on ϕ and $g_{\mu\nu}$ in the combination $\hat{g}_{\mu\nu} \equiv e^{2\phi}\mathcal{L}_0$ and individual derivatives. The field ϕ then only appears with derivative couplings and cannot take the role of a dynamical field in the adjustment mechanism (especially its contribution in the Lagrangian is then always zero for a constant ϕ to preserve translational invariance).

This is the easiest approach to the no-go theorem provided by Weinberg [1]. However, one has to be aware of the "technical" assumptions: all fields are taken constant in the flat space solutions although they may only preserve some combination of translational and gauge invariance. Another assumption is the decomposition of the ψ_n into the σ_a and *phi* for which it is not *a priori* clear that it works in the full field space.

Conformal Anomalies Another approach ("*one example of many failed attempts*" [1]) was given by Peccei, Solà and Wetterich [3], where they break the corresponding symmetry (20) by conformal anomalies. There is an effective Lagrangian density[6] including the conformal anomaly $\Theta^{\mu}{}_{\mu}$

$$\mathcal{L}_{\text{eff}} = \sqrt{\det(g)}\left[e^{4\phi}\mathcal{L}_0(\sigma) + \phi\,\Theta^{\mu}{}_{\mu}\right]\,. \tag{25}$$

Now, Eq. (24) gets modified by the anomalous term

$$\frac{\partial \mathcal{L}}{\partial \phi} = \left(T^{\mu}{}_{\mu} + \Theta^{\mu}{}_{\nu}\right)\sqrt{\det(g)}, \tag{26}$$

with $T^{\mu\nu}$ unchanged. The equilibrium solution solution for the field ϕ at the constant value ϕ_0 is found to be determined by the equation

$$4e^{4\phi_0}\mathcal{L}_0 + \Theta^{\mu}{}_{\nu} = 0\,. \tag{27}$$

This again cannot provide a flat and constant metric which would be determined in contrast by the equation

$$0 = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial g_{\mu\nu}} \propto e^{4\phi}\mathcal{L}_0 + \phi\Theta^{\mu}{}_{\nu}\,. \tag{28}$$

[6] Weinberg traces back the appearance and disappearance of equivalences to this expressions in the preprint and published version of [3] and a paper by Ellis, Tsamis and Voloshin in the same year.

Changing Gravity We have seen that is impossible to define a proper adjustment mechanism where there is an equilibrium solution of all involved fields ("matter" fields as well as the metric field). A modification of the laws of gravity without changing the observable phenomenology may allow to calculate the CC as constant of integration and thus unrelated to the fundamental parameters. The most promising approach maintains general covariance but the determinant of the metric is not a dynamical field anymore. Let us consider the action for matter and gravity

$$I[\psi, g] = \frac{-1}{16\pi G} \int d^4x \sqrt{g} R + I_M[\psi, g], \tag{29}$$

where I_M represents the matter action for the generic matter fields ψ, including a possible cosmological term $-\lambda \int d^4x \sqrt{g}/8\pi G$ since λ can be treated as the vacuum energy caused by the fields ψ.

The variation gives the set of Einstein equations

$$\frac{\delta I}{\delta g_{\mu\nu}} = \frac{1}{8\pi G}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) + T^{\mu\nu}, \tag{30}$$

where $T^{\mu\nu} = \delta I_M / \delta g_{\mu\nu}$. These equations hold for all μ and ν. Although keeping the general covariant formalism, one can treat parts of the metric not as dynamical fields.[7] In a kind of minimal approach, we consider the determinant g of the metric not as such a dynamical field, so the variations keep the determinant fixed, $g^{\mu\nu}\delta g_{\mu\nu} = 0$. Only the *traceless* part then determines the field equations

$$R^{\mu\nu} - \frac{1}{4} g^{\mu\nu} R = -8\pi G\left(T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T^{\lambda}{}_{\lambda}\right), \tag{31}$$

which are the traceless part of the usual Einstein equations. The conservation laws still hold, so energy-momentum conservation $T^{\mu\nu}{}_{;\mu} = 0$, as well as the Bianchi identities

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu} = 0\,.$$

[7] *"For instance, we all learn in childhood how to write the equations of Newtonian mechanics in general curvilinear spatial coordinate systems, without supposing that the 3-metric has to obey any field equations at all."* [1]

The covariant derivative with respect to the coordinate x^μ then gives

$$\frac{1}{4}\partial_\mu R = 8\pi G \frac{1}{4}\partial_\mu T^\lambda{}_\lambda\,, \tag{32}$$

and thus $R - 8\pi G T^\lambda{}_\lambda = -4\Lambda = \text{const}$; finally

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \Lambda g^{\mu\nu} = -8\pi G T^{\mu\nu}, \tag{33}$$

resembling the Einstein field equation with a CC that is not related to a corresponding term in the action! There is actually no CC but merely a constant of integration and thus no peculiar cancellation between vacuum fluctuations and the CC is needed. The fluctuations automatically cancel in Eq. (31) and indeed there are flat-space solutions in the absence of matter and radiation. "*The remaining problem is: why should we choose the flat-space solutions?*" [1][8]

This collection of failures to get rid of the CC shows that there is no feasible way to deal with the CC problem. The problem can be either phrased why the observed CC is much much smaller than the expected one or how to preserve radiative stability of this quantity that seems not to be protected by a symmetry. At the end, the CC problem shows up as a fine-tuning problem. The no-go theorem stated by Weinberg in this review on the CC [1] deals with a very peculiar assumption, namely the request for a translational invariant theory that results in a flat and constant space-time metric. The question is, however, why should we rely on this assumption, especially since we know about the cosmological expansion which appears to be even accelerated. Such constant static solutions do not coincide with current observations of the universe.

Besides those rather irregular attempts to deal with the problem, we shortly motivate two more promising approaches in the following: a symmetry argument to keep the CC small (Supersymmetry) and a probabilistic argument which sets us in one of the most probable states of the universe with vanishing CC (Quantum Gravity). Furthermore, Weinberg devotes a whole section in his review on anthropic considerations which will not be covered here.

[8] A very good question that rather has to be put at the very beginning of the whole discussion.

4 Solution: Supersymmetry (SUSY)

It is known at least since Zumino [4] that supersymmetric field theories have (as long as SUSY is unbroken) a vanishing vacuum energy. The argument is very simple, based on the SUSY algebra: the SUSY generators Q_α, fermionic operators, obey the anticommutation relations

$$\{Q_\alpha, Q_\beta^\dagger\} = (\sigma_\mu)_{\alpha\beta} P^\mu \tag{34}$$

with the Pauli matrices $\sigma_{1,2,3}$ and $\sigma_0 = \mathbb{1}$; P^μ being the 4-momentum operator and $\alpha, \beta = 1, 2$ spinor indices. Unbroken SUSY is characterised by

$$Q_\alpha |0\rangle = Q_\alpha^\dagger |0\rangle = 0 \,, \tag{35}$$

where the ground state $|0\rangle$ is both annihilated by the creation and annihilation operators.

Combining Eqs. (34) and (35), it is easy to find that the vacuum has zero energy and momentum:

$$\langle 0| P^\mu |0\rangle = 0 \,. \tag{36}$$

The scalar potential $V(\phi, \phi^*)$ is given by the superpotential $\mathcal{W}(\phi)$ in terms of its derivatives:

$$V(\phi, \phi^*) = \sum_i \left| \frac{\partial \mathcal{W}(\phi)}{\partial \phi^i} \right|^2 \,. \tag{37}$$

The condition for unbroken SUSY, that $\mathcal{W}$ is stationary in ϕ, apparently yields

$$\langle \rho \rangle = V_{\text{min}} = 0 \,. \tag{38}$$

Quantum effects obviously have no effect on this result since fermionic and bosonic loops cancel. Unfortunately, SUSY is broken in the real world. The story gets even more intricate once gravity is taken into account. Any global SUSY including gravity is a locally supersymmetric supergravity, where the CC is given by the expectation value of the scalar potential, which in turn is determined by the superpotential and the Kähler potential $\mathcal{K}(\phi, \phi^*)$.

One realisation of broken SUSY ($\mathcal{D}_i \mathcal{W} \neq 0$) and $V = 0$ can be found with a Kähler potential of the type

$$\mathcal{K} = -3 \ln |T + T^* - h(C^a, C^{a*})| / (8\pi G) + \tilde{\mathcal{K}}(S^n, S^{n*}) \tag{39}$$

and a superpotential $\mathcal{W} = \mathcal{W}_1(C^a) + \mathcal{W}_2(S^n)$, where T, C^a and S^n are chiral superfields.

5 Solution: Quantum Gravity and Quantum Cosmology

The proposed "solution" of the CC problem (i. e. *Why is it small?*) dealing with a quantum universe relies on the fact that the cosmological term can arise as constant of integration in a modification of gravity. In that way, the observed CC appears as a superposition of all possible choices and either anthropic or probabilistic (or both) arguments take over. The main idea behind the quantum cosmological approach is to provide a theory whose probability density peaks at $\lambda_{\text{eff}} = 0$ as proposed by Hawking [5].

Conceptually, one has to deal with the wave function of the universe satisfying the Wheeler–DeWitt equation in three dimensions (on the spacelike surface)

$$\left[\frac{1}{2\sqrt{h}}\frac{\delta}{\delta h_{ij}}\sqrt{h}\mathcal{G}_{ij,kl}\frac{\delta}{\delta h_{kl}} - {}^{(3)}R - 2\lambda + 8\pi G T_{00}\right]\Psi[h,\phi] = 0\,, \tag{40}$$

with matter fields ϕ, the 3-metric h_{ij} and $\mathcal{G}_{ij,kl} \equiv h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}$.

The solution of Eq. (40) can be expressed as "Euclidean path integral"

$$\Psi \propto \int \mathcal{D}g\ \mathcal{D}\Phi\ \exp\left(-S[g,\Phi]\right)\,, \tag{41}$$

where the 3-metric h_{ij} and the matter fields ϕ appear as boundary of the usual 4-metric $g_{\mu\nu}$ and fields Φ on the 3-manifold $M_3[h,\phi]$. The Euclidean action S is given by

$$S = \frac{1}{16\pi G}\int_{M_4}\sqrt{g}(R + 2\lambda)\ +\ \text{matter terms}\ +\ \text{surface terms}\,. \tag{42}$$

The argument now is the following: Eq. (40) is a differential equation in an infinite-dimensional space; it thus has infinitely many solutions that are determined via the boundary conditions. However, the main result shall not crucially depend on those initial conditions.

There are some technical problems adherent to the formulation above which we are not going to discuss here for brevity. Weinberg's interpretation [1] now takes $|\Psi[h,\phi]|^2$ as probability density. Furthermore, the CC is treated as a dynamical variable (a field) by taking the constant of integration $c = c(x)$. The probability distribution for this scalar field at any point

$c = x_1$ in presence of 3-form gauge field $A_{\mu\nu\lambda}$ tracking its origin[9]

$$P(c) = \langle \delta\,(c(x_1) - c)\rangle \propto \int \mathcal{D}A\,\mathcal{D}g\;\mathcal{D}\Phi\;\delta\,(c(x) - c)\exp(-S[A, g, \Phi])\;; \tag{43}$$

at the stationary point

$$P(c) \propto \exp(-\Gamma[A_c, g_c, \Phi_c])\,, \tag{44}$$

with the field values A_c, g_c and Φ_c that leave $c(x_1) = c$ fixed at a point where the total action Γ is stationary. Setting all other fields to their A- and g-dependent stationary values results in an effective action relevant to large 4-manifolds

$$\Gamma_{\text{eff}}[A, g] = \frac{\lambda}{8\pi G}\int \sqrt{g}\;d^4x + \frac{1}{16\pi G}\int \sqrt{g}R\;d^4x + \frac{1}{48}\int d^4x\,\sqrt{g}F_{\mu\nu\lambda\rho}F^{\mu\nu\lambda\rho} + \dots\,, \tag{45}$$

where there are all terms with more than two derivatives of g and/or A omitted. The stationary condition for $A_{\mu\nu\lambda}$ requires $F_{\mu\nu\lambda\rho}$ to have vanishing covariant divergence and c being constant, which gives

$$\Gamma_{\text{eff}} = \frac{\lambda(c)}{8\pi G}\int \sqrt{g}\;d^4x + \frac{1}{16\pi G}\int \sqrt{g}R\;d^4x + \dots\,, \tag{46}$$

with $\lambda(c) = \frac{c^2}{2} + \lambda$. The stationary solution satisfies Einstein's field equations for $g_{\mu\nu}$ with a CC $\lambda(c)$, thus $R = -4\lambda(c)$ and

$$\Gamma_{\text{eff}} = -\frac{\lambda(c)}{8\pi G}\int \sqrt{g}\;d^4x\,. \tag{47}$$

The solution describes a 4-sphere for $\lambda(c) > 0$ with proper circumference $2\pi r$ with $r = \sqrt{3/\lambda(c)}$ and the probability density $\propto \exp(-\Gamma_{\text{eff}}) = \exp[3\pi/G\lambda(c)]$. For $\lambda(c) < 0$, in contrast, solutions can be made compact with periodicity conditions; in any case they have $\Gamma_{\text{eff}} \geq 0$. The conclusion drawn by Hawking is thus that the probability density peaks towards infinity at $\lambda(c) \to 0+$, which means

$$P(c) = \delta(c - c_0)\,, \tag{48}$$

where c_0 is the value for which $\lambda(c = c_0) = 0$. The quantity $\lambda(c)$ is supposed to be the "*true*

[9]The exterior derivative of $A_{\nu\rho\sigma}$ is given by the totally antisymmetric combination $F_{\mu\nu\rho\sigma} = \partial_{[\mu}A_{\nu\rho\sigma]}$ which can be written as $F^{\mu\nu\rho\sigma} = c\,\varepsilon^{\mu\nu\rho\sigma}/\sqrt{g}$ with $g \equiv -\det(g_{\mu\nu})$ and $\varepsilon^{\mu\nu\rho\sigma}$ the Levi-Cività tensor with $\varepsilon^{0123} \equiv 1$.

effective cosmological constant" λ_{eff}, covering all quantum fluctuations.

This closes the discussion on a quantum approach to the CC problem in this notes: "*Hence the result* (48)*, if valid, really does solve the cosmological constant problem.*" [1] It might be discussed whether or not this is a valid approach, since the derivation outlined above depends on many assumptions which hardly can be proven. On the other hand, the quantum cosmology described in an effective way in Weinberg's review [1] suggests that the infinite peak of any probability density for $\lambda = 0$ arises very naturally.

However, after all, we stay far away from a true solution of the CC problem. There are still open and unanswered questions besides the fact that there are neglected terms in the effective action which might be of relevance:

1. Does Euclidean quantum cosmology have anything to do with the real world?[10]
2. What are the boundary conditions?[11]
3. Are wormholes real?[12]

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[5] S. W. Hawking, "*The Cosmological Constant Is Probably Zero*", Phys. Lett. **134B** (1984) 403.

[10] The arguments crucially depend on a Euclideanised action.

[11] They are not really fixed and may be affected by any perturbation.

[12] The interpretation of quantum cosmology, which is not discussed here, relies on creation and annihilation of baby universes that serve as wormholes.