Lecture by QFT II
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HR 1

* $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$.

* $D_\mu = \partial_\mu - i g A^a_\mu R(T^a)$

$\Rightarrow D_\mu \phi = \partial_\mu \phi - i g A^a_\mu R(T^a)_{em} \phi^m$

* $A^a_\mu \xrightarrow{GT} \hat{A}^a_\mu + \delta A^a_\mu$

$\delta A^a_\mu = -\partial_\mu \epsilon^a(x) + g f_{abc} \epsilon^b A^c_\mu$

* $\psi_s \nearrow \psi_s + \delta\psi_s$

$\delta\psi_s = -i g \epsilon^a(x) (R(T^a))_{st} \psi_t$

* $\delta\phi_s = -i g \epsilon^a(x) (R(T^a))_{st} \phi_t$

* <u>Ghosts:</u>

$\delta c^a(x) = +\frac{1}{2} g f_{abc} \epsilon^b(x) c^c(x)$

$\delta B^a(x) = -\frac{1}{2} F_a(x)$.

QFT II 1

08.08.

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* Lessons from QFT 1 :

* Field theory with a single scalar $\varphi(x)$.

→ All information contained in

$$G(x_1, \dots, x_n) = \langle \Omega | T(\prod_i \hat{\phi}(x_i) | \Omega \rangle / \langle \Omega | \Omega \rangle$$

ground state (vacuum)

operator representing φ

with $x_i = (\vec{x}_i, t_i)$

* Heisenberg Picture :

$$\hat{\phi}(x_i) \to \hat{\phi}(\vec{x}_i, t_i) = e^{i\hat{H}t_i} \hat{\phi}(\vec{x}_i, 0) e^{-i\hat{H}t_i}$$

* Perturbation theory : AIM: Calculating $G(x_1, \dots, x_n)$.

$$H = H_0 + \lambda H_{int}$$

We've a well defined Taylor series expansion of $G(x_1, \dots, x_n)$ in λ

→ Feynman rules.

→ Rederive the rules using 'path integral formulation'.

Why do this ?

⇒ ① Canonical quantization breaks manifest Lorentz invariance

→ special importance on time through Hamiltonian formulation.

Path integral formulation treats space & time on the same footing.

② Path integral formulation is easier to apply to non-abelian gauge theories.

* Path Integral method in QM :

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$$

Lagrangian, $L = \frac{1}{2} m \dot{q}^2 - V(q)$

Action, $S = \int_{t'}^{t''} L \, dt$

Action is always associated with some path $q(t)$.

The quantity that we calculate first :

$$K(q', t'; q'', t'') \equiv \langle q'' | e^{-i\hat{H}(t''-t')} | q' \rangle$$

where, $|q'\rangle, |q''\rangle$ are eigenstates of $\hat{q}$ with eigenvalues q', q'' respectively. Heisenberg $\sim$ $|q\rangle$ does not have time dep.

Path integral formulation :

$$K(q', t'; q'', t'') = \int [\mathcal{D}q] \, e^{iS}$$

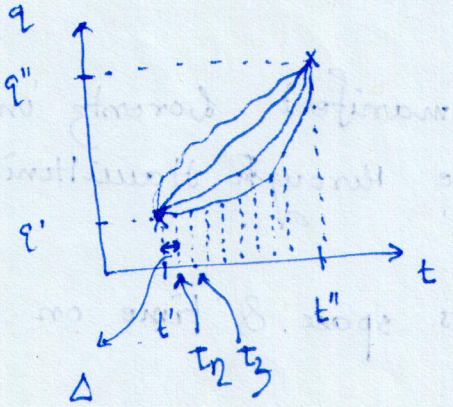
$\Downarrow$

sum over all paths beginning at q' at t' and ending at q'' at t''.

Given any path $S = \int dt \left(\frac{1}{2} m \dot{q}^2 - V(q) \right)$ is well defined :

$q \to$ trajectory

$\dot{q} \to$ slope.



$$\Delta = \frac{t'' - t'}{N}$$

$$\tau_i = t' + \Delta(i-1) \; ; \; \tau_1 = t'$$

$$\tau_{N+1} = t''$$

Denote : $q(\tau_i) \equiv q_i$

So, $\left. \begin{array}{l} q_1 = q' \\ q_{N+1} = q'' \end{array} \right\}$ fixed

$q_2, q_3, \ldots, q_N$ depend on the path.

$$\int [Dq] \xrightarrow{\text{Def}^n} \int dq_2\, dq_3 \ldots dq_N$$

summing over all paths $\longrightarrow$ integrate over $q_2, \ldots, q_N$

$$S \longrightarrow \underbrace{\Delta \sum_{i=1}^{N}}_{\int dt} \left[\frac{1}{2} m \left(\frac{q_{i+1} - q_i}{\Delta} \right)^2 - V(q_i) \right]$$

There is no reason why the paths have to satisfy causality condition or eqn of motn $\Rightarrow$ but ultimately we'll end up with a causal theory.

$$K(q', t'; q'', t'') = \langle q'' | e^{-iH(t''-t')} | q' \rangle$$

Here, $K(q', t'; q'', t'')$

$$= \lim_{N \to \infty} \mathcal{N} \int dq_2 \ldots dq_N \exp\left[i\Delta \sum_{i=1}^{N} \left\{ \frac{1}{2} m \left(\frac{q_{i+1} - q_i}{\Delta} \right)^2 - V(q_i) \right\} \right]$$

Normalization factor independent of q', q''.

* <u>More general quantity</u>:

$$K(q', t'; q'', t''; t_1, \ldots, t_n)$$

$$\equiv \langle q'' | e^{-iHt''} T\left(\prod_{i=1}^{n} \hat{q}(t_i) \right) e^{iHt'} | q' \rangle$$

with $t'' > t_1, \ldots, t_n > t'$

Assume $t_n > t_{n-1} > \cdots > t_1$

Also, $\hat{q}(t) = e^{i\hat{H}t} \hat{q}(0) e^{-i\hat{H}t}$

$$K(q', t'; q'', t''; t_1, \ldots, t_n)$$

$$= \langle q'' | e^{-iH(t''-t_n)} \hat{q}(0) e^{-iH(t_n - t_{n-1})} \hat{q}(0) \ldots e^{-iH(t_2 - t_1)} \hat{q}(0) e^{-iH(t_1 - t')} | q' \rangle$$

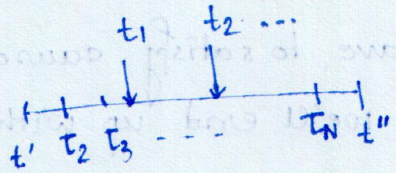
As long as we use time ordered operator $\Rightarrow$ time interval in the exponential is +ve $\Rightarrow -iH\Delta t$; $\Delta t > 0$ $\Rightarrow$ same manipulation.

Path integral description :

$$\tau_i = t' + (i-1)\Delta \quad , \quad i = 1, 2, \ldots, (N+1)$$

$$\Delta = \frac{t'' - t'}{N}$$

$$\tau_1 = t', \quad \tau_{N+1} = t''$$



Let, $t_i = \tau_{k_i}$ for some integer k_i

NOTE: Path integral average of classical quantities $A(t_a)$ & $B(t_b)$ corresponds to the transition matrix element of the time ordered product of these two operators:

$$\langle q'', t'' | T\left(\hat{A}(t_a)\hat{B}(t_b)\right) | q', t' \rangle = \int [\mathcal{D}q(t)]\, A(t_a)\, B(t_b)\, e^{iS[q(t)]}$$

of concern

Now,

$$K(q', t'; q'', t''; t_1, \ldots, t_n)$$

$$= \lim_{N\to\infty} \int dq_1 \ldots dq_N\, e^{iS}\, \underbrace{q_{k_1} q_{k_2} \ldots q_{k_n}}_{\text{eigenvalues}}$$

$\rightarrow$ time order is not a matter here.

$$\equiv \int [Dq]\, e^{iS}\, q(t_1) \ldots q(t_n)$$

$\Updownarrow$

$*$ $K(q', t'; q'', t''; t_1, \ldots, t_n)$

$$= \langle q'' | e^{-iHt''}\, T\left(\prod_{i=1}^{n} \hat{q}(t_i)\right) e^{iHt'} | q' \rangle$$

$\rightarrow$ So, path integral by defult makes time ordered calculation.

$*$ Want to calculate

$$\langle \Omega | T\left(\prod_i \hat{q}(t_i)\right) | \Omega \rangle .$$

Ω : ground state

$$* \quad K(q', t'; q'', t''; t_1, \dots, t_n)$$

$$= \sum_{m,n} \langle q''|m\rangle\langle m| e^{-iE_m t''} T\left(\prod_{i=1}^{n} \hat{q}(t_i)\right) e^{iE_n t'} |n\rangle\langle n|q'\rangle$$

If $t'' \equiv T(1-i\epsilon)$; $t' = -T(1-i\epsilon)$ [analytic continuo

ϵ: small but fixed +ve no.

$$\left.\begin{aligned} e^{-iE_m t''} &= e^{-iE_m T - \epsilon E_m T} \\ e^{iE_n t'} &= e^{-iE_n T - \epsilon E_n T} \end{aligned}\right\} \quad e^{-\epsilon T(E_m + E_n)}$$

$\downarrow \Omega \sim E_\Omega = 0$

So, $t'' = T(1-i\epsilon)$, $t' = -T(1-i\epsilon)$ and in the limit $T \to \infty$ the ground state dominates.

So,

$$K = \lim_{\substack{T\to\infty \\ \epsilon\to 0^+}} e^{-2iE_\Omega T - 2\epsilon E_\Omega T} \langle q''|\Omega\rangle\langle\Omega| \prod_{i=1}^{n} T(\hat{q}(t_i))|\Omega\rangle\langle\Omega|q'\rangle$$

$t'' = T(1-i\epsilon)$

$t' = -T(1-i\epsilon)$

So,

$$\lim_{T\to\infty} \frac{K(q', -T(1-i\epsilon); q'', T(1-i\epsilon), t_1, \dots, t_n)}{K(q', -T(1-i\epsilon), T(1-i\epsilon), q'')}$$ is a better

defined quantity which equals $\langle\Omega| \prod_{i=1}^{n} T(\hat{q}(t_i))|\Omega\rangle / \langle\Omega|\Omega\rangle$

$$* \quad H(\hat{q}^{(1)}, \hat{q}^{(2)}, \dots, \hat{q}^{(n)}, \hat{p}^{(1)}, \dots, \hat{p}^{(n)})$$

$$\longleftrightarrow L(\hat{q}^{(1)}, \dots, \hat{q}^{(n)}, \hat{p}^{(1)}, \dots, \hat{p}^{(n)})$$

<u>Multiple variables</u>

Generalization is straight forward.

08.12

(2)

Goal: $QM \to QFT$

$q^{(1)}, q^{(2)}, \ldots \qquad \phi^{(1)}(\vec{x},t), \phi^{(2)}(\vec{x},t), \ldots$

Consider the case for single field $\phi(\vec{x},t)$.

Action $S[\phi] \underset{eg.}{=} \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right]$

Discretisation (space): Replace $\vec{x} \to h(i_1, i_2, i_3)$ where i_1, i_2, i_3 are integ. $\Rightarrow -\infty <$ integers $< \infty$. and h is a small number.

$$\begin{matrix} \times & \times & \times & \times \\ \times & \times & \times & \times \end{matrix}$$

$$\phi(\vec{x},t) \to \phi_{i_1,i_2,i_3}(t) = \phi_{\vec{i}}(t)$$

Instead of specifying ϕ continuously throughout for all $\vec{x}$, we need to specify only at these discrete points $\vec{i}$. $\to$ discrete $\phi_{\vec{i}}$ as fun. of time $\to$ same as in QM: $q^{(1)}(t), \ldots$

So,

$$S[\phi] = \int dt\, h^3 \sum_{i_1,i_2,i_3} \left[\frac{1}{2}\left(\frac{\partial\phi_{i_1,i_2,i_3}(t)}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\phi_{i_1+1,i_2,i_3} - \phi_{i_1,i_2,i_3}}{h}\right)^2 - \frac{1}{2}\left(\frac{\phi_{i_1,i_2+1,i_3} - \phi_{i_1,i_2,i_3}}{h}\right)^2 - \frac{1}{2}\left(\frac{\phi_{i_1,i_2,i_3+1} - \phi_{i_1,i_2,i_3}}{h}\right)^2 - \frac{1}{2}m^2\phi^2_{i_1,i_2,i_3} - \frac{\lambda}{4!}\left(\phi_{i_1,i_2,i_3}(t)\right)^4 \right]$$

$$* \quad \langle\Omega|T\left(\prod_{k=1}^{n}\phi(\vec{x}_k,t_k)\right)|\Omega\rangle \longrightarrow \langle\Omega|T\left(\prod_{k=1}^{n}\phi_{\vec{i}(k)}(t_k)\right)|\Omega\rangle$$

where $\vec{x}_k = h\vec{i}(k)$

$$\propto \int \prod_{i_1,i_2,i_3} \left[d\phi_{\vec{i}}(t)\right] e^{iS} \prod_{k=1}^{n}\phi_{\vec{i}_k}(t_k)$$

To define this we discretize time :

$$t = \Delta i_0 \qquad -\infty < i_0 < \infty$$

So,

$$S = \Delta \sum_{i_0} h^3 \sum_{i_1, i_2, i_3} \Bigg[\frac{1}{2} \left(\frac{\phi_{i_0+1, i_1, i_2, i_3} - \phi_{i_0, i_1, i_2, i_3}}{\Delta} \right)^2$$

$$- \frac{1}{2} \left(\frac{\phi_{i_0, i_1+1, i_2, i_3} - \phi_{i_0, i_1, i_2, i_3}}{h} \right)^2 - \frac{1}{2} \left(\frac{\phi_{i_0, i_1, i_2+1, i_3} - \phi_{i_0, i_1, i_2, i_3}}{h} \right)^2$$

$$- \frac{1}{2} \left(\frac{\phi_{i_0, i_1, i_2, i_3+1} - \phi_{i_0, i_1, i_2, i_3}}{h} \right)^2 - \frac{1}{2} m^2 \phi^2_{i_0, i_1, i_2, i_3}$$

$$- \frac{\lambda}{4!} \phi^4_{i_0, i_1, i_3, i_4} \Bigg]$$

Now,

$$\langle \Omega | T \left(\prod_{k=1}^{n} \phi(\vec{x}_k, t_k) \right) | \Omega \rangle$$

$$= N \int \prod_{i_0, i_1, i_2, i_3} \left[d\phi_{i_0, i_1, i_2, i_3} \right] e^{iS} \prod_{k=1}^{n} \phi_{i_{(k)0}, i_{(k)1}, i_{(k)2}, i_{(k)3}}$$

* Recall from QM, $e^{iHt'}$

$$\langle q'', t'' | e^{-iHt''} T \left(\prod_i q(t_i) \right) e^{iHt'} | q', t' \rangle$$

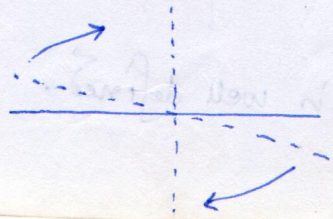
$$t'' = T(1 - i\epsilon)$$

$$t' = -T(1 - i\epsilon), \quad T \to \infty$$

$\Rightarrow$ Slightly tilting

Vacuum dominates.

Now, let's do $t'' = T(-i)$
($\pi/2$ rot^{n}) $t' = -T(-i)$ $\Bigg\} \Rightarrow e^{-iE_m(-iT)}$, $e^{iE_n(iT)}$ $\Rightarrow$ again vacuum



So, $\boxed{t \to -i\tau}$ Euclidean time

So, $t_i \to -i\tau_i$

* As long as were given how to compute a function along an axis in imaginary plane $\xrightarrow[\text{continuation}]{\text{analytic}}$ we can calculate the fun throughout the whole complex plane.

* $t \to -i\tau$

$$\Downarrow$$

$$\delta \to -i\delta$$

So, $t = \Delta i_0 \longrightarrow -i\delta i_0 \Rightarrow \tau = \delta i_0$

$$S = -i\delta h^3 \sum_{i_0, i_1, i_2, i_3} \left[-\frac{1}{2} \left(\frac{\phi_{i_0+1, i_1, i_2, i_3} - \phi_{i_0, i_1, i_2, i_3}}{\delta} \right)^2 \right.$$

$$- \frac{1}{2} \left(\frac{\phi_{i_0, i_1+1, i_2, i_3} - \phi_{i_0, i_1, i_2, i_3}}{h} \right)^2 - \cdots - \frac{1}{2} m^2 \phi^2_{i_0, i_1, i_2, i_3}$$

$$\left. - \frac{\lambda}{4!} \phi^4_{i_0, i_1, i_2, i_3} \right]$$

$$= i\delta h^3 \sum \left[+ \; + \; + \; + \cdots \right]$$

$$\equiv i S_E$$

$$\downarrow$$

$$\boxed{\text{Euclidean Action}}$$

$\downarrow$ continuum limit : $\delta, h \to 0$

$$S_E = \int d\tau \int d^3x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial z} \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

* $e^{iS} = e^{-S_E}$

$\because S_E \to$ each term ≥ 0

$\longrightarrow$ damped $\to$ path integral is well defined.

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So, $\langle \Omega | T_\tau \left(\prod_{k=1}^{n} \phi(\vec{x}_k, -i\tau) \right) | \Omega \rangle$

$$= N \int \prod_{i_0, i_1, i_2, i_3} \left[d\phi_{i_0, i_1, i_2, i_3} \right] e^{-S_E} \prod_{k=1}^{n} \phi_{i_{(k)0}, i_{(k)1}, i_{(k)2}, i_{(k)3}}$$

$$= N \int \prod_i d\phi_i \, e^{-S_E} \prod_{k=1}^{n} \phi_{i_{(k)}}$$

* See in Euclidean action : space & time are perfectly symmetric.

* Alternative to this analytic continuation : [$m^2 \to m^2 - i\epsilon$ prescription is equivalent to Euclidean formalism

$$\langle q'' | e^{-i\hat{H}t''} T \left(\prod_{k=1}^{p} \hat{q}(t_k) \right) e^{i\hat{H}t'} | q' \rangle$$

$$\downarrow \quad \sum_m |m\rangle\langle m| \qquad\qquad \uparrow \quad \sum_n |n\rangle\langle n|$$

$$= \sum_{m,n} \langle q'' | m \rangle \langle n | q' \rangle \langle m | T \left(\prod_{k=1}^{p} \hat{q}(t_k) \right) | n \rangle e^{-iE_m t'' + iE_n t'}$$

Suppose, we manage to get $E_m = E_{m,\text{real}} - i\epsilon_m$ with

$\epsilon_m = 0$ for vacuum

$= +ve$ otherwise.

Take $t'' = T$

$t' = -T$

So, $e^{-iE_m t'' + iE_n t'}$

$$= e^{-i(E_{m,\text{real}} - i\epsilon_m)T} \, e^{i(E_{n,\text{real}} - i\epsilon_n)(-T)}$$

$$= e^{-iT(E_{m,\text{real}} + E_{n,\text{real}})} \, e^{-(\epsilon_m + \epsilon_n)T}$$

Again $T \to \infty \Rightarrow$ ground state dominates. $\longrightarrow$ serves the job.

<u>AIM</u>: acquire small imaginary part in all excited states except the ground state.

$$E = \sum_{s=1}^{N} \sqrt{\vec{p}_s^{\,2} + m_{phy}^2}$$

$$m_{phy}^2 = m^2 + \text{correction}$$

Suppose we replace $m^2 \to m^2 - i\epsilon$

$$\Rightarrow m_{phy}^2 \to m_{phy}^2 - i\epsilon + \text{correc.}$$

So, $$E = \sum_{s=1}^{N} \sqrt{\vec{p}_s^{\,2} + m_{phy}^2 - i\epsilon}$$

$$= \sum_{s=1}^{N} \sqrt{\vec{p}_s^{\,2} + m_{phy_s}^2} - i \times (\text{small +ve no.})$$

$\to$ Exactly what we wanted.

* $$S = \cancel{\tfrac{1}{2}} \int dt\, d^3x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right]$$

$$\xrightarrow{m^2 \to m^2 - i\epsilon} S_{real} + \frac{i}{2}\epsilon \int d^3x\, dt\, \phi^2$$

So, $$e^{iS} \to e^{iS_{real} - \frac{\epsilon}{2}\int dt\, d^3x\, \phi^2}$$

$\to$ as ϕ becomes large, the 2nd part is damped

$\to$ integral is well defined $\to$ path integral is well defined.

* <u>Try</u>: $\int dx\, e^{iax^2 - \epsilon x^2}$ as $\epsilon \to 0$ $\Rightarrow$ well defined

$\int dx\, e^{iax^2}$ $\to$ not well defined.

16.08.1

③

⊠ Functional differentiation :

* $\phi(x)$: some function

$F[\phi]$: Functional of φ

$\frac{\delta F[\phi]}{\delta \phi(x)}$: functional of φ & a function of x.

* Analogy : $f(q_1, \ldots, q_n)$

$\frac{\partial f}{\partial q_i}$ → need to know the form of f and depends also on the index i.

* Discretize :

$(\vec{x}, t) \longrightarrow (\vec{i}, i_0) = (i_1, i_2, i_3, i_0)$ ~ integers

$\left.\begin{array}{l} \vec{x} = h\vec{i} \\ t = \Delta i_0 \end{array}\right\}$ $\vec{i} = \frac{\vec{x}}{h}$ → those values of x are allowed which give integer $\vec{i}$ → doesn't matter upon taking limit.

$i_0 = \frac{t}{\Delta}$

F : functional of $\{\varphi_{i_0, \vec{i}}\}$

$$\boxed{\frac{\delta F[\varphi]}{\delta \phi(x)} \equiv \underset{h,\Delta \to 0}{\mathrm{Lt}} \frac{\partial F}{\partial \varphi_{i_0, \vec{i}}} \times \underbrace{\frac{1}{h^3 \Delta}}_{\text{Normalization}}}$$

Example :

$F[\varphi] = \int d^4x \, J(x) \, \varphi(x)$

↓ Some fixed fun.

Goal : Calculate $\frac{\delta F[\varphi]}{\delta \phi(x)}$

$$F[\varphi] = h^3 \Delta \sum_{i_0, \vec{i}} J_{i_0, \vec{i}} \, \varphi_{i_0, \vec{i}}$$

$$\frac{\delta F[\varphi]}{\delta \varphi(x)} = \underset{h,\Delta \to 0}{\mathrm{Lt}} \frac{\partial}{\partial \varphi_{i_0, \vec{i}}} \sum_{j_0, \vec{j}} J_{j_0, \vec{j}} \, \varphi_{j_0, \vec{j}} = \underset{h,\Delta \to 0}{\mathrm{Lt}} J_{i_0, \vec{i}} = J(x)$$

Take $J(x) = \delta^4(x-x')$

$F[\varphi] = \phi(x')$

$$\frac{\delta F[\varphi]}{\delta\varphi(x)} = \frac{\delta\phi(x')}{\delta\phi(x)} = \delta^4(x-x') \iff \frac{\partial q_i}{\partial q_j} = \delta_{ij}$$

So, $\frac{1}{\hbar^3 \Delta}$ is also necessary to match dimensionally $\Rightarrow [\delta^4] = \frac{1}{[\text{STvol}]}$

* $$\frac{\delta}{\delta\varphi(x)}\left(F[\varphi]\,G[\varphi]\right)$$

$$= \frac{\delta F[\varphi]}{\delta\varphi(x)}\,G[\varphi] + F[\varphi]\,\frac{\delta G[\varphi]}{\delta\varphi(x)}$$

* Take a field theory with fields $\varphi^{(1)}(x), \ldots, \varphi^{(s)}(x)$ and an action $S[\varphi^{(1)}, \ldots, \varphi^{(s)}]$.

<u>Define</u>:

$$Z[J^{(1)}, \ldots, J^{(s)}]$$

$$\equiv \int [\mathcal{D}\varphi^{(1)}] \cdots [\mathcal{D}\varphi^{(s)}]\; e^{iS + i\sum_{\ell=1}^{s}\int d^4x\, J^{(\ell)}(x)\,\varphi^{(\ell)}(x)}$$

Let's consider,

$$\left(-i\frac{\delta}{\delta J^{(k_1)}(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J^{(k_n)}(x_n)}\right) Z[J^{(1)}, \ldots, J^{(s)}]$$

$$= \int [\mathcal{D}\varphi^{(1)}] \cdots [\mathcal{D}\varphi^{(s)}]\; e^{iS + i\sum_{\ell=1}^{s}\int d^4x\, J^{(\ell)}(x)\,\varphi^{(\ell)}(x)}\; \varphi^{(k_1)}(x_1) \cdots \varphi^{(k_n)}(x_n)$$

Now, we need

$$\langle\Omega|\, T\left(\prod_{i=1}^{n} \varphi^{(k_i)}(x_i)\right)|\Omega\rangle \Big/ \langle\Omega|\Omega\rangle$$

$$= \frac{1}{Z(0)}\left(-i\frac{\delta}{\delta J^{(k_1)}(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J^{(k_n)}(x_n)}\right) Z[J^{(1)},\dots,J^{(s)}]\Big|_{J^{(1)}=0,\, J^{(\dots}\, \dots J^{(s)}=}$$

So, everything in QFT is encoded in $Z[J^{(1)},\dots,J^{(s)}]$.

* Free Scalar Field theory: $(\Box - m^2)\varphi(x) = 0 \Rightarrow (-\partial_t^2 + \nabla^2 - m^2)\varphi = 0$.

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\varphi^2\right] \; ; \; -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = \frac{1}{2}\left[(\partial_t\phi)^2 - (\nabla\phi)^2\right]$$

$$= \int d^4x\, \frac{1}{2}\left[\phi(x)(\Box_x - m^2)\phi(x)\right] \; ; \; \Box_x = \eta^{\mu\nu}\partial_\mu\partial_\nu$$

$Z[J]$

$$= \int [\mathcal{D}\varphi]\cdot \exp\left[\frac{i}{2}\int d^4x \left\{\phi(x)(\Box_x - m^2)\phi(x) + 2J(x)\varphi(x)\right\}\right]$$

$\downarrow$ actually

$m^2 \rightsquigarrow m^2 - i\epsilon \rightsquigarrow$ damped

$$= \int [\mathcal{D}\varphi]\, \exp\Big[\frac{i}{2}\int d^4x \left\{\phi(x) + \int d^4x'\, G(x,x')J(x')\right\}$$
$$(\Box_x - m^2)\left\{\phi(x) + \int d^4x''\, G(x,x'')J(x'')\right\}$$
$$- \frac{i}{2}\int d^4x\, d^4x'\, G(x,x')\, J(x')J(x)\Big]$$

where, $(\Box_x - m^2)\, G(x,x') = \delta^4(x-x')$.

Verify: exp term $\Rightarrow$

$$\frac{i}{2}\int d^4x\, \phi(x)(\Box_x - m^2)\phi(x) + \frac{i}{2}\int \phi(x)J(x)\, d^4x$$
$$+ \frac{i}{2}\int J(x)\varphi(x)\, d^4x + \frac{i}{2}\int d^4x\, d^4x'\, \cancel{G(x,x')}\, J(x')J(x)$$
$$- \frac{i}{2}\int d^4x\, d^4x'\, \cancel{G(x,x')}\, J(x)J(x')$$

Same as earlier.

Define $\chi(x) \equiv \varphi(x) + \int d^4x' \, G(x,x') J(x')$

Change variables $\Rightarrow \phi \rightsquigarrow \chi$

$$Z[J] = \int [\mathcal{D}\chi] \exp\left[\frac{i}{2} \int d^4x \; \chi(x) (\Box - m^2) \chi(x) - \frac{i}{2} \int d^4x \, d^4x' \, G(x,x') J(x) J(x') \right]$$

$$\Rightarrow Z[J] = \exp\left[-\frac{i}{2} \int d^4x \, d^4x' \, G(x,x') J(x) J(x') \right]$$

$$\underbrace{\int [\mathcal{D}\chi] \exp\left[\frac{i}{2} \int d^4x \; \chi(x) (\Box - m^2) \chi(x) \right]}_{\text{Independent of } J \rightsquigarrow \text{ does not contribute to } \frac{\delta Z[J]}{\delta J}}$$

$$\longrightarrow \exp\left[-\frac{i}{2} \int d^4x \, d^4x' \, G(x,x') J(x) J(x') \right]$$

$$\boxed{Z[J] = \exp\left[-\frac{1}{2} \int d^4x \, d^4x' \, \Delta(x,x') J(x) J(x') \right]}$$

where, $\Delta(x,x') \equiv i G(x,x')$

* Now, $(\Box_x - m^2) G(x,x') = \delta^4(x-x')$.

$\downarrow$ translational invariance in RHS

$G(x-x')$

$$= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \tilde{G}(k)$$

Substituting,

$$\left.\begin{aligned} \text{LHS} &= \int \frac{d^4k}{(2\pi)^4} (-k^2 - m^2) \tilde{G}(k) e^{ik\cdot(x-x')} \\ \text{RHS} &= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \end{aligned}\right\} \Rightarrow \boxed{\tilde{G}(k) = \frac{1}{-k^2 - m^2}}$$

So, $G(x,x') = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \frac{1}{-k^2 - m^2 + i\epsilon}$ $\longrightarrow$ to well define integration $\rightarrow$ damped or, Euclidean continuation.

So, $\Delta(x,x') \equiv i G(x,x')$

$$= i \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \frac{1}{-k^2 - m^2 + i\epsilon}$$

$\longrightarrow$ Feynman propagator.

* $\langle \Omega | T(\hat{\phi}(x_1)\hat{\phi}(x_2)) | \Omega \rangle / \langle \Omega | \Omega \rangle$

$$\Rightarrow \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta J(x_2)}\right)\left(-i \frac{\delta}{\delta J(x_1)}\right) Z[J] \Big|_{J=0}$$

$$\Rightarrow \frac{1}{Z[0]} \left\{ -\frac{1}{2} \int d^4x\, d^4x' \, \Delta(x,x') \{ \delta^4(x-x_1) J(x') + \delta^4(x'-x_1) J(x) \} \right] Z[J]$$

$$\Rightarrow \frac{1}{Z[0]} \left[-\frac{1}{2} \int d^4x' \, \Delta(x_1,x') J(x') - \frac{1}{2} \int d^4x \, \Delta(x,x_1) J(x) \right] Z[J] \Big|$$

$$\Rightarrow \frac{1}{Z[0]} \left[-\int d^4x' \, \Delta(x',x_1) J(x') \right] Z[J]$$

So, $\left(-i \frac{\delta}{\delta J(x_1)}\right) Z[J] = i \int d^4x' \, \Delta(x',x_1) J(x') \cdot Z[J]$

$\frac{1}{Z[0]}\left(-i \frac{\delta}{\delta J(x_2)}\right) \longrightarrow \frac{1}{Z[0]} \Delta(x_1,x_2) Z[0]$ $\left\{ \begin{array}{l} \text{Other terms} = 0 \\ \text{as } J = 0 \end{array} \right.$

$$= \Delta(x_1,x_2)$$

$\longrightarrow$ Expected.

* Calculate :

$\langle \Omega | T(\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)) | \Omega \rangle / \langle \Omega | \Omega \rangle$

$$= \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta J(x_1)}\right)(\cdots)\left(-i \frac{\delta}{\delta J(x_4)}\right) Z[J] \Big|_{J=0}$$

$$= \Delta(x_1,x_2)\Delta(x_3,x_4) + \Delta(x_1,x_3)\Delta(x_2,x_4) + \Delta(x_1,x_4)\Delta(x_2,x_3).$$

08.2012

2n-point Green's function in KG theory:

$$* \; G^{(2n)}(x_1, \ldots, x_{2n}) = \frac{\langle \Omega | T\left(\prod_{i=1}^{2n} \hat{\phi}(x_i)\right) | \Omega \rangle}{\langle \Omega | \Omega \rangle}$$

$$= \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta J(x_1)}\right) \cdots \left(-i \frac{\delta}{\delta J(x_{2n})}\right) Z[J] \Big|_{J=0}$$

$$* \; Z[J] = \exp\left[-\frac{1}{2} \int d^4x \, d^4x' \, \Delta(x, x') \, J(x) \, J(x')\right]$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \left\{-\frac{1}{2} \int d^4x \, d^4x' \, \Delta(x, x') \, J(x) \, J(x')\right\}^m$$

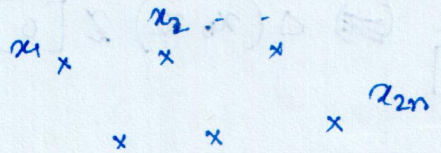
Only $m = 2n$ term contributes.

Diagram:

$-i \frac{\delta}{\delta J(x)}$: $x \Rightarrow \times$ ← vertex

$\Delta(x, x')$: ———

Here, 2n vertex



Rules:

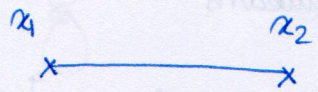
x_1 $\times$: 1

x ——— x' $\Delta(x, x')$

Every cross ($\times$) must be at the end of a line. Every line must begin & end at $\times$. Draw all possible diagrams & add their contribution.

Examples:

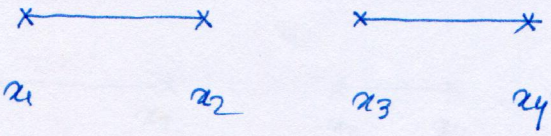
* $G^{(2)}(x_1, x_2)$



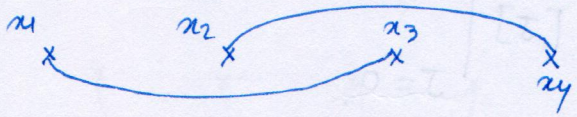
$\Delta(x_1, x_2)$

$\frac{{}^{2n}C_2}{2}$

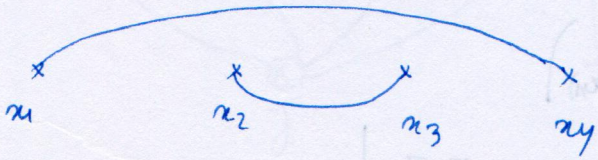
* $G^{(4)}(x_1, x_2, x_3, x_4)$



$\Delta(x_1, x_2)\,\Delta(x_3, x_4)$



$\Delta(x_1, x_3)\,\Delta(x_2, x_4)$



$\Delta(x_1, x_4)\,\Delta(x_2, x_3)$

* $G^{(6)}(x_1, x_2, x_3, x_4, x_5, x_6)$

* $\frac{(2n)!}{2^n\, n!} = (2n-1)!!$ → Free KG theory

$\Rightarrow (2n)! \Rightarrow$ because we've $2n$ # pts. But this $(2n)!$ includes, say for eg $n=2 \Rightarrow 4$ pts $\overset{1}{\bullet}\!\!-\!\!\overset{2}{\bullet}$ $\overset{3}{\bullet}\!\!-\!\!\overset{4}{\bullet}$ mutual interchange $\overset{1}{\bullet}\!\!\rightarrow\!\!\overset{2}{\bullet}$, $\overset{2}{\bullet}\!\!\rightarrow\!\!\overset{1}{\bullet}$ or might be dir^n. $\Rightarrow$ But scalar field theory doesn't need dir^{ns} $\Rightarrow \frac{1}{2^n}$. and reshuffle among propagators: $\frac{1}{n!}$

$\boxed{*}$ Interacting Theory:

* $S = S_{free} + S_{int}$

$$\parallel$$

$$\int d^4x \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right) \qquad \int d^4x \left(-\frac{\lambda}{4!}\phi^4\right)$$

* $$Z[J] = \int [\mathcal{D}\varphi]\, \exp\left[i S_{free}(\varphi) + i S_{int}(\varphi) + i\int d^4x\, \varphi(x) J(x)\right]$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \int [\mathcal{D}\varphi]\, \exp\left[i S_{free}(\varphi) + i\int d^4x\, \varphi(x) J(x)\right] \left(i S_{int}\right)^m$$

$$\Rightarrow \boxed{Z[J] = \sum_{m=0}^{\infty} \frac{1}{m!}\left(i S_{int}\left[-i\frac{\delta}{\delta J}\right]\right)^m Z_{free}[J]}$$

$\longrightarrow$ for φ^4 theory $\left(i S_{int}\left[-i\frac{\delta}{\delta J}\right]\right)^m$ means

$$= \left(-\frac{i\lambda}{4!}\right)^m \int d^4y_1 \cdots d^4y_m \left(-i\frac{\delta}{\delta J(y_1)}\right)^4 \left(-i\frac{\delta}{\delta J(y_2)}\right)^4 \cdots \left(-i\frac{\delta}{\delta J(y_m)}\right)^4$$

and $Z_{free}[J] = \int [\mathcal{D}\varphi]\, e^{i S_{free}[\varphi] + i\int d^4x\, \varphi(x) J(x)}$

So,

* $G^{(n)}(x_1, \ldots, x_n)$

$$= \frac{1}{Z[0]}\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}.$$

$$= \frac{1}{Z[0]}\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right)$$

$$\sum_{m=0}^{\infty}\frac{1}{m!}\left(i S_{int}\left[-i\frac{\delta}{\delta J}\right]\right)^m Z_{free}[J]\Big|_{J=0}.$$

Feynman Rules:

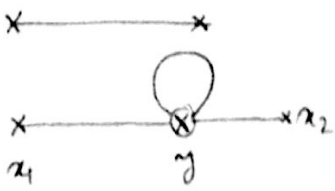
Same as earlier except for the following additions.

y
$\otimes$ $\quad \int d^4y \left(-\frac{i\lambda}{4!}\right)$

a) Every such cross must have 4 lines end on it.

b) Every line can end either at $\otimes$ or at $\times$.

* Examples: Calculate upto order λ & $G^{(2)}(x_1, x_2)$



$\Delta(x_1, x_2)$ [order 1]

$\int d^4y \left(-\frac{i\lambda}{4!}\right) \Delta(x_1, y)\, \Delta(y, y)\, \Delta(y, x_2)$

$\times 4 \times 3$ [order λ]

$\rightarrow$ Leaving exchanging propagators & directions.

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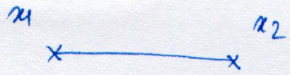


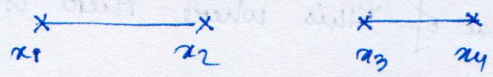
Δ [order λ]

$$\Delta(x_1,x_2)\int d^4y\left(-\frac{i\lambda}{4!}\right)\left(\Delta(y,y)\right)^2 \times 3$$

Now, $\Delta(y,y) \to \infty$

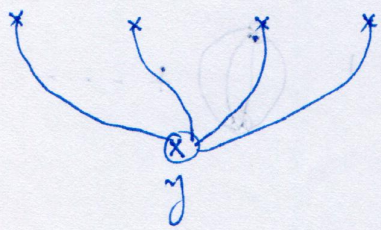
NOTE: "More diagrams" $\equiv$ extra diagrams due to 'x' whereas ⊗ structure is unchanged.
Combinatorial factor $\Rightarrow$ exchange associated with ⊗.

* $G^{(4)}(x_1,x_2,x_3,x_4)$ to order λ.

$\sim$ 

(order 1) + 2 other diagrams

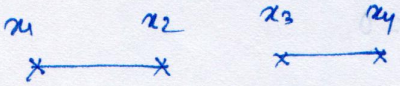
⊗ is not present.
x : $\{(1-2, 3-4), (1-3, 2-$
$(1-4, 2-3)\} = 3$
$(2n-1)!! = 3; n=2$

$\sim$ 

(order λ)

$$\int d^4y\left(-\frac{i\lambda}{4!}\right)\Delta(x_1,y)\,\Delta(x_2,y)\,\Delta(x_3,y)\,\Delta(x_4,y)$$

$\times\ 4!$; No "more diagram" since all of the x are associa with ⊗.

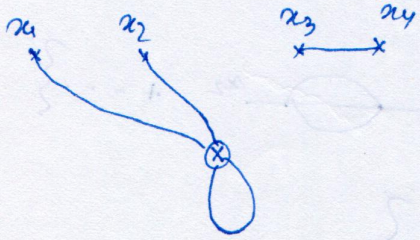
$\sim$ 

(order λ)

$$\int d^4y\left(-\frac{i\lambda}{4!}\right)\Delta(x_1,x_2)\,\Delta(x_3,x_4)\left(\Delta(y,y)\right)^2$$

$\times\ 3$ [from bubbles]

+ 2 more
$\left[\begin{array}{l} 1\to 3,\ 2\to 4 \\ 1\to 4,\ 2\to 3 \end{array}\right]$

$\sim$ 

(order λ)

$$\int d^4y\left(-\frac{i\lambda}{4!}\right)\Delta(x_1,y)\,\Delta(x_2,y)\,\Delta(y,y)\,\Delta(x_3,x_4)$$

$\times\ 4 \times 3$

+ 5 more diagrams

[total 6 : $^4C_2$] $\longrightarrow$ Any two among four x can be chosen for $\times\!-\!\times$ $\Rightarrow {}^4C_2 = 6$.

(5)

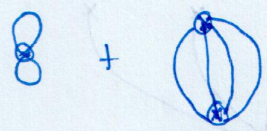
3.08.12

✱ Feynman Rules :

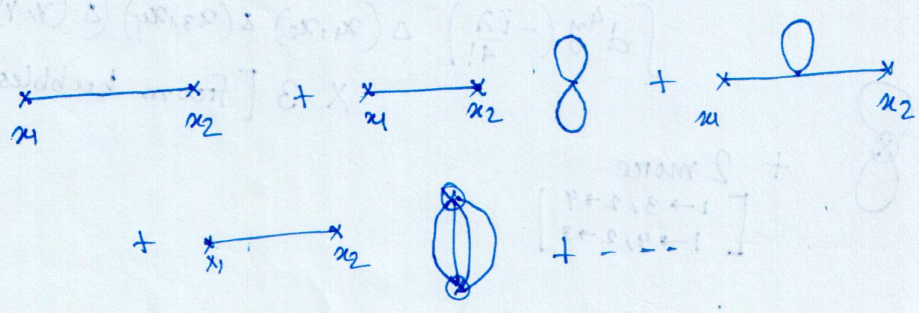
$$G^{(n)}(x_1, \ldots, x_n) = \underbrace{\frac{1}{Z[0]}}\underbrace{\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right)}_{\downarrow} Z[J]\Big|_{J=0}$$

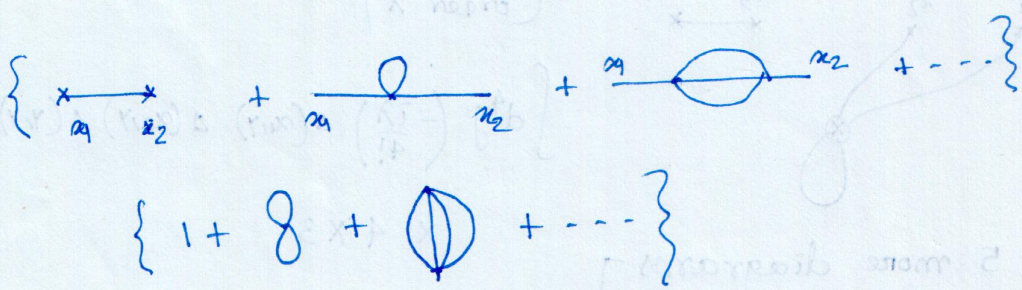
For each $-i\frac{\delta}{\delta J(x)}$ ~~propagators~~ $\Rightarrow$ external vertex

$Z[0]$ is special case of this where there is no propagator & $J=0$. So,

$Z[0]$: $1 +$  $+ \cdots$

eg. $\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right) Z[J]\Big|_{J=0}$

$=$ 

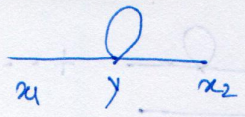
$=$ 

$\{1 +$ $+ \cdots\}$

What about combinatorial factor ?

$\Downarrow$

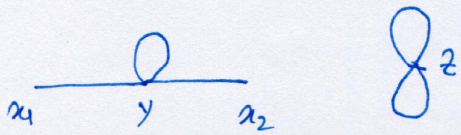
eg.



$$\int d^4y \left(-\frac{i\lambda}{4!}\right) \Delta(y,y)\, \Delta(x_1,y)\, \Delta(x_2,y) \times 4 \times 3$$

$$\int d^4z \left(-\frac{i\lambda}{4!}\right) \left(\Delta(z,z)\right)^2 \times 3$$

and let's do,



$$\int d^4y\, d^4z\ \Delta(x_1,y)\, \Delta(x_2,y)\, \Delta(y,y) \left(\Delta(z,z)\right)^2 \frac{1}{2!}\left(-\frac{i\lambda}{4!}\right)^2$$

$\times\ 8$ $\quad\times\ 3$ $\quad\times\ 1$ $\quad\times\ 3$

$[x_1$ to any $8(y,z)]$ $\quad[x_2$ can go to get same diagram$]$ $\quad[z]$

Both are same.

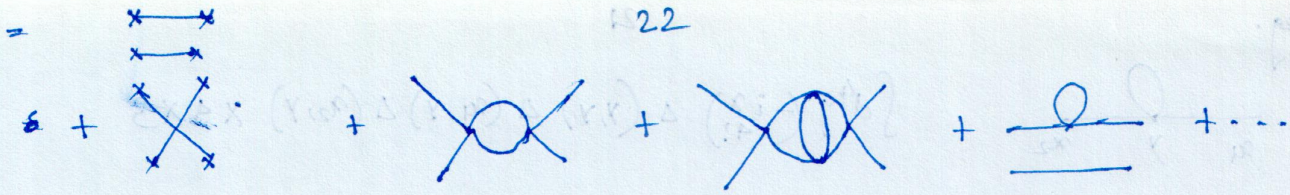
<u>Exercise</u>: Prove that this happens in general.

* $\rightsquigarrow$ Sum of Feynman diagrams $= \left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=}$

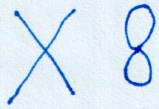
$=$ (1 + sum of bubbles) (sum of diagrams where every line is connected to an external line)

* eg. four point function:

$$\frac{\langle\Omega|\, T\left(\prod_{i=1}^{4} \hat{\phi}(x_i)\right) |\Omega\rangle}{\langle\Omega|\Omega\rangle}$$



But not,



⊠ Momentum Space:

* $\tilde{\phi}(k) = \int d^4x \, e^{-ik\cdot x} \, \phi(x)$

$\Leftrightarrow \phi(x) = \int \frac{d^4k}{(2\pi)^4} \, e^{ik\cdot x} \, \tilde{\phi}(k)$.

where, $k\cdot x = \eta_{\mu\nu} k^\mu x^\nu = -k^0 x^0 + \vec{k}\cdot\vec{x}$.

* Define:

$$\left\langle \prod_{j=1}^{n} \tilde{\phi}(k_j) \right\rangle \equiv \int d^4x_1 \cdots d^4x_n \prod_i e^{-ik_i\cdot x_i} \frac{\langle\Omega| T\left(\prod_{i=1}^{n} \hat{\phi}(x_i)\right)|\Omega\rangle}{\langle\Omega|\Omega\rangle}$$

* We'll derive Feynman rules directly in ϕ^4 theory in momentum space instead of starting with Feynman rules in position space & taking FT of it.

* $\int [\mathcal{D}\phi] \longrightarrow \int [\mathcal{D}\tilde{\phi}(k)]$

Summing over all possible $\phi(x) \equiv$ summing over in momentum space $\tilde{\phi}(k)$

To make sense $\longrightarrow$ discretize $\Rightarrow (k^0, k^1, k^2, k^3) = \tilde{\Delta}(n^0, n^1, n^2, n^3)$.
$\downarrow$ integers

So, $[\mathcal{D}\tilde{\phi}(k)] \longrightarrow \prod_{n^0,n^1,n^2,n^3} d\phi_{n^0,n^1,n^2,n^3}$

* Functional differentiation:

$\longrightarrow$ defined in the same way.

$$\frac{\delta \tilde{\phi}(k')}{\delta \tilde{\phi}(k)} = \delta^4(k-k').$$

* $S = S_f + S_{int}$

with $S_f = \frac{1}{2}\int d^4x \left(-\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right)$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\tilde{\phi}(-k)\left(-k^2 - m^2\right)\tilde{\phi}(k)$$

$$S_{int} = -\frac{\lambda}{4!}\int d^4x\,(\phi(x))^4$$

$$= -\frac{\lambda}{4!}\int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_4}{(2\pi)^4}\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\,\tilde{\phi}(k_1)\,\tilde{\phi}(k_2)\,\tilde{\phi}(k_3)\,\tilde{\phi}(k_4).$$

* Now,

$$\left\langle \prod_{i=1}^{n} \tilde{\phi}(k_i)\right\rangle = \int[\mathcal{D}\tilde{\phi}(k)]\,e^{iS}\prod_{i=1}^{n}\tilde{\phi}(k_i) \Big/ \int[\mathcal{D}\tilde{\phi}(k)]\,e^{iS}$$

* $$Z[J] = \int[\mathcal{D}\tilde{\phi}]\,e^{iS + i\int d^4x\,J(x)\phi(x)}$$

$$= \int[\mathcal{D}\tilde{\phi}]\,e^{iS + i\int\frac{d^4k}{(2\pi)^4}\tilde{J}(-k)\tilde{\phi}(k)} \equiv \tilde{Z}[\tilde{J}]$$

So, $$\left\langle \prod_{i=1}^{n}\tilde{\phi}(k_i)\right\rangle = \frac{1}{\tilde{Z}[0]}\left(-i\frac{\delta}{\delta\tilde{J}(-k_1)}\right)(\cdots)\left(-i\frac{\delta}{\delta J(-k_n)}\right)\tilde{Z}[\tilde{J}]\Big|_{\tilde{J}=0}$$

Now,

$$\tilde{Z}[\tilde{J}] = \int [\mathcal{D}\tilde{\phi}]\, e^{iS_f + iS_{int} + i\int \frac{d^4k}{(2\pi)^4}\tilde{J}(-k)\tilde{\phi}(k)}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\left(i S_{int}\left[-i \frac{\delta}{\delta \tilde{J}(-k)}\right]\right)^m$$

$$\underbrace{\int [\mathcal{D}\tilde{\phi}]\, e^{iS_f + i\int \frac{d^4k}{(2\pi)^4}\tilde{J}(-k)\tilde{\phi}(k)}}_{\tilde{Z}_{free}[\tilde{J}]}$$

Now,

$$\tilde{Z}_{free}[\tilde{J}]$$

$$= \int [\mathcal{D}\tilde{\phi}] \exp\Big[i \cdot \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{\phi}(-k)(-k^2-m^2)\tilde{\phi}(k)$$

$$+ \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left(\tilde{J}(-k)\tilde{\phi}(k) + \tilde{J}(k)\tilde{\phi}(-k)\right)\Big]$$

$$= \int [\mathcal{D}\tilde{\phi}] \exp\Big[+ \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left\{\tilde{\phi}(-k) + \frac{1}{(-k^2-m^2)}\tilde{J}(-k)\right\}(-k^2-m^2)$$

$$\left\{\tilde{\phi}(k) + \frac{1}{(-k^2-m^2)}\tilde{J}(k)\right\}$$

$$- \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{J}(-k)\frac{1}{(-k^2-m^2)}\tilde{J}(k)\Big]$$

<u>Define</u>: $\tilde{\chi}(k) = \tilde{\phi}(k) - \frac{1}{k^2+m^2}\tilde{J}(k)$

So,

$$\tilde{Z}[\tilde{J}] = \int [\mathcal{D}\tilde{\chi}] \underbrace{\exp\left[-\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{\chi}(-k)\frac{(k^2+m^2)}{\cancel{(k^2+m^2)}}\tilde{\chi}(k)\right]}_{\text{Ignorable being indep of } \tilde{J}(k)} \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{J}(-k)\frac{1}{k^2+m^2}\tilde{J}(k)\right]$$

Hence,

$$\tilde{Z}_f[\tilde{J}] = \exp\left[-\frac{1}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\,\tilde{\Delta}(k_1,k_2)\,\tilde{J}(-k_1)\,\tilde{J}(-k_2)\right]$$

where, $\tilde{\Delta}(k_1,k_2) = -\dfrac{i}{k_1^2+m^2-i\epsilon}(2\pi)^4\,\delta^4(k_1+k_2)$.

eg.

$$\langle\tilde{\phi}(k_1)\tilde{\phi}(k_2)\rangle_{free}$$

$$= \left(-i\,(2\pi)^4\frac{\delta}{\delta\tilde{J}(-k_1)}\right)\left(-i\,(2\pi)^4\frac{\delta}{\delta\tilde{J}(-k_2)}\right)\tilde{Z}_{free}[\tilde{J}]\Big/\tilde{Z}_f[0]\Big|_{\tilde{J}=0}$$

$$= \tilde{\Delta}(k_1,k_2).$$

Similarly,

$$\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_{2n})\rangle$$

$$= \tilde{\Delta}(k_1,k_2)\cdot\tilde{\Delta}(k_3,k_4)\cdots\tilde{\Delta}(k_{2n-1},k_{2n})$$

+ inequivalent permutations.

* $S_{int}\left[-i\,(2\pi)^4\dfrac{\delta}{\delta\tilde{J}(-k)}\right]$

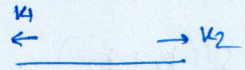
$$= -\frac{\lambda}{4!}\int\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_4}{(2\pi)^4}\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)$$

$$\left\{-i\,(2\pi)^4\frac{\delta}{\delta\tilde{J}(-k_1)}\right\}\cdots\left(-i\,(2\pi)^4\frac{\delta}{\delta\tilde{J}(-k_4)}\right)\cancel{\tilde{Z}_f[\tilde{J}]}$$

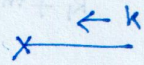
$\longrightarrow$ operate on $\tilde{Z}_f[\tilde{J}]$

* Feynman Rules:

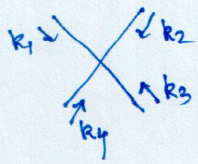
$\rightsquigarrow$ $\Delta(k_1, k_2)$



$\rightsquigarrow$ $\left\{ -i(2\pi)^4 \dfrac{\delta}{\delta \tilde{J}(-k)} \right\}$



So, $S_{int}\left[-(2\pi)^4 i \dfrac{\delta}{\delta \tilde{J}(-k)} \right]$



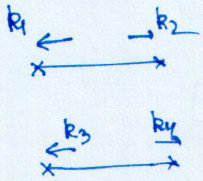
$\rightsquigarrow$ End of every line must be at a 'X' or $\otimes$ and every external vertex must be connected to a line & every internal vertex must be connected to 4-lines.

So, above diagram:

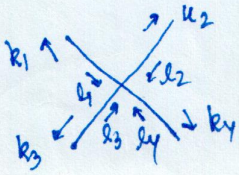
$$-\frac{i\lambda}{4!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_4}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + k_2 + k_3 + k_4).$$

<u>Example:</u>

$\langle \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_4) \rangle$

$=$ 

$\tilde{\Delta}(k_1, k_2)\, \tilde{\Delta}(k_3, k_4)$ + 2 more diagrams.



$$\int \frac{d^4l_1}{(2\pi)^4} \cdots \frac{d^4l_4}{(2\pi)^4} (2\pi)^4 \delta^4(l_1 + l_2 + l_3 + l_4)$$

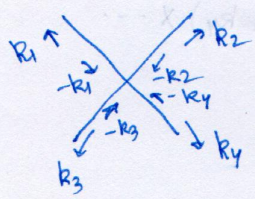
$$\left(-\frac{i\lambda}{4!}\right) (2\pi)^4 \delta^4(k_1 + l_1) \frac{i}{-k_1^2 - m^2 + i\epsilon}$$

$$\times\ (2\pi)^4\,\delta^4(k_2+l_2)\ \frac{i}{-k_2^2-m^2+i\epsilon}\ \times\ (2\pi)^4\,\delta^4(k_3+l_3)\ \frac{i}{-k_3^2-m^2+i\epsilon}$$

$$\times\ (2\pi)^4\,\delta^4(k_4+l_4)\ \frac{i}{-k_4^2-m^2+i\epsilon}\ \times\ 4!$$

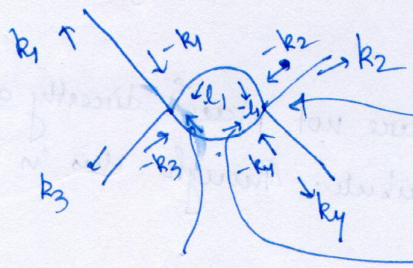
$$= \cancel{\times}\ 4!\left(-\frac{i\lambda}{4!}\right)(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\ \prod_{j=1}^{4}\frac{i}{-k_j^2-m^2+i\epsilon}$$

So, we should have drawn



$\xrightarrow[\text{conserv. at propagators}]{\text{Mom.}}$ $(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\ 4!\left(-\frac{i\lambda}{4!}\right)$ $\prod_{j=1}^{4}\frac{i}{-k_j^2-m^2+i\epsilon}$

*



$$\int\frac{d^4l_1}{(2\pi)^4}\,(2\pi)^4\,\delta^4(-k_1-k_3-k_2-k_4)\ \times\left(-\frac{i\lambda}{4!}\right)^2\times\frac{1}{2!}\ \text{(2nd order)}.$$

$$\prod_{j=1}^{4}\frac{i}{-k_j^2-m^2+i\epsilon}$$

$k_1+k_3-l_1$ $\longrightarrow$ $-(k_1+k_3-l_1)$

[corresponding to $\delta^4(+++)$.

$$\frac{i}{-l_1^2-m^2+i\epsilon}\ \frac{i}{-(k_1+k_3-l_1)^2-m^2+i\epsilon}$$

Combinatorial factor

$\Downarrow$

same expression.

$$\times\ \underset{(k_1)}{8}\ \times\ \underset{(k_2)}{4}\ \times\ \underset{(k_3)}{3}\ \times\ \underset{(k_4)}{3}\ \times\ 2$$

6

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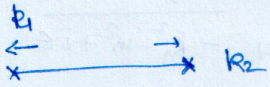
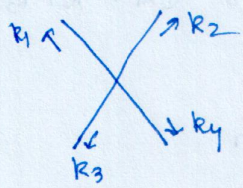
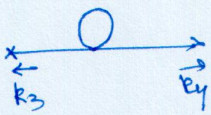
Connected vs Disconnected Diagram:

* ϕ^4:







$$(2\pi)^4 \delta^4(k_1+k_2+k_3+k_4) \times \cdots$$

$$(2\pi)^4 \delta^4(k_1+k_2)$$
$$\times (2\pi)^4 \delta^4(k_3+k_4)$$
$$\times \cdots$$

For generic momentum satisfying total momentum conservation only connected diagrams contribute (in calculating S-matrix).

For PP scattering experiment → detectors are not placed directly along the forward dir^n. disconnected do not contribute. Though this is not necessary for all cases.

* define

$$G^{(n)}(k_1, \ldots, k_n) \equiv \left\langle \prod_{i=1}^{n} \tilde{\phi}(-k_i) \right\rangle$$

$$G^{(2)}(k_1, k_2) = (2\pi)^4 \delta^{(4)}(k_1+k_2)\, G^{(2)}(k_1)$$

$\Rightarrow$ defines $G^{(2)}(k_1)$

If m_{phys} is the physical mass then

$$G^{(2)}(k) \simeq \frac{-iZ}{k^2+m_p^2-i\epsilon} \quad \text{near } k^2 \simeq -m_{phys}^2$$

Z = cons.

$$S(\underbrace{\vec{p}_1, \dots, \vec{p}_n}_{\text{outgoing}}; \underbrace{\vec{k}_1, \dots, \vec{k}_m}_{\text{incoming}})$$

These are 3-momentum because they are on-shell → energy is fixed by 3-momentum

$$= (-\sqrt{z})^{m+n} \prod_{i=1}^{m} \frac{G^{(2)}(k_i)^{-1}}{\sqrt{(2\pi)^3 2\omega_{k_i}}} \prod_{i=1}^{n} \frac{G^{(2)}(p_i)^{-1}}{\sqrt{(2\pi)^3 2\omega_{p_i}}} \, G^{(m+n)}(k_1, \dots, k_m; -p_1, \dots, -p_n) \Big|_{\substack{k_i^0 = \omega_{k_i} \\ p_i^0 = \omega_{p_i}}}$$

with $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m_p^2}$

⊠ Advantage of Path Integral formulation:

→ interaction with derivatives.

* eg. $S_{int} = k \int d^4x \, \eta^{\mu\nu} \phi \, \partial_\mu \phi \, \partial_\nu \phi$

$\pi = \dot{\phi} - 2\kappa\phi\dot{\phi}$

$\Rightarrow \dot{\phi} = \frac{\pi}{-2\kappa\phi + 1}$ → expand in power series.

→ Substitute in H → separate out free & int

→ Many times it's not possible to find closed form of $\dot{\phi}$ in terms of π

→ complicated in Canonical.

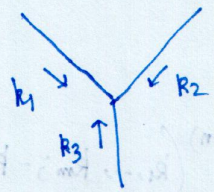
* $$iS_{int} = ik \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \times (2\pi)^4 \delta^4(k_1 + k_2 + k_3) \tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(k_3)$$

$$\times \underbrace{ik_2 \times ik_3}_{-k_2 \cdot k_3 \,=\, -\frac{1}{3}(k_1 \cdot k_2 + k_2 \cdot k_3 + k_1 \cdot k_3)}$$

since rest are symmetric in k_1, k_2, k_3.

→ ~~the~~ apply usual procedure

→ Feynman ↓



$$\int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_3}{(2\pi)^4} \times (2\pi)^4 \delta^4(k_1+k_2+k_3)$$

$$\left(-\frac{i\kappa}{3}\right)\left(k_1\cdot k_2 + k_2\cdot k_3 + k_3\cdot k_1\right)$$

⊠ Multiple bosonic fields:

Suppose, there is a set of fields

$$\phi^{(1)}(x), \ldots, \phi^{(s)}(x)$$

$$S = S_f + S_i$$

↓

required to be quadratic in fields → solvable exactly.
(not cubic & higher).

$$S_f = \frac{1}{2}\int d^4x\, \phi^{(l)}(x)\, D_{ll'}\, \phi^{(l')}(x)$$

↓

Some differential operator

eg. $A_{ll'}\Box + B_{ll'}$

In Momentum space:

$$S_f = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{\phi}^{(l)}(-k)\, M_{ll'}(k)\, \tilde{\phi}^{(l')}(k)$$

↓

replace ∂_μ in $D_{ll'}$ by ik_μ.

We can choose $M_{ll'}(k) = M_{l'l}(-k)$

Proof: $$S_f = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{\phi}^{(l)}(-k)\, \tilde{\phi}^{(l')}(k)\, \underbrace{\frac{M_{ll'}(k) + M_{l'l}(-k)}{2}}_{\text{New } M_{ll'}(k)}$$

→

Now,

$$\tilde{Z}_{free}\left[\tilde{J}^{(1)}, \ldots, \tilde{J}^{(s)}\right]$$

$$= \int \left[\mathcal{D}\tilde{\phi}^{(1)} \ldots \mathcal{D}\tilde{\phi}^{(s)}\right] \exp\left[i S_{free} + i \int \frac{d^4k}{(2\pi)^4} \underbrace{\tilde{J}^{(l)}(-k)\, \tilde{\phi}^{(l)}(k)}_{\sum_l}\right]$$

$$\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left[\tilde{\phi}^{(l)}(-k)\, M_{ll'}(k)\, \tilde{\phi}^{(l')}(k) + 2\tilde{J}^{(l)}(-k)\, \tilde{\phi}^{(l)}(k)\right]$$

$$= \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\Big[\left\{\tilde{\phi}^{(l)}(-k) + \tilde{J}^{(q)}(-k)\, M^{-1}_{ql}(k)\right\}$$

$$M_{ll'}(k)\left\{\tilde{\phi}^{(l')}(k) + M^{-1}_{l'q'}(k)\, \tilde{J}^{(q')}(k)\right\}$$

$$- \tilde{J}^{(q)}(-k)\, M^{-1}_{qq'}(k)\, \tilde{J}^{(q')}(k)\Big]$$

where, $M^{-1}_{ql}(k)\, M_{ll'}(k) = \delta_{ql'}$

and $M^{-1}_{ql}(k) = M^{-1}_{lq}(-k)$

Define:

$$\chi^{(l')}(k) = \tilde{\phi}^{(l')}(k) + M^{-1}_{l'q'}(k)\, \tilde{J}^{(q')}(k)$$

Hence,

$$\tilde{Z}_f\left[\tilde{J}^{(1)}, \ldots, \tilde{J}^{(s)}\right]$$

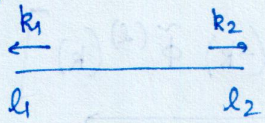
$$= \int \left[\mathcal{D}\tilde{\chi}^{(1)} \ldots \mathcal{D}\tilde{\chi}^{(s)}\right] \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{\chi}^{l}(-k)\, M_{ll'}(k)\, \tilde{\chi}^{(l')}(k)\right]$$

$$\times \exp\left[-\frac{1}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\, \Delta_{l_1 l_2}(k_1, k_2)\, \tilde{J}^{(l_1)}(-k_1)\, \tilde{J}^{(l_2)}(-k_2)\right]$$

$$\Delta_{l_1 l_2}(k_1, k_2) = i\, M^{-1}_{l_1 l_2}(k_1)\, (2\pi)^4\, \delta^4(k_1 + k_2)$$

1st exp is ignorable.

Propagator:



$$\Delta_{l_1 l_2}(k_1, k_2)$$

$$= i\, M^{-1}_{l_1 l_2}(k_1)\,(2\pi)^4\,\delta^4(k_1+k_2)$$

* Interactions:

Most General form $\Rightarrow$

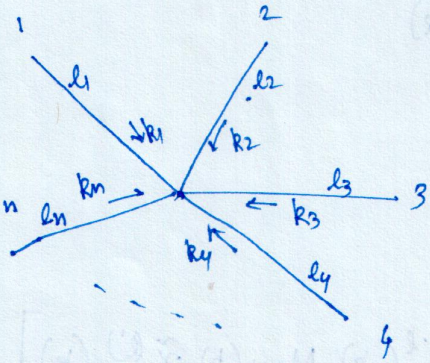
$$S_{int} = \sum_{n=3}^{\infty} \frac{1}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4}\, (2\pi)^4\, \delta^{(4)}(k_1 + \cdots + k_n)\, \tilde{\phi}^{(l_1)}(k_1) \cdots \tilde{\phi}^{(l_n)}(k_n)$$

$$V^{(n)}_{l_1 \cdots l_n}(k_1, \ldots, k_n)$$

$\downarrow$

can be chosen symmetric under simultaneous exchange of

$$\begin{cases} l_i \leftrightarrow l_j \\ k_i \leftrightarrow k_j \end{cases}$$



$i\,\frac{1}{n!}$ (mom. conserving δ-fun.)

$V^{(n)}_{l_1 \cdots l_n}(k_1, \ldots, k_n)$

[*] Path Integral With Fermions:

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(7)

* Canonical commutation rel^{n}:

$$\{\hat{\psi}_\alpha(\vec{x},t), \hat{\psi}_\beta(\vec{y},t)\} = 0 = \{\hat{\psi}^\dagger_\alpha(\vec{x},t), \hat{\psi}^\dagger_\beta(\vec{y},t)\} \qquad \alpha, \beta : 1,2,3,4$$

$$\{\hat{\psi}_\alpha(\vec{x},t), \hat{\psi}^\dagger_\beta(\vec{y},t)\} = \delta_{\alpha\beta}\, \hbar\, \delta^3(\vec{x}-\vec{y})$$

* $\hat{H}$: functional of $\hat{\psi}_\alpha(\vec{x},t), \hat{\psi}^\dagger_\beta(\vec{x},t)$

* Path integral for this is not a limit of ordinary integration, since ~~∄~~ no classical analog of fermionic system.

* Goal: Compute

$$\left\langle \prod_{i=1}^{n} \hat{\psi}_{\alpha_i}(x_i) \prod_{j=1}^{m} \hat{\psi}^\dagger_{\beta_j}(y_j) \right\rangle$$

$$= \frac{\langle\Omega| T\left(\prod_{i=1}^{n} \hat{\psi}_{\alpha_i}(x_i) \prod_{j=1}^{m} \hat{\psi}^\dagger_{\beta_j}(y_j)\right)|\Omega\rangle}{\langle\Omega|\Omega\rangle} \rightsquigarrow \text{compute through Path integral}$$

$$= \frac{\int [\mathcal{D}\psi][\mathcal{D}\psi^\dagger]\, e^{iS} \prod_{i=1}^{n} \psi_{\alpha_i}(x_i) \prod_{j=1}^{m} \psi^\dagger_{\beta_j}(y_j)}{\int [\mathcal{D}\psi][\mathcal{D}\psi^\dagger]\, e^{iS}}$$

NOTE: Fermionic $\Rightarrow$ mutual $\psi_{\alpha_1}, \psi_{\alpha_2}$ exchange $\Rightarrow$ picks up one $-$ve sign.
Ordinary path integral doesn't have this property.

* How to define this?

Step-I: Discretize this.

$$\psi_\alpha(\vec{x},t) \longrightarrow \psi_{\alpha;\, i^0, \vec{i}} \qquad \text{with } x^0 = \Delta i^0,\ x^k = h i^k$$

$$\tilde{\psi}_\alpha(k) \longrightarrow \tilde{\psi}_{\alpha;\, n^0, n^1, n^2, n^3} \qquad \text{with } k^\mu = \tilde{\Delta} n^\mu$$

$$\int [\mathcal{D}\psi\, \mathcal{D}\bar{\psi}] \longrightarrow \int \prod d\tilde{\psi}_{\alpha;n^0,n^1,n^2,n^3}\, d\tilde{\psi}^{\dagger}_{\alpha;n^0,n^1,n^2,n^3}$$

or in terms of ψ instead of $\tilde{\psi}$.

Action :

Since, $\{\psi_\alpha\}$ are not ordinary functions $\Rightarrow$ action won't be an ordinary functionals.

Motivation $\Rightarrow$ define S s.t. 1st variation $=0 \Rightarrow$ DE.

$$S_{free} = \int d^4x\, \bar{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x) \quad \text{with } \bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$$

$$= \int \frac{d^4k}{(2\pi)^4}\, \bar{\tilde{\psi}}(-k)\left(-\gamma^\mu k_\mu - m\right)\tilde{\psi}(k) \qquad \text{[Momentum space]}$$

$\downarrow$ discretize

$$\frac{1}{(2\pi)^4}\, \tilde{\Delta}^4 \sum_{n^\mu} \bar{\tilde{\psi}}_{\alpha;n^0,n^1,n^2,n^3}\left(-\gamma^\mu \eta_\mu \tilde{\Delta} - m\right)_{\alpha\beta}\, \tilde{\psi}_{\beta;n^0,n^1,n^2,n^3}$$

(*)

$$\tilde{\psi}_{\alpha;n^0,n^1,n^2,n^3} = \chi_{\alpha;n^0,n^1,n^2,n^3} + i\,\phi_{\alpha;n^0,n^1,n^2,n^3}$$

$$\tilde{\psi}^{\dagger}_{\alpha;n^0,n^1,n^2,n^3} = \chi_{\alpha;n^0,n^1,n^2,n^3} - i\,\phi_{\alpha;n^0,n^1,n^2,n^3}$$

Finally both numerator & denominator have the form

$$\int \prod_{\alpha,n^0,\dots,n^3} d\chi_{\alpha;n^0,\dots,n^3}\, d\phi_{\alpha;n^0,\dots,n^3} \times \text{some function of } \chi_{\alpha;n^0,\dots,n^3}\, \phi_{\alpha;n^0,\dots,n^3}$$

Step-II : Regard $\chi_{\alpha;n^0,n^1,n^2,n^3}$ and $\phi_{\alpha;n^0,n^1,n^2,n^3}$ as grassmann variables following rules different from those of ordinary numbers.

$$\{\chi_{\alpha;n^0,\dots,n^3}\} \;\&\; \{\phi_{\alpha;n^0,\dots,n^3}\} \equiv \theta_1,\dots,\theta_n$$

We've to evaluate/define an integration of the form:

$$\int d\theta_1 \dots d\theta_n \; F(\theta_1,\dots,\theta_n)$$

$$F(\theta_1,\dots,\theta_n) \equiv \sum_{k=0}^{\infty} A^{(k)}_{i_1\dots i_k}\,\theta_{i_1}\dots\theta_{i_k} \Longleftarrow \text{def}^{n} \text{ of a function of grassman variable}$$

$A^{(k)}_{i_1\dots i_k}$ ↓ ordinary numbers.

& $i_1\dots i_k \to$ summed over.

Rules: $\theta_i\theta_j = -\theta_j\theta_i$

$$\Rightarrow \theta_i^2 = 0.$$

$\Longleftrightarrow A^{(k)}_{i_1\dots i_k}$ is anti-symmetric in $i_1\dots i_k$.

So,

$$f(\theta_1,\dots,\theta_n) = \sum_{k=0}^{n} A^{(k)}_{i_1\dots i_k}\,\theta_{i_1}\dots\theta_{i_k}$$

eg. $e^{A\theta_1\theta_2}$

$$= \sum_{k=0}^{\infty} (A\theta_1\theta_2)^k \frac{1}{k!}$$

$$= 1 + A\theta_1\theta_2 + \tfrac{1}{2}\underbrace{A\theta_1\theta_2 A\theta_1\theta_2}_{0} + 0 \dots$$

$$= 1 + A\theta_1\theta_2$$

$$e^{A\theta_1\theta_2 + B\theta_3\theta_4}$$

$$= 1 + (A\theta_1\theta_2 + B\theta_3\theta_4) + \frac{1}{2}(A\theta_1\theta_2 + B\theta_3\theta_4)^2 + \cdots$$

$$\frac{1}{2}\left(AB\,\theta_1\theta_2\theta_3\theta_4 + AB\,\theta_3\theta_4\theta_1\theta_2\right)$$

$$AB\,\theta_1\theta_2\theta_3\theta_4.$$

$$= 1 + (A\theta_1\theta_2 + B\theta_3\theta_4) + AB\,\theta_1\theta_2\theta_3\theta_4$$

$\Rightarrow$ We need to learn what we mean by

$$\int d\theta_1 \cdots d\theta_n \; f(\theta_1, \ldots, \theta_n)$$

$\Downarrow$

Polynomial in $\theta_1, \ldots, \theta_n$

<u>Rules to perform integration:</u>

$$\boxed{\begin{array}{l} \int d\theta_i = 0 \;, \quad \int d\theta_i\, \theta_i = 1 \\ d\theta_i\, d\theta_j = -d\theta_j\, d\theta_i \;, \quad d\theta_i\, \theta_j = -\theta_j\, d\theta_i \end{array}}$$

$\rightsquigarrow$ This is the defn of integration over Grassman variables.

eg. $\int d\theta_1\, d\theta_2\, e^{A\theta_1\theta_2}$

$$= \int d\theta_1\, d\theta_2\, (1 + A\theta_1\theta_2)$$

$$= 0 + A\int d\theta_1 (-)\theta_1\, d\theta_2\, \theta_2$$

$$= -A$$

Alternative way:

$$A\int d\theta_1\, d\theta_2\, (-\theta_2\theta_1)$$

$$= -A\int d\theta_1\, \theta_1$$

$$= -A.$$

* F is even if 'only' even no. of θ_i's appear in each term

F is odd " " odd " " " " " " "

eg. even fun : $A + B\theta_1\theta_2 + C\,\theta_1\theta_2\theta_3\theta_4 + \cdots$

odd " : $A\theta_1 + B\,\theta_1\theta_3\theta_5 + \cdots$

Any, $F = F_{even} + F_{odd}$

* Differentiation:

$$\boxed{\frac{\partial}{\partial\theta_i} \equiv \int d\theta_i}$$

$$\frac{\partial}{\partial\theta_i} 1 = 0 \quad ; \quad \frac{\partial}{\partial\theta_i}\theta_i = 1$$

$$\frac{\partial}{\partial\theta_i}(\theta_j(\,)) = -\theta_j \frac{\partial}{\partial\theta_i}(\,)$$

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}(\,) = -\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i}(\,)$$

Prop:

$$\frac{\partial}{\partial\theta_i}(FG)$$

$$\overset{?}{=} \frac{\partial F}{\partial\theta_i}G \cancel{+} F\frac{\partial G}{\partial\theta_i}$$

$$= \frac{\partial F}{\partial\theta_i}G + (-1)^F F\frac{\partial G}{\partial\theta_i}$$

$$\frac{\partial}{\partial\theta_j}(\theta_i\theta_j) = -\theta_i$$

$$\neq \frac{\partial\theta_i}{\partial\theta_j}\theta_j + \theta_i\frac{\partial\theta_j}{\partial\theta_j}$$

θ_i

where, $(-1)^F = 1$ if F is even fun.

$= -1$ if F is odd fun.

Proof:

Any $F = A + \theta_i B$; C, D, A, B contain all the other grassmann variables.

$G = C + \theta_i D$

$$LHS = \frac{\partial}{\partial \theta_i}\underbrace{(A+\theta_i B)(C+\theta_i D)}_{AC+\theta_i BC+\theta_i AD(-1)^A}$$

$$= BC + (-1)^A AD$$

Now, if F is even $\rightarrow$ A is even & B is odd

$\Rightarrow (-1)^F = (-1)^A$

$$RHS = \frac{\partial F}{\partial \theta_i} G + (-1)^F F \frac{\partial G}{\partial \theta_i}$$

$$= B(C+\theta_i D) + (-1)^A (A+\theta_i B) D$$

$$= BC + (-1)^A AD + \underbrace{B\theta_i D}_{(-1)^B \theta_i BD} + (-1)^A \theta_i BD$$

& $(-1)^A + (-1)^B = 0$.

$$= BC + (-1)^A AD$$

$$= LHS.$$

* <u>Prove this:</u>

① $\frac{\partial}{\partial \theta_i} F(G(\vec{\theta}))$; $\vec{\theta} = (\theta_1, \ldots, \theta_n)$

$\downarrow$ for ordinary

$$F'(G(\vec{\theta})) \frac{\partial G(\vec{\theta})}{\partial \theta_i}$$

Here, this holds for even $G(\vec{\theta})$.

eg. $\frac{\partial}{\partial \theta_1}(\overset{F}{\cos}\, \overset{G}{\theta_1 \theta_2})$ But $\frac{\partial}{\partial \theta_1} \cos \theta_1 \neq -\sin \theta_1$

$$= -\sin(\theta_1 \theta_2) \cdot \theta_2$$

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② $\int d\theta_i \, F(\vec{\theta}) \frac{\partial}{\partial \theta_i} G(\vec{\theta})$

$= -(-1)^F \int d\theta_i \, \frac{\partial F(\vec{\theta})}{\partial \theta_i} G(\vec{\theta})$

$* \; \partial_{\theta_i} F(\underset{\text{Even}}{G(\vec{\theta})}) = F'(G(\vec{\theta})) \partial_{\theta_i} G$

$F(x) = \sum_n f_n x^n$

$\Rightarrow F'(x) = \sum_n n f_n x^{n-1}$

Now, $\partial_{\theta_i} \sum_n f_n (G(\vec{\theta}))^n \overset{?}{=} \sum_n f_n \, n (G(\vec{\theta}))^{n-1} \partial_\theta G$

$\rightarrow$ Compare coef of f_n

$\rightarrow \partial_{\theta_i} (G(\vec{\theta}))^n = n (G(\vec{\theta}))^{n-1} \partial_{\theta_i} G$

$G(\vec{\theta}) = A + \theta_i B$

$* \; F(x, \theta_1, \theta_2) = a(x) + b_1(x)\theta_1 + b_2(x)\theta_2 + c_{12}(x) \underbrace{\theta_1 \theta_2}_{\text{anti}}$

We'll write it $\Rightarrow$

$F(x, \theta_1, \theta_2) = a(x) + \theta_i b_i(x) + \frac{1}{2} \epsilon_{ij} c_{ij}(x) \theta_i \theta_j$

$* \int d\theta_i F(\theta_i + c(\vec{\theta}'))$

Now, $F(\theta_i + c(\vec{\theta}')) \equiv G(\theta_i, \vec{\theta}') \equiv G(\theta_i, x)$

$= a(x) + \theta_i b_i(x)$

Then,

$\int d\theta_i \, F(\theta_i + c(\vec{\theta}'))$

$= \int d\theta_i (a(x) + \theta_i b_i(x))$

$= \int d\theta_i \, \theta_i b_i(x)$

$= \int d\theta_i \, G(\theta_i)$

$= \int d\theta_i \, F(\theta_i)$

$*$ Shifting of variables in integration :

$\int d\theta_i \, F(\theta_i + c(\vec{\theta}'))$

$\hookrightarrow$ a fun. of other θ's.

$= \int d\theta_i \, F(\theta_i)$

Proof: $F(\theta_i) = A(\vec{\theta}') + \theta_i B(\vec{\theta}')$

$\Rightarrow F(\theta_i + c(\vec{\theta}')) = A(\vec{\theta}') + (\theta_i + c(\vec{\theta}')) B(\vec{\theta}')$

$= A(\vec{\theta}') + c(\vec{\theta}') B(\vec{\theta}') + \theta_i B(\vec{\theta}')$

$\Rightarrow \int d\theta_i \, F(\theta_i + c(\vec{\theta}')) = B(\vec{\theta}') = \int d\theta_i \, F(\theta_i)$

* Scaling a grassman variables:

For ordinary variables if $y = \lambda x$

$$\Rightarrow dy = \lambda\, dx$$

" grassman " if $\varphi_i = \lambda \theta_i$

$$\Rightarrow d\varphi_i = \lambda^{-1} d\theta_i$$

where, λ even fun.

Proof:

Let, $F(\vec{\theta}) = A(\vec{\theta}') + \theta_i B(\vec{\theta}')$

$$\Rightarrow \int d\theta_i\, F(\vec{\theta}) = B(\vec{\theta}')$$

Now, let $\theta_i = \lambda^{-1} \varphi_i$

$$\int d\phi_i\, F(\vec{\theta}) = \int d\phi_i \left\{ A(\vec{\theta}') + \lambda^{-1} \varphi_i B(\vec{\theta}') \right\}$$

$$= \lambda^{-1} B(\vec{\theta}')$$

$$= \lambda^{-1} \int d\theta_i\, F(\vec{\theta})$$

$$\Rightarrow \boxed{d\varphi_i = \lambda^{-1} d\theta_i} \Longleftrightarrow \boxed{\varphi_i = \lambda \theta_i}$$

* Generalization:

For ordinary variables $y_i = A_{ij} x_j$

$$\Rightarrow \prod_i^n dy_i = (\det A) \prod_i^n dx_i$$

For Grassman " : $\phi_i = A_{ij} \theta_j$ ↳ even

$$\Rightarrow \prod_i^n d\phi_i = (\det A)^{-1} \prod_{i=1}^n d\theta_i$$

Proof:

$$F = \sum_{k=0}^{n} \frac{1}{k!} A^{(k)}_{i_1 \cdots i_k} \theta_{i_1} \cdots \theta_{i_k}$$

$$\int d\theta_1 \cdots d\theta_n F = \frac{1}{n!} A^{(n)}_{i_1 \cdots i_n} \int d\theta_1 \cdots d\theta_n \, \theta_{i_1} \cdots \theta_{i_n}$$

$$= A^{(n)}_{n(n-1)\cdots 1} \int d\theta_1 \cdots d\theta_n \, \theta_n \theta_{n-1} \cdots \theta_1$$

$\because$ each integration $\to$ same value $\Rightarrow$ nilpotent

Also, $\because$ $\theta_{i_1} \cdots \theta_{i_k} \rightsquigarrow$ anti-symm

$\Rightarrow A^{(n)}_{i_1 \cdots i_n} \rightsquigarrow$ "

$$= A^{(n)}_{n(n-1)\cdots 1}$$

Let, $\theta_i = B_{ij} \phi_j$ ← even

$$\int d\phi_1 \, d\phi_2 \cdots d\phi_n F$$

$$= \int d\phi_1 \cdots d\phi_n \frac{1}{n!} A^{(n)}_{i_1 \cdots i_n} B_{i_1 j_1} \cdots B_{i_n j_n} \phi_{j_1} \cdots \phi_{j_n}$$

$$\phi_{j_1} \cdots \phi_{j_n} = \epsilon_{j_n \cdots j_1} \phi_n \cdots \phi_1$$

take $(j_1 \cdots j_n) = (n(n-1) \cdots 1)$

$$\text{lhs} = \phi_n \cdots \phi_1$$

$$\text{rhs} = \underbrace{\epsilon_{1 \cdots n}}_{1} \phi_n \cdots \phi_1$$

$$= \phi_n \cdots \phi_1$$

So, $$\int d\phi_1 \cdots d\phi_n F = \int d\phi_1 \cdots d\phi_n \frac{1}{n!} A^{(n)}_{i_1 \cdots i_n} B_{i_1 j_1} \cdots B_{i_n j_n} \epsilon_{j_n \cdots j_1} \phi_n \cdots \phi_1$$

$$= \frac{1}{n!} A^{(n)}_{i_1 \cdots i_n} B_{i_n j_n} \cdots B_{i_1 j_1} \epsilon_{j_n \cdots j_1}$$

$$(\det B)\ \epsilon_{i_n \cdots i_1}$$

$$= A^{(n)}_{n \cdots 1} (\det B)$$

So, $\int d\varphi_1 \cdots d\varphi_n\, F = (\det B) \int d\theta_1 \cdots d\theta_n\, F$

$$\therefore\ d\varphi_1 \cdots d\varphi_n = (\det B)\, d\theta_1 \cdots d\theta_n .$$

* Ordinary Variables :

$$\int dx_1 \cdots dx_n \exp\Big[-\tfrac{1}{2} \underbrace{A_{ij}}_{\text{Symmetric}} x^i x^j\Big] = \frac{(2\pi)^{n/2}}{(\det A)^{1/2}}$$

$\propto (\det A)^{-1/2}$ [Proof → go to diagonal → change variables]

Grassman :

$$\int d\theta_1 \cdots d\theta_n \exp\Big[-\tfrac{1}{2} \underbrace{A_{ij}}_{\text{anti-symmetric}} \theta_i \theta_j\Big] \quad \text{with } (n = \text{even})$$

$\propto (\det A)^{1/2}$ $\left[\begin{array}{l} = 0 \text{ for } n = \text{odd} \\ \text{since } \exp(\text{even } \# \theta) \end{array}\right.$

$\rightsquigarrow$ $A = i\underset{\substack{\uparrow \\ \text{Hermitian}}}{H}$ → imaginary eigenvalues

Not possible ⇒ $RAR^T \longrightarrow$ diagonal $\Leftrightarrow RR^T = \mathbb{1}$

But $RAR^T = \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & 0 & a_2 & \\ & & -a_2 & 0 & \\ & & & & \ddots \end{pmatrix} = \tilde{A}$

$$\theta_j = (R^T)_{jk}\,\varphi_k$$

$$\Rightarrow d\theta_1 \cdots d\theta_n = d\varphi_1 \cdots d\varphi_n$$

$$\int d\varphi_1 \cdots d\varphi_n \underbrace{\exp\left[-\tfrac{1}{2}\tilde{A}_{ij}\,\varphi_i\,\varphi_j\right]}_{\exp\left[-a_1\varphi_1\varphi_2 - a_2\varphi_3\varphi_4 - \cdots\right]}$$

$$\int d\varphi_1 \cdots d\varphi_n \,(1 + a_1\varphi_2\varphi_1)(1 + a_2\varphi_4\varphi_3)\cdots$$

$$= a_1 a_2 \cdots a_{n/2} = \sqrt{(\det A)}$$

$$\therefore \int d\theta_1 \cdots d\theta_n \exp\left[-\tfrac{1}{2}A_{ij}\,\theta_i\,\theta_j\right] = (\det A)^{1/2}$$

* Delta function:

Ordinary variables: $\int dx\, \delta(x)\, F(x) = F(0)$.

Grassman " : $\int d\theta_i\, \delta(\theta_i)\, \underbrace{F(\theta_i, \vec{\theta}')}_{A(\vec{\theta}') + \theta_i B(\vec{\theta}')} = \underbrace{F(0, \vec{\theta}')}_{A(\vec{\theta}')}$.

$$\Downarrow$$

$$\boxed{\delta(\theta_i) = \theta_i}$$

$$\Downarrow$$

$$\delta^{(n)}(\vec{\theta}) = \theta_n \cdots \theta_1$$

* Complex Grassman Numbers:

$$\theta_i = \frac{1}{\sqrt{2}}(\phi_i + i\psi_i)$$

$$\theta_i^\dagger = \frac{1}{\sqrt{2}}(\phi_i - i\psi_i)$$

$$d\theta_i\, d\theta_i^\dagger = \left\{\det\begin{pmatrix} 1/\sqrt{2} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}\right\}^{-1} d\phi_i\, d\psi_i = i\, d\phi_i\, d\psi_i$$

$$\int d\theta_i \, d\theta_i^\dagger \, \theta_i^\dagger \theta_i \qquad (\text{no sum over } i)$$

$$i \, d\phi_i \, d\psi_i \times \frac{1}{2} \, 2i \, \phi_i \psi_i$$

$$= - \, d\phi_i \, d\psi_i \, \phi_i \psi_i$$

$$\Rightarrow \int d\theta_i \, d\theta_i^\dagger \, \theta_i^\dagger \theta_i = \int d\phi_i \, d\psi_i \, \psi_i \phi_i = 1 .$$

$\downarrow$

So, $\theta_i, \theta_i^\dagger$ can be thought of as independent Grassman variables.

<u>Check</u>: $\int d\theta_i \, d\theta_i^\dagger \exp\left(A_{ij} \, \theta_i^\dagger \theta_j\right) \propto \det A$.

$\blacksquare$

RW

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$$* \quad A = \int_{-\infty}^{\infty} dx\, e^{-ax^2/2}$$

$$= \frac{1}{\sqrt{a}} \int dx\, e^{-x^2/2}$$

$$= \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} dx\, e^{-x^2/2}$$

$$\Rightarrow A^2 = \frac{1}{a} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}(x^2+y^2)}$$

$$= \frac{1}{a} \cdot 2\pi \int_0^{\infty} dr\, r\, e^{-r^2/2}$$

$$= \frac{2\pi}{a} \cdot \int_0^{\infty} d(r^2/2)\, e^{-r^2/2}$$

$$= \frac{2\pi}{a}$$

$$\Rightarrow \boxed{A = \sqrt{\frac{2\pi}{a}}}$$

$$* \quad C = \int_{-\infty}^{\infty} dx_1 \cdots dx_n \exp\left(-\frac{xAx}{2} + Jx\right)$$

$A \to$ symmetric matrix

Matrix : $xAx \equiv x^T A x \rightarrow 1\times 1$

$Jx \equiv J^T x \rightarrow 1\times 1$

$J : 4\times 1$

$X : 4\times 1$

So,

$$\left.\begin{aligned} & xAx - 2Jx \\ &\equiv x^T A x - 2J^T x \\ &= (x^T A - 2J^T)\, x \\ &= (x^T - 2J^T A^{-1})\, AX \\ &= (X - 2A^{-1}J)^T AX \end{aligned}\right\} \text{motivation}$$

Consider : $(X - A^{-1}J)^T A\, (X - A^{-1}J)$

$$= X^T A X - J^T A^{-1} A X - X^T A A^{-1} J + J^T A^{-1} A A^{-1} J$$

$$= X^T A X - J^T X - X^T J + J^T A^{-1} J$$

Say $X - A^{-1}J \equiv X'$

$$= X^T A X - 2J^T X + J^T A^{-1} J$$

$$\Rightarrow -X^T A X + 2J^T X = -X'^T A X' + J^T A^{-1} J$$

$$\Rightarrow -\frac{1}{2} X^T A X + J^T X = -\frac{1}{2} X'^T A X' + \frac{1}{2} J^T A^{-1} J$$

So,

$$C = \int dx_1 \cdots dx_n \exp\left(-x^T A x \frac{1}{2} + J^T x\right)$$

$$\Big\downarrow \; x \to x' = x - A^{-1} J \quad (\text{Jacobian } 1)$$

$$C = \int dx_1' \cdots dx_n' \exp\left(-\frac{1}{2} x'^T A x'\right) \exp\left(\frac{1}{2} J^T A^{-1} J\right)$$

Now, $A \to$ symmetric matrix $\to$ diagonalizable through orthogonal transformation. Let A_D be the diagonal form of A. So,

$$A_D = O A O^T \qquad O O^T = \mathbb{1} = O^T O$$

$$\Rightarrow A = O^T A_D O$$

$$C = \int dx_1' \cdots dx_n' \exp\left(-\frac{1}{2} x'^T O^T A_D O x'\right) \exp\left(\frac{1}{2} J^T A^{-1} J\right)$$

let, $Y = O X' \to$ rot^n of the integration variables.

$$\Rightarrow X'^T O^T = Y^T$$

& $dy_1 \cdots dy_n = (\text{Jacobian})\, dx_1' \cdots dx_n' = dx_1' \cdots dx_n'$

since det(orthogonal) $= \pm 1$

$$\therefore C = \int dy_1 \cdots dy_n \exp\left(-\frac{1}{2} Y^T A_D Y\right) \exp\left(\frac{1}{2} J^T A^{-1} J\right)$$

Now, $Y^T A_D Y = (y_1 \cdots y_n) \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ & & \ddots \\ 0 & & a_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ $\quad a_i \to$ eigenvalues of A

$$= a_1 y_1^2 + a_2 y_2^2 + \cdots + a_n y_n^2$$

$$\therefore C = \exp\left(\frac{1}{2} J^T A^{-1} J\right) \int dy_1 \cdots dy_n \prod_{i=1}^{n} \exp\left(-\frac{1}{2} a_i y_i^2\right)$$

$$\boxed{C = \exp\left(\frac{1}{2} J^T A^{-1} J\right) \sqrt{\frac{(2\pi)^n}{(\det A)}}}$$

Harmonic Oscillator:

* We would face for fermions:

$$\int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\, e^{iS}\, \psi_{\alpha_1}(x_1)\ldots\psi_{\alpha_n}(x_n)\,\bar{\psi}_{\beta_1}(y_1)\ldots\bar{\psi}_{\beta_m}(y_m)$$

* $$\psi_\alpha(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\, \tilde{\psi}_\alpha(k)$$

$$\bar{\psi}_\alpha(x) \equiv \psi^\dagger_\alpha(x)\gamma^0$$

$$= \int \frac{d^4k}{(2\pi)^4}\cdot e^{-ik\cdot x}\, \tilde{\bar{\psi}}_\alpha(k)$$

Also, $$\bar{\psi}_\alpha(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\, \tilde{\bar{\psi}}_\alpha(k)$$

$$\Rightarrow \boxed{\tilde{\bar{\psi}}_\alpha(k) = \bar{\tilde{\psi}}_\alpha(-k)}$$

* In momentum basis:

$$\int [\mathcal{D}\tilde{\psi}][\mathcal{D}\tilde{\bar{\psi}}]\, e^{iS}\, \tilde{\psi}_{\alpha_1}(k_1)\ldots\tilde{\psi}_{\alpha_n}(k_n)\,\tilde{\bar{\psi}}_{\beta_1}(p_1)\ldots\tilde{\bar{\psi}}_{\beta_m}(k_m)$$

* Introduce new grassman variables $\tilde{J}_\alpha(k)$, $\tilde{\bar{J}}_\alpha(k)$.

<u>NOTE</u>: $\psi, \bar{\psi}, J, \bar{J}$ $\Rightarrow$ all would mutually anti-commute.

* $\tilde{Z}[\tilde{J}(k), \tilde{\bar{J}}(k)]$

$$\equiv \int [\mathcal{D}\tilde{\psi}][\mathcal{D}\tilde{\bar{\psi}}]$$

$$\exp\left[iS + i\int \frac{d^4k}{(2\pi)^4}\left\{\underbrace{\tilde{\bar{J}}_\alpha(-k)\tilde{\psi}_\alpha(k)}_{\text{relative order among } J \,\&\, \psi \text{ is important.}} + \tilde{\bar{\psi}}_\alpha(-k)\tilde{J}_\alpha(k)\right\}\right]$$

+ve due to the order

* $$\frac{1}{Z[0]}\left(-i(2\pi)^4\frac{\delta}{\delta\tilde{\bar{J}}_{\alpha_1}(-k_1)}\right)\cdots\left(-i(2\pi)^4\frac{\delta}{\delta\tilde{\bar{J}}_{\alpha_n}(-k_n)}\right)\left(i(2\pi)^4\frac{\delta}{\delta\tilde{J}_{\beta_1}(-p_1)}\right)\cdots$$

$$\left(i(2\pi)^4\frac{\delta}{\delta\tilde{J}_{\beta_m}(-p_m)}\right) Z[\tilde{J},\tilde{\bar{J}}]\Big|_{\tilde{J}=0=\tilde{\bar{J}}}$$

Check:

$$i(2\pi)^4 \frac{\delta}{\delta \tilde{J}_{\beta_m}(-p_m)} \Rightarrow \tilde{\bar{\psi}}_{\beta_m}(p_m)$$

Indeed this reproduces

$$\int [\mathcal{D}\tilde{\psi}][D\tilde{\bar{\psi}}]\, e^{iS}\, \tilde{\psi}_{\alpha_1}(k_1)\cdots\tilde{\psi}_{\alpha_n}(k_m)\, \tilde{\bar{\psi}}_{\beta_1}(p_1)\cdots\tilde{\bar{\psi}}_{\beta_m}(p_m).$$

* Free field theory:

$$S_{free} = \int d^4x\, \bar{\psi}(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi(x)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, \tilde{\bar{\psi}}(-k)\left(-\gamma^\mu k_\mu - m\right)\tilde{\psi}(k) \longleftarrow \left[\cancel{\delta x \Rightarrow -x^\mu \gamma_\mu}\right]$$

where, $-\gamma^\mu k_\mu = \not{k} = \gamma^0 k^0 - \vec{k}\cdot\vec{\gamma}$

$$\tilde{Z}_{free}[\tilde{J},\tilde{\bar{J}}] \equiv Z_{free}[J,\bar{J}]$$

$$= \int [\mathcal{D}\tilde{\psi}][\mathcal{D}\tilde{\bar{\psi}}] \exp\Big[i\int \frac{d^4k}{(2\pi)^4}\, \tilde{\bar{\psi}}_\alpha(-k)(\not{k}-m)_{\alpha\beta}\tilde{\psi}_\beta(k)$$

$$+ i\int \frac{d^4k}{(2\pi)^4}\left\{\tilde{\bar{J}}_\alpha(-k)\tilde{\psi}_\alpha(k) + \tilde{\bar{\psi}}_\alpha(-k)\tilde{J}_\alpha(k)\right\}\Big]$$

$$= \int [\mathcal{D}\tilde{\psi}][\mathcal{D}\tilde{\bar{\psi}}] \exp\Big[i\int \frac{d^4k}{(2\pi)^4}\underbrace{\left\{\tilde{\bar{\psi}}(-k) + \tilde{\bar{J}}(-k)(\not{k}-m)^{-1}\right\}}_{\equiv \tilde{\chi}(-k)}(\not{k}-m)$$

$$\underbrace{\left\{\tilde{\bar{\psi}}(k) + (\not{k}-m)^{-1}\tilde{J}(k)\right\}}_{\equiv \tilde{\chi}(k)} - i\int \frac{d^4k}{(2\pi)^4}\, \tilde{\bar{J}}(-k)(\not{k}-m)^{-1}\tilde{J}(k)\Big]$$

So, $\tilde{Z}_{free}[\tilde{J},\tilde{\bar{J}}]$

$$= \int [\mathcal{D}\tilde{\chi}][\mathcal{D}\tilde{\bar{\chi}}] \exp\left[i\int \frac{d^4k}{(2\pi)^4}\, \tilde{\bar{\chi}}(-k)(\not{k}-m)\tilde{\chi}(k)\right]$$

$$\exp\left[-\int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\, \tilde{\bar{J}}_\alpha(-k)\, S_{F\,\alpha\beta}(k,p)\, \tilde{J}_\beta(-p)\right]$$

where,

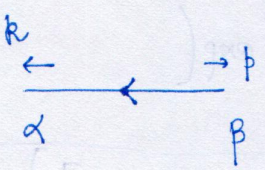
$$S_{F\,\alpha\beta}(k,p) = (2\pi)^4\,\delta^4(k+p)\left(\frac{i}{\not{k}-m}\right)_{\alpha\beta}$$

$$* \ \langle \tilde{\psi}_\alpha(k)\,\tilde{\bar{\psi}}_\beta(p)\rangle$$

$$= \frac{1}{Z[0]}\left(-(2\pi)^4\, i\,\frac{\delta}{\delta\tilde{\bar{J}}_\alpha(-k)}\right)\left((2\pi)^4\, i\,\frac{\delta}{\delta\tilde{J}_\beta(-p)}\right)\tilde{Z}_{free}[\tilde{J},\tilde{\bar{J}}]\Big|_{\tilde{J}=0=\tilde{\bar{J}}}$$

$$= S_{F,\alpha\beta}(k,p)$$

$\Updownarrow$ diagram

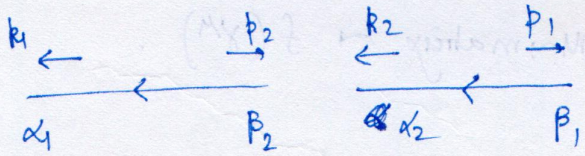


$\psi, \bar{\psi} \Rightarrow$ not same

Convention, to keep track of index $\leftrightarrow$ $\psi, \bar{\psi}$, directed towards $\tilde{\psi}_\alpha$.

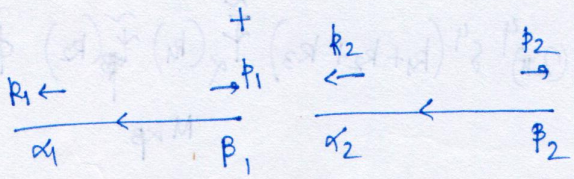
Also note that the arrow goes: $S_{F,\alpha\beta}$

$$* \ \langle \tilde{\psi}_{\alpha_1}(k_1)\,\tilde{\psi}_{\alpha_2}(k_2)\,\tilde{\bar{\psi}}_{\beta_1}(p_1)\,\tilde{\bar{\psi}}_{\beta_2}(p_2)\rangle$$



$$: S_{F\,\alpha_1\beta_2}(k_1,p_2)\,S_{F\,\alpha_2\beta_1}(k_2,p_1)$$

$$+$$



$$: -S_{F\,\alpha_1\beta_1}(k_1,p_1)\,S_{F\,\alpha_2\beta_2}(k_2,p_2)$$

Rules to keep track of sign:

$$\langle \tilde{\psi}_{\alpha_1}(k_1)\,\tilde{\psi}_{\alpha_2}(k_2)\,\tilde{\bar{\psi}}_{\beta_1}(p_1)\,\tilde{\bar{\psi}}_{\beta_2}(p_2)\rangle$$

$$\text{Sign} = (-1)^{\#\text{ crossing} + \#\text{ of }\bar{\psi}\psi\text{ pairs}}$$

2nd one:

$$\tilde{\Psi}_{\alpha_1}(k_1)\,\tilde{\Psi}_{\alpha_2}(k_2)\,\tilde{\bar{\Psi}}_{\beta_1}(p_1)\,\tilde{\bar{\Psi}}_{\beta_2}(p_2)$$

$$(-1)^{1+0} = -1$$

NOTE: Eventually ~~we~~ this rule will hold true even in presence of interaction.

* Interaction:

$$S_{int}\left[\tilde{\Psi}_\alpha(k),\, \tilde{\bar{\Psi}}_\alpha(k)\right]$$

Now,

$$\tilde{Z}[\tilde{J},\tilde{\bar{J}}] = \int [\mathcal{D}\tilde{\Psi}]\,[\mathcal{D}\tilde{\bar{\Psi}}]\; e^{iS_{free}}\; e^{iS_{int}}\; \exp(\quad)$$

$$\Rightarrow \boxed{\tilde{Z}[\tilde{J},\tilde{\bar{J}}] = \exp\left[i\, S_{int}\left[-i(2\pi)^4 \frac{\delta}{\delta \tilde{\bar{J}}_\alpha(-k)},\; i(2\pi)^4 \frac{\delta}{\delta \tilde{J}_\alpha(-k)}\right]\right] \tilde{Z}_{free}[\tilde{J},\tilde{\bar{J}}]}$$

Examples:

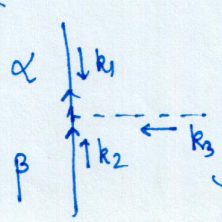
$$iS_{int} = ig\int \bar{\Psi} M \Psi\, \phi\, d^4x$$

scalar field

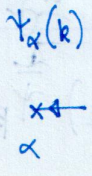
$M \sim$ matrix $\sim f(\gamma^\mu)$.

$$= ig\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\,(2\pi)^4\,\delta^4(k_1+k_2+k_3)\,\underbrace{\tilde{\bar{\Psi}}_\alpha(k_1)\,\tilde{\Psi}_\beta(k_2)}_{M_{\alpha\beta}}\,\tilde{\phi}(k_3)$$

Feynman rules:



$\langle \Psi_\alpha(k)\, \bar{\Psi}_\beta(p) \rangle$



$\Rightarrow$ arrow towards vertex labelling Ψ.

Remember these are not propagator.

Also, arrows must match with propagator's dir^n.

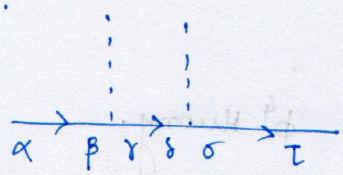
$(2\pi)^4 \delta^4(k_1+k_2+k_3)$ $\quad g\, iM_{\alpha\beta}$

$\xrightarrow{}$ opposite to the dir^n of Feynman diagram

↳ same as in propagator.

↳ cast into form of matrix multiplication.

eg.

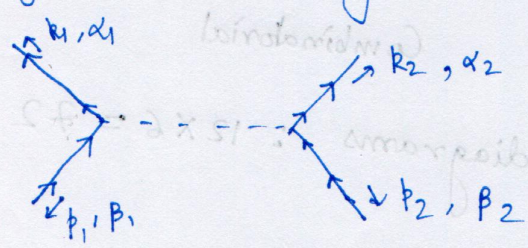


$$\Rightarrow \quad (S_F)_{\tau\sigma} M_{\sigma\delta} (S_F)_{\delta\gamma} M_{\gamma\beta} (S_F)_{\beta\alpha}$$

* $\langle \tilde{\psi}_{\alpha_1}(k_1)\, \tilde{\psi}_{\alpha_2}(k_2)\, \tilde{\bar{\psi}}_{\beta_1}(p_1)\, \tilde{\bar{\psi}}_{\beta_2}(p_2) \rangle$ to order g^2.

[only connected diagrams].

$\Rightarrow$



$$(2\pi)^4 \delta^4(k_1+k_2+p_1+p_2) \cdot (ig)^2 \left(\frac{i}{\not{k}_1 - m} M \frac{i}{-\not{p}_1 - m}\right)_{\alpha_1\beta_1} \left(\frac{i}{\not{k}_2 - m} M \frac{i}{-\not{p}_2 - m}\right)_{\alpha_2\beta_2}$$

$$\frac{i}{-(k_1+p_1)^2 - m^2 + i\epsilon} \quad \frac{1}{2!} \quad \times 2$$

<u>Sign:</u>

$$\langle \tilde{\psi}_{\alpha_1}(k_1)\, \tilde{\psi}_{\alpha_2}(k_2)\, \tilde{\bar{\psi}}_{\beta_1}(p_1)\, \tilde{\bar{\psi}}_{\beta_2}(p_2)\, \tilde{\bar{\psi}}\, \tilde{\psi}\, \phi\, \tilde{\bar{\psi}}\, \tilde{\psi}\, \phi \rangle$$

$(-1)^{7+2} = -1$ [upper]

$(-1)^{4+2} = 1$ [below] → other diagram.

9.12

~~Other diagram~~:

[*] How to test if we've all the Feynman diagrams?

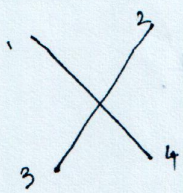
⇒ Take a scalar field theory & given order Feynman diagram.

$2N$ = total no. of lines coming out of external & internal vertices.

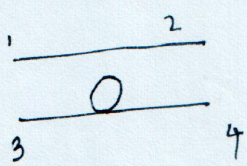
$(2N-1)!!$ = sum of combinatorial factors of all diagrams to this order.

eg. $O(\lambda)$ contribution to $\langle \prod_i^4 \phi(x_i) \rangle$ in ϕ^4 theory.

$2N = 8 \Rightarrow (2N-1)!! = 7 \times 5 \times 3 \times 1 = 105$

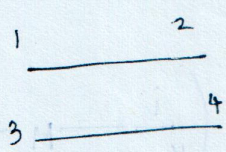


$4! = 24$



$3 \times 4 = 12$ Combinatorial

5 - more diagrams : $12 \times 6 = 72$



Com ⇒ 3

2 - more

$\} \; 3 \times 3 = 9.$

$(24 + 72 + 9) = 105.$

$\boxed{\text{Do it}}$ Find similar rules in a theory containing Dirac fermions & scalars.

Non-Abelian Gauge Theory:

* Recall QED action:

$$S = \int d^4x \, \mathcal{L}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left\{ i\gamma^\mu (\partial_\mu - ieA_\mu) - m \right\} \psi$$

where, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

* Symmetries: S - remains unchanged.

1. $\psi(x) \longrightarrow e^{ie\lambda} \psi(x)$

$A_\mu \to A_\mu$

$\lambda \to$ arbit. const.

2. Gauge symmetry:

$$\psi(x) \longrightarrow e^{ie\lambda(x)} \psi(x)$$

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \lambda(x).$$

* Abelian gauge symmetry:

Successive transformations by $\lambda(x)$ & $\tilde{\lambda}(x)$ commute.

* Covariant derivative:

$$D_\mu \psi \equiv (\partial_\mu - ieA_\mu)\psi$$

$$[D_\mu, D_\nu]\psi = D_\mu D_\nu \psi - D_\nu D_\mu \psi = -ie F_{\mu\nu} \psi$$ do it

$$\Rightarrow \boxed{[D_\mu, D_\nu] = -ie F_{\mu\nu}}$$

* Consider a theory of N- free Dirac fields of same mass:

$$S_f = \int d^4x \, \bar{\psi}^k \left(i\gamma^\mu \partial_\mu - m \right) \psi^k \quad ; \; k = 1, \ldots, N \; ; \text{ summed over.}$$

Symmetry: $\psi^k_{(x)} \longrightarrow U_{kl} \, \psi^l(x)$

$\downarrow$

$N \times N$ unitary matrix i.e. $U^\dagger U = \mathbb{1}$

$$\Rightarrow U_{kl} U^*_{kr} = \delta_{lr}$$

$$\mathcal{L} \rightarrow \underline{U^*_{kl}} \, \bar{\psi}^{(l)} \left(i\gamma_\mu - m \right) \underline{U_{kr}} \, \psi^r$$

$$= \bar{\psi}^l \left(i\gamma - m \right) \psi^l$$

$\psi' = U\psi$

$\Rightarrow (\psi')^{kl} = (U\psi)^{kl} = U_{kl}\psi_{l1}$

$\Rightarrow \psi'^\dagger = \psi^\dagger U^\dagger$

$\Rightarrow (\psi'^\dagger)^{lk} = (\psi^\dagger U^\dagger)^{lk} = (\psi^\dagger)^{lr} (U^\dagger)_{rk}$

$= (\psi^\dagger)^{lr} U^*_{kr}$

Component form:

$$\psi^k_\alpha \longrightarrow U_{kl} \, \psi^l_\alpha$$

$$\bar{\psi}^k_\alpha = \psi^{k*}_\beta (\gamma^0)_{\beta\alpha} \longrightarrow U^*_{kl} \, \psi^{l*}_\beta (\gamma^0)_{\beta\alpha} = U^*_{kl} \bar{\psi}^l_\alpha$$

[$\rightarrow$ NOT a gauge symmetry since $U \neq U(x)$]

$\Rightarrow$

$$\underset{\text{num}}{(\psi'^*)^{kl}} = \underset{\text{num}}{(\psi^*)^{rl}} \, \underset{\text{num}}{U^*_{kr}}$$

$$= U^*_{kr} (\psi^*)^{rl}$$

$$\Rightarrow \boxed{(\psi'^\dagger)^{lk} = U^*_{kr} (\psi^\dagger)^{lr}}$$

whereas

$$\boxed{(\psi')^{kl} = U_{kr} \psi^{rl}}$$

Non-abelian:

$\psi \rightarrow U\psi$ followed by $\psi \rightarrow \tilde{U}\psi$

$\Rightarrow \psi \rightarrow \tilde{U} U \psi$

If $\psi \rightarrow \tilde{U}\psi$ followed by $\psi \rightarrow U\psi$

$\Rightarrow \psi \rightarrow U\tilde{U}\psi$

But in general, $U\tilde{U} \neq \tilde{U}U$

$\longrightarrow$ Non-abelian symmetry with group $U(N)$.

Subset of unitary matrices with $\det U = 1$ are called $SU(N)$ matrices.

* Goal : Make $\underline{\text{this}}$ into a gauge symmetry.

$\downarrow$

$U(N)$ group or $SU(N)$ subgroup.

$\Longrightarrow \psi^k(x) \to U_{kl}(x)\psi^l(x)$ into a symmetry.

* $\mathcal{L}_{free} = \bar{\psi}^k(i\gamma^\mu\partial_\mu - m)\psi^k$

$$\longrightarrow U^*_{kl}\bar{\psi}^l(i\gamma^\mu\partial_\mu - m)U_{kr}(x)\psi^r$$

$$= \underbrace{U^*_{kl}(x)\bar{\psi}^l\, i\gamma^\mu(\partial_\mu U_{kr}(x))\psi^r}_{\text{Extra term}} + \bar{\psi}^l(i\gamma^\mu\partial_\mu - m)\psi^l$$

$\downarrow$

need to be compensated

$\downarrow$

Add to the action : $\int d^4x\, \mathcal{L}_{int}$

with $\mathcal{L}_{int} = \bar{\psi}^l(x)\gamma^\mu\psi^{(r)}(x)\, S_{\mu lr}$ s.t. this compensates the extra term. where $S_{\mu lr}$ is some ~~new~~ function of some new fields.

Suppose, that under the same transformation $S_{\mu lr} \longrightarrow S'_{\mu lr}$.

Then $\mathcal{L}_{int} \longrightarrow U^*_{ll'}\bar{\psi}^{l'}\gamma^\mu\psi^{r'}U_{rr'}S'_{\mu lr}$

$$= \bar{\psi}^{l'}\gamma^\mu\psi^{r'}(U^\dagger S'_\mu U)_{l'r'}$$

$$= \bar{\psi}^l\gamma^\mu\psi^r(U^\dagger S'_\mu U)_{lr}$$

$\hookrightarrow$ should be equal to the $\mathcal{L}_{int}$ + (-ve) of extra term.

Comparing coef of $\bar{\psi}^l\gamma^\mu\psi^r$

$$\boxed{(U^\dagger S'_\mu U)_{lr} = (S_\mu)_{lr} - i(U^\dagger\partial_\mu U)_{lr}}$$

As a matrix equation

$$U^\dagger S'_\mu U = S_\mu - i\, U^\dagger \partial_\mu U$$

$$\Rightarrow \boxed{S'_\mu = U S_\mu U^\dagger - i\,(\partial_\mu U)\, U^\dagger}$$

<u>Proof:</u>

1. If U is unitary, then $i(\partial_\mu U)U^\dagger$ is hermitian.

$$\Rightarrow \quad \partial_\mu (U^\dagger U) = \partial_\mu (I) = 0 = \partial_\mu (U U^\dagger)$$

$$\Rightarrow (\partial_\mu U^\dagger) U + U^\dagger \partial_\mu U = 0$$

$$\Rightarrow (\partial_\mu U) U^\dagger + U \partial_\mu U^\dagger = 0$$

$$\Rightarrow (\partial_\mu U) U^\dagger + (\partial_\mu U\, U^\dagger)^\dagger = 0$$

$\Rightarrow (\partial_\mu U) U^\dagger$ is anti-hermitian.

$\Rightarrow i(\partial_\mu U) U^\dagger$ " hermitian.

2. If $\det U = 1$ i.e. $U \in SU(N)$.

$$\Rightarrow \ln \det U = 0$$

$$\Rightarrow \mathrm{Tr} \ln U = 0$$

$$\partial_\mu \Rightarrow \partial_\mu (\mathrm{Tr} \ln U) = 0$$

$$\Rightarrow \mathrm{Tr}\,(\partial_\mu \ln U) = 0$$

$$\Rightarrow \mathrm{Tr}\,(U^{-1} \partial_\mu U) = 0$$

$$\Rightarrow \mathrm{Tr}\,(\partial_\mu U\, U^\dagger) = 0.$$

Conclusion: For $U(N)$ gauge theory, S_μ is a hermitian matrix.

For $SU(N)$ " " , S_μ " " traceless hermitian matrix.

$\longrightarrow$ Minimal possible modification.

$\longrightarrow$ more general form can be chosen but that has no reason to ~~cal~~ cancel the unwanted term.

* T^a : a linearly independent set of (traceless) hermitian matrices.

$N=2$; $SU(2) \longrightarrow$ Pauli

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \ \mathrm{tr}\, \sigma_i = 0 .$$

& for $U(2)$

additional $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \mathrm{tr}\, I \neq 0$.

We postulate $(S_\mu)_{lr} = \sum_a B^a_\mu(x) (T^a)_{lr}$ $\qquad l, r : 1, 2, \ldots, N$

$a : 1, 2, \ldots, N^2-1 \quad SU(N)$

$1, 2, \ldots, N^2 \quad U(N)$

elementary (real) part of new field (not matrices)

Then $S'_\mu = B'^a_\mu T^a = B^a_\mu \underbrace{U T^a U^\dagger}_{\text{hermitian}} \underbrace{- i (\partial_\mu U) U^\dagger}_{\text{hermitian}}$

$\sum_b T^b R_{ba}(x)$ depends on U. $\qquad = \sum_a \alpha^a_\mu(x) T^a$ depends on U (real).

$$\Rightarrow B'^a_\mu T^a = B^b_\mu T^a R_{ab}(x) + \alpha^a_\mu(x) T^a$$

$$\Rightarrow \boxed{B'^a_\mu(x) = R_{ab} B^b_\mu(x) + \alpha^a_\mu(x)} .$$

Also it: for $U(1)$ this gives us the earlier gauge transformation.

* Gauge invariant action:

$$S = \int d^4x \left[\bar{\psi}^k(x) \left(i \gamma^\mu \partial_\mu - m \right) \psi^k(x) + \bar{\psi}^l(x) \gamma^\mu \psi^r(x) \underbrace{B^a_\mu(x) (T^a)_{lr}}_{S_{\mu lr}} \right]$$

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(1)

2

* Notation:

$a: 1,2,\dots,N^2-1 \Leftarrow SU(N) \Leftarrow$ # of generators

$1,2,\dots,N^2 \Leftarrow U(N) \Leftarrow$ "

$l,m: 1,2,\dots,N$ (# of Dirac fields)

μ,ν: ST indices $\Rightarrow 0,1,2,3$

α,β: Dirac's spinor indices $\Rightarrow 1,2,3,4$

$$S_{\mu\, lm} = B^a_\mu(x)(T^a)_{lm}$$

matrix

$$S_\mu \equiv \underset{\sim}{B}_\mu \equiv B^a_\mu(x)\, T^a$$

number (fun. of x)

$$S = \int d^4x \left[\bar\psi^k (i\gamma^\mu \partial_\mu - m)\psi^k + \bar\psi^l(x)\gamma^\mu \psi^m(x) B^a_\mu(x)(T^a)_{lm} \right]$$

$$= \int d^4x\, \bar\psi \left(i\gamma^\mu \partial_\mu + \gamma^\mu \underset{\sim}{B}_\mu - m \right)\psi$$

where, $\psi \to \psi^k_\alpha \in 4N$ dimension ; $k = 1,2,\dots,N$

α: spinor indices

eg. $(\gamma^\mu\psi)^k_\alpha = \gamma^\mu{}_{\alpha\beta}\psi^k_\beta$

$(\underset{\sim}{B}_\mu\psi)^k_\alpha = (\underset{\sim}{B}_\mu)_{kl}\psi^l_\alpha$

and so, $(\gamma^\mu \underset{\sim}{B}_\nu \psi)^k_\alpha = (\underset{\sim}{B}_\nu \gamma^\mu \psi)^k_\alpha$

because LHS $= \gamma^\mu{}_{\alpha\beta}(\underset{\sim}{B}_\nu)_{kl}\psi^l_\beta$

RHS $= (\underset{\sim}{B}_\nu)_{kl}(\gamma^\mu)_{\alpha\beta}\psi^l_\beta$

} Same

$\gamma^\mu \to$ acts on α,β VS.

$\underset{\sim}{B}_\mu \to$ " " kl VS.

} Commute.

Now, $\boxed{S = \int d^4x\, \bar\psi (i\gamma^\mu D_\mu - m)\psi}$; $D_\mu\psi \equiv (\partial_\mu - i\underset{\sim}{B}_\mu)\psi$

$$\Rightarrow (D_\mu\psi)^k_\alpha = \partial_\mu\psi^k_\alpha - i(\underset{\sim}{B}_\mu)_{kl}\psi^l_\alpha$$

* Gauge transformation Laws:

$$\underset{\sim}{B}_\mu \to U\underset{\sim}{B}_\mu U^\dagger - i\partial_\mu U U^\dagger$$

Now, $\psi \to U\psi \Rightarrow \psi^k \to U_{kl}\psi^l \Rightarrow \psi^k_\alpha \to U_{kl}\psi^l_\alpha$

Then,

$$D_\mu \psi \to \left(\partial_\mu - iU\underset{\sim}{B}_\mu U^\dagger - \partial_\mu U\, U^\dagger\right) U\psi$$

$$= \cancel{\partial_\mu U}\,\psi + U\partial_\mu \psi - iU\underset{\sim}{B}_\mu \psi - \cancel{\partial_\mu U}\,\psi$$

$$= U\left(\partial_\mu \psi - i\underset{\sim}{B}_\mu \psi\right)$$

$$= U D_\mu \psi .$$

Under these 3-transformations simultaneously $S \to S$.

* $D_\nu D_\mu \psi \longrightarrow U D_\nu D_\mu \psi$

$$\Rightarrow [D_\mu, D_\nu]\psi \longrightarrow U[D_\mu, D_\nu]\psi .$$

* $D_\mu D_\nu \psi$

$$= (\partial_\mu - i\underset{\sim}{B}_\mu)(\partial_\nu - i\underset{\sim}{B}_\nu)\psi$$

$$= \partial_\mu\partial_\nu\psi - i(\partial_\mu \underset{\sim}{B}_\nu)\psi - i\underset{\sim}{B}_\nu \partial_\mu \psi - i\underset{\sim}{B}_\mu \partial_\nu \psi - \underset{\sim}{B}_\mu \underset{\sim}{B}_\nu \psi$$

$[D_\mu, D_\nu]\psi$

$$= D_\mu D_\nu \psi - D_\nu D_\mu \psi .$$

$$= -i(\partial_\mu \underset{\sim}{B}_\mu - \partial_\nu \underset{\sim}{B}_\mu)\psi - [\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu]\psi$$

$= -i\,\underset{\sim}{G}_{\mu\nu}\psi$ where, $\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu]$.

doesn't contain any derivative terms acting on ψ.

* Transformation of $\underset{\sim}{G}_{\mu\nu}$:

$$\underset{\sim}{G}_{\mu\nu} \longrightarrow \underset{\sim}{G}'_{\mu\nu}$$

$$[D_\mu, D_\nu]\psi \longrightarrow U[D_\mu, D_\nu]\psi$$

$$\Rightarrow \underset{\sim}{G}_{\mu\nu}\psi \longrightarrow U\underset{\sim}{G}_{\mu\nu}\psi$$

But $\longrightarrow \underset{\sim}{G}'_{\mu\nu} U\psi$

should be same $\Rightarrow \underset{\sim}{G}'_{\mu\nu} U = U\underset{\sim}{G}_{\mu\nu}$

$$\Rightarrow \boxed{\underset{\sim}{G}'_{\mu\nu} = U\underset{\sim}{G}_{\mu\nu}U^\dagger}$$

$\underline{U(1)}$: $U = e^{i\lambda(x)}$

$D_\mu = \partial_\mu - iA_\mu$; $\underset{\sim}{B}_\mu \Rightarrow A_\mu$

$D_\mu\psi \longrightarrow \cdots = UD_\mu\psi$

$[D_\mu, D_\nu]\psi = -iF_{\mu\nu}\psi \longrightarrow U[D_\mu, D_\nu]\psi$

$\underset{\sim}{G}_{\mu\nu} = F_{\mu\nu} \xrightarrow{U} F_{\mu\nu} = e^{i\lambda}F_{\mu\nu}e^{-i\lambda} = F_{\mu\nu}$

electron charge $e = 1$ (say)

Exercise: Verify this directly by writing $\underset{\sim}{G}_{\mu\nu}$ in terms of $\underset{\sim}{B}_\mu$.

* See: $\underset{\sim}{G}^{\dagger}_{\mu\nu} = \underset{\sim}{G}_{\mu\nu}$. use $\underset{\sim}{B}^{\dagger}_{\mu} = \underset{\sim}{B}_\mu$
→ Hermitian

So, $\Rightarrow \underset{\sim}{G}_{\mu\nu} = G^a_{\mu\nu}(x) T^a$

$$= (\partial_\mu B^a_\nu) T^a - (\partial_\nu B^a_\mu) T^a - i B^b_\mu B^c_\nu \underbrace{[T^b, T^c]}_{\text{anti hermitian}}$$

$$= T^a \underbrace{\{\partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{bca} B^b_\mu B^c_\nu\}}_{G^a_{\mu\nu}(x)} \quad \cancel{if^{bca}}$$

$\| \; if^{bca} T^a$ ↓ cons.

→ like $\partial_\mu A_\nu - \partial_\nu A_\mu$

and other term is extra due to non-abelian group nature.

$$\underset{\sim}{G}_{\mu\nu} \equiv G^a_{\mu\nu} T^a \longrightarrow U G^a_{\mu\nu} T^a U^\dagger$$

Remember: $G^a_{\mu\nu}$ → number.

$$= G^a_{\mu\nu} T^b R_{ba}$$

$$= G^b_{\mu\nu} T^a R_{ab}$$

$$\Rightarrow G^a_{\mu\nu} \to R_{ab} G^b_{\mu\nu}$$

Construction of term in S analogous to $F_{\mu\nu} F^{\mu\nu} \Rightarrow$

See $\mathrm{Tr}(\underset{\sim}{G}_{\mu\nu} \underset{\sim}{G}^{\mu\nu}) \longrightarrow \mathrm{Tr}(U \underset{\sim}{G}_{\mu\nu} U^\dagger U \underset{\sim}{G}^{\mu\nu} U^\dagger) = \mathrm{Tr}(\underset{\sim}{G}_{\mu\nu} \underset{\sim}{G}^{\mu\nu})$

i.e. Gauge invariant by itself

NOTE: Here "Trace" is on NXN matrices ⇒ wrt indices l, m

Remember gauge transformation ⇒ in the l, m space $[U_{kl}]$.

Full action

$$\boxed{S = \int d^4x \left[-\frac{1}{2g^2} \mathrm{Tr}(\underset{\sim}{G}_{\mu\nu} \underset{\sim}{G}^{\mu\nu}) + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \right]}$$

g → arbit cons. $\mathcal{L}_{gauge}$ $\mathcal{L}_{fermion}$

* Assumption:

$$\boxed{Tr\,(T^a T^b) = \frac{1}{2}\delta_{ab}}$$

eg. for $SU(2)$

$$T^a = \frac{1}{2}\sigma^a \text{ (Pauli)}.$$

$U(2)$

$$T^4 = \frac{1}{2} I$$

$$\Leftarrow Tr\,(G_{\mu\nu} G^{\mu\nu}) = Tr\,(G^a_{\mu\nu} G^{b\mu\nu} T^a T^b)$$
$$= G^a_{\mu\nu} G^{b\mu\nu} \frac{1}{2}\delta^{ab}$$
$$= \frac{1}{2} G^a_{\mu\nu} G^{b\mu\nu}$$

- $\mathcal{L}_{gauge} \Rightarrow -\frac{1}{4g^2} G^a_{\mu\nu} G^{a\mu\nu}$

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{bca} B^b_\mu B^c_\nu$$

$$D_\mu \psi = (\partial_\mu - i B_\mu)\psi$$

$\longleftrightarrow$ ED

$-ieA_\mu$

important in perturbative expansion in QED

Introduce $\boxed{A^a_\mu = \frac{1}{g} B^a_\mu}$

$$\Rightarrow B^a_\mu = g A^a_\mu$$

Then, $G^a_{\mu\nu} = g\,(\underbrace{\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu}_{F^a_{\mu\nu}})$

So, $\boxed{\mathcal{L}_{gauge} = -\frac{1}{2} F^a_{\mu\nu} F^{a\mu\nu}}$.

We dont want any extra parameter in front of propagator but vertex term.

See, Quadratic $\propto 1$, no parameter g.
Cubic term $\propto g$
quartic " $\propto g^2$

} in $F^a_{\mu\nu} F^{a\mu\nu}$ *

Now, $D_\mu \psi = (\partial_\mu - i g A_\mu)\psi$

$$\bar{\psi}(i\gamma^\mu D_\mu - m)\psi = \bar{\psi}(i\gamma^\mu(\partial_\mu - i g A_\mu) - m)\psi$$

* Unlike abelian gauge theory, the free non-abelian theory contains

interaction terms (cubic, quartic).

* Coupling to scalars:

N-complex scalar fields

$$\phi^k = \frac{1}{\sqrt{2}}\left(\varphi_R^k + i\varphi_I^k\right) \qquad k: 1 \text{ to } N$$

$$\mathcal{L}_{free} = -\frac{1}{2}\partial_\mu \phi_I^k \partial^\mu \phi_I^k - \frac{1}{2}\partial_\mu \phi_R^k \partial^\mu \phi_R^k - \frac{1}{2}m^2\left(\phi_I^k \phi_I^k + \phi_R^k \phi_R^k\right)$$

$$= -\partial_\mu \phi^{k*} \partial^\mu \phi^k - m^2 \phi^{k*}\phi^k$$

$$= -\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$$

where, $\phi = \begin{pmatrix} \phi^1 \\ \vdots \\ \phi^N \end{pmatrix}$

Suppose under $SU(N)$ or $U(N)$ gauge transformation

$$\phi \longrightarrow U(x)\,\phi .$$

$\mathcal{L}_{free}$ is not invariant.

Replace ∂_μ by D_μ.

$$\mathcal{L}_{scalar} = -(D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi$$

where, $D_\mu = \partial_\mu - i\underset{\sim}{B}_\mu = \partial_\mu - ig\underset{\sim}{A}_\mu$

<u>Note</u>: Same coupling constant (no freedom to change coupling constant) for fermionic or scalar interaction unlike abelian gauge theory.

[*] $SU(N) \times SU(M)$ Gauge theory :

$\underset{\sim}{B}_\mu$ (N×N Hermitian matrices) $\longrightarrow \underset{\sim}{G}_{\mu\nu}$

$\underset{\sim}{C}_\mu \longrightarrow \underset{\sim}{H}_{\mu\nu}$.

~~No coup~~

$$\mathcal{L}_{gauge} = -\frac{1}{2g_1^2}\,\mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\underset{\sim}{G}^{\mu\nu}\right) - \frac{1}{2g_2^2}\,\mathrm{Tr}\left(\underset{\sim}{H}_{\mu\nu}\underset{\sim}{H}^{\mu\nu}\right)$$

No coupling between two $\Rightarrow$ trivial.

Introduce coupling : fermions ~~&~~ or scalars

Possibility :

① $D_\mu\psi = \partial_\mu\psi$ (ψ do not transform under GT).

② Transforms under SU(N) but not under SU(M).

$$\psi^k \longrightarrow \underset{\underset{SU(N)}{\wr}}{U_{kl}}\,\psi^l$$

$\psi \to \psi^l_\alpha \Rightarrow 4N$
$k, l : 1, 2, \ldots, N$ and $\alpha = 1, 2, 3, 4$

So, $D_\mu\psi = (\partial_\mu - i\underset{\sim}{B}_\mu)\psi$

③ Reverse of ②. $\psi^r \longrightarrow V_{rs}\psi^s$; V is SU(M) matrix.

$\Rightarrow D_\mu\psi = (\partial_\mu - i\underset{\sim}{C}_\mu)\psi$ $\qquad r, s : 1, 2, \ldots, M$

$\Rightarrow D_\mu\psi^r = \partial_\mu\psi^r - i(\underset{\sim}{C}_\mu)_{rs}\psi^s$

④ ψ transforms under both :

$$\psi^{k,r} \longrightarrow U_{kl}(x)\,V_{rs}(x)\,\psi^{l,s}$$

$$\Rightarrow (D_\mu\psi)^{k,r} = \partial_\mu\psi^{k,r} - i(\underset{\sim}{B}_\mu)_{kl}\psi^{l,r} - i(\underset{\sim}{C}_\mu)_{rs}\psi^{k,s}$$
$$= \partial_\mu\psi^{k,r} - i\left((\underset{\sim}{B}_\mu)_{kl}\,\delta_{rs} + \delta_{kl}(\underset{\sim}{C}_\mu)_{rs}\right)\psi^{l,s}$$

$$D_\mu \psi = \partial_\mu \psi - i \underset{\sim}{B}_\mu \psi - i \underset{\sim}{C}_\mu \psi$$

$$(D_\mu \psi)^{k\ell} \longrightarrow U_{k\ell}(g)\, V_{\ell s}(g)\, (D_\mu \psi)^{ls}$$: $D_\mu \psi$ should transform this way.

and

$$\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i\,[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu]$$

$$\underset{\sim}{H}_{\mu\nu} = \partial_\mu \underset{\sim}{C}_\nu - \partial_\nu \underset{\sim}{C}_\mu - i\,[\underset{\sim}{C}_\mu, \underset{\sim}{C}_\nu] .$$

<u>Summary</u> (So far)

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(12)

⊛ Infinitesimal Gauge Transformation:

* $\mathbb{1}_N \in U(N), SU(N)$

* Infinitesimal gauge transformation : $U = \mathbb{1} - i\epsilon\Omega$

Ω → $N \times N$ matrix

ϵ → infinitesimal

$U^\dagger U = \mathbb{1}_N$ (to order ϵ).

$\Rightarrow \boxed{\Omega^\dagger = \Omega} \Longleftarrow$ unitarity of U demands hermiticity of Ω.

Furthermore, for $SU(N)$: $\det U = 1 \Rightarrow \mathrm{tr}\,\Omega = 0$.

$$U = \begin{pmatrix} 1 - i\epsilon\Omega_{11} & -i\epsilon\Omega_{12} & \cdots \\ -i\epsilon\Omega_{21} & 1 - i\epsilon\Omega_{22} & \cdots \\ \vdots & \cdots & \cdots \end{pmatrix}$$

$$\Rightarrow \det U = (1 - i\epsilon\Omega_{11})(1 - i\epsilon\Omega_{22})\cdots$$

$$= 1 - i\epsilon(\Omega_{11} + \Omega_{22} + \cdots + \Omega_{NN}) \quad \text{(order } \epsilon\text{)}$$

$$\Rightarrow \det U = 1 \Rightarrow \boxed{\mathrm{tr}\,\Omega = 0}$$

So, $\Omega \sim$ hermitian, traceless $\Rightarrow \boxed{\epsilon\Omega = \sum_a \epsilon^a T^a}$
[in l, m space]

$$\Rightarrow U = \mathbb{1} - i\sum_a \epsilon^a T^a$$

Local gauge transf $\sim \boxed{U(x) = \mathbb{1} - i\sum \epsilon^a(x) T^a}$

* $\psi^k \to (U\psi)^k$

$$= \left\{ \left(\mathbb{1} - i\sum \epsilon^a T^a\right)\psi \right\}^k$$

$$= \psi^k \underbrace{- i\sum_a \epsilon^a (T^a)_{kl}\psi^l}_{\delta\psi^k}$$

* Infinitesimal transformation of gauge fields B_μ^a :

Recall: $B_\mu^a \to R_{ab} B_\mu^b + \alpha_\mu^a \quad \Leftarrow \quad \underset{\sim}{B}_\mu \to U \underset{\sim}{B}_\mu U^\dagger - i(\partial_\mu U) U^\dagger$

also, $U T^a U^\dagger = R_{ba} T^b = T^b R_{ba}$

& $-i \partial_\mu U U^\dagger = \alpha_\mu^a T^a$

Infinitesimal transf $\Rightarrow U = \mathbb{1}_N - i \sum_a \epsilon^a T^a$

So, $\Rightarrow (\mathbb{1} - i\epsilon^c T^c) T^a (\mathbb{1} + i\epsilon^b T^b) = R_{ba} T^b$

$$\Rightarrow R_{ba} T^b = T^a - i\epsilon^c T^c T^a + i\epsilon^b T^a T^b + O(\epsilon^2).$$

$$= T^a - i\epsilon^c [T^c, T^a]$$

$$= T^a + \epsilon^c f^{cab} T^b \quad ; \boxed{[T^c, T^a] \equiv i f^{cab} T^b}$$

So, $R_{ba} = \delta_{ba} + \epsilon^c f^{cab}$

$$\Rightarrow \boxed{R_{ab} = \delta_{ab} + \epsilon^c f^{cba}}$$

Now, $\alpha_\mu^a = ?$

$\rightsquigarrow -i \partial_\mu U U^\dagger = \alpha_\mu^a T^a$

$$= -i \left\{ -i \sum_a \partial_\mu \epsilon^a T^a \right\} \left\{ \mathbb{1} + i\epsilon^b T^b \right\}$$

$$= -\partial_\mu \epsilon^a T^a + O(\epsilon^2).$$

$$\Rightarrow \boxed{\alpha_\mu^a = -\partial_\mu \epsilon^a}$$

So, $B_\mu^a \to R_{ab} B_\mu^b + \alpha_\mu^a$

$$= B_\mu^a + \underbrace{\epsilon^c f^{cba} B_\mu^b - \partial_\mu \epsilon^a}_{\delta B_\mu^a}.$$

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* Exercise: Check that

$$\bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \;\&\; -\frac{1}{2g^2}\,\mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\underset{\sim}{G}^{\mu\nu}\right)$$ remain invariant under infinitesimal gauge transformation.

- Need:

$$f^{abc} f^{cde} + f^{cae} f^{bde} + f^{dae} f^{cbe} = 0$$

$\rightsquigarrow$ follows from Jacobi identity

$$\left[\left[T^a, T^b\right], T^d\right] + \left[\left[T^b, T^d\right], T^a\right] + \left[\left[T^d, T^a\right], T^b\right] = 0 .$$

* Why infinitesimal trans?

$\Rightarrow$ Invariance under arbit gauge trans

$\Downarrow$

invariance under infinitesimal gauge trans.

$\Downarrow$

" " arbit gauge trans.

Exceptions are there, for eg. $O(3)$ $\begin{cases} \det = 1 \\ \det = -1 \end{cases}$.

But for $U(N)$ $\rightsquigarrow$ all elements can be built from infinitesimal.

* Suppose, U is $U(N)$ (or $SU(N)$ matrix).

An M dimensional unitary representation is a rule that associates to every $U \in U(N)$ an $M \times M$ unitary matrix $R_G(U)$ s.t.

$$R_G(U_1 U_2) = R_G(U_1)\, R_G(U_2) \quad \forall\; U_1, U_2 \in U(N)$$

$$R_G(\mathbb{1}_N) = \mathbb{1}_M$$

$$R_G(U^{-1}) = \left(R_G(U)\right)^{-1}$$

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eg. ① $R_G(U) = VUV^{-1}$; V fixed matrix

$\equiv U$

② Trivial representation:

$R_G(U) = 1$

③ $(R_G(U))_{ab} = R_{ab}(U)$ $\qquad a, b : 1, 2, \ldots, M$

$U_{kl} \Rightarrow k, l : 1, 2, \ldots, N$

Recall, $UT^aU^{-1} = T^b R_{ba}(U)$

$U_1 T^a U_1^{-1} = T^b R_{ba}(U_1)$

$U_2 T^a U_2^{-1} = T^b R_{ba}(U_2)$

$U_1 U_2 T^a U_2^{-1} U_1^{-1} = T^b R_{ba}(U_1 U_2)$

$U_1 U_2 T^a U_2^{-1} U_1^{-1}$

$T^b R_{ba}(U_2)$

$= U_1 T^b U_1^{-1} \cdot R_{ba}(U_2)$

$= T^c R_{cb}(U_1) R_{ba}(U_2) = T^c R_{ca}(U_1 U_2)$

$\Rightarrow R_{cb}(U_1) R_{ba}(U_2) = R_{ca}(U_1 U_2)$

Other pros are followed from this

$\rightsquigarrow$ Adjoint representation.

Exercise: Prove that R_{ab} is unitary

$UT^aU^{-1} = T^b R_{ba}$; $UT^cU^{-1} = T^d R_{dc}$

$(\dagger)$ $UT^aU^{-1} = T^b R^*_{ba}$

Then $(UT^aU^{-1})(UT^cU^{-1}) = T^b R^*_{ba} T^d R_{dc}$

$\xrightarrow{Tr} Tr(T^aT^c) = R^*_{ba} R_{dc} Tr(T^bT^d)$

$\Rightarrow \cancel{\tfrac{1}{2}} \delta_{ac} = R^*_{ba} R_{dc} \cancel{\tfrac{1}{2}} \delta_{bd} = R^*_{ba} R_{bc}$

matrix (Use $Tr(T^aT^b) = \frac{1}{2}\delta_{ab}$).

$\Downarrow$

cond^n to make sure transformation of basis is unitary.

Also, R_{ba} is real $\rightsquigarrow$ orthogonal matrix.

$\rightsquigarrow$ Real representation.

* Suppose, we've M fermions $\psi^1, \ldots, \psi^M$ transforming in an M-dimensional representation of $U(N)$ or $SU(N)$

$$\psi^k \longrightarrow (R_G(U))_{kl}\, \psi^l \quad \text{under gauge trans } U(\alpha).$$

What would be the action which is gauge invariant?

$\Longrightarrow$

$$* \; R_G(\mathbb{1}_N) = \mathbb{1}_M \qquad (k, l : 1, 2, \ldots, M)$$

$$\Rightarrow \boxed{R_G(\mathbb{1}_N - i\epsilon T^a) \equiv \mathbb{1}_M - \underbrace{i\epsilon R_A(T^a)}_{\text{defn}}}$$

$A \to$ Algebraic represen

$G \to$ Group "

$R_A(T^a)$: $M \times M$ matrix.

Now, we want to calculate

$$R_G\left(\mathbb{1}_N - i\sum_a \epsilon^a T^a\right)$$

$$= R_G\Big(\prod_a (\mathbb{1}_N - i\underbrace{\epsilon^a T^a}_{\text{no sum}})\Big) = \prod_a R_G(\mathbb{1}_N - i\epsilon^a T^a). \quad \Big[\text{using } R_G(U_1U_2) = R_G(U_1)R_G(U$$

$$\Big[\Downarrow O(\epsilon)\Big] \qquad = \prod_a (\mathbb{1}_M - i\epsilon^a R_A(T^a))$$

$$= \mathbb{1}_M - i\sum \epsilon^a R_A(T^a).$$

Exercise:

$$\left[R_A(T^a), R_A(T^b)\right] = i f^{abc} R_A(T^c) \qquad \left[R_A(T) : M \times M\right]$$

follows from order $\epsilon_a \eta_b$ term in

$$R_G(U_1 U_2 U_1^{-1} U_2^{-1}) = R_G(U_1)\, R_G(U_2)\, R_G(U_1^{-1})\, R_G(U_2^{-1})$$

$$U_1 = \mathbb{1}_N - i\epsilon_a T^a$$

$$U_2 = \mathbb{1}_N - i\eta_b T^b.$$

* Define: $\boxed{D_\mu \psi = \partial_\mu \psi - i B_\mu^a R_A(T^a)\psi}$

Exercise: Show that $\bar{\psi}(i\gamma^\mu D_\mu - m)\psi$ is gauge invariant.

$$\left[\Rightarrow : (R_G(U))^*_{kl} (R_G(U))_{kn} = \delta_{ln} \Leftarrow \text{unitarity of } R_G(U)\right]$$

* Weyl fermions:

Left handed $\gamma_5 \psi^k = -\psi^k$

Right " $\gamma_5 \psi^k = \psi^k$

Ex: $\bar{\psi}\psi = 0$ for Weyl fermions.

$$\bar{\psi}\psi = -\bar{\psi}\gamma^5\psi$$
$$= \overline{\gamma^5\psi}\,\psi$$
$$= -\bar{\psi}\psi$$

$\Rightarrow \bar{\psi}\psi = 0$.

Gauge invariant action: $\bar{\psi}\, i\gamma^\mu D_\mu \psi$.

* Majorana fermions:

Suppose, ψ^k denote a set of Mf.

$\psi^k \to$ real. ; $\psi^k \to U_{kl}\psi^l$

Possible to form gau. inv. action { gauge group : $U(N)$ }.

$$\psi^k \to \underbrace{(R_G(U))_{kl}}_{\text{dimension } M \text{ ; real representation.}} \psi^l$$

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Quantization:

* $S_{gauge} = -\int d^4x \, \frac{1}{4g^2} \sum_a G^a_{\mu\nu} G^{a\mu\nu}$

$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{bca} B^b_\mu B^c_\nu$

$S_{gauge} = -\int d^4x \frac{1}{2g^2} Tr(G_{\mu\nu} G^{\mu\nu})$

$= -\int d^4x \frac{1}{2g^2} Tr(G^a_{\mu\nu} G^{b\mu\nu} T^a T^b)$

$= -\int d^4x \frac{1}{2g^2} G^a_{\mu\nu} G^{b\mu\nu} Tr(T^a T^b)$

$= -\int d^4x \frac{1}{4g^2} G^a_{\mu\nu} G^{a\mu\nu}$ (sum over a)

In perturbation theory we want usual kinetic term involving no g. $\Rightarrow$ $\boxed{B^a_\mu \equiv g A^a_\mu}$

So, $S_{gauge} = -\frac{1}{4} \int d^4x \, F^a_{\mu\nu} F^{a\mu\nu}$

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu$

Clearly quadratic term in action is same as was in Maxwell theory. Same as in Maxwell $U(1)$ when $g \to 0$.

* Gauge transformation:

$B^a_\mu \longrightarrow R_{ab} B^b_\mu + \alpha^a_\mu$

$a, b : \begin{cases} 1, 2, \ldots, N^2-1 & SU(N) \\ 1, 2, \ldots, N^2 & U(N) \end{cases}$

$\Rightarrow A^a_\mu \longrightarrow R_{ab} A^b_\mu + \frac{1}{g} \alpha^a_\mu$

Recall: $U T^a U^\dagger = R_{ab} T^b$

$-i \partial_\mu U U^\dagger = \alpha^a_\mu T^a$

Let, Infinitesimal transformation $\Rightarrow$ $\boxed{U \equiv \mathbb{1} - i g \epsilon^a(x) T^a}$ [See the 'g' factor]

Then, Similar to the previous calculation $\Rightarrow R_{ab} = \delta_{ab} + g \epsilon^d f^{dba}$ & $\alpha^a_\mu = -g \partial_\mu \epsilon^a$

Exercise: $A^a_\mu \longrightarrow A^a_\mu + \delta A^a_\mu$

with $\boxed{\delta A^a_\mu = -\partial_\mu \epsilon^a + g f^{bca} \epsilon^b A^c_\mu}$

Exercise: Also, under this U: $D_\mu \psi \to U D_\mu \psi$

Remember: $D_\mu = \mathbb{1} \partial_\mu - i g A^a_\mu T^a$

Exercise: Action remains invariant under this gauge trans.

General: $\begin{cases} D_\mu \phi^b = \partial_\mu \phi^b - i g A^a_\mu (T^a)^{bc} \phi^c \\ D_\mu \phi^l = \partial_\mu \phi^l - i g A^a_\mu (T^a)^{lm} \phi^m \end{cases}$

So, as $g \to 0$, Yang-Mill theory $\rightsquigarrow$ collection of Maxwell theories.

Exercise: Show that if $Tr(T^a T^b) = \frac{1}{2} \delta^{ab}$, then f^{abc} is totally anti-symmetric.

* Quantization :

$$S_{free} = -\frac{1}{4}\int d^4x \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right)$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{A}_\mu^a(-k)\, \underbrace{\left(-k^2\eta^{\mu\nu} + k^\mu k^\nu\right)}_{M^{\mu\nu}(k)}\, \tilde{A}_\nu^a(k)$$

Naive propagator :

$$\langle \tilde{A}_\mu^a(k_1)\, \tilde{A}_\nu^b(k_2)\rangle$$

$$= i(2\pi)^4 \delta^4(k_1+k_2)\, \delta_{ab}\, (M^{-1}(k))_{\mu\nu}$$; Does it work ?

$\Longrightarrow$ No.

But, $M^{\mu\nu}(k)\, k_\nu = 0$

$\downarrow$

eigenvector $\rightarrow$ zero eigenvalue

$\rightarrow$ not invertible.

$\rightarrow$ M^{-1} does not exist.

$\rightarrow$ What is the underlying reason ?

Gauge transf in F.T. $\Rightarrow$ $\delta\tilde{A}_\mu(k) = -ik_\mu \tilde{\epsilon}(k)$ $\left[\begin{array}{c}\text{Since } S_{free} \\ g=0\end{array}\right]$

So, $\delta S_{free} = \int \frac{d^4k}{(2\pi)^4}\, \tilde{A}_\mu(-k)\, M^{\mu\nu}(k)\left(-ik_\nu\, \tilde{\epsilon}(k)\right)$

$\parallel$

0 for arbit $\tilde{A}_\mu(k)$

$\Longrightarrow$ $M^{\mu\nu}(k)\, k_\nu = 0$

$\rightarrow$ so it has zero eigenvalue due to this gauge invariance of the action.

$\rightarrow$ But what is the physical reason ? Go back to the original path integral formulation.

So, Need to go back to path integral.

* Consider a finite dimensional integral:

$$I = \frac{\int \prod_i du_i \exp\left(-\frac{1}{2} A_{ij} u_i u_j\right) u_k u_l}{\int \prod_i du_i \exp\left(-\frac{1}{2} A_{ij} u_i u_j\right)}$$

What's happened when A has zero eigenvalue?

$u_i \longleftrightarrow A^a_\mu(k)$

$\hookrightarrow (\mu, a, k)$

$\hookrightarrow$ discretized.

* Suppose $A = W^\dagger A_d W$

$W \hookrightarrow$ orthogonal

$A_d \downarrow$ diagonal $(a_1, a_2, \ldots)$

Introduce $v_i = w_{ij} u_j \Leftrightarrow v = Wu \Rightarrow u = W^{-1} v = W^T v$

$\Rightarrow u_i = (W^T)_{ij} v_j = w_{ji} v_j$

$\Rightarrow u_i = w_{ji} v_j$

$$I = \frac{\int \prod_i dv_i \exp\left(-\frac{1}{2} \sum_k a_k v_k^2\right) w_{mk} v_m w_{nl} v_n}{\int \prod_i dv_i \exp\left(-\frac{1}{2} \sum a_k v_k^2\right)}$$

$\propto \frac{1}{\sqrt{\det A}}$ i.e. $\frac{1}{\sqrt{a}}$ [Analytically continuable even if a_k's are $-$ve.]

But if some of the a_k's ~~be~~ are zero $\rightarrow$ diverge.

Numerator diverges faster than denominator.

i.e. if $a_{k_0} = 0$ then v_{k_0} integral is problematic.

Action $S \propto \frac{1}{2} \sum A_{ij} u_i u_j = \frac{1}{2} \sum_k a_k v_k^2$

Symmetry: $v_{k_0} \rightarrow v_{k_0} + \epsilon$ (arbit) $\rightarrow$ [as we move along this dir^n action is not ~~dec~~ falling down]

$\rightarrow$ integral won't be finite. (Recall Euclidean action e^{-S} or $i\epsilon$ pres.).

$\delta u_i = w_{ki}\,\epsilon$ $\Longleftrightarrow$ $u_i = w_{ji} v_j \Rightarrow \delta u_i = w_{ji}\,\delta v_j$

$\Rightarrow \delta u_i = w_{ki}\,\epsilon$

Problem will be solved if $w_{mk} v_m w_{nl} v_n \rightarrow$ does not depend on v_{k_0} $\rightarrow$ v_{k_0} integrals are cancelled in num & denominator.

What is in terms of symmetry? [Then we don't have to depend on coordinate system.

$\Rightarrow$ $w_{mk} v_m w_{nl} v_n$ is invariant under the transformation $v_{k_0} \rightarrow v_{k_0} + \epsilon$.

i.e. As long as the integrand is invariant under this symmetry transformation, we can avoid this problem. Hence v_{k_0} integral can be pulled out which will cancel the integral coming from denominator.

Now, this $w_{mk} v_m w_{nl} v_n$ $\equiv$ in the correlation function this corresponds to some operator. So, as long as we choose these operators as gauge invariant, we're done.

* General Case:

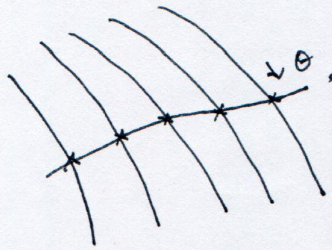
$$I = \frac{\int \prod_i du_i \; e^{iS(\vec{u})} \; f(\vec{u})}{\int \prod_i du_i \; e^{iS(\vec{u})}}$$

Suppose, $S(\vec{u})$ has certain symmetry under $u_i \rightarrow f_i(\vec{u}, \theta)$

set of functions — continuous parameter

$f_i(\vec{u}, \theta = 0) = u_i$

$\vec{u}$ - plane :



"integrate along this dir^n"

$\Leftarrow u_i \to f_i(\vec{u}, \theta)$ as we vary θ

Start from a pt & vary $\theta \to$ generate a curve.

$\to$ Gauge orbits.

$\to$ pts on a curve are related by the symmetry transf.

$\to$ along each orbit S remains constant.

$\to$ along this dir^n S is not changing. But we need suppression to get a well-defined 'integration.

$\to$ nominator as well as denominator ~~can~~ are divergent.

$\to$ best hope whether they cancel. $\to$ equivalently,

$\to$ pick one point from each orbit $\to$ "integrate along this dir^n". [diagram]

$\to$ But it makes sense only if $f(\vec{u})$ is also invariant under the symmetry.

$\to$ One can pick any point along each curve $\to$ ambiguous.

$\to$ We've to cover all ~~new~~ transverse gauge orbits.

$\to$ $\langle F_{\mu\nu} F_{\rho\sigma} \rangle$ well defined but not $\langle A_\mu A_\nu \rangle$

$\downarrow$

not gauge invariant, depends on trans. orbit.

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* Problem:

$$I = \frac{\int \left(\prod_i du_i\right) e^{iS(\vec{u})} F(\vec{u})}{\int \left(\prod_i du_i\right) e^{iS(\vec{u})}}$$

Gauge Symmetry : $u_i \to f_i(\vec{u}, \vec{\theta})$

$$S(\vec{u}) = S(\vec{f}(\vec{u}, \vec{\theta}))$$

$$F(\vec{u}) = F(\vec{f}(\vec{u}, \vec{\theta}))$$

Gauge theory:

$$u_i \to A_\mu^a(x)$$

$$\hookrightarrow \mu, a, x$$

$$\downarrow$$

$$\Delta(i^0, i^1, i^2, i^3)$$

$$\theta_s \to \theta^a(x) \leftarrow \epsilon^a(x).$$

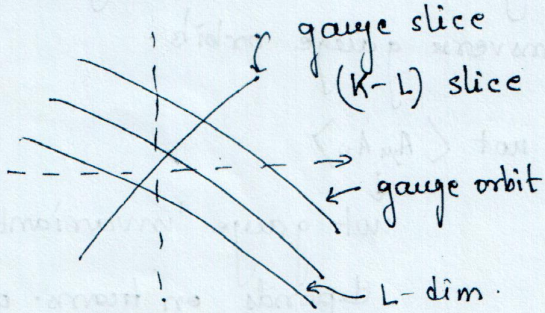
$$\hookrightarrow (a, x)$$

[# $\theta \rightsquigarrow$ actually ∞]

$$U = \exp(i \sum \theta^a T^a)$$

and, $T^{a\dagger} = T^a$ #/# $\Rightarrow UU^\dagger = \mathbb{1}$

U-space:



K : # of u_i's

L : # of θ_s's.

Each gauge orbit L-dimensional.

How do you factor out (K-L) dim subspace out of K dim space?

$\Rightarrow$ need L-constraints: Let, $H_s(\vec{u}) = 0$ for $s = 1, \dots, L$

$\downarrow$

some functions

Change of H_s under infinitesimal gauge transf:

$$f_s(\vec{u},\vec{\theta}) \underset{\substack{\text{small}\\ \vec{\theta}}}{\simeq} f_s(\vec{u},\vec{0}) + \left.\frac{\partial f_s(\vec{u},\vec{\phi})}{\partial \phi_r}\right|_{\vec{\phi}=0} \theta_r + \cdots$$

$\vec{\theta} = \vec{0}$: identity transf ; $f_s(\vec{u},\vec{0}) = \vec{u}_s$.

Now,

$$\delta H_s(\vec{u}) = \frac{\partial H_s(\vec{u})}{\partial u_p}\,\delta u_p$$

under gauge trans with $\vec{\theta}$

$$= \underbrace{\frac{\partial H_s(\vec{u})}{\partial u_p}\left.\frac{\partial f_p(\vec{u},\phi)}{\partial \phi_r}\right|_{\vec{\phi}=\vec{0}}}_{M_{sr}(\vec{u})\,\theta_r} \theta_r + \cdots$$

$$\Rightarrow \boxed{\delta H_s(\vec{u}) = M_{sr}(\vec{u})\,\theta_r} \Leftarrow$$ change of $H_s(\vec{u})$ due to gauge transformation.

Suppose, $M_{sr}(\vec{u})\,n_r = 0$, now if we choose $\theta_r = \epsilon n_r \Rightarrow \delta H_s(\vec{u}) = 0$.

So, if we start with $H_s(\vec{u}) = 0$ at any one of the pts on the gauge orbit $\rightarrow$ as we move along the gauge orbit (due to $\delta H_s(\vec{u}) = 0$) $H_s(\vec{u})$ still remains zero. But, the gauge slice should contain all the pts of $H_s(\vec{u}) = 0$ $\rightarrow$ becomes <u>tangential</u> to the gauge orbit $\rightarrow$ ~~we~~ but we want to avoid this $\Rightarrow H_s(\vec{u}) = 0$ is not a valid constraint when $\det M = 0$.

Another concern $\Rightarrow$ Gribov ambiguity $\rightarrow$ gauge orbit moves a large distance from the origin $\rightarrow$ perturbation around origin is not valid.

Hence, we need $\det M(\vec{u}) \neq 0$ to choose $H_s(\vec{u}) = 0$ constraint.

<u>Naive attempt</u> : Replace I by I' with the def^n:

$$I' = \frac{\int \prod_i du_i \, e^{iS(\vec{u})} \prod_s \delta(H_s(\vec{u})) \, F(\vec{u})}{\int \prod_i du_i \, e^{iS(\vec{u})} \underbrace{\prod_s \delta(H_s(\vec{u}))}_{\text{constraint}}}$$

Check if it works in simple cases:

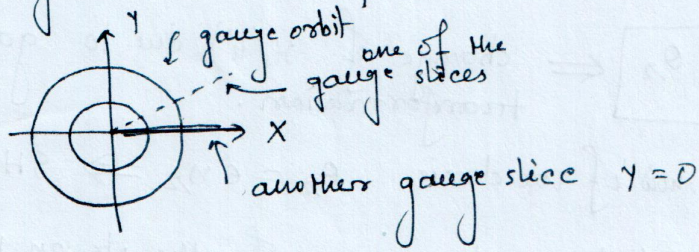
$$I = \frac{\int dx\, dy \; e^{-a(x^2+y^2)} (x^2+y^2)}{\int dx\, dy \; e^{-a(x^2+y^2)}}$$

Symmetry: $x \longrightarrow f_x(x,y) = x\cos\theta + y\sin\theta$

$y \longrightarrow f_y(x,y) = -x\sin\theta + y\cos\theta$.

$$\Rightarrow I = \frac{1}{a}$$

Let's try to do another way $\rightarrow$ 1st fix the symmetry $\Rightarrow$



So,

$$I' = \frac{\int dx\, dy \; e^{-a(x^2+y^2)} (x^2+y^2)\, \delta(y)}{\int dx\, dy \; e^{-a(x^2+y^2)} (x^2+y^2)\, \delta(y)} = \frac{1}{2a} \neq \frac{1}{a}$$

because of the different integration measure $\rightarrow$ $r\,dr$ (I). dr or dx (I').

$\longrightarrow$ it would be corrected if we ~~can~~ start with I & try to factor out θ-integration. We'll do this.

$$* \quad I = \frac{N}{D}$$

$$N = \int \prod_k du_k \, e^{iS(\vec{u})} F(\vec{u})$$

$$D = \int \prod_k du_k \, e^{iS(\vec{u})}$$

Goal: factor out θ-integral.

Use

$$\boxed{\int \left(\prod_r d\theta_r\right) \prod_s \delta\left(H_s\left(\vec{f}(\vec{u},\vec{\theta})\right)\right) \times \det\left(\frac{\partial H_p\left(\vec{f}(\vec{u},\vec{\theta})\right)}{\partial \theta_q}\right) = 1 .}$$

Identity

$$\longleftrightarrow \int d\theta \; \delta(f(\theta))\, f'(\theta) = 1 \qquad (1D) .$$

Now, let, $\theta_s \rightsquigarrow \phi_r = H_r(\vec{f}(\vec{u},\vec{\theta}))$

$$\Rightarrow \int \left(\prod_r d\phi_r\right) \prod_r \delta\phi_r = 1 .$$

$\{ s, r : 1, \ldots, L \rightsquigarrow$ # gauge trans. param.

So,

$$N = \int \left(\prod_k du_k\right) e^{iS(\vec{u})} F(\vec{u})$$

$$\times \int \left(\prod_r d\theta_r\right) \prod_s \left(\delta\, H_s\left(\vec{f}(\vec{u},\vec{\theta})\right)\right) \det\left(\frac{\partial H_p\left(\vec{f}(\vec{u},\vec{\theta})\right)}{\partial \theta_q}\right)$$

Now, ~~in~~ 1st perform u_k integral.

Change variable from $u_k \to v_k = f_k(\vec{u},\vec{\theta})$ | done at fixed $\vec{\theta}$.

Since, this is the gauge transformation $\Rightarrow$

$S(\vec{u}) = S(\vec{v})$: Gauge symmetry

~~since~~ & $F(\vec{u}) = F(\vec{v})$: " since $f_k \rightsquigarrow$ gauge tran.

<u>Assume</u> : $\boxed{\prod_k du_k = \prod_k dv_k}$

[Integration measure is gauge invariant $\Leftarrow$ theory is gauge invariant.] .

So,

$$N = \int \prod_s d\theta_s \int \prod_k dv_k \, e^{iS(\vec{v})} f(\vec{v}) \prod_s \delta\left(H_s(\vec{v})\right) \det\left(\frac{\partial H_p\left(\vec{f}(\vec{u},\vec{\theta})\right)}{\partial \theta_q}\right)$$

1st perform differentiation $\rightarrow$ some fun of $\vec{v}, \vec{\theta}$.

So, $$N = \underbrace{\int \prod_s d\theta_s \ \det(s(\vec{\theta}))}_{\text{is cancelled by } \mathcal{D}} \int \prod_k dv_k \ e^{iS(\vec{v})} \ F(\vec{v}) \prod_s \delta(H_s(\vec{v})) \det(M(\vec{v}))$$

<u>Exercise</u>: Check that when applied to previous problem $\det M = x$

Hence, $$\delta H_s = \frac{\partial H_s(\vec{u})}{\partial u_p} \delta u_p$$

$$= \frac{\partial H_s(\vec{u})}{\partial u_p} \frac{\partial f_p(\vec{u},\vec{\phi})}{\partial \phi_r}\Big|_{\vec{\phi}=0} \theta_r$$

actually $\delta\theta_r = \theta_r - 0 = \theta_r$

$$\Rightarrow \boxed{\delta H_s \equiv M_{sr}(\vec{u})\,\theta_r}$$

$$\Rightarrow M_{sr}(\vec{u}) = \frac{\partial H_s(\vec{u})}{\partial u_p} \frac{\partial f_p(\vec{u},\vec{\phi})}{\partial \phi_r} = \frac{\partial H_s(\vec{f}(\vec{u},\vec{\phi}))}{\partial \phi_r}\Big|_{\vec{\phi}=\vec{0}}$$

and, now we've got

$$\boxed{\frac{\partial H_p(\vec{v})}{\partial \theta_q} = M_{pp'}(\vec{v})\, S_{p'q}(\vec{\theta})}$$

Therefore,

$$N = \underbrace{\int \prod_s d\theta_s \ \det s(\vec{\theta})}_{\text{cancelled against denominator}} \int \prod_k dv_k \ e^{iS(\vec{v})} F(\vec{v}) \cdot \prod_s \delta(H_s(\vec{v})) \det M(\vec{v})$$

effectively

$$\Rightarrow \boxed{N' = \int \prod_k du_k \ e^{iS(\vec{u})} F(\vec{u}) \prod_s \delta(H_s(\vec{u})) \det M(\vec{u})}$$

where, $H_s(\vec{u}) = 0 \rightsquigarrow$ constraints ; $s = 1, \dots, L$

$$M_{sr}(\vec{u}) \equiv \frac{\partial H_s(\vec{u})}{\partial u_p} \frac{\partial f_p(\vec{u},\vec{\phi})}{\partial \phi_r}\Big|_{\phi=0} \theta_r$$

with $u_k \longrightarrow f_k(\vec{u},\vec{\theta})$ are gauge transformation.

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$$* \quad I = \frac{\mathcal{N}}{\mathcal{D}}$$

$$\mathcal{N} = \int \prod_k du_k \, e^{iS(\vec{u})} F(\vec{u})$$

$$\mathcal{D} = \int \prod_k du_k \, e^{iS(\vec{u})}$$

and we arrived

$$\mathcal{N} = \int \prod_s d\theta_s \det s(\vec{\theta}) \, \mathcal{N}' \qquad \text{Haar measure}$$

$$\mathcal{D} = \int \quad - \quad - \quad - \quad \mathcal{D}'$$

where, $\mathcal{N}' = \int \prod_k du_k \, e^{iS(\vec{u})} \prod_s s(H_s(\vec{u})) \det(M(\vec{u})) \, F(\vec{u})$

$\mathcal{D}'$ = same expression with $F(\vec{u}) \to 1$.

→ **NOTE:** Parametrize $SU(2)$: $e^{i\theta^a T^a}$

or

$$e^{i\phi T^3} e^{i\theta T^1} e^{i\psi T^3}$$

$$\prod_a d\theta^a \neq d\phi \, d\theta \, d\psi \quad \text{in general}$$

But $\underbrace{\prod_a d\theta^a \det(\vec{\xi}_1)}_{\text{more natural to integrate over this}} = d\phi \, d\theta \, d\psi \det(\vec{\xi}_2)$.

Here, $\det(s)$ plays the similar role as Jacobian.

This essentially moves $\prod_a d\theta^a$ near origin.

→ This form of $\mathcal{N}'$ is not appropriate to do perturbation theory.

Recall, $\int \prod_r d\theta_r \, d\phi_r \, e^{-M_{rs}\theta_r\phi_s} \propto \det M$; $\theta, \phi \to$ grassman.

Similarly, $\det M(\vec{u}) = \text{const.} \int \prod_r dB_r \, dC_r \exp\left(-i M_{rs}(\vec{u}) B_r C_s\right)$

$B_r, C_r \to$ grassmann variable

field theory $\to$ $B^a(x)$, $C^a(x)$: Fadeev-Popov Ghost fields

NOTE: Here this doesn't imply ~~ϕ~~ fermionic field is appearing. These are not physical → just an intermediate stage to evaluate det M.

[∵ $\beta^a(x)$, $c^a(x)$ do not carry spino index

⇝ Whereas det M wasn't a polynomial, here $M_{sr}(\vec{u})$ is a polynomial.

* N', $\mathcal{D}'$ → independent of $H_s(\vec{u})$. ⇒ H_s's introduced by δ-fun || must be compensated by det M.

* So, we've successfully handled det M converting this into a polynomial. But what about $H_s(\vec{u})$? How to deal with $\delta H_s(\vec{u})$?

* Hence, let's choose

How do we ensure this separation?

$$\boxed{H_s(\vec{u}) = \underbrace{\tilde{H}_s(\vec{u})}_{\text{functions of } \vec{u}} - \underbrace{a_s}_{\text{cons. in the sense it's indep. of } \vec{u}}}$$

ST as well as gauge indep

N' is indep. of a_s.

* $$M_{rs}(\vec{u}) = \left.\frac{\partial H_r(\vec{f}(\vec{u},\vec{\phi}))}{\partial \phi_s}\right|_{\vec{\phi}=0}$$

$$= \frac{\partial H_r(\vec{u})}{\partial u_k} \left.\frac{\partial f_k(\vec{u},\vec{\phi})}{\partial \phi_s}\right|_{\vec{\phi}=0}$$

AIM: $\delta(H_s(\vec{u}))$ ⇒ convert this contribution to some other function which can be handled successfully.

$$N'' = \underset{\text{cancel against denominator}}{\text{cons.}} \underbrace{\int \prod_s da_s \exp\left(-\frac{i}{2\alpha_0}\sum_s a_s^2\right)}_{\text{constant}} N'$$

$$= \int \prod_s da_s \int \prod_k du_k \exp\left(-\frac{i}{2\alpha_0}\sum_s a_s^2\right) e^{iS(\vec{u})} \prod_s \delta\left(\tilde{H}_s(\vec{u}) - a_s\right) f(\vec{u}) \det M(\vec{u}).$$

∵ finite dimensional integration

⇓

order of integ can be swapped.

$$N_{effective} = \int \prod_k du_k \exp\left(-\frac{i}{2\alpha_0}\sum_s \tilde{H}_s(\vec{u})^2\right) \exp\left(iS(\vec{u})\right) \int \prod_r dB_r\, dC_r \exp\left(-iM_{rs}(\vec{u})\, B_r C_s\right)$$

$F(\vec{u})$

$\downarrow$

polynomial

Hence,

$$N_{effective} = \int \prod_k du_k \prod_r dB_r\, dC_r \exp\left[iS(\vec{u}) - \frac{i}{2\alpha_0}\sum_s \tilde{H}_s(\vec{u})^2 - i\, M_{rs}(\vec{u}) B_r C_s\right] F(\vec{u})$$

and everything else is exponential

$\rightarrow$ perturbation theory could be done.

* Full Action:

$$S = \underbrace{S_{gauge\ inv}}_{\text{original}} + S_{gauge\ fixing} + S_{ghost}$$

$$S_{gauge\ fixing} = -\frac{1}{2\alpha_0}\sum_s \tilde{H}_s(\vec{u})^2$$

$$S_{ghost} = -M_{rs}(\vec{u})\, B_r C_s \quad (\text{sum over } r, s)$$

$\downarrow$

this breaks gauge invariance which is compensated by S_{ghost}.

* Back to gauge theory:

$k \rightarrow \mu, a, x$

$s \rightarrow a, x$

Choice of gauge $\tilde{H}_s(\vec{u}) = \partial^\mu A^a_\mu(x)$ $\rightsquigarrow$ remember this should not be gauge invariant.

$$S_{gauge\ fixing} = -\frac{1}{2\alpha_0}\sum \left(\tilde{H}_s(\vec{u})\right)^2$$

$$= -\frac{1}{2\alpha_0}\sum_{a,x} \left(\partial^\mu A^a_\mu(x)\right)\left(\partial^\nu A^a_\nu(x)\right)$$

$\hookrightarrow$ sum over i^0, i^1, i^2, i^3

$x = s \quad (i^0, i^1, i^2, i^3)$

$$= -\frac{1}{2\alpha_0}\frac{1}{\Delta^4}\int d^4x\,(\partial^\mu A^a_\mu(x))(\partial^\nu A^a_\nu(x)) \quad (\text{sum over } a)$$

$$\equiv -\frac{1}{2\alpha}\int d^4x\,(\partial^\mu A^a_\mu(x))(\partial^\nu A^a_\nu(x)) = -\frac{1}{2\alpha}\int d^4x\, H(x,A)H(x,A)$$

Now,

$$\boxed{M_{rs}(\vec{u}) = \frac{\partial H_r}{\partial \phi_s}\left(\vec{f}(\vec{u},\vec{\phi})\right)\Big|_{\vec{\phi}=0}}$$

$$\therefore \boxed{S_{gf} = -\frac{1}{2\alpha}\int d^4x\, H(x,A)\,H(x,A)}$$

$$\Rightarrow M(a,x;b,y)\Big|_{\vec{\phi}=\vec{0}} = \frac{\partial}{\partial\phi_b(y)}\left[\eta^{\mu\mu'}\frac{\partial}{\partial x^{\mu'}}\left(A^a_\mu(x) - \frac{\partial}{\partial x^\mu}\phi^a(x) + g f^{dca}\phi^d(x)A^c_\mu(x)\right)\right]$$

$\Downarrow$

$\phi_{b;j^0,j^1,j^2,j^3}$

$$= \Delta^4\frac{\delta}{\delta\phi_b(y)}\left[\ldots\right]$$

NOTE: Form of M depends on choice of gauge $\tilde{H}_s(\vec{u})$. HERE, $\tilde{H}_s(\vec{u}) = \partial^\mu A^a_\mu$

$$\Rightarrow M(a,x;b,y) = \Delta^4\left[\eta^{\mu\mu'}\frac{\partial}{\partial x^{\mu'}}\left\{-\frac{\partial}{\partial x^\mu}\delta^{(4)}(x-y)\,\delta_{ab} + g f^{dca}\delta^4(x-y)\,\delta_{bd}A^c_\mu\right.\right.$$

$\rightarrow$ at $\vec{\phi}=\vec{0}$.

So, $S_{ghost} = -\sum_{r,s} B_r M_{rs}(\vec{u})\, C_s$

$$= -\frac{1}{\Delta^4}\frac{1}{\Delta^4}\int d^4x\, d^4y\; M(a,x;b,y)\, B^a(x)\, C^b(y).$$

$$= -\frac{1}{\Delta^4}\int d^4x\, B^a(x)\left[\eta^{\mu\mu'}\frac{\partial}{\partial x^{\mu'}}\left\{-\frac{\partial}{\partial x^\mu}C^b(x)\,\delta_{ab} + g f^{dca}C^b(x)\,\delta_{bd}A^c_\mu(x)\right\}\right]$$

absorb in being $B^a(x)$;

$$= -\int d^4x\, B^a(x)\left[-\partial^\mu\partial_\mu C^a(x) + g f^{bca}\partial^\mu\left(C^b(x)A^c_\mu(x)\right)\right]$$

$$= \int d^4x\left[+B_a(x)\,\partial_\mu\partial^\mu C^a(x) + g f^{bca}(\partial^\mu B^a(x))C^b(x)A^c_\mu(x)\right]$$

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Total gauge fixed action:

$$S = S_{\text{gauge-inv.}} + S_{\text{gauge fixing}} + S_{\text{ghost}}$$

where,

$$S_{\text{gauge-inv.}} = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu} F^{a\mu\nu}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu$

$$S_{\text{gauge fixing}} = -\frac{1}{2\alpha}\int d^4x\, \partial_\mu A^{a\mu}\, \partial_\nu A^{a\nu}$$ for gauge choice $\partial^\mu A^a_\mu$

$$S_{\text{ghost}} = \int d^4x \left(-\partial^\mu B^a \partial_\mu C^a + g f^{bca} (\partial_\mu B^a) C^b A^{c\mu}\right)$$

for gauge choice $\partial_\mu A^{\mu a}$

* Goal: derive feynman rules.

* Propagator: set $g=0$. (free)

Gauge field kinetic term:

$$-\frac{1}{4}\int d^4x \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right)$$

$$-\frac{1}{2\alpha}\int d^4x\, \partial_\mu A^{a\mu}\, \partial_\nu A^{a\nu}$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{A}^a_\mu(-k) \underbrace{\left(-k^2\eta^{\mu\nu} + k^\mu k^\nu - \frac{1}{\alpha}k^\mu k^\nu\right)}_{M^{\mu\nu}(k)} \tilde{A}^a_\nu(k).$$

So, $\langle \tilde{A}^a_\mu(k_1)\, \tilde{A}^b_\nu(k_2)\rangle$

$$= (2\pi)^4 \delta^4(k_1+k_2)\, i\, (M^{-1}(k_1))_{\mu\nu}\, \delta_{ab}$$

Now, LI $\Rightarrow$

$$\left(M^{-1}(k)\right)_{\mu\nu} = f(k^2)\,\eta_{\mu\nu} + g(k^2)\,k_\mu k_\nu \quad \text{[Most general form]}.$$

Now, $\left(M^{-1}(k)\right)_{\mu\nu} M^{\nu\gamma} = \delta^{\gamma}_{\mu}$.

$$\Rightarrow \left\{ f(k^2)\eta_{\mu\nu} + g(k^2)k_\mu k_\nu \right\}\left\{ -k^2\eta^{\nu\gamma} + \left(1-\frac{1}{\alpha}\right)k^\nu k^\gamma \right\} = \delta^{\gamma}_{\mu}$$

$$\Rightarrow -k^2 f\,\delta^{\gamma}_{\mu} + \left(1-\frac{1}{\alpha}\right) f\, k_\mu k^\gamma - k^2 g\, k_\mu k^\gamma + \left(1-\frac{1}{\alpha}\right) g\, k^2 k_\mu k^\gamma = \delta_\mu{}^\gamma .$$

Comparing :

$$\delta_\mu{}^\gamma \Rightarrow -k^2 f = 1 \Rightarrow f = -\frac{1}{k^2}$$

$$k_\mu k^\gamma \Rightarrow \left(1-\frac{1}{\alpha}\right) f - \frac{1}{\alpha} g\, k^2 = 0$$

$$\Rightarrow g = \frac{\alpha-1}{k^2} f = -\frac{\alpha-1}{k^4}.$$

So, $\boxed{\left(M^{-1}(k)\right)_{\mu\nu} = -\frac{1}{k^2}\left\{\eta_{\mu\nu} + \frac{\alpha-1}{k^2} k_\mu k_\nu\right\}}$ for gauge choice $\partial^\mu A^a_\mu$

See : $\alpha \to \infty \Rightarrow M^{-1}$ is 'ill defined'.

Case-I : $\alpha = 1$ (Feynman gauge)

$$\left(M^{-1}(k)\right)_{\mu\nu} = -\frac{1}{k^2}\eta_{\mu\nu}$$

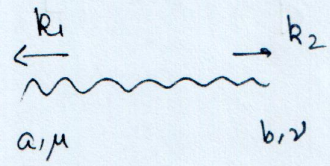
Case-II : $\alpha = 0$ (Lorentz gauge).

$$\left(M^{-1}(k)\right)_{\mu\nu} = -\frac{1}{k^2}\left\{\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right\}.$$

Diagram:

$$M^{-1}(k)_{\mu\nu} = -\frac{1}{k^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \quad ; \quad \langle \tilde{A}^a_\mu(k_1)\, \tilde{A}^b_\nu(k_2)\rangle$$

for general case: $\quad$ ‖ [Free part → propagator]



$$\Rightarrow (2\pi)^4\, \delta^4(k_1+k_2)\, \frac{i}{(-k_1^2)}\left(\eta_{\mu\nu} + (\alpha-1)\frac{k_{1\mu}k_{1\nu}}{k_1^2}\right)\delta_{ab}$$

→ propagator is gauge dependent.

→ expected since A^a_μ are not gauge invariant.

* In the limit $g=0 \Rightarrow F^a_{\mu\nu}$ is gauge invariant.

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x).$$

$$\Rightarrow \tilde{F}^a_{\mu\nu} = i\left[k_\mu \tilde{A}^a_\nu(k) - k_\nu \tilde{A}^a_\mu(k)\right] = i\, k_{[\mu}\tilde{A}^a_{\nu]}(k)$$

$$\langle \tilde{F}^a_{\mu\rho}(k_1)\, \tilde{F}^b_{\nu\sigma}(k_2)\rangle = -\langle k_{1[\mu}\tilde{A}^a_{\rho]}(k_1)\, k_{2[\nu}\tilde{A}^b_{\sigma]}(k_2)\rangle$$

$$= -k_{1[\mu}\langle \tilde{A}^a_{\rho]}(k_1)\, \tilde{A}^b_{[\sigma}(k_2)\rangle k_{2\nu]}$$

$$= (2\pi)^4\delta^4(k_1+k_2)\left(-\frac{i}{k_1^2}\right) i \times i$$

$$k_{1[\mu}\left\{\eta_{\rho][\sigma} + (\alpha-1)\frac{k_{1\rho]}k_{1[\sigma}}{k^2}\right\} k_{2\nu]}\ \delta_{ab}$$

where: $A_{[\mu}B_{\nu]} \equiv A_\mu B_\nu - A_\nu B_\mu$

$$= (2\pi)^4\delta^4(k_1+k_2)\left(-\frac{i}{k_1^2}\right) i^2\ k_{1[\mu}\eta_{\rho][\sigma}k_{2\nu]}\ \delta_{ab}$$

$$= \left[k_{1\mu}\eta_{\rho\sigma}k_{2\nu} - k_{1\rho}\eta_{\mu\sigma}k_{2\nu} - k_{1\mu}\eta_{\rho\nu}k_{2\sigma} + k_{1\rho}\eta_{\mu\nu}k_{2\sigma}\right]$$

→ Clearly α-independent → expected.

* Suppose, $\alpha = 1$

Propagator $\propto \dfrac{i}{-k^2}\, \eta_{\mu\nu}\, \delta_{ab}$.

Poles : at $k^2 = 0$ $\Rightarrow$ massless particles.

$a \to$ gauge inv
$U(N) : N^2$.
$SU(N) = N^2 - 1$
$\to$ # of generat

\# massless particles = # of a's $\times$ 4

$[\eta_{00}, \eta_{11}, \eta_{22}, \eta_{33}]$

But in QED # massless states = 2 $\to$ ~~not mat~~ whereas. This is giving 4

$\to$ not matching !

$\to$ all massless poles do <u>not</u> contribute to the gauge invariant correlation function.

$\to$ Question : Which of these poles contribute to $\langle \tilde{F}^a_{\mu\nu}(k_1)\, \tilde{F}^b_{\sigma\rho}(k_2) \rangle$?

$\Downarrow$

$\rightsquigarrow$ In Euclidean 4D space

$$\delta_{ij} = \sum_{k=1}^{4} n_i^{(k)} n_j^{(k)}$$

Think : $\langle \vec{e}_i | \vec{e}_j \rangle = \delta_{ij}$ ($\vec{e}_i, \vec{e}_j$: O.N.)

n^1, n^2, n^3, n^4 are 4 O'N. vectors.

$\rightsquigarrow$ Here, $k_\mu = (k_0, \vec{k})$, $\bar{k}_\mu \equiv (k_0, -\vec{k})$

$\epsilon^{(i)}_\mu = (0, \vec{\epsilon}^{(i)})$ with $\vec{\epsilon}^{(i)} \cdot \vec{k} = 0$ $\quad i = 1, 2$.

$\vec{\epsilon}^{(i)} \cdot \vec{\epsilon}^{(j)} = \delta_{ij}$

<u>Claim</u> :

$$\boxed{\eta_{\mu\nu} = \frac{k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu}{k \cdot \bar{k}} + \sum_{i=1}^{2} \epsilon^{(i)}_\mu \epsilon^{(i)}_\nu + \underbrace{O(k^2)}_{\text{not contribute to pole.}}}$$

<u>Check</u>: Go to a frame where $k = (k_0, k_1, 0, 0)$

[Rot^{n} s.t. momentum along 1 axis].

$$k^2 = -k_0^2 + k_1^2$$

$$\bar{k} = (k_0, -k_1, 0, 0)$$

$$k \cdot \bar{k} = -(k_0)^2 - (k_1)^2$$

$$\epsilon^{(1)} = (0,0,1,0) \; ; \; \epsilon^{(2)} = (0,0,0,1).$$

00 comp: $\text{LHS} = -1$

$$\text{RHS} = \frac{2(k_0)^2}{-k_0^2 - k_1^2}$$

$$\text{LHS} - \text{RHS} = -1 + \frac{2(k_0)^2}{k_0^2 + k_1^2}$$

$$= \frac{k_0^2 - k_1^2}{k_0^2 + k_1^2} \propto \cancel{k^2}\, k^2 \longrightarrow \text{does not contribute to pole}$$

<u>Exercise</u>: Check others.

* Put back this $\eta_{\mu\nu}$ to correlation function $\Rightarrow$

$$\langle \tilde{F}^a_{\mu\rho}(k_1)\, \tilde{F}^b_{\nu\sigma}(k_2) \rangle \qquad (k \equiv k_1)$$

$$= (2\pi)^4 \delta^4(k_1 + k_2) \left(\frac{-i}{k_1^2}\right) i \times i$$

$$k_{[\mu} \left\{ \frac{k_{\rho]} \bar{k}_{[\sigma} + \bar{k}_{\rho]} k_{[\sigma}}{k \cdot \bar{k}} + \sum_{i=1}^{2} \epsilon^{(i)}_{\rho]} \epsilon^{(i)}_{[\sigma} + O(k^2) \right\} (-1) k_{\nu]}$$

do not contribute

Part of the propagator that contributes to poles in $\langle \tilde{F}^a_{\mu\rho}(k_1)\, \tilde{F}^b_{\nu\sigma}(k_2) \rangle$

is $(2\pi)^4 \delta^4(k_1 + k_2) \left(\frac{i}{-k_1^2}\right) \left\{ \sum_{i=1}^{2} \epsilon^{(i)}_\mu \epsilon^{(i)}_\nu \right\}$

→ only transverse components contribute to the gauge invari Green's fu

→ 2 - ~~ve~~ indep. in QED

→ $2N^2$ in $U(N)$.

→ in calculating S-matrix only transverse polariz. contribute.

* <u>Ghost action as $g=0$</u> :

$$-\int d^4x\, \partial^\mu B^a(x)\, \partial_\mu C^a(x).$$

$$= +\int \frac{d^4k}{(2\pi)^4}\, \tilde{B}^a(-k)(-k^2)\tilde{C}^a(k).$$

{ Remember they are Grassman variables.
↓
just like Dirac.

$\tilde{C}^a \rightarrow$ plays role of ψ_α

$\tilde{B}^a \rightarrow$ " " " $\bar{\psi}_\alpha$

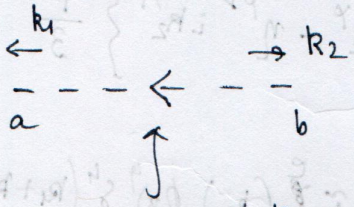
$-k^2 \rightarrow$ " " " $(\not{k} - m)$.

<u>Recall</u>: Dirac

$$S_{free} = \int \frac{d^4k}{(2\pi)^4}\, \tilde{\bar{\psi}}(-k)(-\gamma^\mu k_\mu - m)\tilde{\psi}(k)$$

So, $\langle \tilde{C}^a(k_1)\, \tilde{B}^b(k_2)\rangle$

$$= (2\pi)^4 \delta^4(k_1+k_2)\, \delta_{ab}\left(\frac{i}{-k_1^2}\right)$$



convention ; needed since C, B are not equivalent ($\bar{\psi}$ to ψ)

F)
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$$* \quad S_{int} = -\frac{g}{2}\int d^4x\, f^{abc} A^a_\mu A^b_\nu \left(\eta^{\nu\rho}\partial^\mu A^c_\rho - \eta^{\mu\rho}\partial^\nu A^c_\rho\right)$$

$$- \frac{g^2}{4}\int d^4x\, f^{abc} f^{dec} A^a_\mu A^b_\nu A^d_\rho A^e_\sigma \eta^{\mu\rho}\eta^{\nu\sigma}$$

$$+ g f^{bca}\int d^4x\, \partial^\mu B^a(x)\, A^c_\mu(x)\, C_b(x)$$

$$= -\frac{g}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\, \tilde{A}^a_\mu(k_1)\tilde{A}^b_\nu(k_2)\tilde{A}^c_\rho(k_3)\, f^{abc}$$

$$(2\pi)^4\delta^4(k_1+k_2+k_3)\left\{\eta^{\nu\rho}\, i k_3^\mu - \eta^{\mu\rho}(i k_3^\nu)\right.$$

$\{(a,\mu,k_1),(b,\nu,k_2),(c,\rho,k_3)\} \to$ symm. $\} \Rightarrow (\mu,k_1),(\nu,k_2),(\rho,k_3) \Rightarrow$ anti

$f^{abc} \to$ anti in a,b,c

our obtained express $\Rightarrow$ $\mu,\nu \to$ anti.

$k_1, k_2 \to$ indep.

So, we need

$\mu \to \nu \to \rho$

$k_1 \to k_2 \to k_3$

$$\left. + \eta^{\rho\mu}(i k_1^\nu) - \eta^{\nu\mu}(i k_1^\rho) + \eta^{\mu\nu}\, i k_2^\rho - \eta^{\rho\nu}\, i k_2^\mu \right\}\frac{1}{3}$$

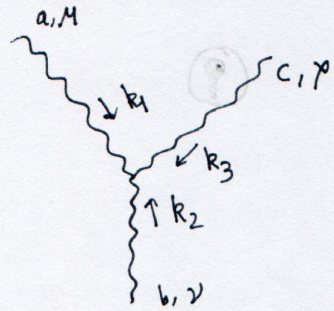
$$-\frac{g^2}{4}\int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_4}{(2\pi)^4}\, \underbrace{\tilde{A}^a_\mu(k_1)\tilde{A}^b_\nu(k_2)\tilde{A}^d_\rho(k_3)\tilde{A}^e_\sigma(k_4)}_{\hookrightarrow\, {}^4C_2 = \frac{4\cdot 3}{2} = 6}\, (2\pi)^4\delta^4(k_1+k_2+k_3+k_4)$$

$$\left\{\eta^{\mu\rho}\eta^{\nu\sigma} f^{abc} f^{dec} + \underbrace{23 \text{ more terms}}_{5 \text{ more}}\right\}\underbrace{\frac{1}{24}}_{\frac{1}{6}} \Longrightarrow$$

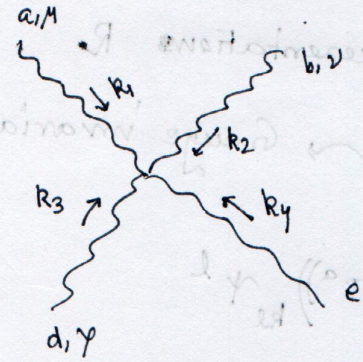
$$+ g\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\, (2\pi)^4\delta^4(k_1+k_2+k_3)\, \tilde{B}^a(k_1)\tilde{A}^c_\mu(k_2)\tilde{C}^b(k_3)\, f^{bca}(i k_1^\mu)$$

[$\because$ $A_\mu, B, C \to$ different
$\to$ no need of symmetrize]

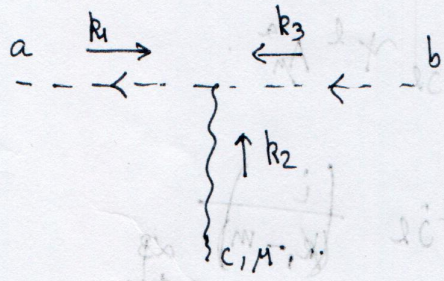
Diagram: Vertex



$$= i(2\pi)^4 \delta^4(k_1+k_2+k_3)\left(-\frac{g}{2}\right) f^{abc}$$
$$\frac{i}{3}\left\{\eta^{\nu\rho}k_3^{\mu} - \eta^{\mu\rho}k_3^{\nu} + \cdots\right\}.$$

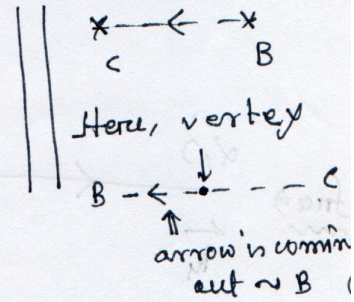


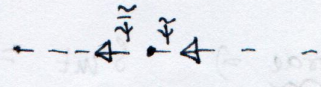
$$= i(2\pi)^4 \delta^4(k_1+k_2+k_3+k_4)\left(-\frac{g^2}{4}\right)$$
$$\frac{1}{24}\left\{f^{abc} f^{dec}\, \eta^{\mu\rho}\eta^{\nu\sigma} + 23\text{-more term}\right.$$



$$= i(2\pi)^4 \delta^4(k_1+k_2+k_3)\; g\, f^{bca}\left(i k_1^{\mu}\right)$$

Convention





Convention: towards $\tilde{\Psi}$
away from $\tilde{\Psi}$.

$\Rightarrow$ (a, μ, k_1), (b, ν, k_2), (d, ρ, k_3), (e, σ, k_4)

Any mutual exchange bet^n any pair leaves $\tilde{A}_{\mu}^{a}(k_1) \cdots \tilde{A}_{\sigma}^{e}(k_4)$ invariant

$\Rightarrow$ ${}^4C_2 = 6$ ways

Remember: What we did $\Rightarrow$

$$\vec{u} \to \vec{v} = \vec{f}(\vec{u}, \vec{\theta})$$

Being gau-trans: $S(\vec{u}) = S(\vec{v})$

Renamed: $\vec{v}$ as $\vec{u}$.

Coupling to fermionic fields:

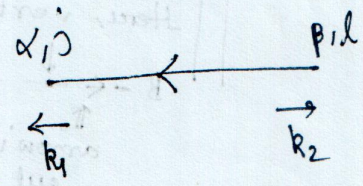
Take fermion fields in some representations R.

$$S_{fermion} = \int d^4x \, \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \quad \rightsquigarrow \text{Gauge invariant.}$$

with $(D_\mu \psi)^k = \partial_\mu \psi^k - i g A_\mu^a (R_A(T^a))_{kl} \psi^l$

$$S_{fermion} = \int d^4x \, \bar{\psi}^l (i\gamma^\mu \partial_\mu - m)\psi^l$$
$$+ g \int d^4x \, \bar{\psi}^j \gamma^\mu R_A(T^a)_{jl} \psi^l A_\mu^a .$$

free $\Rightarrow$

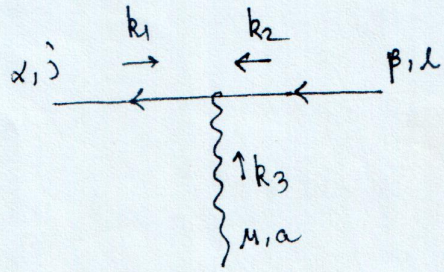


$$(2\pi)^4 \delta^4(k_1+k_2) \, \delta_{jl} \left(\frac{i}{k_1 - m}\right)_{\alpha\beta}$$

$\{$ arrow : $\bar{\psi}$ to $\psi \}$

interac $\Rightarrow$ $S_{int} = g \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} (2\pi)^4 \delta^4(k_1+k_2+k_3)$

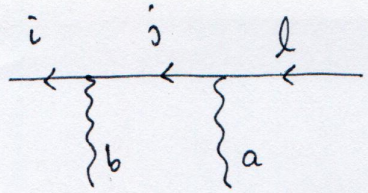
$$\tilde{\bar{\psi}}^j_\alpha(k_1) (\gamma^\mu)_{\alpha\beta} (R_A(T^a))_{jl} \tilde{\psi}^l_\beta(k_2) \tilde{A}_\mu^a(k_3)$$



$$i (2\pi)^4 \delta^4(k_1+k_2+k_3) \, g (\gamma^\mu)_{\alpha\beta} (R_A(T^a))_{jl}$$

arrow

eg.



$i \to j \to l$ [opposite of arrow]

$$\left(R_A(T_b)\right)_{ij} \left(R_A(T_a)\right)_{jl} .$$

⊛ Coupling to Scalar fields:

$$S_{scalar} = \int d^4x \left[-\eta^{\mu\nu} (D_\mu \phi)^\dagger D_\nu \phi - m^2 \phi^\dagger \phi \right] + \cdots$$

$$= \int d^4x \left[-(D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi \right] + \cdots$$

with $(D_\mu \phi)^k = \partial_\mu \phi^k - i g A_\mu^a \left(R_A(T^a)\right)_{kl} \phi^l$.

$$\Rightarrow (D_\mu \phi) = \partial_\mu \Phi - i g A_\mu^a R_A(T^a) \phi \quad \text{(Matrix form)}$$

and $(D_\mu \phi)^\dagger = \partial_\mu \phi^\dagger + i g A_\mu^a \phi^\dagger \underbrace{\left(R_A(T^a)\right)^\dagger}_{= R_A(T^a)}$

$$\Rightarrow \left(D_\mu \phi^\dagger\right)^{*k} = \partial_\mu \phi^{k*} + i g A_\mu^a \phi_j^* \left(R_A(T^a)\right)_{jk} .$$

S_{scalar}

$$= \int d^4x \left[-(D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi \right]$$

$$= \int d^4x \Big[-\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi + i g \partial^\mu \phi^\dagger A_\mu^a R_A(T^a) \phi$$
$$- i g \phi^\dagger R_A(T^a) A_\mu^a \partial^\mu \phi$$
$$- g^2 \phi^\dagger R_A(T^a) R_A(T^b) \phi A_\mu^a A_\nu^b \eta^{\mu\nu} \Big]$$

$$= \int d^4x \Big[-\partial_\mu \phi_k^* \partial^\mu \phi - m^2 \phi_k^* \phi_k + i g \partial^\mu \phi_j^* A_\mu^a \left(R_A(T^a)\right)_{jl} \phi^l$$
$$- i g \phi_j^* \left(R_A(T^a)\right)_{jl} A_\mu^a \partial^\mu \phi_l - g^2 \phi_k^* \left(R_A(T^a) R_A(T^b)\right)_{kl} \phi^l A_\mu^a A_\nu^b \eta^{\mu\nu} \Big]$$

*

Let, $\phi_k = \frac{1}{\sqrt{2}}(\xi_k + i\chi_k)$

$\phi_k^* = \frac{1}{\sqrt{2}}(\xi_k - i\chi_k)$

Then,

$$\delta_{free} = -\frac{1}{2}\left[\sum_k \left(\partial_\mu \xi_k \partial^\mu \xi_k + m^2 \xi_k \xi_k\right) + \sum_k \left(\partial_\mu \chi_k \partial^\mu \chi_k + m^2 \chi_k \chi_k\right)\right]$$

Propagator:

[OR, directly using path integral we could do it treating φ, φ^* as indep.]

$$\langle \tilde{\xi}_j(k_1)\, \tilde{\xi}_l(k_2)\rangle$$

$$= (2\pi)^4\, \delta^4(k_1+k_2)\, \frac{i}{-k_1^2-m^2}\, \delta_{jl}$$

$$\langle \tilde{\chi}_j(k_1)\, \tilde{\chi}_l(k_2)\rangle = \text{same}.$$

$$\langle \tilde{\xi}_j(k_1)\, \tilde{\chi}_l(k_2)\rangle = 0.$$

Using these

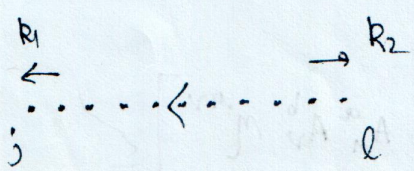
$$\langle \tilde{\varphi}_j(k_1)\, \tilde{\varphi}_l(k_2)\rangle$$

$$= \frac{1}{2}\langle (\tilde{\xi}_j(k_1) + i\tilde{\chi}_j(k_1)) \cdot (\tilde{\xi}_l(k_2) + i\tilde{\chi}_l(k_2))\rangle$$

$$= 0.$$

$$\langle \tilde{\varphi}_j^*(k_1)\, \tilde{\varphi}_l^*(k_2)\rangle = 0$$

$$\langle \tilde{\varphi}_j(k_1)\, \phi_l^*(k_2)\rangle = (2\pi)^4\, \delta^4(k_1+k_2)\, \frac{i}{-k_1^2-m^2}\, \delta_{jl}.$$



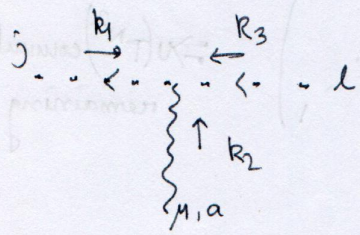
[Convention: arrow: φ^* to φ]

$$= (2\pi)^4\, \delta^4(k_1+k_2)\, \frac{i}{-k_1^2-m^2}\, \delta_{jl}$$

(Propagator for complex scalar field)

$$* \quad S_{int} = ig \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \; \tilde{\Phi}_j^*(k_1)\, \tilde{A}_\mu^a(k_2)\, \tilde{\Phi}^l(k_3)\, (2\pi)^4 \delta^4(k_1+k_2+k_3)$$

$$(R_A(T^a))_{jl} \left\{ ik_1^\mu - ik_3^\mu \right\} \quad + \text{ quadratic term}$$



// vertex

$$i\,(2\pi)^4 \delta^4(k_1+k_2+k_3) \quad ig\,(R_A(T^a))_{jl} \quad i(k_1^\mu - k_3^\mu).$$

arrow

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1.09.12

✱ Note on U(N) gauge groups:

* $\underset{\sim}{B}_\mu = B_\mu^a T^a$

$$\underbrace{T^1, \ldots, T^{N^2-1}}_{\substack{SU(N) \text{ generators} \\ \downarrow \\ \text{traceless, hermitian}}}, \quad T^{N^2} = \frac{1}{\sqrt{2N}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \Rightarrow U(T^{N^2}) \text{ commutes with remaining } T^i\text{'s}.$$

We could think of U(N) as multiplication of two groups:

$$U(N) = SU(N) \otimes U(1)$$

$$\uparrow$$

$$e^{i\phi T^{N^2}} \propto \begin{pmatrix} e^{i\phi} & & \\ & \ddots & \\ & & e^{i\phi} \end{pmatrix}$$

$$\Downarrow$$

can be thought of as two coupling constants.

Given U(N) ⇝ how do we know/identify two coupling constants?

$$\Rightarrow \quad \underset{\sim}{G}_{\mu\nu} = G_{\mu\nu}^a T^a$$

$$= \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i\left[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu\right]$$

$$-\frac{1}{2g^2} \mathrm{Tr}\left(\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu}\right) \rightarrow \text{Gauge invariant}$$

Gauge invariant $\Rightarrow$ $-\frac{1}{2g'^2} \mathrm{Tr}(\underset{\sim}{G}_{\mu\nu}) \mathrm{Tr}(\underset{\sim}{G}^{\mu\nu})$

→ vanishes for SU(N)

→ non-vanishing " U(1).

So, $\frac{1}{g^2} \rightarrow SU(N)$

$$\frac{1}{g^2} + \frac{1}{g'^2} \equiv \frac{1}{g''^2} \rightarrow U(1)$$

See, $\mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\, \underset{\sim}{G}^{\mu\nu}\right) \longrightarrow$ cross terms vanishes.

$\longrightarrow$ non-zero for

⊠ Standard Model Lagrangian:

* $SU(3) \times SU(2) \times U(1) \iff$ Experimental i/p ; $SU(3) \to$ strong, $SU(2) \to$ weak, $U(1) \to$ EM

$B_\mu^a \qquad C_\mu^A \qquad S_\mu$

$a = 1, \ldots, 7, 8 \qquad A = 1, \ldots, 3$

$$\underset{\sim}{B}_\mu = B_\mu^a T^a$$

$$\underset{\sim}{C}_\mu = C_\mu^A L^A$$

$$\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i\left[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu\right]$$

$$\underset{\sim}{H}_{\mu\nu} = \partial_\mu \underset{\sim}{C}_\nu - \partial_\nu \underset{\sim}{C}_\mu - i\left[\underset{\sim}{C}_\mu, \underset{\sim}{C}_\nu\right]$$

$$K_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu .$$

Hence gauge invariance $\Rightarrow$ kinetic term

$$-\frac{1}{2g_1^2}\mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\underset{\sim}{G}^{\mu\nu}\right) - \frac{1}{2g_2^2}\mathrm{Tr}\left(\underset{\sim}{H}_{\mu\nu}\underset{\sim}{H}^{\mu\nu}\right) - \frac{1}{4g_3^2}K_{\mu\nu}K^{\mu\nu} .$$

* Hypercharge: $Y \rightsquigarrow U(1)$ charge

If a field ψ has hypercharge Y, then

$$D_\mu \psi \equiv \left(\partial_\mu \psi - iY S_\mu \psi + \cdots\right)$$

$\Rightarrow$ how does ψ transform under $U(1)$.

depends on the transf. of ψ under $SU(3)$, $SU(2)$.

eg. $U(1)$ gauge trans $\Rightarrow$

$$S_\mu \longrightarrow S_\mu + \partial_\mu \alpha(x)$$

$$\psi \longrightarrow e^{i\alpha(x)Y}\psi$$

Recall: $U(1) \Rightarrow \psi(x) \to e^{ie\lambda(x)}\psi(x)$

$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x)$

$D_\mu \equiv \partial_\mu - ieA_\mu$

* Quark fields: (Weyl).

→ Left-handed quarks ⇒ fundamental of $SU(3)$

" " $SU(2)$

hypercharge $\frac{1}{3}$

So, $U_{i\alpha}$ (suppressing Dirac index).

$i \to SU(3)$ index; $\alpha \to SU(2)$ index.

Lagrangian $\Rightarrow i\,\bar{U}_L \gamma^\mu D_\mu U_L$

where, $D_\mu = \partial_\mu - i B^a_\mu T^a - i C^A_\mu L^A - \frac{i}{3} S_\mu$

[There is no explanation of charge quantization in standard model]

So, $i\,\bar{U}_{Li\alpha}\gamma^\mu\left(\delta_{ij}\delta_{\alpha\beta}\partial_\mu - i B^a_\mu (T^a)_{ij}\delta_{\alpha\beta} - i C^A_\mu (L^A)_{\alpha\beta}\delta_{ij} - \frac{i}{3} S_\mu \delta_{ij}\delta_{\alpha\beta}\right) U_{Lj\beta}$

No mass term since Weyl.

$i \to$ color index $\Rightarrow 1,2,3$ [$SU(3)$ index]

$\alpha \to 1,2$ (fundamental repres. → dim of matrix $SU(2) \to$ Pauli).

$1 \to u_L$, $2 \to d_L$ [Experimental ⇒ up, down quark].

→ Two more sets of fields with identical transf. pro^s:

$C_{Li\alpha}$: $1 \to$ charm, $2 \to$ strange

$T_{Li\alpha}$: $1 \to$ top, $2 \to$ bottom.

→ Right handed quarks:

U_{Ri}: Fundamental of $SU(3)$

Singlet (trivial repre) of $SU(2) \Rightarrow R_G(U) = \mathbb{1}$

hypercharge $\frac{4}{3}$

d_{Ri} : fundamental of $SU(3)$
Singlet of $SU(2)$
hypercharge $-2/3$

So, $i\bar{u}_R \gamma^\mu D_\mu u_R$ (kinetic term)

$$= i\bar{u}_{Ri}\gamma^\mu\left(\delta_{ij}\partial_\mu - iB_\mu^a (T^a)_{ij} - \frac{4}{3} i S_\mu \delta_{ij}\right) u_{Rj}$$

and

$$i\bar{d}_{Ri}\gamma^\mu\left(\delta_{ij}\partial_\mu - iB_\mu^a (T^a)_{ij} + \frac{2}{3} i S_\mu \delta_{ij}\right) d_{Rj}$$

Similarly introduce (c_{Ri}, s_{Ri}), (t_{Ri}, b_{Ri}).

Remember at this stage there is no difference betw various quarks

~ Mass term ?

$m_u \bar{U}_{L1} u_R + m_d \bar{U}_{L2} d_R + h.c.$ → not gauge invariant.
→ not possible.

[Left-handed → fundamental represen. of $SU(2)$
Right- " → singlet of $SU(2)$.

* Leptons:

Left-handed: singlet of $SU(3)$
fundamental of $SU(2)$
hypercharge -1

3-generations : $E^{(1)}_{L\alpha}$, $E^{(2)}_{L\alpha}$, $E^{(3)}_{L\alpha}$ $\alpha \to SU(2)$ indep.

So, $i\bar{E}^{(1)}_{L\alpha}\gamma^\mu\left(\delta_{\alpha\beta}\partial_\mu - i c_\mu^A (L^A)_{\alpha\beta} + i\delta_{\alpha\beta} S_\mu\right) E^{(1)}_{L\beta} + (1\to 2) + (1\to 3)$

$E^{(1)}_{L1} \to \nu_{eL}$; $E^{(2)}_{L1} \to \nu_{\mu L}$; $E^{(3)}_{L1} \to \nu_{\tau L}$

$E^{(1)}_{L2} \to e_L$; $E^{(2)}_{L2} \to \mu_L$; $E^{(3)}_{L2} \to \tau_L$

<u>Right-handed</u>: singlet of $SU(3)$
" " $SU(2)$
hypercharge -2.

$$\mathcal{L}\ \bar{e}_R \gamma^\mu (\partial_\mu + 2i S_\mu) e_R + (e \to \mu) + (e \to \tau)$$

$\longrightarrow$ no neutrino since no mass of neutrino according to standard model.

* <u>Exercise</u>: Physical electric charge $= T^3 + Y/2$

T^3: 3rd generator of $SU(2)$

$$\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

* $u_{Li} \to u_{Li1}$, $Y = 1/3$; $T^3 + Y/2$

$$= \frac{1}{2} + \frac{1}{6}$$

$$= \frac{2}{3}.$$

$\boxed{\text{up quark}}$

* Term to incorporate mass $\Rightarrow$

<u>Higgs field ϕ</u> (ϕ_α) : Complex scalar in the singlet of $SU(3)$
fundame. of $SU(2)$
hypercharge 1

So, $\mathcal{L} = -(D_\mu \phi)^\dagger (D^\mu \phi) \longrightarrow$ Gauge invariant

$\phi \to \phi_\alpha$

$= -(D_\mu \phi_\alpha)^* (D^\mu \phi_\alpha)$

Now,

$$D_\mu \phi_\alpha = \left(\delta_{\alpha\beta}\, \partial_\mu - i\, c^A_\mu (L^A)_{\alpha\beta} - i\, S_\mu\, \delta_{\alpha\beta} \right) \phi_\beta$$

Add gauge invariant: $-\mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$ → stop at this order due to renormalisation.

<u>Coupling</u>:

fun SU(3), fun SU(2) ; fun SU(3), sing SU(2) ; sing SU(3), fun SU(2)

$$\lambda_d\, \bar{U}_{i\alpha L}\, d_{iR}\, \phi_\alpha + h.c. \quad \Rightarrow \text{invariant under } SU(3) \,\&\, SU(2)$$

λ_d → com. ; $\bar{U}_{i\alpha L}$ → $-1/3$; d_{iR} → $-2/3$; ϕ_α → $Y=1$ } add = 0 → u .. under hypercharge

under hypercharge ⇒

We may change d to s and $\bar{U}$ ⇒ λ → changes in general from generation to generation ⇓

sing SU(3), fun SU(2) ; sing SU(3), fun SU(2) ; sing SU(3), sing SU(2)

$$* \quad \lambda_e\, \bar{E}^{(1)}_{\alpha L}\, \phi_\alpha\, e_R$$

$\bar{E}^{(1)}_{\alpha L}$ → $Y=1$; ϕ_α → $Y=1$; e_R → $Y=-2$ & other gener.

$$* \quad \lambda_u\, \bar{U}_{i\alpha L}\, u_{iR}\, \epsilon_{\alpha\beta}\, \phi^*_\beta + h.c.$$

$\bar{U}_{i\alpha L}$ → $Y=-\frac{1}{3}$; u_{iR} → $Y=4/3$; ϕ^*_β → $Y=-1$

See, if χ_α, ψ_α fundamental of SU(2)

$\epsilon^{\alpha\beta} \chi_\alpha \psi_\beta$ is SU(2) invariant $\left[\because \epsilon_{\alpha\beta} U_{\alpha\gamma} U_{\beta\delta} = \epsilon_{\gamma\delta} \det U = \epsilon_{\gamma\delta} \right]$

(19)

10.10.12

Mid-Sem : Prob 1 : $SU(2) \times U(1)$

What's the gauge inv & LI Lagrangian?

$$\Rightarrow -\frac{1}{2g_1^2} Tr\left(F_{\mu\nu}F^{\mu\nu}\right) - \frac{1}{4g_2^2} G_{\mu\nu}G^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi$$

$$-\left(D_\mu \phi\right)^\dagger \left(D^\mu \phi\right) - m^2\left(\phi_a \phi^a\right) + \lambda\left(\phi_a\phi_a\right)^2 + \bar{\psi}\left(\phi^a T^a\right)\psi$$

$$\phi^a T^a \to U \phi^a T^a U^\dagger$$

OR, you can think in terms of Clebsh Gordan coef.

Never use A^μ in writing a gauge invariant object. If you use that then that should come with covariant derivative.

Spontaneously Symmetry Breaking

[See Ryder too → fantastic discussion]

Consider a scalar field theory:

$$* \quad S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \underbrace{\frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4}_{V(\phi)} \right]$$

Symmetry: $\phi \to -\phi$

$$\Rightarrow \left\langle \prod_{i=1}^{n} \phi(x_i) \right\rangle = \left\langle \prod_{i=1}^{n} (-\phi(x_i)) \right\rangle$$

$\Rightarrow$ vanishes for odd n.

$\Downarrow$

* Formal proof:

$$\left\langle \prod_{i=1}^{n} \phi(x_i) \right\rangle = \frac{\int [\mathcal{D}\varphi]\, e^{iS[\varphi]} \prod_i \phi(x_i)}{\int [\mathcal{D}\varphi]\, e^{iS[\varphi]}}$$

Let $\chi \to -\phi$; Jacobian $= -1$ → cancel against denominator

$$= \frac{\int [\mathcal{D}\chi]\, e^{iS[-\chi]} \prod_i (-\chi(x_i))}{\int [\mathcal{D}\chi]\, e^{iS[-\chi]}}$$

$$\underset{S}{\underline{\underline{\text{Symmetry}}}} \text{ of } \frac{\int [\mathcal{D}\chi]\, e^{iS[\chi]} \prod_i (-\chi(x_i))}{\int [\mathcal{D}\chi]\, e^{iS[\chi]}}$$

$$\underset{\chi \to \phi}{\underline{\underline{\text{Rename}}}} \quad \frac{\int [\mathcal{D}\varphi]\, e^{iS[\varphi]} \prod_i (-\phi(x_i))}{\int [\mathcal{D}\varphi]\, e^{iS[\varphi]}}$$

$$= \left\langle \prod_{i=1}^{n} -\phi(x_i) \right\rangle$$

Canonical formalism $\Rightarrow$ Symmetry means $\exists\, U$ s.t.

(i) $\hat{U}\hat{\phi}(x_i)\hat{U}^{-1} = -\hat{\phi}(x_i)$

(ii) $\hat{U}\hat{H}\hat{U}^{-1} = \hat{H}$. with $\hat{U}|\Omega\rangle = |\Omega\rangle$

Then,

$$\langle \prod_i \phi(x_i)\rangle = \langle\Omega| T\left(\prod_i \phi(x_i)\right)|\Omega\rangle$$

$$= \langle\Omega|\hat{U}\, T\left(\prod_i \phi(x_i)\right)\hat{U}^{-1}|\Omega\rangle$$

$$= \langle\Omega| T\left(\prod_i \hat{U}\hat{\phi}(x_i)\hat{U}^{-1}\right)|\Omega\rangle$$

$$= \langle\Omega| T\left(\prod_i -\hat{\phi}(x_i)\right)|\Omega\rangle$$

$$= \langle \prod_i (-\phi(x_i))\rangle.$$

Hence, Lagrangian $\Rightarrow$ symmetry under $\phi \to -\phi$

then correlation function is also symmetric under this.

$$* \quad S = \int d^4x \left[-\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$$

$$V(\phi) = -\tfrac{1}{2}\mu^2\phi^2 + \tfrac{\lambda}{4}\phi^4$$

Change wrt earlier: $m^2 \to -\mu^2$

mass becomes imaginary.

Previous: $E^2 - \vec{p}^2 = m^2 \to$ from this we identified m as mass of single particle.

Now, $E^2 - \vec{p}^2 = -\mu^2$

$\Rightarrow E = \sqrt{\vec{p}^2 - \mu^2} \Rightarrow \vec{p}^2 \geq \mu^2 \Rightarrow$ momentum can never be zero.

Think of a classical particle with $H = \sqrt{\vec{p}^2 - \mu^2}$

Then $v^i = \dot{x}^i = \dfrac{\partial H}{\partial p^i} = \dfrac{p^i}{\sqrt{\vec{p}^2 - \mu^2}}$

$\Rightarrow |\vec{v}| = \dfrac{|\vec{p}|}{\sqrt{\vec{p}^2 - \mu^2}} > 1 \Rightarrow$ Tachyon.

→ Seems strange!

→ Let's see whether perturbation theory is applicable here or not. Higher order terms should be subdominant for perturbation.

→ Let's examine: $\lambda = 0$; then whether a small fluctuation remains small.

Classical Eqn. with $\lambda = 0$:

$$-\Box\phi = \mu^2\phi$$

Consider a slowly space-varying $\phi \Rightarrow |\vec{\nabla}^2\phi|$ is small.

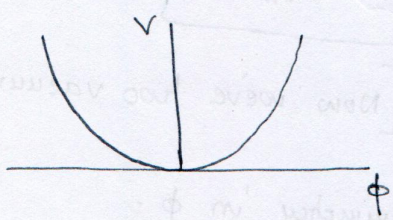
Then eqn of motn: $\dfrac{\partial^2\phi}{\partial x^{02}} = \mu^2\phi$

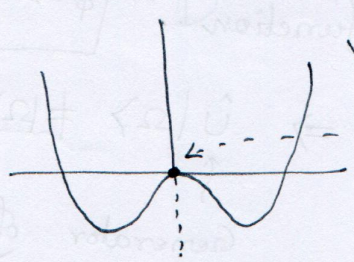
$$\Rightarrow \boxed{\phi = \underbrace{Ae^{\mu x^0}}_{\text{grows as } x^0\uparrow} + Be^{-\mu x^0}}$$

$\Rightarrow$ signals the breakdown of perturbation theory.

→ initial small perturbation (due to quantum fluctuation) grows as time proceeds. For perturbation theory to be valid this should not grow over.

$V = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$





$V = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4$

small perturbation around this $\Rightarrow$ particle moves away unlike the other one.

$\Downarrow$

this perturbation theory around origin is not stable. But one can do perturbation theory around the minimum.

Let's do it $\Rightarrow$

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4$$

$$\Rightarrow V'(\phi) = -\mu^2\phi + \lambda\phi^3$$

$$= \phi(\lambda\phi^2 - \mu^2)$$

$$= 0 \quad \text{for} \quad \underset{\substack{\uparrow \\ \text{Max}}}{\phi = 0}, \quad \underset{\substack{\uparrow \\ \text{Minima}}}{\phi = \pm\frac{\mu}{\sqrt{\lambda}}}$$

Perturbation theory around $\phi = \frac{\mu}{\sqrt{\lambda}}$

$$\chi \equiv \phi - \sqrt{\frac{\mu^2}{\lambda}}$$

$$V(\phi) = V\left(\chi + \sqrt{\frac{\mu^2}{\lambda}}\right) = -\frac{\lambda}{4}\left(\frac{\mu^2}{\lambda}\right)^2 + \mu^2\chi^2 + \lambda\sqrt{\frac{\mu^2}{\lambda}}\,\chi^3 + \frac{\lambda}{4}\chi^4$$

So,

$$S[\chi] = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\chi\,\partial_\nu\chi - \tilde{V}(\chi)\right]$$

We now have a particle of

$(\text{mass})^2 = 2\mu^2 +$ quantum correction

$+$

interactions.

<u>NOTE</u>: No $\chi \to -\chi$ symmetry. No visible symmetry in the correlation function. $\boxed{\phi \xrightarrow{U} -\phi \Rightarrow \chi \to -2\sqrt{\frac{\mu^2}{\lambda}} - \chi}$

<u>Canonical formalism</u> $\Rightarrow$ $\hat{U}|\Omega\rangle \neq |\Omega\rangle$ [Now we've two vacuum]

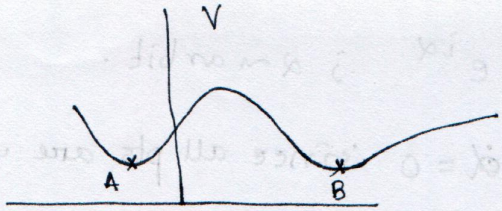
$\uparrow$ Generator of the symmetry in ϕ.

<u>Path Integral</u> $\Rightarrow$ We use B.C. field $\to 0$ at $\pm\infty$.

Now, if $\phi \to 0$ at $\pm\infty$, then $\not\Rightarrow$ $\chi \to 0$ at $\pm\infty$.

$\rightsquigarrow$ <u>Spontaneous Symmetry Breaking</u>.

*

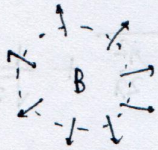


Two minima ; no symmetry ;

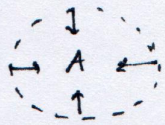
→ two vacuum are not equivalent.

→ two different phases of same theory.

→ It can be shown → B is stable (non-perturbatively) (lower one)



Bubble



Bubble

Analogy:

B ~ ~~water~~ ice.

A ~ Supercooled water.

Start with B ; A bubble is formed due to either classical perturbation / quantum fluctuation But that eventually dies down. Though A could be metastable.

⊠ Spontaneous breaking of continuous global symmetry :

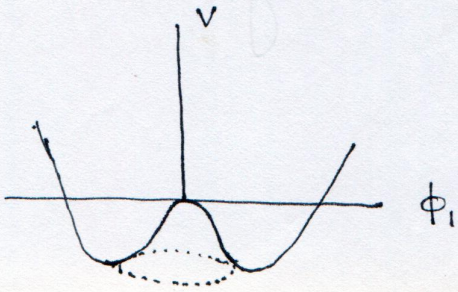
* Complex scalar : $\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$S = \int d^4x \left[-\eta^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - V(\phi) \right]$$

with $V(\phi) = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$

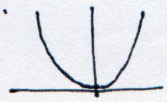
* Symmetry under $\phi \to e^{i\theta} \phi$; θ ~ arbit constant.

*



Rotational symmetry ⇒ set of minimum on circle.

[Unlike :



Only one pt → min

$\boxed{\text{H.W.}}$ Minimum of $V(\phi)$ is at $\sqrt{\frac{\mu^2}{2\lambda}}\, e^{i\alpha}$; $\alpha \sim$ arbit.

* No perturbation theory around $\dot{\alpha} = 0$ since all pts are equiv.

$$\phi = \sqrt{\frac{\mu^2}{2\lambda}}.$$

Introduce new complex field $\chi \equiv \phi - \sqrt{\frac{\mu^2}{2\lambda}}$

$$\Rightarrow \phi = \sqrt{\frac{\mu^2}{2\lambda}} + \chi$$

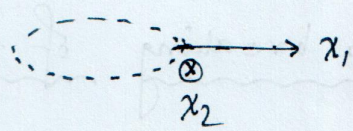
$$= \sqrt{\frac{\mu^2}{2\lambda}} + \frac{1}{\sqrt{2}}\left(\chi_1 + i\chi_2\right)$$

So,

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu} \cancel{\partial_\mu \phi \partial_\nu \phi}\ \partial_\mu \chi_1 \partial_\nu \chi_1 - \frac{1}{2}\partial_\mu \chi_2 \partial^\mu \chi_2 - V(\chi_1, \chi_2)\right]$$

where, $V(\chi_1, \chi_2) = -\frac{\lambda}{4}\left(\frac{\mu^2}{\lambda}\right)^2 + \mu^2 \chi_1^2 + (\text{cubic} + \text{quartic in } \chi_1, \chi_2)$.

$\Longrightarrow \chi_2$ is massless. This is because



$\rightarrow$ potential remains flat along χ_2.

$\rightsquigarrow (\text{mass})^2 =$ 2nd derivative of potential

If flat potential $\Rightarrow$ 2nd deri $= 0$.

<u>Goldstone Theorem</u>: There is a massless scalar for every spontaneously broken global symmetry.

$\Longrightarrow$ particle remains massless to all order in perturbation theory.

* Suppose, we've a scalar field theory with symmetry group G. 11.10.

$$\phi \longrightarrow R(g)\phi \quad ; \quad g \in G$$

Symmetry $\Rightarrow$ $V(\phi) = V(R(g)\phi)$

Suppose. $V(\phi)$ has a minimum at $\phi = \phi_0$.

$\Rightarrow$ $R(g)\phi_0$ is also a minimum with same energy, $\forall\, g \in G$

In general, $R(g)\phi_0$ may or may not be equal to ϕ_0.

Suppose, H is a subset of G s.t. if $h \in H$ then,

$$R(h)\phi_0 = \phi_0 .$$

If $h_1, h_2 \in h$, then $h_1 h_2 \in H$.

$\left\{ \begin{array}{l} G \rightarrow \text{Symmetry group of } \mathcal{L}. \\ H \rightarrow \text{symmetry group of vacuum} \end{array} \right.$

<u>Proof</u>:

$$R(h_1)\phi_0 = \phi_0$$
$$R(h_2)\phi_0 = \phi_0$$

then $R(h_1 h_2)\phi_0 \cancel{\neq \phi_0}$

$$= R(h_1)R(h_2)\phi_0$$
$$= R(h_1)\phi_0$$
$$= \phi_0 .$$

$\Rightarrow h_1, h_2 \in H$.

Similarly one finds that if $h \in H$, then $h^{-1} \in H$ etc.

$\Rightarrow$ H is a group. H is a subgroup of G.

In general, one says that G is spontaneously broken to H by the "vacuum expectation value" of ϕ. Vacuum ex $\Rightarrow$ classical result at the lowest order.

G is not broken.

G is completely broken. <u>U(1)</u>: $R(g)\phi = e^{i\alpha}\phi$

$$R(g)\phi_0 = e^{i\alpha}\phi_0 \neq \phi_0$$

unless $\alpha = 2\pi$.

Consider infinitesimal transf

$$g = \mathbb{1} + i \sum_{a=1}^{n_G} \epsilon_a T^a$$

n_G : # of generators of the group.

for $g \in G$.

For $h \in H$, $h = \mathbb{1} + i \sum_{a=1}^{n_H} \epsilon_a T^a$

We choose # n_H generators of G are first n_H generators of G.

T^k : $k = n_H + 1, \ldots, n_G$ are the generators which are in G but not in H.

We know,

$$R(h)\phi_0 = \phi_0 \quad \forall\ h \in H$$

$$\Rightarrow R\left(\mathbb{1} + i \sum_{a=1}^{n_H} \epsilon_a T^a\right)\phi_0 = \phi_0$$

$$\Rightarrow \left(\mathbb{1} + i \sum_{a=1}^{n_H} \epsilon_a R(T^a)\right)\phi_0 = \phi_0$$

$R(T^a) \equiv R_A(T^a)$

$R(\mathbb{1} + \cdots) \equiv R_G(\mathbb{1} + \cdots)$.

$$\Rightarrow i \sum_{a=1}^{n_H} \epsilon_a R(T^a)\phi_0 = 0$$

$$\Rightarrow \boxed{R(T^a)\phi_0 = 0 \quad \text{for } a = 1, \ldots, n_H}$$

For $k > n_H$: $R(T^k)\phi_0 \neq 0$ and in general (no linear combination) $\phi_0 \neq 0$

If $\sum_{k=n_H+1}^{n_G} a_k R(T^k)\phi_0 = 0$.

$$\Rightarrow R\left(\sum_{n_H+1}^{n_G} a_k T^k\right)\phi_0 = 0$$

$\Rightarrow \sum_{n_H+1}^{n_G} a_k T^k$ is a generator of H.

Since $V(\phi)$ is constant along these dir^{ns} there will be $n_G - n_H$ massless particles.

$\longrightarrow$ General Goldstone thm.

<u>Example</u>: Real scalar field. [See Ryder]

$S(\phi_1, \phi_2, \phi_3)$

$$= \int d^4x \left[-\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a + \tfrac{1}{2}\mu^2\phi^a\phi^a - \tfrac{\lambda}{4}(\phi^a\phi^a)^2 \right] \quad ; \; a = 1,2,3$$

$$V(\vec{\phi}) = -\tfrac{1}{2}\mu^2(\phi^a\phi^a) + \tfrac{\lambda}{4}(\phi^a\phi^a)^2$$

$\longrightarrow$ SO(3) symmetry.

Let, $y^2 = \phi^a\phi^a$

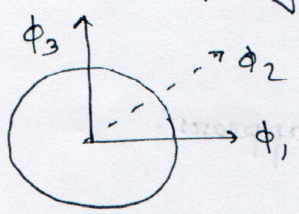
then $v(\phi) = -\tfrac{1}{2}\mu^2 y^2 + \tfrac{\lambda}{4}y^4$

$y' = y - \sqrt{\mu^2/\lambda} = y - y_0 \Rightarrow y = y' +$

$V = -\tfrac{1}{2}\mu^2(y' + y_0)^2 + \tfrac{\lambda}{4}(y' + y_0)^4$

$=$

Minimum at $y = \sqrt{\dfrac{\mu^2}{\lambda}}$



$\dfrac{N(N-1)}{2}$

$G : SO(3) \xrightarrow{n_G} 3$

$H : SO(2) \xrightarrow{n_H} 1$

$\Big\}$ $3 - 1 = 2$ # Goldstone massless particles

$$\phi_0 = \sqrt{\frac{\mu^2}{\lambda}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\boxed{*}$ Take U(1) gauge field A_μ coupled to a complex scalar field ϕ

$$S = \int d^4x \left[-(D_\mu\phi)^*(D_\nu\phi)\eta^{\mu\nu} - V(\phi) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \right]$$

where,

$$F_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D_\mu\phi = (\partial_\mu\phi - igA_\mu\phi)$$

Take $v(\phi) = -\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2$

$V(\phi)$ has a minimum at $\phi = \sqrt{\dfrac{\mu^2}{2\lambda}}\, e^{i\alpha}$

Take $\chi = \dfrac{1}{\sqrt{2}}(\chi_1 + i\chi_2) = \phi - \sqrt{\dfrac{\mu^2}{2\lambda}}$

$$S = \int d^4x \Big[-\partial_\mu \phi^* \partial^\mu \phi - ig\,\partial_\mu \phi^* A^\mu \phi - ig\,\phi^* \partial_\mu \phi A^\mu \Big\} - g^2 A_\mu A^\mu \phi^* \phi$$

$$+ \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Big]$$

$$= \int d^4x \Big[-\tfrac{1}{2} \partial_\mu \chi_1 \partial^\mu \chi_1 - \tfrac{1}{2} \partial_\mu \chi_2 \partial^\mu \chi_2 + g\sqrt{\frac{\mu^2}{\lambda}}\, \eta^{\mu\nu} A_\mu \partial_\nu \chi_2$$

$$- \frac{1}{2} g^2 \frac{\mu^2}{\lambda} A_\mu A^\mu + \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda}\right)^2 - \mu^2 \chi_1^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$+ \text{cubic} + \text{quartic} \Big]$$

Define : $B_\mu = A_\mu - \frac{1}{g}\sqrt{\frac{\lambda}{\mu^2}}\, \partial_\mu \chi_2$

So,

$$S = \int d^4x \Big[-\tfrac{1}{2} \partial_\mu \chi_1 \partial^\mu \chi_1 - \tfrac{1}{2} \overbrace{g^2 \frac{\mu^2}{\lambda}}^{M^2} \eta^{\mu\nu} B_\mu B_\nu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \mu^2 \chi_1^2$$

$$+ \text{cons} + \text{cubic} + \text{quartic} \Big)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu$$

<u>Peculiar things.</u>

(i) Mass term of gauge field appears.

(ii) χ_2 disappears

* <u>Gauge transformation laws</u>:

$$A_\mu \to A_\mu + \partial_\mu \theta(x)$$

$$\phi \to e^{ig\theta(x)} \phi$$

Infinitesimal $\theta(x)$: $A_\mu \to A_\mu + \partial_\mu \theta$

$$\phi \to \phi + ig\,\theta(x)\,\phi$$

$$\phi = \sqrt{\frac{\mu^2}{2\lambda}} + \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2)$$

So, $\frac{1}{\sqrt{2}}(\delta\chi_1 + i\delta\chi_2) = ig\,\theta(x)\left[\sqrt{\frac{\mu^2}{2\lambda}} + \frac{1}{\sqrt{2}}\chi_1 + \frac{i}{\sqrt{2}}\chi_2\right]$

Compare both sides $\Rightarrow$

$$\delta \chi_1 = - g\,\theta(x)\,\chi_2$$

$$\delta \chi_2 = g\,\theta(x)\sqrt{\frac{\mu^2}{\lambda}} + g\,\theta(x)\,\chi_1$$

$$= M\,\theta(x) + g\,\theta(x)\,\chi_1$$

$$\delta B_\mu = \partial_\mu \theta - \frac{1}{g}\sqrt{\frac{\lambda}{\mu^2}}\,\partial_\mu\left(g\,\theta(x)\sqrt{\frac{\mu^2}{\lambda}} + g\,\theta(x)\,\chi_1\right) = -\sqrt{\frac{\lambda}{\mu^2}}\,\partial_\mu(\theta(x)\chi_1)$$

$$= -\frac{g}{M}\,\partial_\mu(\theta(x)\chi_1)$$

Now,

as $g \to 0$ at fixed M, μ

$$\delta B_\mu = 0$$

$$\delta \chi_1 = 0$$

$$\delta \chi_2 = M\,\theta(x)$$

Gauge fixing : $H(x) = \chi_2(x)$ (Unitary gauge)

Insert into path integral $\underbrace{\prod_x \delta(\chi_2(x))}_{\Rightarrow \text{ drop } \chi_2}$

<u>H.W.</u> Ghost action $\int d^4x\; B(x)\left(\sqrt{\frac{\mu^2}{\lambda}} + \chi_1(x)\right) C(x)$.

B.C. propagator : $i\sqrt{\frac{\lambda}{\mu^2}}\,(2\pi)^4\,\delta^4(k_1+k_2)$

<u>H.W.</u> Check that the B_μ propagator is :

$$(2\pi)^4\,\delta^4(k_1+k_2)\cdot\frac{i}{-k^2-M^2}\left(\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)$$

10.12

* We had seen: scalar field coupled to U(1) Gauge theory. ~~If gauge sym~~ non-zero vacuum expectation value ⇒ symmetry breaking ⇒ massive gauge particles (bosons).

* $i(2\pi)^4 \delta^4(k_1+k_2) \dfrac{1}{-k^2-M^2}\left(\eta_{\mu\nu} + \dfrac{k_\mu k_\nu}{M^2}\right)$

No. of independent poles at $k^2 = -M^2$

i.e. near $k = (M,0,0,0)$

[Choosing specific Lorentz frame].

Now, Residue at $k^2 = -M^2$

$\eta_{\mu\nu} + \dfrac{k_\mu k_\nu}{M^2}$

$\rightarrow \text{diag}(-1,1,1,1) + (1,0,0,0) + O(k^2+M^2)$

$= \text{diag}(0,1,1,1) \rightsquigarrow$ eigenvalues [do not change if we go to other Lorentz frames.]

$\rightsquigarrow$ one of the residues is zero.

$\rightsquigarrow$ 3-independent polarizations.

$\rightsquigarrow$ in general non-zero ~~polar~~ eigenvalues correspond to SL polarization vector.

* Broken phase ($\phi_0 \neq 0$)

1 massive gauge field $\rightarrow$ 3 states
1 massive scalar $\chi_1 \rightarrow$ 1 state.
} 4 states

Generically, Total # states do not change as we change the phases.

Unbroken phase ($\phi_0 = 0$)

1 massless gauge field $\rightarrow$ 2 states
2 scalar $\phi_1, \phi_2 \rightarrow$ 2 states.
} 4 states

* General Case:

Gauge group G broken to H.

→ n_H gauge fields correspond to H remain massless.

→ $n_G - n_H$ gauge fields become massive.

→ T^a : generators of G for $a = 1, \dots, n_G$

T^α : " " H " $\alpha = 1, \dots, n_H$

T^k : " " G not in H for $k = n_H + 1, \dots, n_G$.

Then, A_μ^α for $\alpha = 1, \dots, n_H$ are massless.

$B_\mu^k = A_\mu^k$ for $k = n_H + 1, \dots, n_G$ are massive $[n_G - n_H]$

↓

become massive by absorbing the goldstone bosons. $n_G - n_H$ goldstone scalar bosons disappear

Back to the Standard Model:

* Gauge group $SU(3) \times SU(2) \times U(1)$

↓

hypercharge Y

* Higgs field: singlet of $SU(3)$

fundamental of $SU(2)$

hypercharge $Y = 1$

ϕ_α

↓

$SU(2)$ index

$1, 2$

$(A : 1, 2, 3) \leftarrow SU(2)$

Infinitesimal transf: $\delta\phi_\alpha = i\epsilon_A (L^A)_{\alpha\beta}\phi_\beta + i\epsilon\phi_\alpha$

$$-\int d^4x \; V(\phi)$$

$\uparrow$

Higgs potential

$$V(\phi) = m^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

$\hookrightarrow$ min at $\phi = 0$ $\rightarrow$ unbroken gauge symmetry.

Experiment $\rightarrow$ symmetry has to be broken (weak interac is mediated by massive bosons) $\rightarrow$ so we take

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$
$$= \lambda (\phi^\dagger \phi - v^2)^2 - \lambda v^4 \quad \text{with } v = \sqrt{\frac{\mu^2}{2\lambda}}$$

Now, $V(\phi)$ has minimum at $\phi^\dagger \phi = v^2$.

We could have also chosen

$\begin{pmatrix} v \cos\theta \; e^{i\alpha} \\ v \sin\theta \; e^{i\beta} \end{pmatrix}$. But all ϕ's are related by $SU(2) \times U(1)$.

$\Rightarrow$ same physics

$\rightarrow$ choose $\phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$

Here, # gener. of $SU(2) \times U(1)$: 4
parameters θ, α, β : 3

$\Big\}$ $(4-3)$ should be unbroken

Unbroken gauge trans $\delta \phi_\alpha = 0$.

$$\Rightarrow i \epsilon_A (T^A)_{\alpha\beta} \phi_{0\beta} + i\epsilon \phi_{0\alpha} = 0.$$

$$\Rightarrow i \left\{ \frac{\epsilon_1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\epsilon_2}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{\epsilon_3}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0$$

$$\Rightarrow \quad i \begin{pmatrix} \frac{\epsilon_3 + 2\epsilon}{2} & \frac{\epsilon_1 - i\epsilon_2}{2} \\ \frac{\epsilon_1 + i\epsilon_2}{2} & -\frac{\epsilon_3}{2} + \epsilon \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0$$

$$\Rightarrow \quad (\epsilon_1 - i\epsilon_2)\, v = 0 \quad \Rightarrow \epsilon_1 = 0 = \epsilon_2$$

$$\left(-\frac{\epsilon_3}{2} + \epsilon\right) = 0 \quad \Rightarrow \epsilon = \epsilon_3/2 \quad \rightsquigarrow$$ vacuum expectation value does not change.

$\rightsquigarrow$ 1 parameter.

Unbroken generator $\epsilon_A T^A + \epsilon Y$

$$= \epsilon_3 T^3 + \frac{\epsilon_3}{2} Y$$

$$= \epsilon_3 \left(T^3 + Y/2\right)$$

Unbroken symmetry is generated by $T^3 + \frac{1}{2} Y$

Check for Higgs: $\begin{pmatrix} 0 \\ v \end{pmatrix} \sim$ $T^3 = -1/2$ (ev.)

$Y = 1$

$\Rightarrow$ indeed $T^3 + \frac{1}{2} Y = 0$

$\therefore$

1 massless $\rightarrow$ photon (neutral gauge boson).

3 - massive $\longrightarrow$ $W^{\pm}, Z$.

We identify $T^3 + Y/2$ as physical electric charge.

$\boxed{*}$ Fermions :

(1) Left-handed $\rightarrow$ fundamental of $SU(3)$ & $SU(2)$ and hypercharge $Y = 1/3$.

$$U_{i\alpha L} \quad \swarrow SU(3) \quad \searrow SU(2)$$

(2) Right-handed $\rightarrow$ fundamental of $SU(3)$, singlet of $SU(2)$, hypercharge Y

$$\begin{array}{cc} 4/3 & -2/3 \\ u_{iR} & d_{iR} \end{array}$$

Repeated twice more.

Yukawa coupling : $\lambda_d \bar{U}_{i\alpha L} \phi_\alpha d_{iR} + \lambda_u \bar{U}_{i\alpha L} u_{iR} \epsilon_{\alpha\beta} \phi^*_\beta + h.c.$

Expand. $\phi = \begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

$$= \lambda_d v \bar{U}_{i2L} d_{iR} + \lambda_u v \bar{U}_{i1L} u_{iR} + h.c. + \text{cubic}$$

Physical charge : $T_3 + Y/2$: d_{iR} : $-1/3$

U_{i2L} : $-1/2 + \frac{1}{6} = -1/3$.

$\downarrow$ identify

d_{iL}

and $\lambda_d v \equiv m$

$\boxed{H.W.}$ Check for $\begin{pmatrix} v \\ 0 \end{pmatrix} \rightarrow -T^3 + Y/2$.

u_{iR} : $0 + 2/3 = 2/3$.

u_{iL} : $1/2 + 1/6 = 2/3$.

↳ identify u_{iL}

and $\lambda_u v \equiv m_u \Rightarrow$ mass of up quark.

[*] (3) Left-handed fermions → fundamental of $SU(2)$, singlet of $SU(3)$, $Y = -1$

$\rightsquigarrow E_{\alpha L}$

(4) right-handed fermions → singlet of $SU(3)$ & $SU(2)$, $Y = -2$

$\rightsquigarrow e_R$

Repeated twice more.

Yukawa coupling:

$$\lambda_e \underset{Y=1}{\bar{E}_{\alpha L}} \underset{Y=1}{\phi_\alpha} \underset{Y=-2}{e_R}$$

Expand $\phi = \begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

Quadratic term: $\lambda_e v \bar{E}_{2L} e_R$

$T_3 + \frac{Y}{2}$: $e_R \rightsquigarrow 0 + (-1) = -1$

$E_{2L} \rightsquigarrow -1/2 - 1/2 = -1$

↳ identify as $\bar{e}_L$

$\Rightarrow \lambda_e v \rightsquigarrow$ mass of electron.

E_{1L} : $T_3 + Y/2 = 1/2 - 1/2 = 0$

$\hookrightarrow$ ν_{eL} $\rightsquigarrow$ no mass term. We've already written most general Yukawa potential.

$\lambda_\nu v \bar{\nu}_{eL} \nu_{eR} \rightarrow T_3 + Y/2 = 0$ for ν_{eR}

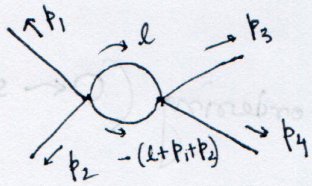
$\Rightarrow$ $Y = 0$ & since $T_3 = 0$

Coupling: $\lambda_\nu \bar{E}_{\alpha L} \nu_R \epsilon_{\alpha\beta} \phi_\beta$

$$\begin{array}{ccc} \downarrow & | & | \\ 1 & 0 & -1 \end{array}$$

UV divergences

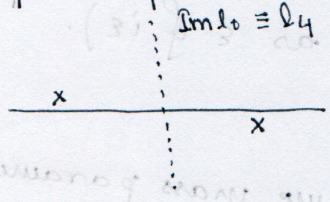
* **Example 1** ϕ^4 theory



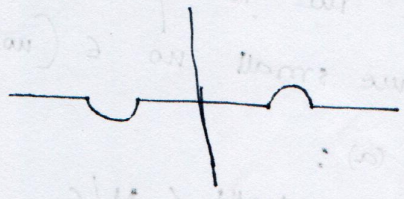
$$(\text{constant}) \prod_{k=1}^{4} \frac{i}{-p_k^2 - m^2 + i\epsilon} (2\pi)^4 \delta^4(p_1+p_2+p_3+p_4)$$

$$\underbrace{\int \frac{d^4 l}{(2\pi)^4} \frac{i}{-l^2 - m^2 + i\varepsilon} \frac{i}{-(l+p_1+p_2)^2 - m^2 + i\epsilon}}_{A}$$

Complex l_0 plane :



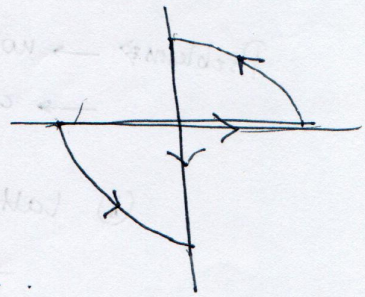
$\equiv$



$\rightarrow$ finite integral.

So, $l^2 + m^2 - i\epsilon \rightarrow$ is not a problem.

But, divergence at large l is problematic.



$$\sim \int \frac{d^4 l}{l^2 + m^2 - i\varepsilon}$$

$$\xrightarrow{l_0 \to i l_4} \int \frac{d^4 l_1 \cdots d^4 l_4 \; i}{l_1^2 + \cdots + l_4^2 + m^2}$$ no need of $i\epsilon$.

$$\therefore A \overset{\text{Large}}{\underset{l}{\sim}} \int \frac{r^3 dr \, \Omega_3 \, i}{(r^2+m^2)^2}$$

$dx\,dy = 2\pi r\, dr = \Omega_1 dr\, r$

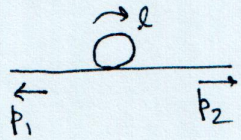
$dx\,dy\,dz = 4\pi r^2 dr = \Omega_2 r^2 dr$

with $r = \sqrt{l_1^2 + \cdots + l_4^2}$

Ω_3 : vol of unit 3-sphere

$$\sim \Omega_3 \, i \int \frac{r^3 dr}{(r^2+m^2)^2} \xrightarrow{r \to \infty} \int \frac{dr}{r}$$ $\rightarrow$ logarithmically divergent

* <u>Example 2</u>:



$$\sim \int \frac{d^4\ell}{-\ell^2 - m^2 + i\epsilon} \rightsquigarrow \text{quadratically divergent.}$$

[NOTE: This divergence, though, arises in normal ordering (O ← same vertex)].

* How do we deal with these?

⇓

Two steps:

① Regularization:

Make the theory finite by using a "cut-off" parametrized by some small no ϵ (not the same as ε of $i\varepsilon$).

Example (a):

$$|\ell^\mu| < M/\epsilon \quad , M \text{ is some mass parameter.}$$

Problems ⇒ → not LI

→ can't shift ℓ^μ to $(\ell+k)^\mu$ ⇝ range should be changed too.

(b) Lattice cut-off

→ lower bound in the ~~momentum~~ wavelength

→ upper " " " momentum.

(c). Pauli - Villars

$$\frac{1}{\ell^2} \to \left(\frac{1}{\ell^2} - \frac{1}{\ell^2 + M^2/\epsilon} \right) \overset{\ell \to \infty}{\sim} \frac{1}{\ell^4}$$

(d) Dimensional regularization

Work in $4-\epsilon$ dimensions instead of in 4 dim and take $\epsilon \to 0$ limit at the end.

② Renormalization:

Make the parameters of the theory, eg. m, λ in ϕ^4 theory, ϵ-dependent & divergent in the $\epsilon \to 0$ limit such that these cancel the divergences from the loop integrals.

→ parameters are not observables → not a problem if we make them divergent → observables should be finite.

→ finite # parameters → adjusting these one has to reance divergences coming from all the loop diagrams of all order!

In general, by adjusting finite # parameters we cant cancel the UV-divergences in the ∞-# of S-matrix elements. Theories for which this is possible are called renormalizable QFT.

Alternative viewpoint ⇒ replace, first, n-parameters in terms of n-observables which are finite → then express other observables in terms of these n-observables → comes out to be finite.

✱ Dimensional regularization:

✱ Typical integral we encounter → denominator factor:

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + \underbrace{L}_{\text{constant}})^\alpha}$$

→ divergent for $D \geq 2\alpha$.

Rotate k^0 to ik^D →

$$\int \frac{dk^1 \cdots dk^D}{(2\pi)^D} \frac{i}{(k^{1\,2} + \cdots + k^{D\,2} + L)^\alpha}$$

Define $r \equiv \sqrt{k^{1^2} + \cdots + k^{D^2}}$

So,

$$i \int \frac{r^{D-1}\, dr\, \Omega_{D-1}}{(2\pi)^D} \frac{1}{(r^2 + L)^\alpha}$$

Let, $r = \sqrt{L}\, u$

$$= i\, L^{D/2 - \alpha} \frac{\Omega_{D-1}}{(2\pi)^D} \int_0^\infty u^{D-1}\, du\, (1+u^2)^{-\alpha}$$

Def^n of beta fun

$$\int_0^\infty \frac{dt\, t^{m-1}}{(t+a^2)^n} = \frac{1}{(a^2)^{n-m}} \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}$$

So, $\int_0^\infty du \frac{u^{D-1}}{(1+u^2)^\alpha} = \frac{1}{2}\int_0^\infty \frac{du^2}{u} \frac{(u^2)^{(D-1)/2}}{(u^2+1)^\alpha}$

Now, $\boxed{\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}}$

Checking: $D = 2 \rightsquigarrow \Omega_1 = \frac{2\pi}{\Gamma(1)} = 2\pi$

$D = 3 \rightsquigarrow \Omega_2 = \frac{2\pi^{3/2}}{\Gamma(3/2)} = \frac{2\pi^{3/2}}{\frac{1}{2}\sqrt{\pi}} = 4\pi$

$D = 4 \rightsquigarrow \Omega_3 = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2$.

Though for $D \neq$ ~~non~~ integer $\rightsquigarrow$ it doesn't make sense of talking area of $D-1$ sphere, nevertheless we analytically continue this & replace Ω_{D-1} by this for any real D.

So,

$$i\, L^{D/2 - \alpha} \frac{1}{(2\pi)^D} \frac{\cancel{2}\pi^{D/2}}{\cancel{\Gamma(D/2)}} \frac{\cancel{\Gamma(D/2)}\, \Gamma(\alpha - D/2)}{\cancel{2}\, \Gamma(\alpha)}$$

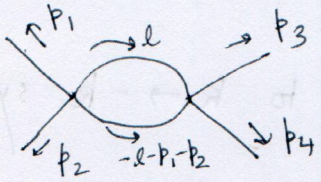
$$= i\, L^{D/2 - \alpha} \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(\alpha - D/2)}{\Gamma(\alpha)}$$

Check: $D = 4 \rightsquigarrow$ should diverge for $\alpha = 0, 1, 2$

See, $\alpha = 0 \rightsquigarrow \Gamma$ diverges for negative integer & at 0.

So, make D as $D-\epsilon$.

* Dealing with more denominators :



Typical situation

$$\frac{1}{\{(l+k_1)^2+m^2\}\{(l+k_2)^2+m^2\}\cdots\{(l+k_n)^2+m^2\}}$$

other loop momentum / external momentum.

$$\frac{1}{A_1\cdots A_n} = \int_0^1 dx_1 \cdots \int_0^1 dx_n\, \delta(1-x_1-x_2-\cdots-x_n)\, \frac{(n-1)!}{(x_1A_1+x_2A_2+\cdots+x_nA_n)^n}$$

$\xrightarrow{\frac{d}{dA_1}}$

$$\frac{1}{A_1^2A_2\cdots A_n} = \int_0^1 dx_1\cdots\int_0^1 dx_n\, \delta(1-x_1-\cdots-x_n)\, \frac{n!}{(x_1A_1+\cdots+x_nA_n)^{n+1}}$$

Using this we get,

$$\int_0^1 dx_1\cdots\int_0^1 dx_n\, \delta(1-x_1-\cdots-x_n)\, \frac{(n-1)!}{\{\sum_i x_i(l+k_i)^2+m^2\sum x_i\}^n}$$

Loop integral $= \int_0^1 dx_1\cdots\int_0^1 dx_n\, \delta(1-x_1-\cdots-x_n)$

$$\int \frac{d^Dl}{(2\pi)^D}\, \underbrace{\frac{(n-1)!}{(l^2+2l\cdot\sum x_ik_i+\sum x_ik_i^2+m^2)^n}}$$

$$\frac{(n-1)!}{\{(\underbrace{l+\sum x_ik_i}_{\tilde{l}})^2-(\sum x_ik_i)^2+\sum x_ik_i^2+m^2\}^n}$$

$$= \int_0^1 dx_1\cdots dx_n\, \delta(1-\sum x_i)\int \frac{d^D\tilde{l}}{(2\pi)^D}\, \frac{(n-1)!}{(\tilde{l}^2+*)^n}$$

$\longrightarrow$ same form as earlier.

Or, use partial decomposition to recast into the above form.

* Numerator factors:

$$\int \frac{d^D k}{(2\pi)^D} \frac{k_\mu}{(k^2+L)^\alpha} = 0 \quad \text{due to } k \to -k \text{ symmetry.}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2+L)^\alpha} = A\, \eta_{\mu\nu} \quad (\mathcal{L}I)$$

$$\Rightarrow \underbrace{\eta^{\mu\nu}\eta_{\mu\nu}}_{D} A = \frac{1}{(2\pi)^D} \int \frac{d^D k}{(k^2+L)^\alpha} (k^2 + L - L)$$

$\longrightarrow$ Done

* Multiple loops:

$$\int \frac{d^D l_1}{(2\pi)^D} \cdots \frac{d^D l_N}{(2\pi)^D} \underbrace{\prod_{i=1}^{n} \frac{1}{D_i}}$$

$\downarrow$ Feynman reparametriz.

$$\int_0^1 dx_1 \cdots \int_0^1 dx_n \; \delta(1 - x_1 - \cdots - x_n) \frac{(n-1)!}{(x_1 D_1 + \cdots + x_n D_n)^n}$$

$$A_{ij}\, l_i \cdot l_j + 2 l_i B_i + C$$

$\Leftrightarrow$ Symm.

$\downarrow$

diagonalize A. i.e.

$A = U^T A_d U$

define $U_{ij} l_j = \tilde{l}_i$

So,

$$\sum_i a_i(\vec{x}) \tilde{l}_i^2 + 2 \sum_i b_i(\vec{x}) \cdot \tilde{l}_i + c(\vec{x})$$

define $a_i^{1/2} \tilde{l}_i \equiv \hat{l}_i$

Hence,

$$\int dx_1 \cdots dx_n \; \delta(1 - x_1 - \cdots - x_n) \int \frac{d^D \hat{l}_1}{(2\pi)^D} \cdots \frac{d^D \hat{l}_N}{(2\pi)^D} \frac{\prod_i a_i^{-D/2}}{\left(\underbrace{\sum_i \hat{l}_i^2 + 2 \sum \hat{l}_i \hat{b}_i + \tilde{c}}_{\sum_i (\underbrace{\hat{l}_i + \hat{b}_i}_{\bar{l}_i})^2 + \tilde{c} - \sum_i \hat{b}_i^2} \right)^n}$$

$$\longrightarrow \int \frac{d^D \bar{l}_1}{(2\pi)^D} \cdots \frac{d^D \bar{l}_N}{(2\pi)^D} \frac{1}{\left(\sum_i \bar{l}_i^2 + \bar{c} \right)^n}$$

** At this stage do not put $D = 4$

At the end of the whole calculation put $D = 4$. Only after renorm. process put $D = 4$.

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General Replacement Rules:

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \int \frac{d^Dk}{(2\pi)^D} \qquad D = 4-\epsilon$$

$$(2\pi)^4 \delta^4(k) \longrightarrow (2\pi)^{4-\epsilon} \delta^{4-\epsilon}(k)$$: when it appears under $\int \frac{d^Dk}{(2\pi)^D}$

$\rightarrow$ gives 1.

$$\eta^{\mu\nu}\eta_{\mu\nu} = D = 4-\epsilon$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\gamma_\mu \gamma^\mu = D = 4-\epsilon$$

$$\gamma_\mu \gamma^\nu \gamma^\mu = \{\gamma_\mu, \gamma^\nu\}\gamma^\mu - \gamma^\nu \gamma_\mu \gamma^\mu = 2\gamma^\nu - D\gamma^\nu = (2-D)\gamma^\nu$$

$$\underline{\mathrm{Tr}(\gamma^\mu \gamma^\nu)} = D\eta^{\mu\nu}$$

$$\frac{1}{2}\mathrm{Tr}(\underline{\{\gamma^\mu, \gamma^\nu\}}) = \frac{1}{2}\underline{2\eta^{\mu\nu}}\ \underbrace{\mathrm{Tr}(I)}_{2 \text{ for } D=3}$$

One could take $4 \times$ fun. of ϵ but we take 4

$\rightarrow$ final result won't be affected.

$\gamma_5 \rightarrow$ can't be regularised using dimensional regularisation.

⊠ RENORMALIZATION:

* Consider a field theory with fields $\phi_1, \ldots, \phi_N$
  → scalars, comp of vectors, fermions etc.

* Parameters $g_1, \ldots, g_M$
  → mass, coupling constants, gauge fixing parameters α etc.

* Introduce new sets of fields & parameters (same #)

$$\phi_{iR}, \; g_{sR} \qquad i = 1, \ldots, N \; ; \; s = 1, \ldots, M \; ; \; R \equiv \text{Renormalized fields \& param}$$

satisfying

$$g_s = f_s(g_{1R}, g_{2R}, \ldots, g_{MR}, \epsilon) \; ; \; s = 1, \ldots, M$$

$$\phi_i = K_{ij}(g_{1R}, g_{2R}, \ldots, g_{MR}, \epsilon)\, \phi_{jR} \; ; \; i, j = 1, \ldots, N$$

(sum over j)

f_s, K_{ij} : some functions.

* $$\left\langle \prod_{l=1}^{n} \phi_{i_l R}(x_l) \right\rangle$$

$$= \sum_{j_1, j_2, \ldots, j_l} \prod_{l=1}^{n} (K^{-1})_{i_l j_l} \left\langle \prod_{l=1}^{n} \phi_{j_l}(x_l) \right\rangle$$

→ a function $g_{1R}, \ldots, g_{MR}, \epsilon, \{x_l\}$.

Ques: Is this quantity finite as $\epsilon \to 0$ with fixed $g_{1R}, \ldots, g_{MR}$?

Correct question: Is it possible to choose K_{ij} & f_s s.t.

$$\left\langle \prod_{l=1}^{n} \phi_{i_l R}(x_l) \right\rangle \text{ is finite for all Green's fun?}$$

$\Longrightarrow$ S-matrix is finite.

[NOTE: Though for some cases it could happen that Green's fun. are not finite but S-matrix are.]

$\Longrightarrow$ Renormalizable theory.

* To lowest order in perturbation theory, we do not "need to" renormalize.

$$g_s = F_s(\bar{g}_R, \epsilon) \qquad \text{where } \bar{g}_R = \{g_{1R}, \dots, g_{MR}\}$$

$$= g_{sR} + G_s(\bar{g}_R, \epsilon)$$

$$K_{ij} = \delta_{ij} + L_{ij}(\bar{g}_R, \epsilon)$$

contain at least one power of the perturbation parameter

So, $\phi_i = \phi_{iR} + L_{ij}(\bar{g}_R, \epsilon)\,\phi_{jR}$

Lagrangian density:

$$\mathcal{L}(\{\phi_i\}, \{g_s\})$$

$$= \mathcal{L}(\{\phi_{iR} + L_{ij}\phi_{jR}\}, \{g_{sR} + G_s\})$$

$$= \mathcal{L}(\{\phi_{iR}\}, \{g_{sR}\}) + \mathcal{L}_{ct}$$

counter term. Given in terms of $\phi_{iR}, \bar{g}_R$ depending on L_{ij}, G_s.

Adjust L_{ij}, G_s to cancel divergences arising from $\mathcal{L}$.
Though it's not guaranteed for any arbit theory.

* Renormalization of ϕ^4 Theory:

* $$S = \int d^{4-\epsilon}x \left[-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 \right]$$

dimen

$$[L] = [M]^{-1}$$

$$M^0 = M^{\epsilon-4}\left[M^2 M^{2\,\mathrm{dim}(\phi)} + M^{2\,\mathrm{dim}(m)} M^{2\,\mathrm{dim}(\phi)} + M^{\mathrm{dim}(g) + 4\,\mathrm{dim}(\phi)}\right]$$

So, $2\,d(\phi) = 4 - \epsilon - 2 = 2 - \epsilon \Rightarrow d(\phi) = 1 - \epsilon/2$

$$d(m) = \frac{1}{2}\{4 - \epsilon - 2\,d(\phi)\} = 1$$

$$d(g) = 4 - \epsilon - 4\,d(\phi) = 4 - \epsilon - 4 + 2\epsilon = \epsilon$$

~~Let~~ Now,

$$\phi = K\phi_R \equiv \tilde{Z}_\phi^{1/2}(g_R, m_R, \epsilon)\,\phi_R$$

$$m^2 = m_R^2 + \cdots \equiv m_R^2 Z_m(g_R, m_R, \epsilon)^2$$

To make g_R dimensionless we introduce some mass parameter μ^ϵ : $g_R\mu^\epsilon$.

$$g = g_R \mu^\epsilon Z_g(g_R, m_R, \epsilon) \rightsquigarrow \text{leading order } Z_g = 1 \text{ s.t. } g = g_R.$$

So,

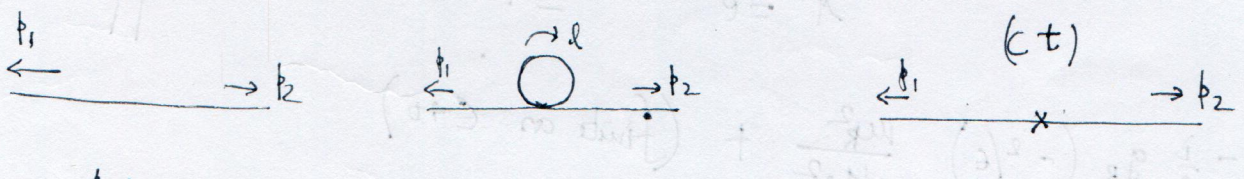
$$S = \int d^{4-\epsilon}x \left[-\frac{1}{2}\tilde{Z}_\phi \partial_\mu\phi_R \partial^\mu\phi_R - \frac{1}{2}\tilde{Z}_\phi Z_m^2 m_R^2 \phi_R^2 - \frac{1}{4!}\mu^\epsilon g_R \tilde{Z}_\phi^2 Z_g \phi_R^4 \right]$$

$$= \int d^{4-\epsilon}x \left[-\frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}m_R^2\phi_R^2 - \frac{g_R\mu^\epsilon}{4!}\phi_R^4 \right]$$

$$+ \int d^{4-\epsilon}x \left[-\frac{1}{2}(\tilde{Z}_\phi - 1)\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}(\tilde{Z}_\phi Z_m^2 - 1)m_R^2\phi_R^2 - \frac{1}{4!}\mu^\epsilon g_R(Z_g\tilde{Z}_\phi^2 - 1)\phi_R^4 \right]$$

$\because \tilde{Z}_\phi \rightsquigarrow$ at least $O(g_R)$ $\Rightarrow$ 2nd part as perturbation.

* <u>Two point fun</u> $\langle \tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)\rangle$: $O(g_R)$



$$(2\pi)^{4-\epsilon}\delta^{4-\epsilon}(p_1+p_2)\frac{i}{-p_1^2 - m_R^2 + i\epsilon}$$

$$+ (2\pi)^{4-\epsilon}\delta^{4-\epsilon}(p_1+p_2)\frac{i}{-p_1^2 - m_R^2 + i\epsilon}\frac{i}{-p_2^2 - m_R^2 + i\epsilon}\int\frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}}\frac{i}{-l^2 - m_R^2 + i\epsilon}$$

$$\left(-\frac{i}{4!}\right)\mu^\epsilon g_R \times 4 \times 3$$

$$+ (2\pi)^{4-\epsilon}\delta^{4-\epsilon}(p_1+p_2)\frac{i}{-p_1^2 - m_R^2 + i\epsilon}\frac{i}{-p_2^2 - m_R^2 + i\epsilon}\, i\left[-\frac{1}{2}(\tilde{Z}_\phi - 1)p_1^2 \times 2 - \frac{1}{2}(\tilde{Z}_\phi Z_m^2 - 1)m_R^2 \times 2\right]$$

$\cancel{= (2\pi)^4 \delta^4}$ 2nd & 3rd $\rightsquigarrow$

$$\Rightarrow (2\pi)^{4-\epsilon} \delta^{4-\epsilon}(p_1+p_2) \frac{i}{-p_1^2 - m_R^2 + i\varepsilon} \frac{i}{-p_2^2 - m_R^2 + i\varepsilon}$$

$$\cdot \left[\int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} \frac{i}{-l^2 - m_R^2 + i\varepsilon} \times \left(-\frac{i}{2}\right) \mu^\epsilon g_R \right.$$

$$\left. - i(\tilde{Z}_\varphi - 1) p_1^2 - i(\tilde{Z}_\varphi Z_m^2 - 1) m_R^2 \right]$$

Now, $[\ ] \rightsquigarrow l_0 \to i l_4$

$\Downarrow$

$$\left(-\frac{i}{2}\right) \mu^\epsilon g_R (-i)(i) \int \frac{d^{4-\epsilon} l^\epsilon}{(2\pi)^{4-\epsilon}} \frac{1}{l^2 + m_R^2} - i(\tilde{Z}_\varphi - 1) p_1^2 - i(\tilde{Z}_\varphi Z_m^2 - 1) m_R^2$$

Recall $\int \frac{d^D k}{(2\pi)^D} \frac{i}{(k^2+L)^\alpha}$ for $L \neq 0$

$$= i L^{D/2 - \alpha} \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(\alpha - D/2)}{\Gamma(\alpha)}$$

$$= -\frac{i}{2} \mu^\epsilon g_R \frac{\Gamma(-1+\epsilon/2)(m_R^2)^{1-\epsilon/2}}{(4\pi)^{2-\epsilon}}$$

$$- i(\tilde{Z}_\varphi - 1) p_1^2 - i(\tilde{Z}_\varphi Z_m^2 - 1) m_R^2$$

now, $\Gamma(-1+\epsilon/2) = \frac{\Gamma(\epsilon/2)}{(-1+\epsilon/2)} \simeq -2/\epsilon$

$$\mu^\epsilon = e^{\epsilon \ln \mu} = 1 + \epsilon \ln \mu + \cdots$$

$$-\frac{2}{\epsilon}\left(1 - \frac{\epsilon}{2}\right)$$

$$= -\frac{i}{2} g_R (-2/\epsilon) \frac{m_R^2}{(4\pi)^2} + (\text{finite as } \epsilon \to 0)$$

$$- i(\tilde{Z}_\varphi - 1) p_1^2 - i(\tilde{Z}_\varphi Z_m^2 - 1) m_R^2$$

(b)

We choose $\tilde{Z}_\varphi = 1 + O(g_R^2)$

$$\tilde{Z}_\varphi Z_m^2 = 1 + \cancel{\tfrac{1}{2}} g_R \frac{\cancel{m_R^2}\, 1}{(4\pi)^2} \frac{1}{\epsilon} + O(g_R^2)$$

$$\Rightarrow Z_m^2 = 1 + \frac{1}{16\pi^2 \epsilon} g_R + O(g_R^2)$$

$$\Rightarrow Z_m = 1 + \frac{1}{32\pi^2 \epsilon} g_R + O(g_R^2)$$

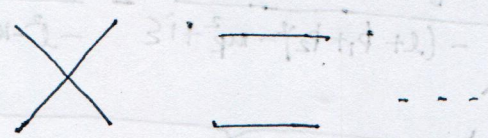
Always think perturb. in terms of g_R, not in terms of ϵ.

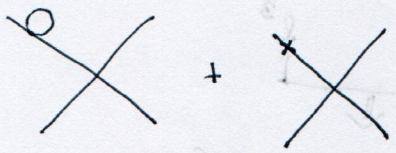
* $$S = \int d^4x \left[-\frac{1}{2} \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} m_R^2 \phi_R^2 - \frac{g_R \mu^\epsilon}{4!} \phi_R^4 \right]$$

$$+ \int d^4x \left[-\frac{1}{2} (\tilde{Z}_\varphi - 1) \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} (\tilde{Z}_\varphi Z_m^2 - 1) m_R^2 \phi_R^2 \right.$$

$$\left. - \frac{1}{4!} \mu^\epsilon g_R (\tilde{Z}_\varphi^2 Z_g - 1) \phi_R^4 \right]$$

* <u>4-point function in ϕ^4</u>:





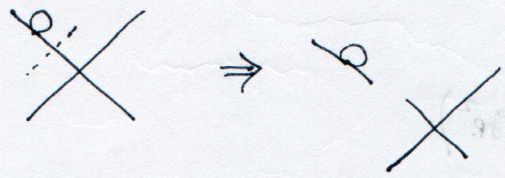
→ finite; since we've already done for two pt. fun.

So, which diagrams? (we ~~don't~~ have to worry about).

⇓

One particle irreducible diagrams (1PI)

→ diagram cannot be divided into two parts by cutting a single line.

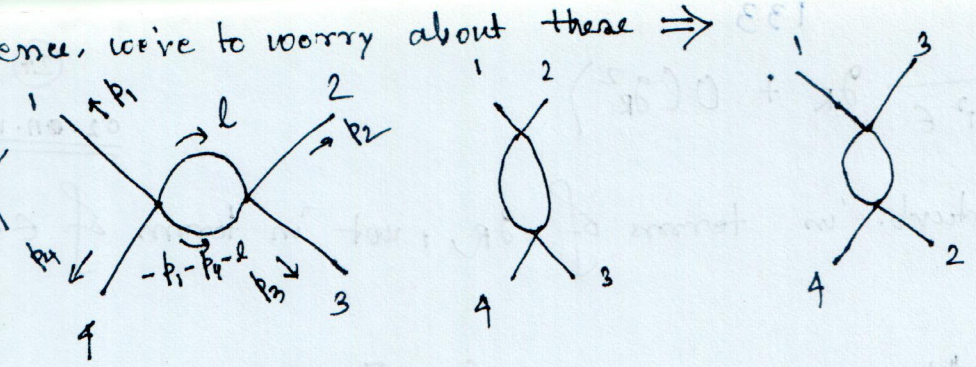


By cutting a line we divide it into two non-trivial parts

⇒ Reducible

⇒ don't worry about this.

ence, we've to worry about these $\Rightarrow$



(A)

$$\Downarrow$$

$$\int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} \frac{i}{-l^2 - m_R^2 + i\varepsilon} \frac{i}{-(l+p_1+p_4)^2 - m_R^2 + i\varepsilon}$$

Even without doing Feynman parametrization we could solve it.

$$\Downarrow$$

$$= \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} \frac{i}{-l^2 - m_R^2 + i\varepsilon} \left\{ \frac{i}{-l^2 - m_R^2 + i\varepsilon} + \underbrace{\left(\frac{i}{-(l+p_1+p_2)^2 - m_R^2 + i\varepsilon} - \frac{i}{-l^2 - m_R^2 + i\varepsilon} \right)}_{\frac{1}{l^3}} \right\}$$

$$\Downarrow$$

$$\frac{1}{l^5} \Rightarrow \text{finite.}$$

Hence,

(A) $\Rightarrow$

$$(2\pi)^{4-\epsilon} \delta^{4-\epsilon}(p_1 + \cdots + p_4) \prod_{k=1}^{4} \frac{i}{-p_k^2 - m_R^2 + i\varepsilon}$$

$$\left\{ -\frac{i}{4!} g_R \mu^{\epsilon} \left(\tilde{Z}_\varphi^2 Z_g - 1 \right) \times 4! + i \frac{3 g_R^2}{16\pi^2 \epsilon} + \text{finite} \right\}$$

$$\Rightarrow \quad Z_g \tilde{Z}_\varphi^2 = 1 + \frac{3 g_R}{16\pi^2 \epsilon} + O(g_R^2)$$

$$\Rightarrow \quad Z_g = 1 + \frac{3 g_R}{16\pi^2 \epsilon} + O(g_R^2).$$

Hence,

$$Z_m = 1 + \frac{g_R}{32\pi^2\epsilon} + O(g_R^2)$$

$$\tilde{Z}_\varphi = 1 + O(g_R^2)$$

$$Z_g = 1 + \frac{3g_R}{16\pi^2\epsilon} + O(g_R^2)$$

These are not unique. We could add

$$Z_m = 1 + \frac{g_R}{32\pi^2\epsilon} + O(g_R^2) + \underset{\substack{\uparrow \\ \text{finite cons.}}}{K_m g_R}$$

$$\tilde{Z}_\varphi = 1 + O(g_R^2) + K_\varphi g_R$$

$$Z_g = 1 + \frac{3g_R}{16\pi^2\epsilon} + O(g_R^2) + K_g g_R$$

Normalization constant ambiguity.

Extra part $\Downarrow$ non-minimal parts $\Downarrow$

Previous $\Downarrow$ Minimal subtraction.

More parameters. But one can show these are not new param. These can be reexpressed in terms of existing parameters.

$$Z_m = 1 + \frac{g_R}{32\pi^2\epsilon} + K_m g_R + O(g_R^2)$$

$$\Rightarrow m = Z_m m_R = \left(1 + \frac{g_R}{32\pi^2\epsilon} + K_m g_R\right) m_R + O(g_R^2)$$

$$\equiv \left(1 + \frac{g_R}{32\pi^2\epsilon}\right) m_R' + O(g_R^2)$$

$$m_R' = m_R(1 + K_m g_R)$$

So, either in terms of m_R, g_R or m_R', g_R' etc. So, in any scheme dependence on only two parameters.

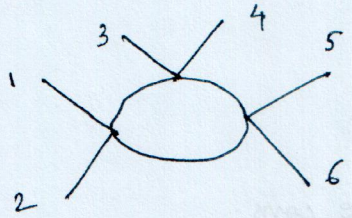
Similarly,

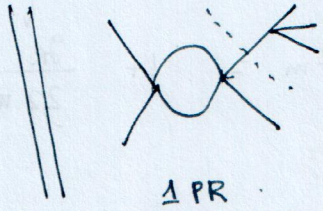
K_φ can be absorved into a redefinition of φ.

K_g " " " " " " " g_R.

* <u>6-point fcn</u>:

We need to look at 1PI diagrams:





$$\sim \int \frac{d^{4-\epsilon} l}{(l^2)^3} \longrightarrow \text{finite}.$$

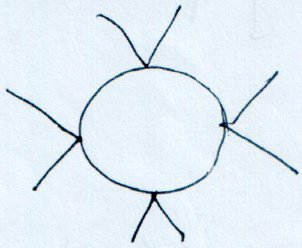
In 6-dimension this diagram is divergent.

Let's add $\lambda \phi^6$ to $\mathcal{L}$.

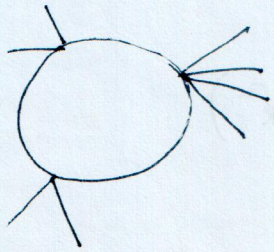
$\rightsquigarrow \lambda = Z_\lambda \lambda_R$

Adjust Z_λ to cancel divergence.

* <u>8-pt-fcn</u>:



$$\int \frac{d^{6-\epsilon} l}{(l^2)^4} \longrightarrow \text{finite}$$



$$\int \frac{d^{6-\epsilon} l}{(l^2)^3} \longrightarrow \text{divergent}$$

So, it doesn't help to add $\lambda \phi^6$.

$\longrightarrow$ it will go to infinite # parameters.

$\rightsquigarrow$ ϕ^4 in 6 dim is not renormalizable.

Q. Under what cond^{n} is a theory renormalizable?

$\Rightarrow$ Consider a theory in D-dim.

$$S = \frac{1}{2}\int d^D x \, \partial_\mu \phi \partial^\mu \phi + \cdots$$

Mass dimension : $2\,d(\phi) + 2 - D = 0$

$$\Rightarrow \boxed{d(\phi) = \frac{D-2}{2}}$$

$$\tilde{\phi}(k) = \int d^D x \, \phi(x) e^{ik\cdot x}$$

$$\boxed{d[\tilde{\phi}] = d(\phi) - D = -\frac{D+2}{2}}$$

Suppose, the action has a term $g^{(n)} \int d^D x \, \phi^n$

$$d[g^{(n)}] - D + n\,d(\phi) = 0$$

$$\Rightarrow d[g^{(n)}] = D - n\,d(\phi) = \underbrace{D - n\frac{D-2}{2}}_{\lambda_n}$$

Now,

$$\left\langle \prod_{i=1}^{N} \tilde{\phi}(p_i) \right\rangle$$

$$= \left(\prod_{i=1}^{N} \frac{i}{-p_i^2 - m_R^2 + i\varepsilon} \right) (2\pi)^D \delta^D\left(\sum p_i\right) \Gamma^{(N)}(p_1, \ldots, p_N)$$

$\longrightarrow$ look only 1PI.

Now,

$$N \times \left(-\frac{D+2}{2}\right) = -2N - D + d\left(\Gamma^{(N)}\right).$$

$$\Rightarrow d\left(\Gamma^{(N)}\right) = 2N + D - N\left(\frac{D+2}{2}\right) = D + N\left(\frac{2-D}{2}\right).$$

If $\Gamma^{(N)} = \prod_n \underbrace{(g^{(n)})^{\alpha_n}}_{} \underbrace{\hat{\Gamma}_N^{(N)}}_{}$

Coming from vertices

devoid of ~~the~~ coupling cons.

$$d(\hat{\Gamma}_N^{(N)}) = d(\Gamma^{(N)}) - \sum_n \alpha_n \lambda_n$$

+ve integ. or 0.

for convergence $d[\hat{\Gamma}^{(N)}] < 0$

$$\Rightarrow d(\Gamma^{(N)}) - \sum \alpha_n \lambda_n < 0$$

If any of the λ_n are negatives then all $\Gamma^{(N)}$'s diverge at some order $\Rightarrow$ non-renormalizable theory.

[$\alpha_n \uparrow$ as we go on higher order.]

$\Rightarrow$ for renormalizability $d[\lambda_n] \geq 0$.

$$\Rightarrow n \frac{D-2}{2} \leq D$$

$$\Rightarrow \boxed{n \leq \frac{2D}{D-2}}$$

General rule: mass dimension of the coupling cons ≥ 0

$\rightarrow$ necessary cond^n.

If $\lambda_n \geq 0 \ \forall n$, then the <u>divergent</u> $\Gamma^{(N)}$'s are those for which $d(\Gamma^{(N)}) \geq 0$. [these are possible; remaining which could be divergent.

$$\Rightarrow D + N\frac{2-D}{2} \geq 0 \quad \Rightarrow D \geq \frac{N}{2}(D-2)$$

$$\Rightarrow \boxed{N \leq \frac{2D}{D-2}} \quad \therefore \text{ Convergent } N > \frac{2D}{D-2}$$

Now, are there ~~as~~ enough counterparts to cancel divergences?
But we've seen $n \leq \cancel{\frac{D/2}{2}}\ \frac{2D}{D-2}$ too, same # ! So, if we introduce renorm. cons. corresponding to each coupling cons
$\Longrightarrow$ we're done.

Hence,

- Only a finite # diagrams should be divergent
  $\downarrow$
  You've to make sure we've enough coupling cons. to adjust this.

5

.11.12

* A QFT is "power counting renormalizable" if all coupling constants have dimension ≥ 0 and all such couplings are present.

* Hidden assumption: No power of M^2 in the denominator from the propagators.

Scalar: $\dfrac{i}{-k^2-m^2+i\epsilon} = \dfrac{i}{-k^2}\left(1+\dfrac{m^2}{k^2}\right)^{-1}$ for large k

$\longrightarrow$ doesn't have $\frac{1}{m^2}$ $\longrightarrow$ no prob

fermions: $\dfrac{i}{\not{k}-m} = \dfrac{i(\not{k}+m)}{-k^2-m^2}$

$= \dfrac{i\not{k}}{k^2}\left(1+\dfrac{m^2}{k^2}\right)^{-1}\left(1+\dfrac{m}{\not{k}}\right)$

$\longrightarrow$ no problem.

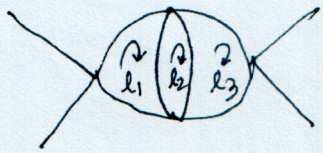
* Massive gauge fields: $\dfrac{i}{-k^2-M^2+i\epsilon}\left\{(\ )+\dfrac{k^\mu k^\nu}{M^2}\right\}$

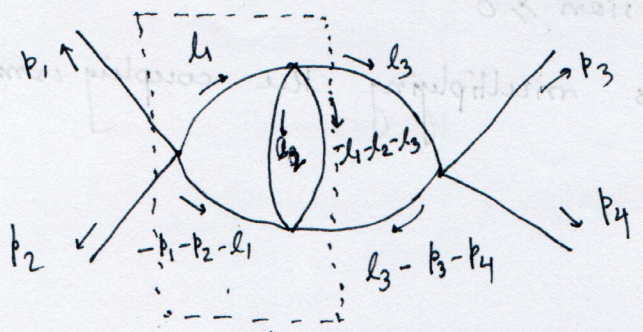
$\longrightarrow$ have $\frac{1}{M^2}$ $\rightarrow$ usual formulation won't work.

$\rightarrow$ indeed nonrenormalizable in this gauge.

* <u>Overlapping divergences</u>:

$\longrightarrow$ log divergent $\left(\dfrac{12 \text{ power}}{12\ n}\right)$

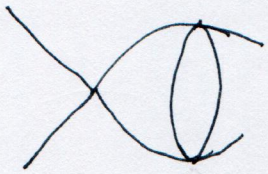




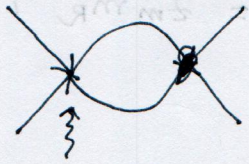
Subdiagram $\Rightarrow$ divergent (8/8).

Let, l_1, l_2 - Large but l_3 is finite.

Subdiagram $\Rightarrow$ 4- external lines $\Rightarrow$ divergence of a 4-pt. fun
$[p_1, p_2, l_3, -l_3-p_3-p_4]$

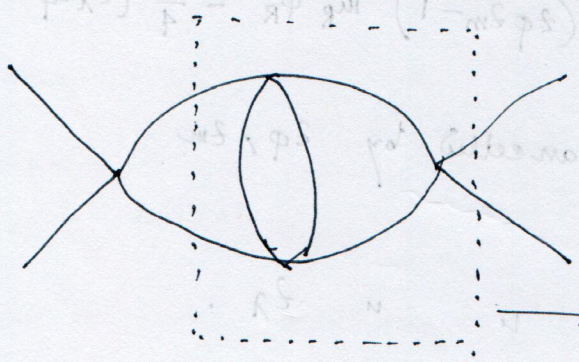


divergence would be cancelled against a counterterm like this:



(lower order)

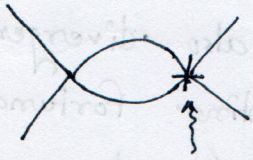
Now consider,



Take l_2, l_3 large and l_1 finite (region of integration)

$\longrightarrow$ subdivergent

Counterterm



This is not always obvious that overlapping divergences could be removed. But it can be shown that once the theory is renormal. from dimension $\Rightarrow$ this also could be done.

* Coupling constants have dimension ≥ 0

$\Longleftrightarrow$ corresponding operators multiplying the coupling constant have dimension $\leq D$.

$\boxed{*}$ Go back:

① Operators of dimension $\leq D$

② We must add all possible operators of dimension $\leq D$. (to remove divergences) [consistent with the symmetries of the theory].

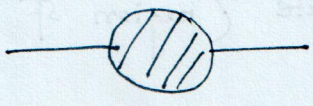
<u>Example</u>: ϕ^4- theory.

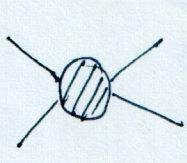
$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

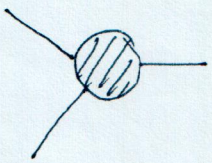
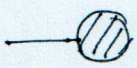
$$\phi = Z_\phi^{1/2}\phi_R, \quad m = Z_m m_R, \quad \lambda = Z_\lambda \lambda_R \mu^\epsilon$$

Counter term:

$$-\frac{1}{2}(Z_\phi - 1)\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}(Z_\phi Z_m^2 - 1)m_R^2\phi_R^2 - \frac{1}{4}(Z_\lambda Z_\phi^2 - 1)\lambda_R\mu^\epsilon\phi_R^4$$

Div () is cancelled by Z_ϕ, Z_m

Div () " " " Z_λ.

 &  are also divergent by power counting. Fortunately these vanish by $\phi \to -\phi$ symmetry. So, we're ok.

* A non-trivial example:

Complex scalar ϕ:

$$\mathcal{L} = -\frac{\partial_\mu \phi^* \partial^\mu \phi}{4} - \frac{m^2}{2}\frac{\phi^*\phi}{2} - \lambda\frac{(\phi^*\phi)^2}{4}$$

We can have (with $D \le 4$) additional terms in the $\mathcal{L}$:

$$(\phi^3 + c.c.),\ (\phi^*\phi^*\phi + c.c.),\ (\phi^*\phi^*\phi^*\phi + c.c.) \text{ etc.}$$

$\mathcal{L}$ has additional symmetry: $U(1) \sim \phi \to e^{i\theta}\phi$

None of the additional terms respect this symmetry.

$$\left\{\begin{array}{l} U(1) \text{ invariance} \Rightarrow \text{equal \# of } \phi, \phi^* \\ D \le 4 \\ \mathcal{L}I \end{array}\right.$$

$\to$ our Lagrangian is the unique.

* $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

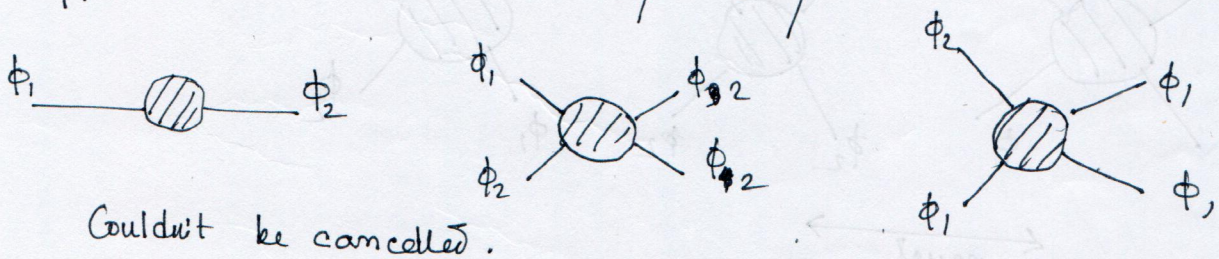
$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) - \frac{1}{2}m^2(\phi_1^2+\phi_2^2) - \frac{\lambda}{4}(\phi_1^4+\phi_2^4+2\phi_1^2\phi_2^2)$$

Symmetry: $\phi_1 \to \phi_1\cos\theta + \phi_2\sin\theta$

$\phi_2 \to -\phi_1\sin\theta + \phi_2\cos\theta$.

Many terms are missing: $\phi_1\phi_2$, $\phi_1\phi_2^3$, $\phi_1^3\phi_2$ etc.

Suppose we don't know the symmetry.



Couldn't be cancelled.

But explicit calculation $\Rightarrow$ these vanish because of the trivial reason $\phi_1 \to -\phi_1$, $\phi_2 \to -\phi_2$ $\Rightarrow$ Each vertex with even # of ϕ_1, ϕ_2.

Using Feynman rules $\Rightarrow$ no way one can draw these diagrams.

$\Rightarrow$ zero.

$\Rightarrow$ consequence of the symm $\phi_1 \to -\phi_1$, $\phi_2 \to -\phi_2$ individually.

$$\phi_1 = \tilde{Z}_{\varphi_1}^{1/2} \phi_{1R} \text{ etc.}$$

$$\mathcal{L}_{ct} = -\frac{1}{2}(\tilde{Z}_{\varphi_1} - 1)\partial_\mu \phi_{1R} \partial^\mu \phi_{1R} - \frac{1}{2}(\tilde{Z}_{\varphi_2} - 1)\partial_\mu \phi_{2R} \partial^\mu \phi_{2R}$$
$$- \frac{1}{2}(Z_m^2 \tilde{Z}_{\varphi_1} - 1)\phi_{1R}^2 - \frac{1}{2}(Z_m^2 \tilde{Z}_{\varphi_2} - 1)\phi_{2R}^2$$
$$- \frac{1}{4}\lambda_R \mu^\epsilon \Big\{ (Z_\lambda \tilde{Z}_{\varphi_1}^2 - 1)\phi_{1R}^4 + (Z_\lambda \tilde{Z}_{\varphi_2}^2 - 1)\phi_{2R}^4$$
$$+ 2(Z_\lambda \tilde{Z}_{\varphi_1} \tilde{Z}_{\varphi_2} - 1)\phi_{1R}^2 \phi_{2R}^2 \Big\}$$

$\rightsquigarrow$ 2-pts

fix Z_m, $\tilde{Z}_{\varphi_1}$

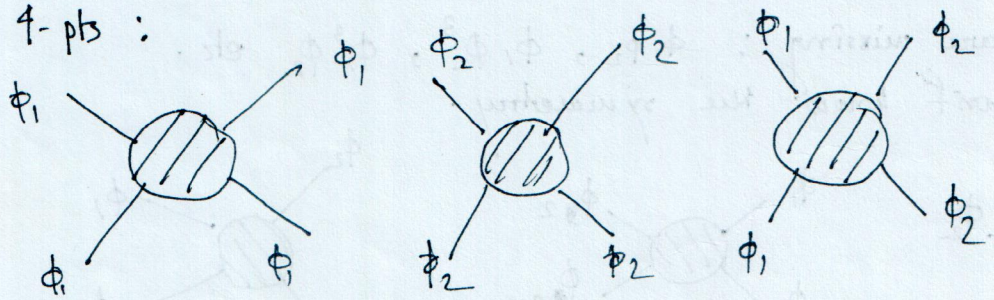
fix $\tilde{Z}_{\varphi_2}$, Z_m independently.

Are these two Z_m same?

$\Rightarrow$ Since $\phi_1 \leftrightarrow \phi_2$ symmetry $\Rightarrow$ $\tilde{Z}_{\varphi_1} = \tilde{Z}_{\varphi_2}$ and consistency of Z_m computed from the two.

$\rightsquigarrow$ No 3-pt diagram.

$\rightsquigarrow$ 4-pts :



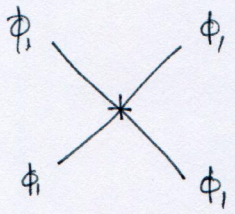
equal

$\Rightarrow$ 2-independent diagrams.

$\Rightarrow$ but we're left with only one Z_λ !

~~If we~~ Actually this is possible due to $U(1)$ symmetry.

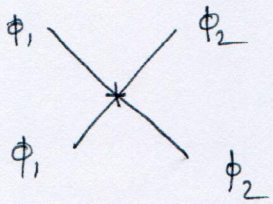
Let's see : counterterms



$$-\frac{i}{4}\lambda_R \mu^{\epsilon}\left(Z_\lambda \tilde{Z}_{\phi_1}^2 - 1\right) \times 4! \times (\text{common factors})$$

(δ fun, external prop.)

① $\longrightarrow \frac{4!}{4} = 6$

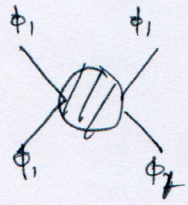
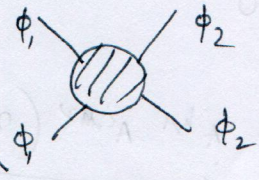


$$-\frac{i}{4} \times 2\, \lambda_R \mu^{\epsilon}\left(Z_\lambda \tilde{Z}_{\phi_1}^2 - 1\right) \times 2 \times 2 \longrightarrow 2 \text{, rest}$$

② are identical.

$\Downarrow$

Divergence part ① = 3 div ②

Div () = 3 Div ()

Now, we'll see this is a consequence of symmetry.

$\rightarrow$ Infinitesimal $U(1)$

$$\phi_1 \rightarrow \phi_1 + \theta\phi_2$$
$$\phi_2 \rightarrow \phi_2 - \theta\phi_1$$

$$\langle \tilde{\phi}_1(p_1)\, \tilde{\phi}_1(p_2)\, \tilde{\phi}_1(p_3)\, \tilde{\phi}_2(p_4)\rangle = 0 \qquad [3\phi_1, 1\phi_2]$$

$$= \langle (\tilde{\phi}_1 + \theta\tilde{\phi}_2)(\tilde{\phi}_1 + \theta\tilde{\phi}_2)(\tilde{\phi}_1 + \theta\tilde{\phi}_2)(\tilde{\phi}_2 - \theta\tilde{\phi}_1)\rangle$$

$$= \underbrace{\langle \tilde{\phi}_1 \tilde{\phi}_1 \tilde{\phi}_1 \tilde{\phi}_2\rangle}_{0} + \theta\Big\{ \langle \tilde{\phi}_2 \tilde{\phi}_1 \tilde{\phi}_1 \tilde{\phi}_2\rangle + \langle \tilde{\phi}_1 \tilde{\phi}_2 \tilde{\phi}_1 \tilde{\phi}_2\rangle + \langle \tilde{\phi}_1 \tilde{\phi}_1 \tilde{\phi}_2 \tilde{\phi}_2\rangle$$
$$- \langle \tilde{\phi}_1 \tilde{\phi}_1 \tilde{\phi}_1 \tilde{\phi}_1\rangle \Big\} + O(\theta^2).$$

3 × Divergence part of $\langle \tilde{\phi}_1 \tilde{\phi}_1 \tilde{\phi}_2 \tilde{\phi}_2\rangle - \text{Div}\langle \tilde{\phi}_1 \tilde{\phi}_1 \tilde{\phi}_1 \tilde{\phi}_1\rangle = 0$

[UV Divergence does not bother about ~~ext~~ labelling of external momentum $\Rightarrow$ we need large limit.]

11.12

Quantization of Gauge Theories

* Bosons:

$$S_{tot} = S_{gauge} + S_{g.f.} + S_{ghost}$$

$$-\frac{1}{4}\int d^4x\, F^a_{\mu\nu} F^{\mu\nu a}$$

$$\& \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abca} A^b_\mu A^c_\nu$$

So, $d(F_{\mu\nu}) = 2$

$\Rightarrow d(A^a_\mu) = 1$

$\Rightarrow$ ~~Now~~ $d(g) = 0$

$$2g f^{abc} \int d^4x \underbrace{A^{b\mu} A^{c\nu} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)}_{4}$$

$$g^2 f^{bca} f^{dea} \int \underbrace{A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}}_{4} d^4x$$

$S_{g.f.}$: ① $-\frac{1}{2\alpha}\int d^4x \underbrace{(\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu)}_{4}$ → renormalizable at the level of power counting.

② $-\frac{1}{2\alpha}\int d^4x \left(\partial^\mu A^a_\mu + \lambda A^b_\nu A^{b\nu} \partial^\tau A^a_\tau\right)\left(\partial^\gamma A^a_\gamma + \lambda A^c_\sigma A^{c\sigma} \partial^\xi A^a_\xi\right)$

→ non-renormalizable gauge.

$$S_{ghost} = \int d^4x\, b_a(x)\, \Box\, c_a(x)$$

$$+ g f^{bca} \int d^4x \underbrace{\partial^\mu b_a(x) A^c_\mu(x) c_b(x)}_{4}$$ → renormalizable

$d[b_a] = 1-\lambda$, $d[c_a] = 1+\lambda$

Set $\lambda = 0$: $d[b_a] = 1$, $d[c_a] = 1$

This shows at the power counting it satisfies 1st criterion.

But there are many terms ≤ 4 which should be added $\rightarrow$ 2nd criteria.

$\downarrow$

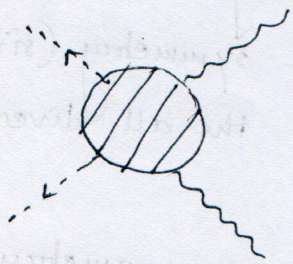
guided by BI not Gauge invariance ($\because$ S_{gf}, S_{gh} are not gauge inv

eg. $m^2 A^a_\mu A^{\mu a}$ $\rightarrow$ we don't have. $\rightarrow$ no Z_m $\Rightarrow$ How would you cancel the divergence arising from 2-pt Green's fun ?

[Feynman rules $\Rightarrow$ calculating Green's fun.
Renormaliz $\Rightarrow$ making them finite.

$\int d^4x \; b_a C_a A^a_\mu A^{b\mu}$ $\rightarrow$ absent $\rightarrow$ no renorm. cons. corresp to this.

So, Question :



Subdiagram $\nearrow$ (8/8)

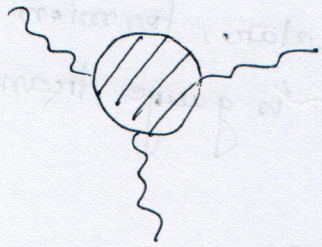
$\rightarrow$ log divergence. (4- external lines)

What counter term cancels this logarithmic divergence ?

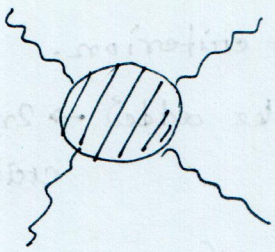
* Consider cubic & quartic terms in gauge fields.

$$-\frac{1}{2} g f^{abc} A^a_\mu A^b_\nu (\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu}) - \frac{1}{4} g^2 f^{abc} f^{ade} A^a_\mu A^b_\nu A^{d\mu} A^{e\nu}$$

$$+ g f^{bca} \partial^\mu b_a(x) A^c_\mu C_b(x)$$

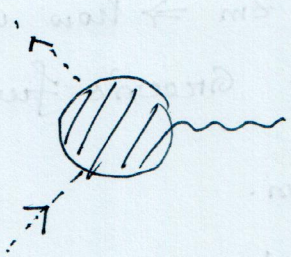


$\rightarrow$ fixes Z_g



$\longrightarrow$ fixes Z_g again.

$\Rightarrow$ so



$\rightsquigarrow$ Z_g !!

$\Rightarrow$ so all divergences should be related if gauge theory ~~has~~ is renormalizable. Certain symmetry may rescue ~~this~~ us.

[Matter $\Rightarrow$ $g\bar{\psi}\gamma^\mu A^a_\mu R(T^a)\psi \longrightarrow$ same Z_g !] But gauge inv can't. Since

S_{gf}, S_{gh}, Feynman rules are not gauge invariant. So, what?

Does the gauge fixing action have a symmetry (since it does not have gauge inv. symm) which could relate the all divergences ?

Q. Does the gauge fixed action have a symmetry which fixes the form of the action ?

$\Rightarrow$ Indeed this has $\rightsquigarrow$ BRST symmetry !

Transformation parameter is a grassman number ξ (constant).

$\longrightarrow$ not fun of x $\longrightarrow$ global symmetry.

* Infinitesimal Symmetry transf :

For ordinary fields (gauge fields, scalars, fermions etc.) the infinitesimal BRST symmetry transformation is gauge transf with parameter $\theta^a(x) = \xi c^a(x)$.

$$\delta A^a_\mu(x) = -\partial_\mu(\xi c^a(x)) + g f^{bca} \xi c^b(x) A^c_\mu(x)$$

$$\delta \psi_s(x) = -ig\,(\xi C^a(x))\,(R_f(T^a))_{st}\,\psi^t(x)$$

$$\delta \phi_k(x) = -ig\,\xi C^a(x)\,(R_s(T^a))_{kl}\,\phi^l(x)$$

If the gauge fixing term is $-\frac{1}{2\alpha}\int d^4x\, F^a(x) F^a(x)$

then

$$\begin{cases} \delta c^a(x) = -\frac{1}{2} g f^{bca}\, \xi C^a(x)\, c^b(x) \\ \delta b^a(x) = -\frac{1}{\alpha} F_a(x)\, \xi \end{cases}$$

↳ eg. $\partial_\mu A^{a\mu}(x)$

Now,

Claim: $\delta S_{gauge} + \delta S_{matter} + \delta S_{gf} + \delta S_{ghost} = 0$ under combined transform mentioned above.

Now, S_{gauge}, S_{matter} automatically satisfy this since trans δ ψ, A, ϕ are exactly same as gauge transf. and these are individu gauge invariant.

Also it can be shown that no more terms consistent with BRST symmetry could be added.

Assuming this $\delta(\cdots) = 0$ to be true, the theory (including fermions & scalars) will be renormalizable as long as we add to the action all possible gauge invariant terms of dim ≤ 4.

$$-\frac{1}{2}\int d^4x\, (\partial_\mu \phi - ig A^a_\mu R_s(T^a)\phi)^\dagger (\partial_\mu \phi - ig A^b_\mu R_s(T^b)\phi)$$

↑ repres. in scalar ~~repr~~ sector

$$+ \int d^4x\, \bar{\psi}\,(i\gamma^\mu \partial_\mu - m + g\gamma^\mu A^a_\mu R_f(T^a))\,\psi$$

Now, $d[\phi_k] = 1$, $d[\psi_s] = 3/2$.

If ψ & ϕ are ~~in~~ in spin 2 represen. of $SU(2)$, then coupling

$$C^{222}_{m'' m m'} \quad \bar{\psi}_{m''} \psi_m \phi_{m'}$$