08.08.2014

[*] What is QFT?

1. QFT as a framework originally devised to deal with relativistic particles. Though it is applicable in non-relativistic case also.

1. QFT as a tool for computing cross-sections & other observables which is encoded in correlation functions.

Limited to perturbation theory (an expansion around a non-interacting theory in powers of coupling constant.

2. Quantum unification of particle & wave (field) notions of classical mechanics.

QFT describes

ⓐ the quantum behaviour of classical fields (eg. EM)

ⓑ " " " of many particle systems especially when the particle no. is not conserved.

But the framework is broader and can describe any quantum system with an infinite # of DOF which does not necessarily have a limit where it behaves either like a classical field or a collection of weakly ~~non~~- interacting particles (like in QCD quark/gluon fields).

[*] QFT from MANY PARTICLE QM :-

Start with N- indistiguishable non-interacting particles (eg. bosons).

not interacting with each-other but may be moving in external potential.

Since non-interacting, the multi-particle wave function is determined in terms of single particle ones, $\psi_i(x)$. ["i" can be continuous, eg. free particles $\psi_i(x) \sim e^{ik\cdot x}$ i.e. $i \leftrightarrow k$].

Multiparticle wave function

$$\psi(x_1, \ldots, x_N) = \left(\frac{N_1! \cdots N_n!}{N!}\right)^{1/2} \sum_P \psi_{P_1}(x_1) \cdots \psi_{P_N}(x_N)$$

$\sum_P$: sum over all permutations

If we have N_1 # particles in state 1

$\vdots$

N_n # " " " "

and $N = \sum_n N_n$

then symmetrization is taken automatically, so we ~~as~~ overcount by taking sum over all permutation. That's why we've this co-f ⇒ normalization factor.

$$\int dx_1 \ldots dx_N \; |\psi(x_1, \ldots, x_N)|^2 = 1$$

The state is completely characterised by its occupation # $\{N_i\}$, we can't say which particle is sitting in which state. So,

$$\psi(x_1, \ldots, x_N) \longrightarrow |N_1, \ldots, N_i, \ldots\rangle = |\{N_i\}\rangle$$

The operators / observables on this system can be viewed as 1-body operator, 2-body operator, . . .

1-body operator : $\hat{F}^{(1)}(x_1,\dots,x_N) = \sum_{a=1}^{N} \hat{f}_a^{(1)}(x_a)$

eg. Total mom $\hat{P} = \sum_{a=1}^{N} \hat{p}_a$ where $\hat{p}_a = -i\frac{\partial}{\partial x_a}$

2-body operator : Something involving $|\vec{x}_a - \vec{x}_b|$.

$$\hat{F}^{(2)}(x_1,\dots,x_N) = \sum_{a,b=1}^{N} \hat{f}_{ab}^{(2)}(x_a, x_b)$$

eg. $\hat{f}_{ab}^{(2)}(x_a, x_b) = \frac{1}{|\vec{x}_a - \vec{x}_b|}$.

Similarly $\hat{F}^{(3)}(x_1,\dots,x_N) = \sum_{a,b,c=1}^{N} \hat{f}_{abc}^{(3)}(x_a, x_b, x_c)$

Since in multi-particle theory we can't distinguish among particles, one must talk about $\hat{F}^{(i)}(x_1,\dots,x_N)$, one can't talk with $\hat{f}^{(i)}$.

* Matrix elements of 1-body operator :-

$$\underbrace{\langle N_1', \dots, N_n', \dots |}_{\Psi_{\{N_i'\}}(x_1,\dots,x_N)} \hat{F}^{(1)} \underbrace{| N_1, \dots, N_n, \dots \rangle}_{\Psi_{\{N_i\}}(x_1,\dots,x_N)}$$

$$\hat{f}_a^{(1)}(x_a)\,\psi_i(x_a) = \sum_j f_{ji}\,\psi_j(x_a)$$

$\hat{f}_a^{(1)}$: at most removes one particle from state i and adds one to another state j.

The non-zero matrix elements are those for which only $N_i' = N_i - 1$ and $N_j' = N_j + 1$ for some pair (i,j). [and all other N_k's are equal].

OR, $N_i' = N_i$ for $\forall\ i$.

$$\langle \cdots (N_i - 1) \cdots (N_j + 1) \cdots | \hat{F}^{(1)} | \cdots N_i \cdots N_j \cdots \rangle$$

dots means they are same in both sides.

$$= f_{ji}^{(1)} \sqrt{(N_j+1) N_i} \quad (i \neq j)$$

$$\langle \{N_i\} | \hat{F}^{(1)} | \{N_i\} \rangle = \sum_i f_{ii}^{(1)} N_i$$

$$f_{ji}^{(1)} \equiv \int dx \; \psi_j^*(x) \; \hat{f}^{(1)} \; \psi_i(x)$$

$$= \langle j | \hat{f}^{(1)} | i \rangle$$

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* $\langle \{N_i'\} | \hat{F}^{(1)} | \{N_i\} \rangle \neq 0$

if ⓐ $N_i' = N_i - 1$

$N_j' = N_j + 1$ for some pair (i, j) $[i \neq j]$ $\rightsquigarrow f_{ji}^{(1)} \sqrt{N_i (N_j+1)}$

and all other $N_k' = N_k$, $k \neq (i, j)$

ⓑ $N_k' = N_k \quad \forall \; k$

$$\rightsquigarrow \sum_i f_{ii}^{(1)} N_i$$

with $f_{ji}^{(1)} = \int dx_a \, \psi_j^*(x_a) \, \hat{f}_a^{(1)} \, \psi_i(x_a)$

where $\hat{F}^{(1)} = \sum_{a=1}^{N} \hat{f}_a^{(1)}\left(x_a, \frac{\partial}{\partial x_a}\right)$

Proof :

$$\langle \dots (N_i - 1) \dots (N_j+1) \dots | \; \hat{f}_a^{(1)} \; | \dots N_i \dots N_j \dots \rangle$$

$$= \left(\frac{N_1! \dots (N_i - 1)! \dots N_j!}{N!} \right) \sqrt{N_i \, (N_j+1)}$$

$$\times \int \psi_j^*(x_a) \, f_a^{(1)} \, \psi_i(x_a) \underbrace{\left[\frac{(N-1)!}{N_1! \dots (N_i - 1)! \dots (N_j)! \dots} \right]}$$

From the remaining (N-1) wave functions which are distributed with occ. #s $\{N_1, \dots, N_i - 1, \dots, N_j \dots\}$.

$$= \frac{1}{N} \sqrt{N_i (N_j+1)} \quad f_{ji}^{(1)}$$

* Recall in a single SHO, the Hilbert space is generated by creation / annihilation operators with

$$[a, a^+] = 1$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad (n = 0, 1, \cdots)$$

$$a^+|n\rangle = \sqrt{(n+1)}\,|n+1\rangle$$

This algebra can be used to give a simple representation of the many body operators in the occupation number basis of the many body Hilbert space.

Claim: The non-zero matrix elements of $\hat{F}^{(1)}$ can be reproduced by representing it as $\hat{F}^{(1)} = \sum_{i,j} f_{ji}^{(1)} a_j^{+} a_i$.

We've introduced a_i, a_i^+ $(i = 1, 2, \cdots)$ with

$$[a_i, a_j^+] = \delta_{ij}$$

$$a_i|\{N_i\}\rangle = \sqrt{N_i}\,|\cdots (N_i-1)\cdots\rangle$$

$$a_i^+|N_1\cdots N_i\cdots\rangle = \sqrt{N_i+1}\,|\cdots (N_i+1)\cdots\rangle$$

$$a_i|\cdots 0\cdots\rangle = 0$$

Creation/annihilation operate on the bigger Hilbert space

$$\underbrace{\mathcal{H} = \mathcal{H}_1}_{\text{single particle- HS}} \oplus \underbrace{\mathcal{H}_2}_{\text{two particle HS}} \oplus \cdots \oplus \mathcal{H}_n \oplus \cdots \;\; ..$$

$$\langle\{N_i'\}|\hat{F}^{(1)}|\{N_i\}\rangle = \langle\{N_i'\}|\sum_{i,j} f_{ji}^{(1)} a_j^+ a_i|\{N_i\}\rangle$$

$$= \sum_{i,j} f_{ji}^{(1)}\langle\{N_i'\}|a_j^+ a_i|\{N_i\}\rangle$$

$$= \sum_{j \neq i} f^{(1)}_{ji} \langle \{N_i'\} | \ldots (N_j+1) \ldots (N_i-1) \ldots \rangle \sqrt{N_i(N_j+1)}$$

$$+ \sum_{i} f^{(1)}_{ii} \langle \{N_i'\} | \{N_i\} \rangle \, N_i$$

* Consider the Hamiltonian for a system of non-interacting particles

$$\hat{H} = \sum_{a=1}^{N} \hat{h}_a$$

In the Fock space representation

$$\hat{H} = \sum_{j,i} h_{ji} \, a_j^{\dagger} a_i$$

$$\downarrow$$

$$\int \psi_i^{*}(x) \, \hat{h} \, \psi_i(x) \, dx$$

If $\{\psi_i(x)\}$ are eigenstates of $\hat{h}$ i.e. $\hat{h}\psi_i(x) = \varepsilon_i \psi_i(x)$,

then $\hat{H} = \sum \epsilon_i \underbrace{a_i^{\dagger} a_i}_{\hat{N}_i}$

* Generalize to 2-body operators $\Rightarrow$

$$\hat{F}^{(2)} = \sum_{a,b} \hat{f}^{(2)}_{ab}(x_a, x_b)$$

The matrix elements of $\hat{F}^{(2)}$ are non-zero iff

$$\langle \{N_i'\} | \hat{F}^{(2)} | \{N_i\} \rangle$$

ⓐ $N_k' = N_k - 1$

$N_i' = N_i - 1$

$N_l' = N_l + 1$ and all others are same.

$N_j' = N_j + 1$

for (i, j, k, l) ~~and~~

ⓑ $N_i' = N_i - 1$

$N_j' = N_j + 1$

and all others same

ⓒ $N_i' = N_i$

$\forall \, i$

ⓓ $N_k' = N_k - 2$

$N_i' = N_i + 2$

Exercise : We can represent $\hat{F}^{(2)} = \frac{1}{2} \sum_{\substack{i_1, i_2 \\ j_1, j_2}} f^{(2)}_{j_1 j_2 i_1 i_2} a^\dagger_{j_1} a^\dagger_{j_2} a_{i_1} a_{i_2}$

with $f^{(2)}_{j_1 j_2 i_1 i_2} = \int dx_1 dx_2 \, \psi^*_{j_1}(x_1) \psi^*_{j_2}(x_2) \overset{\hat{f}^{(2)}}{\hat{}} \psi_{i_1}(x_1) \psi_{i_2}(x_2)$

* Quantum fields :

Introduce $\Phi(x) = \sum_i \psi_i(x) a_i$

ψ_i → one-particle wave function basis

This is an operator acting on the Fock space $\mathcal{H}$, since it is built out of $\{a_i\}$ but also a field - operator valued field.

$\Phi^\dagger(x) = \sum_i \psi_i^*(x) a_i^\dagger$ ⇝ prob amplitude for "creating" a particle at x.

Ex : $[\Phi(x), \Phi^\dagger(x')] = \delta^D(x - x')$

$[\Phi(x), \Phi(x')] = 0$

Take,

$$\hat{F}^{(1)} = \sum_{i,j} f^{(1)}_{ji} a_j^\dagger a_i = \sum_{j,i} \int dx \, \psi_j^*(x) \underbrace{\hat{f}^{(1)}}_{\hat{f}^{(1)}(x, \partial/\partial x)} \psi_i(x) a_j^\dagger a_i$$

$$= \int dx \, \Phi^\dagger(x) \hat{f}^{(1)}(x, \partial/\partial x) \Phi(x)$$

⇝ 1-body operators are quadratic in the field $\Phi(x)$.

eg. $$\hat{f}_a^{(1)} = \frac{1}{2m} \vec{P}_a^{\,2}$$

$$= -\frac{1}{2m} \vec{\nabla}_a^{\,2}$$

Then $$\hat{K} \equiv \sum \hat{f}_a^{(1)} = -\frac{1}{2m} \int \Phi^{+}(x) \vec{\nabla}^2 \Phi(x) \, dx$$

$$= \frac{1}{2m} \int (\vec{\nabla} \Phi^{+})(\nabla \Phi) \, dx$$

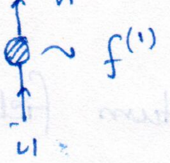
Non-interacting Hamiltonian = sum of 1-body operators, is always quadratic in the quantum fields $\Phi(x)$.

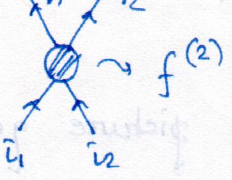
For more general interactions (2-body, 3-body, ...), it is represented by higher powers of Φ.

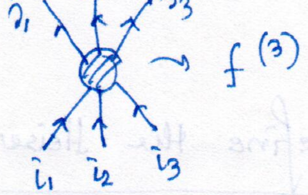
eg. $$\hat{F}^{(2)} = \frac{1}{2} \sum_{\substack{i_1, i_2 \\ j_1, j_2}} f^{(2)}_{j_1 j_2 i_1 i_2} \, a^{+}_{j_1} a^{+}_{j_2} a_{i_1} a_{i_2}$$

$$= \frac{1}{2} \int dx_1 \, dx_2 \, \psi^{*}_{j_1}(x_1) \psi^{*}_{j_2}(x_2) \, f^{(2)} \, \psi_{i_1}(x_1) \psi_{i_2}(x_2) \, a^{+}_{j_1} a^{+}_{j_2} a_{i_1} a_{i_2}$$

$$= \frac{1}{2} \int dx_1 \, dx_2 \, \Phi^{+}(x_1) \Phi^{+}(x_2) \, f^{(2)}(x_1, x_2) \, \Phi(x_1) \Phi(x_2)$$







* Once one has introduced the fock space and describe the quantum mechanical operators in the space by quantum fields, these can describe more general processes which do not necessarily conserve particle no.

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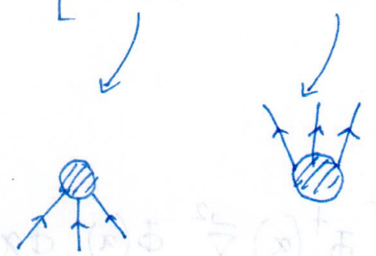
eg. $\lambda \int dx \left[\Phi^{+}(x)^{3} + \Phi^{3}(x) \right]$



$a_i |0 \cdots 0 \cdots \rangle = 0$

↙

vacuum state $\in \mathcal{H}$.

* TIME-EVALUATION OF QUANTUM FIELDS :-

The fields $\Phi(\vec{x}) = \sum_i \psi_i(\vec{x}) a_i$ are defined at a given time instant (say $t=0$).

If $\psi_i(x)$ are energy basis/eigenstates, then natural time evolution of $\Phi(\vec{x})$ can be defined to be

$$\Phi(\vec{x},t) = \sum_i \psi_i(\vec{x}) e^{-iE_i t} a_i$$

Equivalently, $\boxed{\Phi(\vec{x},t) = e^{iH_0 t} \Phi(\vec{x}) e^{-iH_0 t}}$ Exercise

with $H_0 = \sum_i \varepsilon_i a_i^{+} a_i$

This quantum fields behave as operators do in the Heisenberg picture.

In general, define the Heisenberg picture quantum fields

$$\boxed{\Phi_H(\vec{x},t) = e^{iHt} \Phi(\vec{x}) e^{-iHt}} \quad , \quad H = H_0 + \underset{\swarrow \atop \text{2-body, 3-body, ...}}{H_{int}}$$

Interaction picture :

We take the time evolution of the fields to be with the free Hamiltonians H_0

$$\Phi_I(t,\vec{x}) = e^{iH_0 t}\,\Phi(\vec{x})\,e^{-iH_0 t}$$

then $$\Phi_H(\vec{x},t) = U^\dagger(t)\,\Phi_I(t)\,U(t)$$

Exercise : Compute $U(t)$ and show that it obeys

$$i\frac{\partial U}{\partial t} = (H_{int})_I\,U(t)$$

and $$|\psi_I(t)\rangle = U(t)\,|\underbrace{\psi_H}_{\text{time-independent}}\rangle$$

[Section 4.2 of Peskin].

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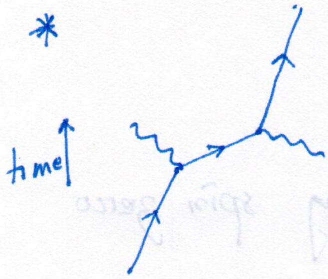
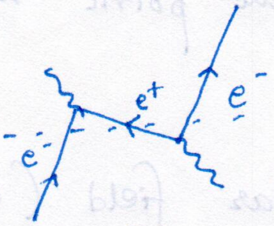




One has to allow all possible paths ~~to take into account~~ in a relativistically covariant way

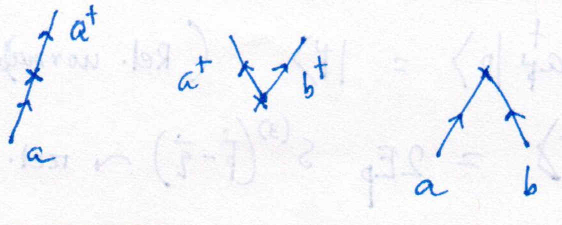
⇓

⇑

instead of particle going back in time, one should think anti-particle going forward in time. These two pictures are on the equal footing. So, one is motivated to rewrite →

$$\Phi(\vec{x}) = \sum\Big(\underbrace{\psi_i(\vec{x})a_i}_{\text{particle annihilation}} + \underbrace{\chi_i(\vec{x})\,b_i^\dagger}_{\text{anti-particle creation}}\Big)$$

Now, $\Phi^\dagger\Phi$ has terms not just $aa^\dagger$ but also $b^\dagger a^\dagger$ & ab.



Quartic interactions like $(\Phi^\dagger\Phi)^2$ contain not just $2 \to 2$ but also $3 \to 1$, $1 \to 3$ etc. All of these are on the same footing.

At least for relativistic invariant QFTs, H_{int} is built from Local fields i.e.

$$H_{int} = \int d^3x \; \mathcal{H}_{int}(x)$$

- local interactions. | built from $\Phi(x)$, $\Phi^\dagger(x)$ and a finite number of derivatives.

Not all interactions can be written like this. All long range interactions (Quolomb) are arise from mediation of massless gauge fields.

Local $\Rightarrow$ evaluated at same point x

* SCALAR FIELDS :-

Start w/ complex scalar field (describing spin zero particles & anti-particles).

In the interaction picture, the quantum field

$$\Phi_I(\vec{x},t) = \sum_{\vec{p}} \Big[e^{i(\vec{p}\cdot\vec{x} - p^0 t)} \underbrace{a_p}_{\text{particle annihilation}} + e^{-i(\vec{p}\cdot\vec{x} - p^0 t)} \underbrace{b_{\vec{p}}^\dagger}_{\text{anti-particle creation}} \Big]$$

w/ $p^0 = \sqrt{\vec{p}^2 + m^2} = E_{\vec{p}}$

$$\sum_{\vec{p}} \; : \; \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}}$$

Normalization $\Rightarrow$ $\sqrt{2E_p}\, a_{\vec{p}}^\dagger |0\rangle = |\vec{p}\rangle$ (Rel. normalization)

$\langle \vec{p} | \vec{q} \rangle = 2E_p \, \delta^{(3)}(\vec{p} - \vec{q})$ ~ rel. invariant

$$\int \frac{d^3p}{2E_p} = \int d^4p \, \delta^4(p^2 - m^2)\, \theta(p^0) \quad ; \quad [a_{\vec{p}}, a^\dagger_{\vec{q}}] = [b_{\vec{p}}, b^\dagger_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$$

* **DIRAC FIELDS** :-

$$\psi_\alpha(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[\sum_{s=1,2} \left\{ u_\alpha^{(s)}(\vec{p})\, e^{ip\cdot x} a_{\vec{p}}^{(s)} + v_\alpha^{(s)}(\vec{p})\, e^{-ip\cdot x} b_{\vec{p}}^{\dagger(s)} \right\} \right]$$

One has to sum over spin-states to get a complete basis.

$\alpha = 1, 2, 3, 4 \rightsquigarrow$ Dirac label

$$\{ a_{\vec{p}}^{(r)}, a_{\vec{q}}^{\dagger(s)} \} = \{ b_{\vec{p}}^{(r)}, b_{\vec{q}}^{\dagger(s)} \} = (2\pi)^3 \delta_{rs}\, \delta^{(3)}(\vec{p} - \vec{q})$$

$$E_p = p^0 = \sqrt{\vec{p}^2 + m^2}$$

* **EM FIELDS** :-

$$A_\mu(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \left[\sum_{r=0}^{3} \left\{ \varepsilon_\mu^{(r)}(\vec{p})\, e^{ip\cdot x} a_{\vec{p}}^{(r)} + \varepsilon_\mu^{(r)*}(\vec{p})\, e^{-ip\cdot x} a_{\vec{p}}^{\dagger(r)} \right\} \right]$$

Photon is ~~its~~ its own anti-particle.

$$E_p = p^0 = |\vec{p}|$$

* **PROPAGATOR** :-

In non-relativistic QM, a basic observable is the amplitude

$$\langle \vec{x}_2, t_2 | \vec{x}_1, t_1 \rangle = K(\vec{x}_2, t_2; \vec{x}_1, t_1) \qquad , \quad t_2 > t_1$$

$$= 0 \qquad , \quad t_2 < t_1$$

$$\parallel$$

$$\langle \vec{x}_2 | e^{-iH(t_2 - t_1)} | \vec{x}_1 \rangle$$

$$= \sum_i \langle \vec{x}_2 | i \rangle \, e^{-iE_i(t_2 - t_1)} \langle i | \vec{x}_1 \rangle$$

$$= \sum_i \psi_i(\vec{x}_2)\, \psi_i^*(x_1)\, e^{-iE_i(t_2 - t_1)} \qquad (t_2 > t_1)$$

This can also be obtained as the "2-pt fun" of the quantum fields.

$$A \equiv \langle 0 | T \{ \Phi(\vec{x}_2, t_2) \Phi^{\dagger}(\vec{x}_1, t_1) \} | 0 \rangle$$

$$= \begin{cases} \langle 0 | \Phi(\vec{x}_2, t_2) \Phi^{\dagger}(\vec{x}_1, t_1) | 0 \rangle & \text{if } t_2 > t_1 \\ \langle 0 | \Phi^{\dagger}(\vec{x}_1, t_1) \Phi(\vec{x}_2, t_2) | 0 \rangle & \text{if } t_2 < t_1 \end{cases}$$

CLAIM $K(\vec{x}_2, t_2 ; \vec{x}_1, t_1)$

Here, $\Phi(\vec{x}) = \sum \psi_i(\vec{x}) a_i$, $\Phi(\vec{x}, t) = \sum_i \psi_i(\vec{x}) a_i e^{-iE_i t}$

Proof: $A = \sum_{i,j} \langle 0 | a_j a_i^{\dagger} | 0 \rangle \psi_j(\vec{x}_2) \psi_i^{*}(\vec{x}_1) e^{iE_i t} e^{-iE_j t}$

$$= \sum_i \psi_i^{*}(x_1) \psi_i(\vec{x}_2) e^{-iE_i(t_2 - t_1)}$$

For $t_2 > t_1$

For $t_2 < t_1$: $A = \sum_{i,j} \langle 0 | a_i^{\dagger} a_j | 0 \rangle \cdots$

$$= 0$$

Exercise: Show that $\left(i \frac{\partial}{\partial t_2} - H \right) K(\vec{x}_2, t_2 ; \vec{x}_1, t_1) = i \delta^{(3)}(\vec{x}_2 - \vec{x}_1) \delta(t_2 - t_1)$

Hint: Start from the def^{n} of $K(\vec{x}_2, t_2 ; \vec{x}_1, t_1)$

$\Longrightarrow$ K is the Green's function of the operator $\left(i \frac{\partial}{\partial t_2} - H \right)$.

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[*] RELATIVISTIC PROPAGATOR :

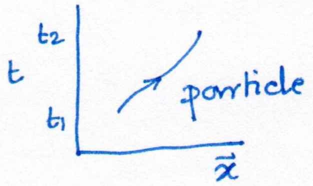
In the relativistic theory, in the presence of anti-particles, the rel invariant propagator is <u>not</u> the retarded one, but instead the Feynman propagator.

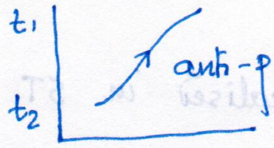
$$K_F(x_2,t_2;x_1,t_1) = \underset{\substack{\text{particle}\\\text{states}\\\text{i.e. "}+E_i\text{"}}}{\sum} \psi_i^*(x_1)\,\psi_i(x_2)\,e^{-iE_i(t_2-t_1)} \qquad [t_2>t_1]$$

$$= \pm\underset{\substack{\text{anti-particle}\\\text{states i.e.}\\\text{"}-E_i\text{"}}}{\sum} \chi_i^*(x_2)\,\chi_i(x_1)\,\underbrace{e^{-iE_i(t_2-t_1)}}_{\Downarrow \; e^{i|E_i|(t_2-t_1)}} \qquad [t_2<t_1]$$

$+$: bosons

$-$: fermions





anti-particle moving forward in time

This is given by the time order product of two relativistic fields:

$$K_F(x_2,t_2;x_1,t_1) = \langle 0|T\{\Phi(x_2,t_2)\,\Phi^\dagger(x_1,t_1)\}|0\rangle$$

$$\Phi(\vec{x}_1,t_1) = \sum_i \left[a_i\,\psi_i(x_1)\,e^{-iE_it_1} + b_i^\dagger\,\chi_i^*(x_1)\,e^{iE_it_1}\right]$$

$t_2>t_1$: then $K_F = \sum_i \langle 0|a_i a_i^\dagger|0\rangle\,\psi_i^*(x_1)\,\psi_i(x_2)\,e^{-iE_i(t_2-t_1)}$

$$= \sum_i \psi_i^*(x_1)\,\psi_i(x_2)\,e^{-iE_i(t_2-t_1)}$$

$t_2<t_1$: $K_F = \pm\sum_i \langle 0|b_i b_i^\dagger|0\rangle\,\chi_i^*(x_2)\,\chi_i(x_1)\,e^{-iE_i(t_1-t_2)}$

$$= \pm\sum_i \chi_i^*(x_2)\,\chi_i(x_1)\,e^{+iE_i(t_2-t_1)}$$

In momentum space (for free fields)

$$K_F(p) = \frac{1}{p^2 - m^2 + i\varepsilon}$$ ~ rel. invariance is prominent.

In an interacting theory the corresponding 2-pt function also contains information about the propagation of the interacting particle states.

$$\langle \Omega_H | T\{ \phi_H(x_2) \phi_H^\dagger(x_1) | \Omega_H \rangle$$

denotes the vacuum of the interacting theory [as opposed to $|0\rangle$ for free theory]

interacting fields in Heisenberg picture

$$x_1 = (\vec{x}_1, t_1)$$
$$x_2 = (\vec{x}_2, t_2)$$

for $H = H_0 + H'_{int}$

H'_{int} is localised in ST

by going to the interaction picture

$$\langle \Omega_H | T\{ \phi_H(x_2) \phi_H(x_1) \} | \Omega_H \rangle$$

$$= \lim_{T \to 0} \frac{\langle 0 | T\{ \phi_I(x_2) \phi_I^\dagger(x_1) \exp[-i \int_{-T}^{T} H^I_{int}(t') dt'] | 0 \rangle}{\langle 0 | T\{ \exp[-i \int_{-T}^{T} H^I_{int}(t') dt'] \} | 0 \rangle}$$

$\Phi_I(x)$ ~ free fields in the interaction picture

$$\sim \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_p e^{ip \cdot x} + b_p^\dagger e^{-ip \cdot x} \right]$$

H^I_{int} ~ expressed in terms of $\phi_I(x)$.

Generalises to more general corr. function

$$\boxed{\begin{aligned}&\langle \Omega_H | T\{ \phi_H(x_1) \cdots \phi_H(x_n)\} | \Omega_H \rangle \\ = \lim_{T\to\infty} \ & \frac{\langle 0 | T\{ \phi_I(x_1) \cdots \phi_I(x_n) \exp[-i \int_{-T}^{T} H_{int}^{I}(t')dt'] \} | 0 \rangle}{\langle 0 | T\{ \exp[-i \int_{-T}^{T} H_{int}^{I}(t')dt'] }\end{aligned}}$$

In local QFTs,

$$\int_{-T}^{T} dt \ H_{int}(t)$$

$$= \int_{-T}^{T} dt \int d^3x \ \mathcal{H}_{int}(\vec{x}, t)$$

$$= - \int_{-T}^{T} dt \int d^3x \ \mathcal{L}_{int}(\vec{x}, t)$$

$$\Big\downarrow T \to \infty$$

$$-\int d^4x \ \mathcal{L}_{int}(x).$$

Wick's theorem organizes the computation of the RHS in terms of free field Wick contractions [expressed in terms of $\langle 0 | T\{ \phi_I(x_1) \phi_I(x_2)\} | 0 \rangle$]. Each of these contributions can be given a diagramatic interpretation:

- → External lines for each of the n- fields in the n-pt fun.
- → Vertices for each term in $\mathcal{H}_{int}$ (or equiv. $\mathcal{L}_{int}$) w/ # of legs (lines) = # of fields in $\mathcal{H}_{int}$ (or $\mathcal{L}_{int}$)
- → internal lines for prop. betn vertices.

[*] QED :

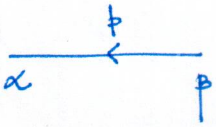
$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_M + \mathcal{L}_{int}$$

Dirac: $\bar\psi\,(i\cancel{\partial}-m)\,\psi$

Maxwell: $-\frac{1}{4}F_{\mu\nu}^2$

$-e\,\underbrace{\bar\psi(x)\,\gamma^\mu\psi}_{J^\mu_{EM}}\,A_\mu(x)$

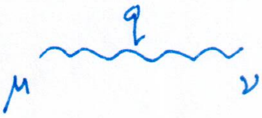
Feynman rules :



$$\longleftrightarrow \quad \frac{i(\cancel{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon} = \left(\frac{i}{\cancel{p}-m}\right)_{\alpha\beta} \equiv S_F(p)$$

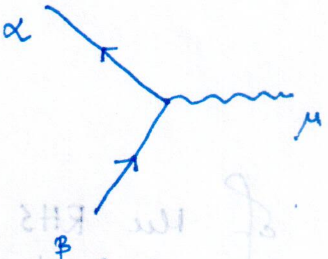
spinor

$$\langle 0|T\{\psi_\alpha(x_1)\,\bar\psi_\beta(x_2)\,|0\rangle \xrightarrow{FT}$$



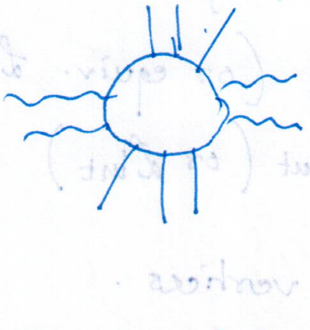
$$\longleftrightarrow \quad \frac{-i\eta_{\mu\nu}}{q^2+i\varepsilon}\,; \quad \eta_{\mu\nu} = \mathrm{diag}\,[1,-1,-1,-1]$$

$$\equiv D_F(q)$$



$$\longleftrightarrow \quad -ie(\gamma_\mu)_{\alpha\beta}$$

In computing the contributions to any correllation function [with $2n_E$ e^+e^- + n_γ photons] we draw all connected Feynman diagrams. with ($2n_E$ e^+e^- + n_γ photons) external lines and vertices  . Assign momenta to each internal ~~momenta~~ lines consistent with 4-mom conservation at each vertex. The independent loop momenta (unconstrained) have to be integrated. For every fermionic loop there is (-1) factor. There may be symmetry factors to divide by.



⑤

20.08.2014

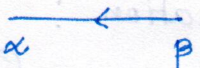
ELECTRON SELF ENERGY :–

Calculate the leading quantum correction to the prop of the electron.

$$\langle \Omega_H | T\{ \psi_H(x)\, \bar\psi_H(y)\} | \Omega_H \rangle$$

$$= \frac{\langle 0 | T\{ \psi_I(x)\, \psi_I(y)\, \exp[-\int_{-T}^{T} H^I_{int}(t')dt'\,]\} | 0 \rangle}{\langle 0 | T\{ \exp[-\int_{-T}^{T} H^I_{int}(t')dt'\,\} | 0 \rangle}$$

Here. $H^I_{int} = -ie \int d^3x \, (\bar\psi_I \gamma^\mu \psi_I)\, A^I_\mu$

The leading order contribution : 

$= \frac{i(\not p + m_0)_{\alpha\beta}}{p^2 - m_0^2 + i\varepsilon}$

[m_0 : parameter appearing in Lagrangian.

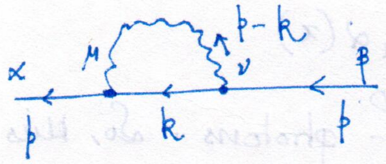
Now, e^2 contribution ⇒

$$\psi(x_1)\, \bar\psi(x_2) \int \bar\psi \gamma^\mu \psi \, A_\mu(x) \int \bar\psi \gamma^\nu \psi \, A_\nu(y)$$

and there are 2 ways one can connect these ⇒ $\frac{2}{2!} = 1$

coming from exponential

Diagrams like  are cancelled against denominator.



$$= \left[\frac{i(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon} \cdot \left[-i\Sigma_2(p) \right] \cdot \frac{i(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon} \right]_{\alpha\beta}$$

$$\left[-i\Sigma_2(p) \right]_{\alpha'\beta'} = (-ie)^2 \int \left[\gamma^\mu \cdot \frac{i(\not k + m_0)}{k^2 - m_0^2 + i\varepsilon} \cdot \gamma^\nu \right]_{\alpha'\beta'} \frac{(-i\eta_{\mu\nu})}{(p-k)^2 + i\varepsilon} \, \frac{d^4k}{(2\pi)^4}$$

Electron self energy

Need to evaluate $\Sigma_2(p)$. But it is actually divergent!
Look at the large k in the integrand:

$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{k}{(k^2)^2} \longrightarrow \text{diverges linearly} \left[\begin{array}{c}\text{actually log}\\ \text{divergence}\end{array}\right]$$

UV divergence / short distance divergence.

We have to systematically deal with divergences and extract computable corrections. [Can do this because the structure of divergence is special].

Strategy:

1. Regularisation : isolate the divergent part by making it finite using a "regulator".

2. Renormalization : interpret the divergence part/piece as a ~~to~~ redefinition of the parameters (e, m) of the theory. Remaining finite piece leads to definite predictions (unambiguous).

First need to regularize $\Sigma_2(p)$. Nice way to regularize is to impose ~~a put~~ a UV cut off on momenta $|k| \leq \Lambda$.
In QED (gauge theory) this is not very suitable since it spoils the gauge invariance :

$$A_\mu(x) \sim A_\mu(x) + \partial_\mu \alpha(x)$$

Gauge invariance $\longleftrightarrow$ 2-pol of photons. So, this gauge inv. is important [Gauge transfo mixes modes with high + low fourier momenta]. We'll instead use a regulator which respects gauge invariance.

Though this cut-off breaks translational invariance which is global symmetry, but this is not very dangerous like gauge invariance. Breaking gauge inv. may lead to violation of unitarity.

To regulate $\Sigma_2(p)$ rewrite the integrand

$$\boxed{\frac{1}{a\cdot b} = \int_0^1 dx \, \frac{1}{[xa+(1-x)b]^2}}$$

In particular

$$\frac{1}{k^2-m_0^2+i\varepsilon} \cdot \frac{1}{(p-k)^2 \cancel{-m_0^2} + i\varepsilon}$$

$$= \int_0^1 dx \, \frac{1}{\underbrace{\{(1-x)[k^2-m_0^2] + (x)\,(p-k)^2\}^2}_{(k-px)^2 + p^2x(1-x) - m_0^2(1-x)}}$$

Define $k^\mu - p^\mu x = k'^\mu$

So, denom $\to k'^2 + p^2x(1-x) - m_0^2(1-x)$

Then

$$-i\Sigma_2 = (-ie)^2 \int_0^1 \int \frac{d^4k'}{(2\pi)^4} \left[\gamma^\mu \frac{(\not{k}' + \not{p}x + m_0)\,\gamma^\nu\,(-i\eta_{\mu\nu})}{[k'^2 + \underbrace{p^2x(1-x) - m_0^2(1-x)}_{\Delta}]^2} \right] dx$$

k' integral is of the form $\displaystyle\int \frac{d^4k'}{(k'^2+\Delta)^2}$

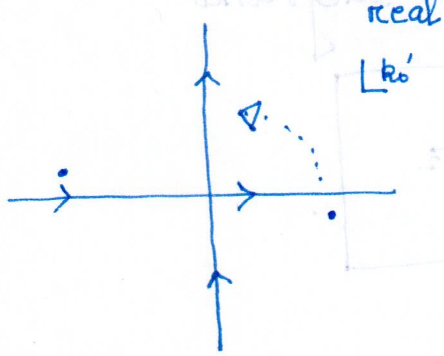
$$\text{So, } -i\Sigma_2 = (-ie)^2 \int_0^1 dx \, \gamma^\mu (\not{p}x + m_0)\gamma^\nu \eta_{\mu\nu} \underbrace{\int \frac{d^4k'}{(2\pi)^4} \frac{1}{[k'^2+\Delta^2]}}_{\text{log-divergent}}$$

$$\Delta \equiv x(1-x)p^2 - m_0^2(1-x) + i\epsilon$$

⑥

22.08.2014

We'll rotate the contour of the k_0' integral to Euclidean space. Take $k_0' = i\,\underset{\text{real}}{\underset{\swarrow}{k_{0E}'}}$ [Wick rotation]



Since we're not crossing any poles ~~as~~ when we're rotating the contour, everything will be unchanged except the factor of i. If we cross any poles, we've to take residue.

$$k_0'^2 - \vec{k}'^2 = -k_{0E}'^2 - \vec{k}'^2 = -k_E^2$$

So, the integral becomes (after 4-dim to d-dim)

$$i\int \frac{d^d k_E}{(2\pi)^d}\,\frac{1}{[k_E^2-\Delta]^2}$$

Dimensional reg ⇒ to isolate the divergence / to know the nature of the divergence.

STUDY THE INTEGRAL IN ARBITRARY CONTINUOUS DIM 'd':

$$\int \frac{d^d k_E}{(2\pi)^d}\,\frac{1}{[k_E^2-\Delta]^2}$$

$$= \int \frac{d\Omega_{d-1}}{(2\pi)^d}\int_0^\infty dk_E\,\frac{k_E^{d-1}}{[k_E^2-\Delta]^2}$$

dk_E is actually $d|k_E|$.
k_E^{d-1} " " $|k_E|^{d-1}$

Now, $\mathrm{vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)}\,\frac{1}{(2\pi)^d}\int_0^\infty \frac{dk_E\;k_E^{d-1}}{[k_E^2-\Delta]^2}$$

Define $y = -\frac{\Delta}{k_E^2 - \Delta}$

$$\Rightarrow k_E^2 = -\Delta\frac{(1-y)}{y}$$

$$\int_0^\infty \frac{dk_E\; k_E^{d-1}}{[k_E^2-\Delta]^2} = \frac{1}{2}\frac{1}{(-\Delta)^{\frac{4-d}{2}}} \underbrace{\int_0^1 dy\; y^{1-d/2}\,(1-y)^{\frac{d}{2}-1}}_{\beta\text{- fun}}$$

$$\frac{\Gamma(2-d/2)\,\Gamma(d/2)}{\Gamma(2)}$$

Hence,

$$\boxed{\int \frac{d^dk_E}{(2\pi)^d}\frac{1}{[k_E^2-\Delta]^2} = \frac{1}{(4\pi)^{d/2}}\Gamma\left(2-d/2\right)(-\Delta)^{\frac{d-4}{2}}}$$

At $d=4 \sim \Gamma(2-d/2) = \Gamma(0) \sim$ has pole

If we define $\varepsilon = 4-d \to 0$, then

$$\Gamma(2-d/2) = \Gamma(\varepsilon/2)$$
$$= \frac{2}{\varepsilon} - \gamma_E + O(\varepsilon)$$

$$a^\varepsilon = e^{\varepsilon \ln a} = 1 + \varepsilon \ln a + O(\varepsilon^2)$$

can contribute upon multiplication with $2/\varepsilon$ coming from $\Gamma(2-d/2)$.

In the transition 4 to d-dimensions, dimension of the integrand is changed. Here dimensionless $\rightarrow$ $d-4$ dim. To keep it dimensionless multiply with μ^{d+4}. μ is some arbit mass scale. Then

$$\boxed{\mu^{4-d}\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{(k_E^2-\Delta)^2} = \frac{1}{(4\pi)^{d/2}}\Gamma\left(2-d/2\right)\left(-\frac{\Delta}{\mu^2}\right)^{\frac{d-4}{2}}}$$

To consistently define the Feynman amplitude integral in d-dim, we must also give rules for $\eta_{\mu\nu}$, γ^μ etc. in d-dim.

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad , \text{ but } \eta^{\mu\nu}\eta_{\mu\nu} = d$$

$$\Rightarrow \gamma^\mu \gamma^\nu g_{\mu\nu} = d = 4 - \varepsilon$$

Similarly : $\gamma^\mu \gamma^\nu \gamma_\mu = -(2-\varepsilon)\gamma^\nu$

So,

$$-i\Sigma_2(p) = i(-ie)^2 \int_0^1 dx \, \gamma^\mu (\not{p}x + m_0)\gamma_\mu \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{[k_E^2 - \Delta]^2}$$

$$\xrightarrow{d \to 4} \frac{(-ie)^2}{(4\pi)^2} i(-2) \int_0^1 dx \, (\not{p}x - 2m_0) \frac{2}{\varepsilon}$$

$$+ \frac{(-ie)^2}{(4\pi)^2} i \Big\{ 2\int_0^1 dx \, (\not{p}x - m_0)\gamma_E + 2\int_0^1 (\not{p}x - m_0) dx$$

$$+ 2\int_0^1 dx \, (\not{p}x - 2m_0) \ln\left[\frac{m_0^2(1-x) - p^2 x(1-x)}{\mu^2}\right]\Big\}$$

<u>Recap</u>: After regularizing the self-energy contribution at 1-loop 

$$-i\Sigma_2(p) = \frac{ie^2}{8\pi^2\varepsilon}(\not{p} - 4m_0) + \frac{ie^2}{16\pi^2}\Big\{\not{p}(1+\gamma_E) - 2m_0(1+2\gamma_E)$$

$$+ 2\int_0^1 dx \, [\not{p}x - 2m_0] \ln\left\{\frac{m_0^2(1-x) - p^2 x(1-x)}{4\pi\mu^2}\right\}\Big\}$$

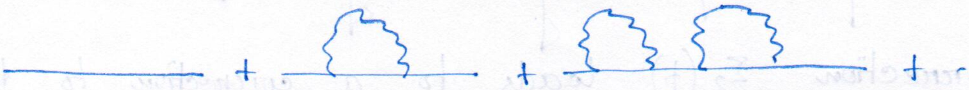
We've isolated a divergent piece $\sim \frac{ie^2}{8\pi^2\varepsilon}(\not{p} - 4m_0)$

[In a cut-off like regularization the divergent piece $\sim \frac{e^2}{8\pi^2}(\not{p} - 4m_0)\ln\frac{\Lambda}{m_0}$]

26.08.2014

and separated out some finite pieces.

This 1-loop contribution is actually a part of a large class of corrections to the 2-pt function which can be summed up.



$$S_F(p) = \frac{i(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon} \qquad S_F(p)\left(-i\Sigma_2(p)\right)S_F(p)$$

$$= S_F(p)\left[1 + (-i\Sigma_2)S_F + \{(-i\Sigma_2)S_F\}^2 + \dots\right]$$

$$= S_F(p)\left[1 + i\Sigma_2(p)S_F(p)\right]^{-1}$$

$$= \frac{i(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon}\left[1 - \Sigma_2(p)\frac{(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon}\right]^{-1}$$

$$= i(\not p + m_0)\left[\underbrace{(p^2 - m_0^2 + i\varepsilon) - \Sigma_2(p)(\not p + m_0)}_{}\right]^{-1}$$

$$(\not p - m_0)(\not p + m_0) - \Sigma_2(p)(\not p + m_0)$$

$$= \left((\not p - m_0) - \Sigma_2(p)\right)(\not p + m_0) \qquad \Big\| \; (AB)^{-1} = B^{-1}A^{-1}$$

$$= \frac{i}{\not p - m_0 - \Sigma_2(p)} \equiv i\left(\not p - m_0 - \Sigma_2(p)\right)^{-1}$$

Similarly if ▨ some arbit n-loop FD then

$$-i\Sigma(p)$$

$$= \frac{i}{\not p - m_0 - \Sigma(p)}$$

If one can't divide diagram into two parts by cutting a single line ⇒ 0PI diagram.

$\Sigma(p)$: $\overset{\text{sum of}}{{}_{\wedge}}$ 1PI diagrams (diagrams which can't be cut into two parts by cutting one single line).

Since the correction $\Sigma_2(p)$ leads to a correction to the propagator $\frac{i}{\not{p} - m_0} \longrightarrow \frac{i}{\not{p} - m_0 - \Sigma_2(p)}$,

we see that $\Sigma_2(p)$ can shift the location of the poles + residue of the poles of the propagator.

In free theory the poles of prop correspond to the energy of physical particle ~ self-energy of the particle. Due to correction the pole is shifted ~ self energy is changed.

We need to look for the pole in the modified prop. That will be at $\not{p}$ where

$$\left(\not{p} - m_0 - \Sigma_2(p)\right)\Big|_{\not{p} = m} = 0 .$$

$$\Rightarrow m - m_0 - \Sigma_2(\not{p} = m) = 0 \Rightarrow m \text{ in terms of } m_0, e^2, \varepsilon .$$

Since we're doing perturbation theory, $(m - m_0) = \delta m \sim e^2$ $\left[\because \Sigma_2 \sim e^2\right]$

So, in the leading order $(\sim e^2)$

$$\boxed{\delta m = \Sigma_2(\not{p} = m_0)}$$

So, $$\delta m = \frac{3e^2 m_0}{8\pi^2 \varepsilon} + \underset{\downarrow\; O(e^2)}{\text{finite}} + O(e^4)$$

$m_0 \Rightarrow$ "bare mass" (in the absence of interaction).

$m \Rightarrow$ "physical mass".

$\delta m = \Sigma_2(\not p = m_0) \longrightarrow$ mass is renormalized.

We'll express our 2-pt function in terms of the physical mass m (rather than the unphysical bare mass m_0).

In classical ED, electron self-energy (correction)

$$\sim \frac{e^2}{r_0^2}$$

take $r_0 \sim \frac{1}{\Lambda}$ then Λ^2. But here our correction $\sim \ln\Lambda \sim$ milder than that of classical $\sim$ due to presence of anti-particle $\sim$ particle + anti cancel some.

Classical $\Rightarrow \exists$ only particle. But in QFT addition to this contribution $\exists$ anti-par. is also there. This reduces the divergence. Symm. reduces divergence here.

$$\not p - m_0 - \Sigma_2(\not p)$$

$$= (\not p - m_0) + \frac{e^2}{8\pi^2\varepsilon}(\not p - 4m_0) + \text{finite} + O(\epsilon^4)$$

$$= \left(1+\frac{e^2}{8\pi^2\varepsilon}\right)\not p - \left(1+\frac{e^2}{8\pi^2\varepsilon}\right)\underbrace{\left(m_0 + \frac{3e^2}{8\pi^2\varepsilon}m_0\right)}_{m_0+\delta m = m} + \text{fin} + O(\epsilon^4)$$

$$= \left(1+\frac{e^2}{8\pi^2\varepsilon}\right)(\not p - m) + \text{fin} + O(\epsilon^4)$$

The pole at $\not p = m$ has a residue $\propto \left(1+\frac{e^2}{8\pi^2\varepsilon}\right)^{-1}$. So we need to take care of the divergence. $\longrightarrow Z_2$

The corrected two pt. function behaves near $\not p = m$ as

$$\frac{i Z_2}{\not p - m}$$

The factor of Z_2 can be absorbed into a redefinition of our 'interacting' fields

$$\langle \Omega_H | T\{ \psi_H \bar{\psi}_H \} | \Omega_H \rangle \xrightarrow{\text{near } \not{p} = m} \frac{i Z_2}{\not{p} - m}$$

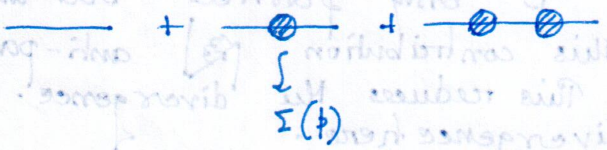
Define $\psi'_H(x) \equiv Z_2^{-1} \psi_H(x)$

$\bar{\psi}'_H(x) \equiv Z_2^{-1} \bar{\psi}_H(x)$

Then the two pt fun of $\psi'_H(x)$ has unit ~~[illegible]~~ residue ~~at~~ at the pole ($\not{p} = m$)

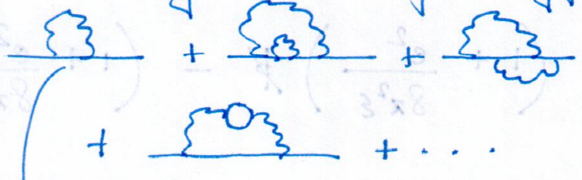
$$\langle \Omega_H | T\{ \psi'_H \bar{\psi}'_H \} | \Omega_H \rangle \xrightarrow{\not{p} \to m} \frac{i}{\not{p} - m}$$

RECAP :



$$= \frac{i}{\not{p} - m - \Sigma(p)}$$

$\Sigma(p)$ = sum of all 1PI diagrams contributing to the self-energy



$\Sigma_2(p)$

$$\frac{i}{\not{p} - m} = \frac{i(\not{p} + m)}{p_0^2 - \vec{p}^2 - m^2 + i\varepsilon} = \frac{i(\not{p} + m)}{p_0^2 - \vec{p}^2 - m^2 + i\varepsilon}$$

$\Big\downarrow$ FT $e^{ip\cdot(x-y)}$

$e^{iE_p(x^0 - y^0)}$ [p^0- integral]

with $E_p = \sqrt{\vec{p}^2 + m^2}$

dispersion rel^n

27.08.14

Two "infinite" redefinitions to make the 2-pt function finite:

① Physical mass $m = m_0 \left(1 + \frac{e^2}{8\pi^2 \varepsilon}\right)$ – as the location of the pole in the corrected propagator.

② "Wave function/ field" renormalization" ~ $\psi'_H(x) = Z_2^{-1/2} \psi_H(x)$

$Z_2 \sim \left(1 + \frac{e^2}{8\pi \varepsilon}\right)^{-1}$ – which made the residue of the 2-pt function unit at the pole.

With these two redefinitions $\langle \Omega_H | T\{ \psi'_H(x) \psi'_H(y) \} | \Omega_H \rangle$ is finite (expressed in terms of m). to this order in perturbation theory ($\sim e^2$).

The redefined 2-pt fun is

$$\frac{i}{\not{p} - m - \Sigma_2^{fin}(p)}$$ where m is the physical mass.

$\Sigma_2^{fin}(p)$ is s.t. $\cancel{\not{p}}$ $\boxed{\Sigma_2^{fin}(\not{p} = m) = 0}$.

$$\frac{i}{\not{p} - m_0 - \Sigma_2(p)}$$ where $\not{p} - m_0 - \Sigma_2(p) = Z_2^{-1}(\not{p} - m) - \Sigma_2^{fin}(p)$

with $$\Sigma_2^{fin}(p) = -\frac{ie^2}{16\pi^2} \Big\{ \not{p}(1+\gamma_E) - 2m(1+\gamma_E) + 2\int_0^1 (\not{p}x - m) \ln \left[\frac{m^2(1-x) - p^2 x(1-x)}{4\pi \mu^2} \right] \Big\}$$

– (same expression evaluated at $\not{p} = m$)

Here we've replaced m_0 by m as the correction is at $O(e^2)$.

So, $$\frac{i}{\not{p} - m_0 - \Sigma_2(p)} = \frac{i}{Z_2^{-1}(\not{p} - m) - \Sigma_2^{fin}(p)} = \frac{iZ_2}{\not{p} - m - \Sigma_2^{fin}(p)} + O(e^4)$$

$$\xrightarrow{\psi'_H = Z_2^{-1} \psi_H} \frac{i}{\not{p} - m - \Sigma_2^{fin}(p)}$$

NOTE :

① We could not have done this redefinitions and gotten a finite two pt. function if the structure of the divergence had been any different from it was namely as

$$\sim (\,)\frac{e^2}{\varepsilon}\left(\not{p} - (\,)m_0\right)$$

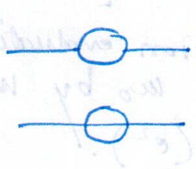
If we had a divergence like $\frac{e^2 p^2}{\varepsilon}$ or $\frac{e^2 \ln p^2}{\varepsilon}$ etc. then we would not have been able to redefine it.

$$\left[\frac{i}{\not{p} - m_0 - \frac{e^2 p^2}{\varepsilon}} \text{ etc.}\right] \rightarrow$$ wave fun. renormalization by some constant factor is insufficient.

The divergent term is analytic in external momentum. (unlike finite part ~ contains log p type term → not analytic ~ contains branch cut).

② This redefinition ("renormalization") is not empty of content. There is an unambiguous finite corrections after the redefinition $\Sigma_2^{\,fin}(p)$. This is the piece of $\Sigma_2^{\,fin}(p)$ that could not have been absorbed by the above redefinitions. These are not analytic in the ext. momenta.

* For scalar theory



~ ϕ^3 theory

~ ϕ^4 theory

$$\frac{i}{p^2 - m^2 - \Sigma_2(p)}$$

↳ $\frac{p^2}{\varepsilon} + \frac{(\,)}{\varepsilon}$ of this form.

The branch cut in $\Sigma_2^{fin}(p)$ occurs at the point where the argument of log becomes -ve. This happens when $p^2 x > m^2 \Rightarrow p^2 > m^2/x$ i.e. for $p^2 \geq m^2$ [at $x=1$] a branch cut opens. $\Sigma_2^{fin}(p)$ gets an imaginary part for $p^2 \geq m^2$.

for an onshell (e^-, γ) pair $(p_e + p_\gamma)^2 \geq p_e^2 = m^2$

Claim : The branch cut is associated with the (hence imaginary part) amplitude to produce a real (e^-, γ) pair.

Im () $\propto$ $\Big|$  $\Big|^2$.

$$\frac{1}{x+i\varepsilon} = P\left(\frac{1}{x}\right) + i\pi\delta(x)$$

* Next Thursday 9:30 ; Tuesday & Wed no class.

29.08.2014

* SPECTRAL REPRESENTATION OF THE PROPAGATOR (Callen-Lehmann)

[Non-perturbative] STR + QM (General principles)

$$\langle \Omega_H | T\{ \phi_H(x)\, \phi_H(y) \} | \Omega_H \rangle$$

Take $x^0 > y^0$
(fixed time-ordering)

$$= \langle \Omega_H | \phi_H(x)\, \phi_H(y) | \Omega_H \rangle$$

$$= \sum_n \langle \Omega_H | \phi_H(x) | n \rangle \langle n | \phi_H(y) | \Omega_H \rangle$$

Take $|n\rangle$ to be a complete set – to be eigenstates of the full Hamiltonian H.

$$H = H_0 + H_{int}$$

In a relativistically invariant theory $[H, \vec{P}] = 0$
(Part of Poincare symm)

So, $|n\rangle$ can also be taken to be simultaneous eigenstates of $\vec{P}$.

$p^\mu p_\mu = (p^0)^2 - (\vec{p})^2$ is an invariant $\rightarrow m_\lambda^2 \rightsquigarrow$ need not be the mass of a single particle, could be bound state

$$p^0 = E_{\vec{p}} = \sqrt{\vec{p}^2 + m_\lambda^2}$$

Denote this state by $|\vec{p}\rangle_\lambda$
↳ all other quantum nos. labelling the state

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

$$= \underset{(\vec{p}=0,\ H=0)}{|\Omega_H\rangle\langle\Omega_H|} + \int \sum_\lambda |\vec{p}\rangle_\lambda \, {}_\lambda\langle \vec{p}| \frac{d^3\vec{p}}{(2\pi)^3\, 2E_{\vec{p}}(\lambda)}$$

Now, let's resume our analysis of the propagator.

$$\langle \Omega_H | \phi_H(x) | \Omega_H \rangle \langle \Omega_H | \phi_H(y) | \Omega_H \rangle$$

$$+ \sum_\lambda \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}(\lambda)} \langle \Omega_H | \phi_H(x) | \vec{p} \rangle_\lambda \cdot {}_\lambda\langle \vec{p} | \phi_H(y) | \Omega_H \rangle$$

1st term: $\langle \Omega_H | e^{ip\cdot x} \phi_H(0)\, e^{-ip\cdot x} | \Omega_H \rangle$

$= \langle \Omega_H | \phi_H(0) | \Omega_H \rangle$

$= \langle \phi \rangle$ independent of x (Translational inv.)

Let $\langle \phi \rangle = 0$ always possible by shifting

$$\phi'_H(x) = \phi_H(x) - \langle \phi \rangle$$

2nd term:

$$\sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}(\lambda)} \underbrace{\langle \Omega_H | \phi_H(x) | \vec{p} \rangle_\lambda}\, {}_\lambda\langle \vec{p} | \phi_H(y) | \Omega_H \rangle$$

$$\downarrow$$

$$\langle \Omega_H | e^{ip\cdot x} \phi_H(0)\, e^{-ip\cdot x} | \vec{p} \rangle_\lambda$$

$$= \underbrace{\langle \Omega_H | \phi_H(0) | \vec{p} \rangle_\lambda}_{\text{simplify further}} e^{-i(E_{\vec{p}}(\lambda)x^0 - \vec{p}\cdot\vec{x})}$$

$$\hookrightarrow \langle \Omega_H | u^{-1}(p)\, u(p)\, \phi_H(0)\, u^{-1}(p) | \vec{p}=0 \rangle_\lambda$$

where $u^{-1}(p) | \vec{p}=0 \rangle_\lambda = | \vec{p} \rangle_\lambda$

for any state w/ $m_\lambda^2 > 0$ - can be boosted to the rest frame. $u(p)$ is the unitary operator which does the boost.

Now, $u^{-1}(p)\, \phi_H(0)\, u(p) = \phi_H(0)$ since, $u^{-1}(\Lambda)\, \phi_H(x)\, u(\Lambda) = \phi_H(\Lambda x)$

and $\langle \Omega_H | u^{-1}(p) = \langle \Omega_H |$

So, $\langle \Omega_H | \phi_H(0) | \vec{p} \rangle_\lambda = \langle \Omega_H | \phi_H(0) | \vec{p}=0 \rangle_\lambda$

Hence,
$$\begin{cases} \langle \Omega_H | \phi_H(x) | \vec{p} \rangle_\lambda = \langle \Omega_H | \phi_H(0) | \vec{p}=0 \rangle_\lambda\, e^{-ip\cdot x} \\ {}_\lambda\langle \vec{p} | \phi_H(y) | \Omega_H \rangle = {}_\lambda\langle \vec{p}=0 | \phi_H(0) | \Omega_H \rangle\, e^{ip\cdot y} \end{cases}$$

$$\therefore \langle \Omega_H | \phi_H(x)\, \phi_H(y) | \Omega_H \rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3\, 2E_{\vec{p}}(\lambda)} \left| \langle \Omega_H | \phi_H(0) | \vec{p}=0 \rangle \right|^2 e^{-ip\cdot(x-y)} \Big|_{p^0 = E_{\vec{p}}(\lambda)}$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}(\lambda)} e^{-ip\cdot(x-y)} \Big|_{p^0 = E_{\vec{p}}} = \int \frac{d^4p}{(2\pi)^4} \frac{1\cdot i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

A similar argument for $x^0 < y^0$. So,

$$\langle \Omega_H | T\{\phi_H(x)\,\phi_H(y)\} | \Omega_H\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\, \frac{i\, e^{-i p\cdot(x-y)}}{p^2 - m_\lambda^2 + i\varepsilon}\, \left| \langle \Omega_H | \phi_H(0) | \vec{p}=0\rangle_\lambda \right|^2$$

The momentum space propagator

$$\int d^4x\, \langle \Omega_H | T\{\phi_H(x)\,\phi_H(0)\} | \Omega_H\rangle\, e^{i p\cdot x} \equiv D_F(p)$$

So. $$D_F(p) = \sum_\lambda \frac{i}{p^2 - m_\lambda^2 + i\varepsilon}\, \left| \langle \Omega_H | \phi_H(0) | \vec{p}=0\rangle_\lambda \right|^2$$

$$= \int_0^\infty i\, \frac{dM^2}{2\pi}\, \frac{\rho(M^2)}{p^2 - M^2 + i\varepsilon} \quad , \quad \rho(M^2) \equiv \sum_\lambda 2\pi\, \delta(M^2 - m_\lambda^2)\, \left| \langle \Omega_H | \phi_H(0) | \vec{p}=0\rangle_\lambda \right|^2$$

$\downarrow$ spectral density.

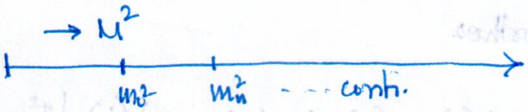
For a typical theory $\rho(M^2) = ?$

$\left\{ \begin{array}{l} \text{Set of single particle states} \\ + \text{ multiparticle states} \end{array} \right.$

$$\rho(M^2) = 2\pi\, \delta(M^2 - m_0^2)\, Z_0 + 2\pi \sum_{n\geq 1} \delta(M^2 - m_n^2)\, Z_n + (\text{conti.})$$

(conti.) $\downarrow$ multiparticle states $\downarrow$ full complexity of QFT

with $Z_n \equiv \left| \langle \Omega_H | \phi_H(0) | \vec{p}=0 \rangle_{m_n^2} \right|^2$



$$D_F(p) = \frac{i Z_0}{p^2 - m_0^2 + i\varepsilon} + \sum_{n\geq 1} \frac{i Z_n}{p^2 - m_n^2 + i\varepsilon} + (\text{multiparticle})$$

$\searrow$ poles $\swarrow$

- single particle $\Rightarrow$ fundamental excitations / bound states $\to$ poles in the prop

  locations of the poles $\Rightarrow$ masses (m_0^2, m_n^2)

  Residues : $Z_0 / Z_n = \left| \langle \Omega_H | \phi_H(0) | \vec{p}=0 \rangle_n \right|^2 \sim$ prob to produce the states from vacuum

  $\phi_H \to \boxed{\text{vacuum}} \rightsquigarrow$ particles

- Multiparticle states give contributions to $\rho(M^2)$ which have branch cuts starting at some threshold (eg. $4m_0^2$, $9m_0^2$, .. etc.)

  eg. $\rho(M^2) = (\,)\sqrt{M^2 - 4m_0^2}$ $\quad M^2 > 4m_0^2$

  $= 0$ $\quad M^2 \leq 4m_0^2$

- Real part of $D_F(p)$ contains info about the continuous part of $\rho(p^2)$.

⊠ PHOTON PROPAGATOR :~

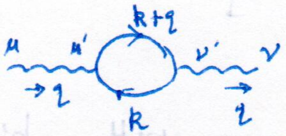
04.09.14

$$P_{\mu\nu} \equiv \langle \Omega_H | T\{ A^H_\mu(x)\, A^H_\nu(y) \} | \Omega_H \rangle$$

$$= \frac{-i\eta_{\mu\nu}}{q^2+i\varepsilon} \quad \text{(Feynman gauge)}$$

$$\equiv D^F_{\mu\nu}(q)$$

$$P_{\mu\nu} = D^F_{\mu\nu} +$$



↑

Leading quantum correction
[vacuum polarization]

$$D^F_{\mu\mu'}(q)\, \underbrace{\left[i\Pi_2^{\mu'\nu'}(q) \right]}_{\text{Contribution from the } e^- \text{ loop}} D_{\nu'\nu}(q)$$

Now,

$$i\Pi_2^{\mu\nu}(q) = (-ie)^2(-1) \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[S_F(k+q)\, \gamma\, S_F(k)\, \gamma \right]$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}\left[(\not{k}+\not{q}+m_0)\gamma^\nu(\not{k}+m_0)\gamma^\mu \right]}{\left[(k+q)^2 - m_0^2 + i\varepsilon \right]\left[k^2 - m_0^2 + i\varepsilon \right]}$$

This integral appears to be quadratically divergent for large k.

$$\because \sim \int d^4k\, \frac{k^2}{k^4} \sim \Lambda^2 \quad \text{where } \Lambda \text{ is a cut-off.}$$

Actually, only logarithmatically divergent [because of gauge invariance].

Need to regularize and renormalize.

i) Need to simplify the form of the integral

ii) Wick rot^{n} to Euclidean space

iii) A. Continue the evaluation in d-dimensions.

$$\frac{1}{\left[(k+q)^2 - m^2 + i\varepsilon\right]\left[k^2 \underset{-m^2}{+} i\varepsilon\right]}$$

$$= \int_0^1 dx \frac{1}{\left[(k+q)^2 x + k^2(1-x) - m^2 + i\varepsilon\right]^2}$$

$$= \int_0^1 dx \frac{1}{\left[k^2 + 2x\, k\cdot q + xq^2 - m^2 + i\varepsilon\right]^2}$$

$$= \int_0^1 dx \frac{1}{\left[k'^2 + x(1-x)q^2 - m^2 + i\varepsilon\right]^2} \quad \text{with} \quad k'^\mu = k^\mu + xq^\mu$$

Hence,

$$i\Pi_2^{\mu\nu}(q) = -e^2 \int_0^1 dx \int \frac{d^4k'}{(2\pi)^4} \frac{tr\left[(\not{k}' + (1-x)\not{q} + m_0)\gamma^\nu(\not{k}' - x\not{q} + m_0)\gamma^\mu\right]}{\left[k'^2 + x(1-x)q^2 - m_0^2 + i\varepsilon\right]^2}$$

Numerator can be evaluated using

$$tr\left[\gamma^\mu\gamma^\nu\right] = \cancel{\tfrac{4}{2}}\, g^{\mu\nu}\, tr\left[\mathbb{1}\right]$$

$$tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] = \left[g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right] tr(\mathbb{1})$$

d-dimensional

$$tr\left[\text{odd}\ \#\ \gamma^\mu\right] = 0$$

The numerator becomes

$$\left[2k'^\mu k'^\nu - k'^2 g^{\mu\nu} - 2x(1-x)q^\mu q^\nu + x(1-x)q^2 g^{\mu\nu} + m_0^2 g^{\mu\nu}\right] tr(\mathbb{1})$$

We'll be left with integrals of the form (after Wick rot^n + AC to d-dimensions)

ⓐ $\displaystyle\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{\left[k^2 - \Delta\right]^2}$ ⓑ $\displaystyle\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{\left[k^2 - \Delta\right]^2}$ ⓒ $\displaystyle\int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 - \Delta\right]^2}$

Integral ⓐ : Note that unless $\mu = \nu$, ⓐ vanishes since it is odd : ~~wrt~~ $\rightsquigarrow$ $(k^\mu \rightarrow -k^\mu)$ then denom is unchanged, num changes sign

for $\mu = \nu$,
$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{[k^2-\Delta]^2} = \frac{g^{\mu\nu}}{d} \int \frac{d^dk}{(2\pi)^d} \frac{k^2}{[k^2-\Delta]^2}$$

$$\left[\text{eg. } \int x^2 f(r)\, d^3x = \int y^2 f(r)\, d^3x = \int z^2 f(r)\, d^3x = \frac{1}{3} \int r^2 f(r)\, d^3x \right]$$

So, the 1st two terms in the numerator give a contributions

$$= -i d e^2 \mu^{4-d} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(-2/d + 1)\, k^2 g^{\mu\nu}}{[k^2-\Delta]^2}$$

where, $\Delta \doteq x(1-x)q^2 - m^2$

The remaining term in the numerator

$$\left[2x(1-x)(q^2 g^{\mu\nu} - q^\mu q^\nu) - \Delta g^{\mu\nu} \right]$$

Now,
$$\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{[k^2-\Delta]^2} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(1-d/2)}{\Gamma(2)} (-\Delta)^{\frac{d-2}{2}}$$

1st pole at d = 2

However,
$$-d/2(-2/d + 1)\, \Gamma(1-d/2) = \Gamma(2-d/2)$$

as
$$\Gamma(2-d/2) = (1-d/2)\,\Gamma(1-d/2)$$
$$= -\frac{d}{2}\left(-\frac{2}{d} + 1\right)\Gamma(1-d/2)$$

So, the 1st two terms

$$\frac{4ie^2\mu^{4-d}}{(4\pi)^{d/2}} g^{\mu\nu} \int_0^1 dx\, (-\Delta)^{\frac{d-4}{2}} (-\Delta)\, \Gamma(2-d/2)$$

1st pole is at d = 4, not d = 2
$\rightsquigarrow$ signature of log-diver in d = 4

The remaining terms

$$\frac{i4e^2\mu^{4-d}}{(4\pi)^{d/2}}\int_0^1 dx\,(-\Delta)^{\frac{d-4}{2}}\,\Gamma(2-d/2)\Big[-2x(1-x)(q^2g^{\mu\nu}-q^\mu q^\nu)+\Delta g^{\mu\nu}\Big]$$

Recall,

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-\Delta]^2}=\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}(-\Delta)^{\frac{d-4}{2}}$$

On adding both contributions

$$i\Pi_2^{\mu\nu}(q^2)=-\frac{8ie^2\mu^{4-d}}{(4\pi)^{d/2}}\int_0^1 dx\,x(1-x)(-\Delta)^{\frac{d-4}{2}}\,\Gamma(2-d/2)(q^2g^{\mu\nu}-q^\mu q^\nu)$$

$$\Rightarrow\boxed{i\Pi_2^{\mu\nu}(q^2)\equiv i\Pi_2(q^2)\big[q^2g^{\mu\nu}-q^\mu q^\nu\big]}$$

with $$\boxed{\Pi_2(q^2)=-\frac{8e^2\mu^{4-d}}{(4\pi)^{d/2}}\int_0^1 dx\,x(1-x)(-\Delta)^{\frac{d-4}{2}}\,\Gamma(2-d/2)}$$

1st pole at d = 4

⇝ logarithimatic diverg.

When $d\to4$, in terms of $\epsilon=4-d$

$$\Gamma(2-d/2)\underset{\epsilon\to0}{\cong}\frac{2}{\epsilon}-\gamma_E+O(\epsilon^2)$$

$$\left(-\frac{\Delta}{4\pi\mu^2}\right)^{-\epsilon/2}\sim1-\frac{\epsilon}{2}\ln\left(-\frac{\Delta}{4\pi\mu^2}\right)+O(\epsilon^2)$$

Then, $$\Pi_2(q^2)\xrightarrow{\epsilon\to0}-\frac{8e^2}{(4\pi)^2}\int_0^1 dx\,x(1-x)\left[\frac{2}{\epsilon}-\ln\left(-\frac{\Delta}{4\pi\mu^2}\right)-\gamma_E+O(\epsilon)\right]$$

05.09.2014

$$\equiv \Pi_2^{div}(q^2) + \Pi_2^{fin}(q^2)$$

$$\swarrow$$

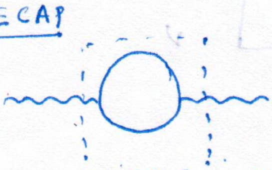
$$-\frac{2\alpha}{3\pi\epsilon}$$ with $\alpha \equiv \frac{e^2}{4\pi}$, fine structure cons.

$$\swarrow$$

$$\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left\{\ln\left(\frac{m_0^2 - x(1-x)q^2}{4\pi\mu^2}\right) + \gamma_E\right\}$$

Similar behaviour as earlier ⇒ divergent term is analytic in q and finite term is not analytic in q.

RECAP



1PI ⇒ $i\Pi_2^{\mu\nu}(q) = i\pi_2(q^2)\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]$

$$\Pi_2(q^2) = \Pi_2^{div}(q^2) + \Pi_2^{fin}(q^2)$$

with $\Pi_2^{div}(q^2) \overset{\epsilon\to 0}{=\!=} -\frac{2\alpha}{3\pi\epsilon}$, $\alpha \equiv e^2/4\pi$

$$\Pi_2^{fin}(q^2) = \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left\{\ln\left(\frac{m_0^2 - x(1-x)q^2}{4\pi\mu^2}\right) + \gamma_E\right\}$$

Note: $\Pi_2^{div}(q^2) \sim$ simple analytic dependence on q^2, in fact const wrt q^2.

$\Pi_2^{fin}(q^2) \sim$ complicated dependence on $q^2 \sim$ branch cut.

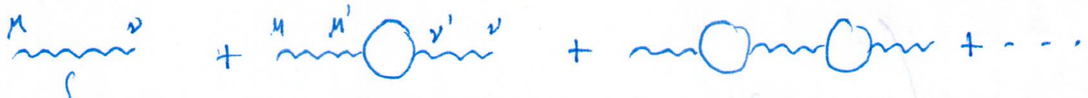
$\Pi_2^{\mu\nu}(q^2)$ obeys $q_\mu \Pi_2^{\mu\nu}(q^2) = 0$

⇒ $(q^2 g^{\mu\nu} - q^\mu q^\nu)$ is transverse to q^μ.

$$\Pi_2^{\mu\nu}(q^2) = q^2\Pi_2(q^2)\Delta^{\mu\nu}(q) \Rightarrow \Delta^{\mu\nu}(q) \equiv \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)$$

$\rightsquigarrow$ projector since

$$\Delta^{\mu}{}_{\nu'}(q)\,\Delta^{\nu'}{}_{\nu}(q) = \Delta^{\mu}{}_{\nu}(q)$$



classical prop

1st quantum correction

$$= -\frac{i g_{\mu\nu}}{q^2} + \frac{(-i g_{\mu\mu'})\, i\Pi_2^{\mu'\nu'}(q)\,(-i g_{\nu'\nu})}{q^2 \quad q^2} + \cdots$$

$$= -\frac{i g_{\mu\nu}}{q^2} + \frac{(-i g_{\mu\mu'})}{q^2}\Delta^{\mu'}{}_{\nu}\,\Pi_2(q^2) + \frac{(-i g_{\mu\mu'})}{q^2}(\Pi_2(q^2))^2 \underbrace{(\Delta\cdot\Delta)}_{\text{matrix products}}{}^{\mu'}{}_{\nu}$$

$$+ \cdots$$

$$= -\frac{i g_{\mu\nu}}{q^2} + \frac{(-i g_{\mu\mu'})}{q^2}\left[\Pi_2\Delta + (\Pi_2)^2\Delta^2 + \Pi_2^3\Delta^3 + \cdots\right]^{\mu'}{}_{\nu}$$

$\{$ Now, $\Delta^2 = \Delta$

$$= -\frac{i g_{\mu\nu}}{q^2} + \frac{-i g_{\mu\mu'}}{q^2}\Delta^{\mu'}{}_{\nu}\,\frac{\Pi_2(q^2)}{1-\Pi_2(q^2)}$$

$$= -\frac{i g_{\mu\nu}}{q^2} + \frac{(-i g_{\mu\mu'})}{q^2}\left(\delta^{\mu'}{}_{\nu} - \frac{q^{\mu'}q_{\nu}}{q^2}\right)\frac{\Pi_2(q^2)}{1-\Pi_2(q^2)}$$

$$= -\frac{i g_{\mu\nu}}{q^2} + \frac{(-i)\left(g_{\mu\nu} - q_\mu q_\nu/q^2\right)}{q^2}\,\frac{\Pi_2(q^2)}{1-\Pi_2(q^2)}$$

$$= \frac{-i g_{\mu\nu}}{q^2(1-\Pi_2(q^2))} + \underbrace{\frac{i(q_\mu q_\nu/q^2)\,\Pi_2}{q^2(1-\Pi_2(q^2))}}$$

This is gauge dependent → does not contribute to any physical amplitude.

→ Quantum corrected propagator.

Instead of feynman gauge if we ~~st~~ do the calculation in some other gauge, the 1st term will be same whereas the

2nd term will be changed.

focus on the 1st term :

$$\frac{-ig_{\mu\nu}}{q^2(1-\Pi_2(q^2))} \rightarrow \text{gauge 'indep. contribution to the corrected propagator.}$$

Since $\Pi_2(q^2)$ has divergence piece which is constant (i.e. indep. of q^2), we can write

$$\Pi_2(q^2) = \underbrace{\Pi_2(0)}_{\substack{\text{diver.}\\ \text{at } \epsilon\to 0}} + \underbrace{\left[\Pi_2(q^2) - \Pi_2(0)\right]}_{\text{finite}}$$

$$\Pi_2(q^2) \sim O(\alpha) = O(e^2)$$

So, $$\frac{-ig_{\mu\nu}}{q^2(1-\Pi_2(0))\left(1-(\Pi_2(q^2)-\Pi_2(0))\right)}$$

Now, ~~at~~ the pole of the prop, we see is still at $q^2=0$. Since $\Pi_2(q^2)$ is regular at $q^2=0$. This is 'important →

IT SHOWS THAT PHOTON IS STILL MASSLESS EVEN AFTER QUANTUM CORRECTION.

Recall this is unlike the electron case where pole was shifted → mass renormalization for electron. There is no mass renormalization for photon.

For example if we had $\Pi_2(q^2) \overset{q^2\to 0}{\sim} \frac{M^2}{q^2}$

then $q^2(1-\Pi_2(q^2)) = q^2 - M^2$

→ pole would have shifted.

Due to quantum correction there is no additional DOF 'introduced to photon.

However the residue changes due to quantum correction.

The residue is at $q^2=0$ $\frac{1}{1-\Pi_2(0)} \equiv Z_3$

↳ divergent at $\epsilon \to 0$

In $\langle \Omega_H | A^\mu A^\nu | \Omega_H \rangle$, the residue at the pole $q^2=0$ $\propto \left| \langle \Omega_H | A^\mu | 1 \text{ Photon} \rangle \right|^2$

⇑ From spectral decomposition

↪ prob to produce a single particle state (photon) from vacuum

This suggests that we can absorb the divergent factor in the quantum corrected photon propagator by redefining the gauge field $A_\mu(x)$:

$$A'_\mu(x) \equiv Z_3^{-1/2} A_\mu(x)$$

Then $A'_\mu(x)$ will produce the 1-particle state of the photon w/ unit prob amplitude. The two pt. fun $\langle \Omega_H | A'A' | \Omega_H \rangle$

$$= Z_3^{-1} \langle \Omega_H | AA | \Omega_H \rangle$$

$$= \frac{-ig_{\mu\nu}}{q^2\left(1-\left(\Pi_2(q^2)-\Pi_2(0)\right)\right)} + (\text{gauge dep. pieces})$$

is finite

Everything is possible due to simple and very specific form of divergent piece. This is extremely crucial to have this divergent piece in this form. Otherwise renormalization would have failed to make two pt. fun. finite.

Redefining $A'_\mu(x) \equiv Z_3^{-1/2} A_\mu(x)$

$$\Rightarrow \vec{B}' = Z_3^{-1/2} \vec{B}$$

$$= \underbrace{\left(1 + \frac{\alpha}{3\pi\epsilon}\right)}_{\text{like some kind of magnetic susceptibility of the vacuum}} \overset{\text{"Bare field"}}{\vec{B}}$$

$$Z_3 \equiv \frac{1}{1-\Pi_2(0)}$$

$$= 1 - \underbrace{\frac{2\alpha}{3\pi\epsilon}}_{\text{div. at } \epsilon \to 0} + \text{cons.}$$

$$\vec{E}' = \underbrace{\left(1 - \frac{\alpha}{3\pi\epsilon}\right)\vec{E}}$$

Electric polarizablity – screening.

$\rightarrow$ that's why it is called vacuum polarization.

Hence, the finite prop.

$$\frac{-i g_{\mu\nu}}{q^2\left(1 - \hat{\Pi}_2(q^2)\right)} \qquad \hat{\Pi}_2(q^2) \equiv \Pi_2(q^2) - \Pi_2(0)$$

$$= \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2 - x(1-x)q^2}{m^2}\right)$$

$\rightarrow$ indep. of arbit scale μ.

The usual classical $\frac{1}{q^2}$ prop is a reflection of the long range Coulomb's law ($\frac{1}{|\vec{x}|}$ potential). The 'interactions bet^n e^- and gauge fields $\sim \int J^\mu A_\mu$. In the presence of a source $J_\mu^0(x)$, the A_μ^{cl} produced by it

$$A_\mu^{cl} = \int d^3x' \cancel{J_\mu^0(x')}\, D^F_{\mu\nu}(x, x')\, J_0^\nu(x')$$

$\hookrightarrow$ Green's fun.

$$\Rightarrow \Box A_\mu^{cl} = J_\mu^0(x)$$

for a static source

$$A_0^{cl}(x) = \int d^3x\, D_{00}(x, x')\, J^0(x')$$

$$D_{00}(x, x') \propto \int \frac{d^3\vec{q}}{|\vec{q}|^2}\, e^{i\vec{q}\cdot(\vec{x}-\vec{x}')} = \frac{1}{|\vec{x}-\vec{x}'|}$$

$\rightarrow$ Coulomb's law.

So, quantum correction modifies Coulomb's law.

⑫ 09.09.2014

The 1-loop corrected and renormalized photon prop $\sim \dfrac{-ig_{\mu\nu}}{q^2\left(1-\hat{\pi}_2(q^2)\right)}$ ↳ finite

$A'_\mu = Z_3^{-1/2} A_\mu$

$Z_3 = \dfrac{1}{1-\pi_2(0)}$

$$\hat{\pi}_2(q^2) \equiv \pi_2(q^2) - \pi_2(0)$$
$$= \frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x) \ln\left(\frac{m^2 - x(1-x)q^2}{m^2}\right)$$

$$H_{int} = e \int d^3x\, A^{EM}_\mu(x)\, J^\mu(x)$$
$$\sim e \int d^3x\, d^3x'\, J^{\nu(0)}(x')\, D^F_{\mu\nu}(x,x')\, J^\mu(x)$$
$$\sim e \int d^3q\, J^{(0)}_0(-q) \frac{1}{|\vec{q}|^2} J_0(q)$$
$$\sim e \int d^3x\, d^3x' \frac{J^{(0)}_0(x')\, J_0(x)}{|\vec{x}-\vec{x}'|}$$

$\square A^{EM}_\mu = J^{(0)}_\mu$ in $\partial_\mu A^\mu = 0$

$$\Rightarrow A^{EM}_\mu = \int D^F_{\mu\nu}(x,x')\, J^{(0)\nu}(x')\, dx'$$

↓ static source

$J^{(0)\nu}_0 = (*,0,0,0)$

~ Coulomb's law

The 8tm corrected propagator modifies Coulomb's law.

The modified electrostatic potential due to a static point charge is now the Fourier transform of
$$\frac{1}{|\vec{q}|^2\left(1-\hat{\pi}_2(-|\vec{q}|^2)\right)}$$

$$V_{el}(\vec{x}) \sim \frac{e^2}{4\pi} \int d^3\vec{q}\, \frac{e^{i\vec{q}\cdot\vec{x}}}{|\vec{q}|^2\left(1-\hat{\pi}_2(-|\vec{q}|^2)\right)}$$

Consider the modification at distances much larger than the Compton radius of the e^- $(\hbar/mc = \frac{1}{m})$ [Bohr radius $\frac{\hbar}{\alpha mc} \gg \frac{\hbar}{mc}$]

$\Rightarrow$ Consider $|\vec{q}| \ll m$ $\left[|\vec{x}| \sim \frac{1}{|\vec{q}|} \gg \frac{1}{m}\right]$

Then $\ln\left(1 - \dfrac{x(1-x)\,q^2}{m^2}\right) \cong \dfrac{x(1-x)|\vec{q}|^2}{m^2}$

$q^2 \to -|\vec{q}|^2$

So, $\hat{\pi}_2(-|\vec{q}|^2) \sim \frac{2\alpha}{\pi}\int_0^1 dx\; x^2(1-x)^2\,\frac{|\vec{q}|^2}{m^2} = \frac{2\alpha}{30\pi}\,\frac{|\vec{q}|^2}{m^2}$

Hence,

$$V_{el}(x) \sim -\frac{e^2}{4\pi}\int d^3q\; e^{i\vec{q}\cdot\vec{x}}\,\frac{1}{|\vec{q}|^2}\left[1+\frac{\alpha}{15\pi}\,\frac{|\vec{q}|^2}{m^2}+\dots\right]$$

$$\Rightarrow \boxed{V_{el}(\vec{x}) = -\frac{\alpha}{|\vec{x}|} - \frac{4\alpha^2}{15m^2}\,\delta^3(\vec{x})} \quad ; \quad \alpha = e^2/4\pi$$

So the correction gives rise to an additional attractive force.

QM: $\Delta E^{(nlm)} = \langle nlm|\Delta H|nlm\rangle$

for $\Delta H \propto \delta^3(\vec{x}) \Rightarrow \Delta E^{nlm} \propto |\psi_{nlm}(\vec{x}=0)|^2$

The only non-zero contributions are to the states $(n\,0\,0)$ [only $\psi_{n00}(x=0)\neq 0$] i.e. $\Delta E^{n00}\neq 0$.

Even in the Dirac eq^n $\underset{n}{2}\,\underset{l=0}{S}_{\underset{j}{1/2}}$ and $2P_{1/2}$ states are degenerate. But this additional QED correction splits this two states ~ in fact measurable $(-1.123\times 10^{-7}\text{ eV})$!! Combined with others → Lamb shift. This what we've calculated is the vacuum polarization contribution.

One can make a better estimate of the long distance modification to Coulomb's law (See P+S Sec 7.5).

$$\underset{|\vec{x}|}{V_{el}(r)} = -\frac{\alpha}{r}\Big(1 \underset{\text{sign is crucial}}{+} \frac{\alpha}{4\sqrt{\pi}}\,\underbrace{\frac{e^{-2mr}}{(mr)^{3/2}}}_{\text{Uehling potential}} + \dots\Big)$$

Now, look at the opposite limit : $|\vec{q}|^2 \gg m^2$

$$\hat{\pi}_2(-|\vec{q}|^2) \sim \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left(\frac{x(1-x)|\vec{q}|^2}{m^2}\right).$$

$$\sim \frac{2\alpha}{\pi}\int dx\, x(1-x)\Big[\underbrace{\ln\frac{|\vec{q}|^2}{m^2}}_{\text{dominant part}} + \ln x(1-x)\Big]$$

$$\sim \frac{\alpha}{3\pi}\left[\ln\left(\frac{|\vec{q}|^2}{m^2}\right) - 5/3\right]$$

So,

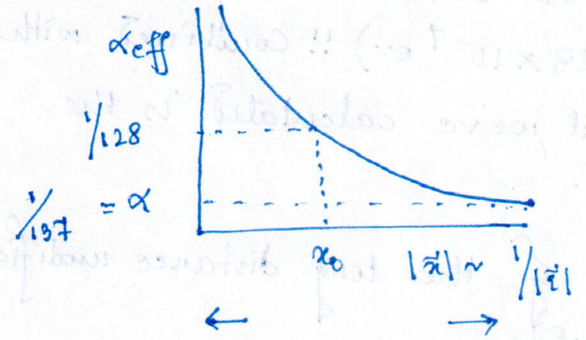
$$D_{\mu\nu}(-|\vec{q}|^2) \sim \frac{i g_{\mu\nu}}{|\vec{q}|^2\left[1-\frac{\alpha}{3\pi}\ln\left(\frac{|\vec{q}|^2}{m^2}\right)\right]}$$

This can be described in terms of an effective coupling (EM) ("running coupling")

$$\alpha_{eff}(q) \sim \frac{\alpha}{\left[1-\frac{\alpha}{3\pi}\ln\left(\frac{|\vec{q}|^2}{m^2}\right)\right]} > \alpha$$

$$\Rightarrow e^2/4\pi\; D_{\mu\nu}(q) \sim \frac{\alpha_{eff}(\vec{q})}{|\vec{q}|^2}.$$

Short distance modification



Now a days people have measured $\alpha_{eff} \sim \frac{1}{128}$ at $x_0 \sim (1\text{ Tev})^{-1}$.

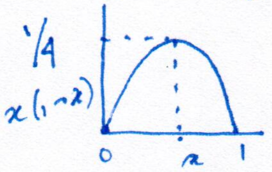
Effective EM coupling ↑ as we go closer and closer. As we come closer, we see more & more bare charge.

" Effective EM coupling grows at shorter distances — less of the "bare charge" is screened by the e^+e^- virtual pair.

— less screening.

Now look at $\hat{\pi}_2(q^2)$ for $q^2 > 0$ [Earlier $q^2 = -|\vec{q}|^2 < 0$].

For $q^2 > 0$, $\hat{\pi}_2(q^2)$ can get an 'imaginary part (when the argument of log becomes negative). This occurs first at $q^2 = 4m^2$. $\left[\geq \frac{m^2}{x(1-x)} \right]$. For $q^2 \geq 4m^2$, $\text{Im}\, \hat{\pi}_2(q^2) \neq 0$.



$\text{Im}\, \hat{\pi}_2(q^2) \propto \rho(q^2)$

spectral density of photon prop

continuous part coming from 2-particle states etc.

For fixed $q^2 \geq 4m^2$, the 'imaginary part will come from those x's s.t. $q^2 \geq m^2/x(1-x)$ $\Rightarrow x(1-x) \geq m^2/q^2 \left(\leq 1/4\right)$.

$$\Rightarrow x \in \left[\frac{1}{2} - \beta/2 , 1/2 + \beta/2 \right]$$

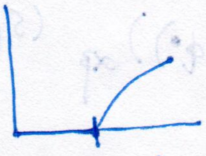
where $\beta = \sqrt{1 - 4m^2/q^2}$

So, $$\text{Im}\, \hat{\pi}_2(q^2) = \frac{2\alpha}{\pi} \pi \int_{1/2(1-\beta)}^{\frac{1}{2}(1+\beta)} x(1-x)\, dx$$

$$= \frac{\alpha}{3} \sqrt{1 - 4m^2/q^2} \left(1 + 2m^2/q^2 \right) \qquad (q^2 \geq 4m^2)$$

branch cut at $q^2 = 4m^2$

The 'imaginary part reflects the amplitude that a virtual photon w/ $q^2 \geq 4m^2$ can produce a real pair of e^+e^-.

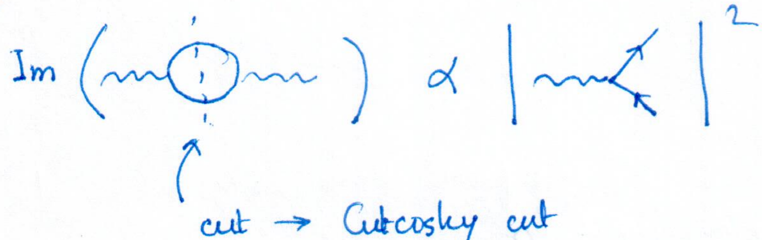


$q^2 = 4m^2$ → min energy to produce e^+e^- pair!

Ex: The $|\vec{p}_{tot}=0\rangle$ state of e^+e^- can be parametrised as $p_1 = (E/2, \vec{k}/2)$, $p_2 = (E/2, -\vec{k}/2)$ w/ $q^2 = (p_1+p_2)^2 = E^2$.

~ show that the factor of $\sqrt{1-4m^2/q^2}$ comes from the phase space vol. of the 2-particle state.

*

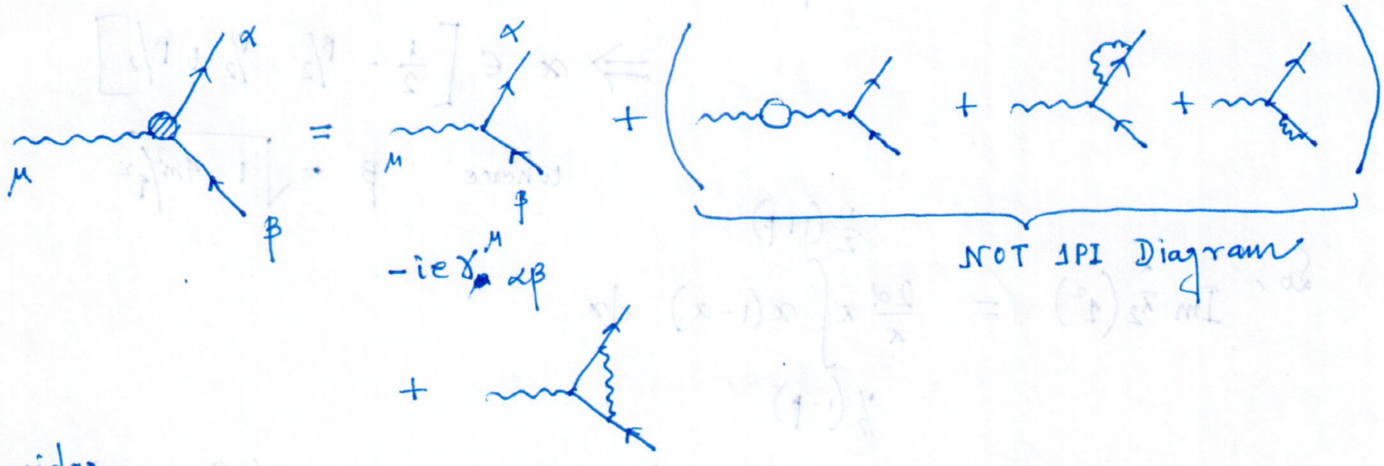


cut → Cutcosky cut

* ELECTRON - PHOTON VERTEX :

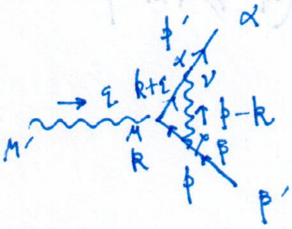
The quantum corrections to the 3-pt fun

$$\langle \Omega_H | T\{ \psi^H_\beta(x)\, \psi^H_\alpha(y)\, A^H_\mu(z) \} | \Omega_H \rangle$$



$-ie\gamma^\mu_{\alpha\beta}$

Consider,



$$q^\mu = (p' - p)^\mu$$

$$= D^F_{\mu'\mu}(q)\left(-ie\Gamma^\mu_{(2)}(p,q)\right)_{\alpha\beta} (S_F)_{\alpha'\alpha} (S_F)_{\beta\beta'}$$

(13)

10.09.2014

Need to compute

$$-ie(\Gamma^{\mu}_{(2)})_{\alpha\beta} = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{(-ig_{\nu\gamma})}{(p-k)^2+i\varepsilon}$$

$$\left[\gamma^{\nu} \frac{i[(\not{k}+\not{q})+m]}{(k+q)^2-m^2+i\varepsilon} \gamma^{\mu} \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon} \gamma^{\gamma}\right]_{\alpha\beta}$$

The integral is potentially log-divergent

$$\sim \int \frac{d^4k}{(k^2)^3} (\not{k})^2 \sim \ln\Lambda$$

We need to regularize & renormalize. for regularize we need to simplify the form of the integrand.

$$\frac{1}{ABC} = \int_0^1 dx \int_0^{1-x} dy \frac{2}{[xA+yB+(1-x-y)C]^3}$$

For us $A = k^2-m^2$, $B = (k+q)^2-m^2$, $C = (p-k)^2$

Den. $xA+yB+(1-x-y)C$

$$= (k')^2 + y(1-y)q^2 - (x+y)m^2 + p^2(x+y)(1-x-y) + 2p.q\,y(1-x-y)$$

$$= (k')^2 + \Delta$$

with $k' = k + yq + (-1+x+y)p$

Simplify the numerator - writing in terms of k'

$$\gamma^{\nu}[\not{k}' + (1-y)\not{q} + (1-x-y)\not{p} + m]\gamma^{\mu}[\not{k}' - y\not{q} + (1-x-y)\not{p} + m]\gamma_{\nu}$$

$$= \underbrace{\gamma^{\nu}\not{k}'\gamma^{\mu}\not{k}'\gamma_{\nu}} - \underbrace{\gamma^{\nu}[(1-y)\not{q} + (1-x-y)\not{p} + m]\gamma^{\mu}[y\not{q} - (1-x-y)\not{p} + m]\gamma_{\nu}}$$

$+$ (terms linear in k' which do not contribute in the integral)

will lead to log div contribution in UV

$$\int \frac{d^4k}{(k^2)^3} \sim \frac{1}{\Lambda^2} \sim \text{finite in UV}$$

The divergent piece

$$\sim \int d^4k \, \frac{k_\sigma k_\beta}{(k^2+\Delta)^3} \rightsquigarrow \frac{1}{d}\, i g_{\sigma\beta} \int d^d k_E \frac{k_E^2}{(k_E^2 - \Delta)^3}$$

i) Rot^{n} to Eucli.
ii) AC to d-dim.

$$= + \frac{i}{d}\, g_{\sigma\delta} \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(3)} \frac{d}{2} \Gamma(2 - d/2) (-\Delta)^{\frac{d-4}{2}}$$

if we take μ $\to \left(\frac{\Delta}{4\pi\mu^2}\right)^{\frac{d-4}{2}}$
and $\frac{1}{(4\pi)^{d/2}} \to \frac{1}{(4\pi)^2}$

The Dirac matrices simplify

$$\gamma_\nu \gamma^\sigma \gamma^\mu \gamma^\delta \gamma^\nu = (2-d)\gamma^\sigma \gamma^\mu \gamma^\delta + 2(\gamma^\mu \gamma^\delta \gamma^\sigma - \gamma^\sigma \gamma^\delta \gamma^\mu)$$

$$g_{\sigma\delta} \gamma_\nu \gamma^\sigma \gamma^\mu \gamma^\delta \gamma^\nu = (2-d)^2 \gamma^\mu$$

and recall $\gamma_\alpha \gamma^\alpha = 4 - \epsilon = d$

$$\gamma_\alpha \gamma^\mu \gamma^\alpha = (2-d)\gamma^\mu$$

This gives a contribution to $(-ie\Gamma^\mu_{(2)})_{\alpha\beta}$

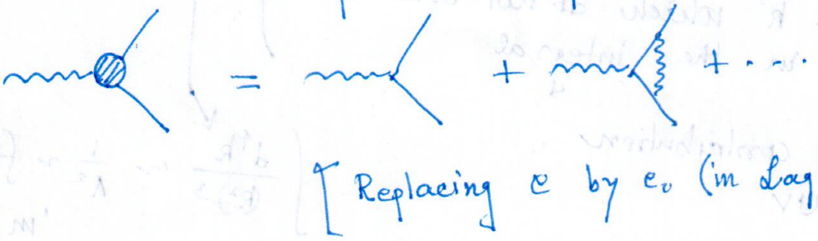
$$-ie(\Gamma^\mu_{(2)})_{\alpha\beta} = (-ie)^3 \frac{i^2}{(4\pi)^2} \frac{1}{2} \frac{1}{2} \frac{2}{\varepsilon} (4\gamma^\mu)_{\alpha\beta} \int_0^1 dx \int_0^{1-x} dy \; 2$$

$$+ (\text{finite}) \gamma^\mu_{\alpha\beta} + (\text{others finite})$$

Ex: Compute the (finite) γ^μ term verifying all the steps/identities used.

$$\Gamma^\mu_{(2)}(p,q) = \underbrace{\frac{e^2}{8\pi^2 \varepsilon}}\gamma^\mu + (\text{finite})\gamma^\mu + (\text{other finite})$$

↓ div. part → indep of external momenta → analytic & simple



[Replacing e by e_0 (in Lag)]

$$= -ie_0 \underbrace{\left(1 + \frac{e_0^2}{8\pi^2 \varepsilon}\right)}_{Z_1^{-1}} \gamma^\mu_{\alpha\beta} + (\text{finite})\gamma^\mu + (\text{other fin})$$

$$= -ie_0 Z_1^{-1} \gamma^\mu_{\alpha\beta} + \dots$$

~~Define $e = e_0 Z_1^{-1}$~~

Since the two pt fun of $\psi', \bar{\psi}', A'$ are the finite objects, we should really be considering

$$\langle \Omega_H | T\{ \psi'^{H}_{\beta} \, \bar{\psi}'^{H}_{\alpha} \, A'^{H}_{\mu} \} | \Omega_H \rangle \qquad \Bigg\| \quad \begin{array}{l} \psi' = Z_2^{-1/2} \psi \\ A'_\mu = Z_3^{-1/2} A_\mu \end{array}$$

$$= Z_2^{-1} Z_3^{-1/2} \langle \Omega_H | T \{ \psi^H_\beta \, \bar{\psi}^H_\alpha \, A^H_\mu \} | \Omega_H \rangle$$

In the original Lagrangian, we had $e_0 \int \bar{\psi} \gamma^\mu \psi A_\mu$

$$= \underbrace{e_0 Z_2 Z_3^{1/2}} \int \bar{\psi}' \gamma^\mu \psi' A'_\mu$$

This is the combination that appears in the vertex for the rescaled fields

$\Rightarrow$ in our computation, replace e_0 by $e_0 Z_2 Z_3^{1/2}$.

Then the divergent piece is

$$-i e_0 Z_1^{-1} \gamma^\mu_{\alpha\beta} \rightarrow -i e_0 Z_1^{-1} Z_2 Z_3^{1/2} \gamma^\mu_{\alpha\beta}$$

Z's also contain e_0 ~ but the modifi. to those e_0 will contribute to higher order.

bare coupling ↘

The combination $e_0 \dfrac{Z_2 Z_3^{1/2}}{Z_1}$ can be redefined to be the physical coupling $e_{phy} \equiv e$ (finite)

$$Z_2\big|_{div} = 1 - e^2/8\pi^2\epsilon + \cdots$$

$$Z_3\big|_{div} = 1 - e^2/6\pi^2\epsilon + \cdots$$

$$Z_1\big|_{div} = 1 - e^2/8\pi^2\epsilon + \cdots$$

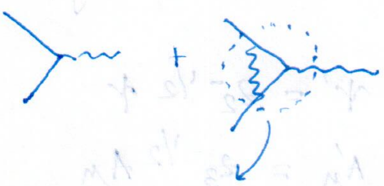
Z_1 is equal to Z_2 at this order. $\Rightarrow e_{phy} = e_0 Z_3^{1/2}$

The net effect of the charge/coupling renormalization comes from the vacuum polarization () diag.

$$e_{phy} \equiv e = e_0 Z_3^{1/2}.$$

⑭ 12.08.2014

<u>RECAP</u>:



$$(\Gamma^{\mu}_{(2)})_{\alpha\beta} = \underbrace{e^2/8\pi^2\epsilon\ \gamma^{\mu} + (\text{finite})\,\gamma^{\mu}}_{\Uparrow\ \int d^4k \frac{k^2}{(k^2+\Delta)^3}} + \underbrace{\text{other finite}}_{\Uparrow\ \int d^4k \frac{(\text{indep. of } k)}{(k^2+\Delta)^3}}$$

Changing notation e to e_0

Adding the above two ; $-ie_0 \underbrace{\left(1+\frac{e_0^2}{8\pi^2\epsilon}\right)}_{Z_1^{-1}}\gamma^{\mu}$

Also, $\langle \Omega_H | \psi' \bar{\psi}' A'^{\mu} | \Omega_H \rangle$

Replace e_0 by $e_0 Z_2 Z_3^{1/2}$ for the Green's fun of rescaled fields.

The effect of qtm correction to coupling at 1-loop/effective coupling at 1-loop is $e_{phys} = e_0 \frac{Z_2 Z_3^{1/2}}{Z_1}$

Notice that $Z_2 = Z_1$. Actually $Z_2 = Z_1$ to all orders.

$\Rightarrow e_{phy} = e_0 Z_3^{1/2}$

$\Rightarrow$ effective coupling is determined only by the vacuum polarization effect.

We computed the Z_1, Z_2 for a given charged particle of mass m, eg. m_e for e^-. The Z_1, Z_2 (finite pieces) depend on m_e. The Z_1, Z_2 are not universal. They could be different for muon.

For e^- : $e^{el}_{phy} = e^{el}_0 \frac{Z_2(m_e)}{Z_1(m_e)} Z_3^{1/2}$

For μ^- : $e^{m\mu}_{phy} = e^{m\mu}_0 \frac{Z_2(m_\mu)}{Z_1(m_\mu)} Z_3^{1/2} \longrightarrow$ indep of the particle mass one is considering

$\Rightarrow$ loop $\rightarrow$ sum of contributions from all charged particles

If z_1 & z_2 were not equal, then

$$\frac{e^{el}_{phy}}{e^{mu}_{phy}} \neq \frac{e_0^{el}}{e_0^{mu}}$$

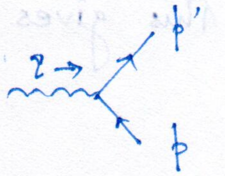
$\Rightarrow$ would have been conflicting with the fact that the charges of e^-, μ^- are the same.

We'll now extract out the correction to the magnetic dipole moment. The interaction of an EM current w/ the gauge field (classical EM field) is

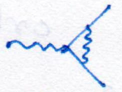
$$H_{int} = e\int j^\mu A^{cl}_\mu \, dx$$

$$= e\int \bar\psi \gamma^\mu \psi A^{cl}_\mu \, dx$$

$$J^\mu = -i\,\bar u(p')\gamma^\mu \underset{\downarrow}{u(p)}$$ to leading order



on-shell Dirac wave fn.

The quantum correction  leads to $\gamma^\mu \to (\gamma^\mu + \Gamma^\mu_{(2)})$.
The effective corrected coupling to the EM field is $\left(\bar u(p')\left(\gamma^\mu + \Gamma^\mu_{(2)}(p,q)\right)u(p)\right)$

The divergent piece of $\Gamma^\mu_{(2)}(p,q)$ was what went into the factor z_1^{-1} → redefined e_{phy}. The finite pieces of $\Gamma^\mu_{(2)}(p,q)$ lead to correction to mag. moment.

The on-shell matrix element of $\Gamma^\mu(p,q)$ lead to after simplification the form:

$$\bar u(p')\Gamma^\mu(p,q)u(p) = \bar u(p')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2) + q^\mu F_3(q^2) + (p'+p)^\mu F_4(q^2)\right]u(p)$$

with $(p'-p)^\mu = q^\mu$

$$\left.\begin{aligned} \bar u(p')\not{p}' &= m\bar u(p') \\ \not{p}u(p) &= m u(p) \end{aligned}\right\}$$ on-shell cond^{ns}.

$\sigma^{\mu\nu} q_\nu \rightsquigarrow \sigma^{\mu\nu}(p_\mu + p'_\mu)$ won't contribute..

$1, \gamma^\mu, \sigma^{\mu\nu}, \underbrace{\gamma^\mu\gamma_5, \gamma_5}$

~~ps~~ pseudo, but EM current is not pseudo. So, these are not allowed.

Also, $p^2 = m^2 = p'^2$. So, only LI (apart from m^2) is $q^2 \sim$ param. in terms of q^2.

Since, this matrix element is proportional to EM current, this must obey $q_\mu \bar{u}(p') \Gamma^\mu(p,q) u(p) = 0$ [conservation]

$$\left\{ \begin{array}{l} \partial_\mu(q) A^\mu(q) \\ q_\mu \partial^\mu(-q) = 0 \end{array} \right.$$

This gives $\bar{u}(p') \not{q}\, u(p) = 0$

$$\bar{u}(p') \underset{\text{anti}}{\sigma^{\mu\nu}} q_\mu q_\nu u(p) = 0$$

$q^\mu (p'+p)_\mu = 0$ orthogonality of $p'+p$ & $p'-p$.

$= p'^2 - p^2$

So, $q^2 F_3(q^2) \bar{u}(p') u(p)$ has to be zero.

$\Longrightarrow F_3(q^2) = 0$

Also, F_4 is redundant.

$$\therefore \quad \bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[\frac{(p'+p)^\mu}{2m} + i \frac{\sigma^{\mu\nu} q_\nu}{2m} \right] u(p)$$

(Gordon identity)

$\vec{\sigma} \cdot (\vec{q} \times \vec{A}) = \vec{\sigma} \cdot \vec{B}$

This is a coupling that a spinless particle also has: $\phi^*(p') \overleftrightarrow{\partial^\mu} \phi(p)$

$\rightsquigarrow \phi^*(p') (p'+p)^\mu \phi(p)$.

Also, since $(p'+p)^\mu \to$ redundant.

The magnetic moment of the e^- is the term in the coupling

$$g \frac{e}{2m} \vec{S} \cdot \vec{B} \quad , \quad \vec{S} \equiv \vec{\sigma}/2 .$$

g-factor . $g = 2$ in the Dirac equ^{n}.

Now, $\gamma^\mu \longrightarrow \gamma^\mu + \Gamma^\mu_{(2)} \implies$ correction to $g = 2$.

So,

$$J^\mu = \bar{u}(p') \left[\gamma^\mu F_1(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right] u(p)$$

$$= \bar{u}(p') \left[\frac{(p'+p)^\mu}{2m} F_1(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2m} \left(F_1(q^2) + F_2(q^2) \right) \right] u(p) .$$

Now, $j^\mu = e_0 \left[\bar{u}(p) \left(\gamma^\mu + \Gamma^\mu_{(2)}(p,q) \right) u(p) \right]$

$$= e_0 \bar{u}(p') \gamma^\mu \left(1 + \frac{e_0^2}{8\pi^2 \epsilon} \right) u(p) + e_0 \bar{u}(p) \Gamma^\mu_{(2)\,fin} u(p)$$

$$\longrightarrow e_0 \bar{u}(p') \left(\gamma^\mu + \Gamma^\mu_{(2)\,fin} \right) u(p) .$$

For the magnetic dipole moment we need the leading contribution to $F_1(q^2) + F_2(q^2)$ as $q^2 \to 0$.

$$\underset{\text{electric charge}}{\underbrace{f_1(0)}} = 0$$

For the magnetic moment we've $\bar{u}(p') i \frac{\sigma^{\mu\nu} q_\nu}{2m} u(p) \left[F_1(0) + F_2(0) \right]$

So, $g = 2 \left[F_1(0) + F_2(0) \right]$

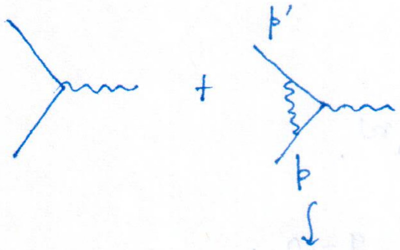
$= 2 \left[1 + F_2(0) \right]$

$F_1 \to$ electric form factor

$F_2 \to$ magnetic " "

⑮ 17.09.2014

RECAP :



$$\Gamma^{\mu}_{(2)}(p,p') = \underbrace{\frac{e^2}{8\pi^2\epsilon}\gamma^{\mu} + (\text{finite})\gamma^{\mu}}_{\int d^4k \frac{k^2}{(k^2+\Delta)^3}} + \underbrace{\text{other finite}}_{\int d^4k \frac{(\dots)}{(k^2+\Delta)^3}}$$

$$\bar{u}(p')\,\Gamma^{\mu}_{fin}\,u(p)$$
$$= \bar{u}(p')\left[\gamma^{\mu}F_1(q^2) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}F_2(q^2)\right]u(p)$$
$$\xlongequal[\text{Id.}]{\text{Gordon}} \bar{u}(p')\left[\frac{(p'+p)^{\mu}}{2m}F_1(q^2) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\left(F_1(q^2)+F_2(q^2)\right)\right]u(p)$$

The magnetic dipole moment term ~ $g\,\frac{\vec{\sigma}\cdot\vec{B}}{2m}$ gets its contribution from the 2nd term in the limit $q^2 \to 0$.

$$g = 2\left[F_1(0) + F_2(0)\right] = 2\left(1 + F_2(0)\right)$$

$$\Rightarrow \quad g-2 = 2F_2(0)$$

~ correction to the Dirac result.

We need to read off the $F_2(0)$ contribution from the "other finite" piece: of $\Gamma^{\mu}_{(2)}$.

The piece of $\Gamma^{\mu}_{(2)}$ which contributes to $F_2(0)$ is

$$2(-ie)^2 i^2 \,\frac{\bar{u}(p')\gamma^{\nu}\left[(1-y)\not{q} + (1-x-y)\not{p} + m\right]\gamma^{\mu}\left[y\not{q} - (1-x-y)\not{p} - m\right]\gamma_{\nu}u(p)}{(k_E^2-\Delta)^3}$$

where, $\Delta = (1-y)yq^2 + (x+y)(1-x-y)p^2 + 2p\cdot q\,y(1-x-y) - (x+y)^2m^2$

Numerator simplifies : $\gamma^{\nu}\not{a}\gamma^{\mu}\not{b}\gamma_{\nu} = -2a_{\alpha}b_{\beta}\left(\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta} - \gamma^{\alpha}\gamma^{\beta}\gamma^{\mu} + \gamma^{\mu}\gamma^{\beta}\gamma^{\alpha}\right)$.

Also, $\gamma^{\nu}\not{a}\gamma^{\mu}\gamma_{\nu} = \gamma^{\nu}\gamma^{\mu}\not{a}\gamma_{\nu} = 4a^{\mu}$.

Then use Dirac eqn for onshell spinors: $\not{p}u(p) = mu(p)$
$\bar{u}(p')\not{p}' = m\bar{u}(p')$

Numerator : $\bar{u}(p')\left[P\gamma^\mu + Q(p'+p)^\mu + Rq^\mu\right]u(p)$

with $$P = 2(1-x)(1-y)q^2 - 2\left[(x+y)^2 - 4(x+y) + 2\right]m^2$$
$$Q = 2m(x+y)(x+y-1)$$
$$R = 2m(y-x)(x+y+1)$$

The term $\propto q^\mu$ vanishes when one does the integral over (x,y) (anti-symm in $x \leftrightarrow y$).

Since we are only interested only in $F_2(0)$, we need to only consider the middle term in the numerator which will give a piece $\propto \frac{\sigma^{\mu\nu}q_\nu}{2m}$ using Gordan identity.

Therefore, the contribution proportional to $F_2(q^2)$ is the term (after the convergent k- integral)

$$-e^2/8\pi^2 \int_0^1 dx \int_0^{1-x} dy\; \bar{u}(p')\, \frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[\frac{1}{\Delta}\, 2m^2(1-x-y)(x+y)\right]u(p)$$

Using $(p+q)^2 = \underset{m^2}{\cancel{p^2}} + q^2 + 2p\cdot q = p'^2 = m^2$

$$\Delta = xyq^2 - (x+y)^2m^2$$

As we take $q^2 \to 0$,

$$F_2(0) = -e^2/8\pi^2 \int_0^1 dx \int_0^{1-x} dy\; \frac{1}{\underset{\hookrightarrow\, -(x+y)^2m^2}{\Delta}}\, 2m^2(1-x-y)(x+y)$$

$$= \frac{\alpha}{2\pi}\int_0^1 dx \int_0^{1-x} dy\; \frac{2(1-x-y)}{(x+y)}$$

$$= \frac{\alpha}{2\pi}$$

$$\approx 0.0011614$$

So, $\boxed{\dfrac{g-2}{2} = F_2(0) = \dfrac{\alpha}{2\pi}}$ (Schwinger)

The "other finite" pieces are actually not all finite. There is a divergence in the term ('q') for the on-shell matrix elements $\bar{u}(p')\,\Gamma^{\mu}_{fin}(p',p)\,u(p)$ [$u(p)$, $\bar{u}(p')$ satisfy Dirac eqn.].

This term is

$$\frac{e^2}{8\pi^2}\ \bar{u}(p')\gamma^{\mu}u(p)\int_0^1 dx\int_0^{1-x} dy\ \frac{m^2\left[(x+y)^2+2(x+y)-2\right]+q^2(1-x)(1-y)}{\underbrace{m^2(x+y)^2-xyq^2}_{\Delta}}$$

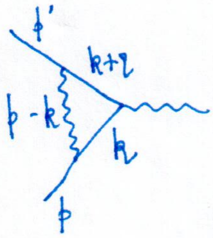
This integral (x,y) is divergent at small (x,y) !

For small y the integrand $\sim \dfrac{1}{x\left[m^2(x+2y)-q^2y\right]}$

$$\xrightarrow[\text{over } y]{\text{int.}}\ \frac{\text{finite}}{x}.$$

When we do the x-integral $\displaystyle\int_0^1 \frac{dx}{x} \rightarrow$ divergent! at lower limit.

Physically we want to understand the origin of this divergence. This can be seen by going back to the original integral.



$$\sim \int d^4k\ \frac{(\text{Num})}{(p-k)^2\,(k^2-m^2)\left[(k+q)^2-m^2\right]}$$

Define $p-k \equiv q'$

$$\sim \int d^4q'\ \frac{(\text{Num})}{q'^2\left((p+q')^2-m^2\right)\left[(p-q')^2-m^2\right]}$$

When $p^2=m^2$, $p'^2=m^2$

$$\sim \int d^4q'\ \frac{1}{q'^2\,(2p\cdot q'+q'^2)\left[-2p'\cdot q'+q'^2\right]}$$

As $q'\to 0$, Numerator is finite. But den $\sim q'^2\,(2p\cdot q')(2p'\cdot q')$

Then the integral $\displaystyle\int d^4q'\ \frac{1}{q'^2\,(2p\cdot q')\,(2p'\cdot q')}$ is log div. as $q'\to 0$

$\rightsquigarrow$ infrared divergence $\rightsquigarrow$ loop mom. becomes small.

This divergence arises only for on-shell. Because if $p^2 \neq m^2$, $p'^2 \neq m^2$, then as $q'\to 0$ the integral

$$\sim \int \frac{d^4q'}{q'^2\,(p^2-m^2)\,(p'^2-m^2)}$$ is convergent as $q'\to 0$

$\boxed{**}$ This is unlike UV divergences which occur ⓐ for large loop momenta (UV/short distance). ⓑ for any value of external momenta.

IR div. are absent for generic ext. momenta. Occur only for on-shell ext. momenta.

The origin of this divergence is from the long distance behavior of EM interaction: prop $\sim \frac{1}{q^2} \longleftrightarrow \frac{1}{r}$ potential.

Would have been absent if $\frac{1}{q^2} \to \frac{1}{q^2-\mu^2}$, $\mu \sim$ small photon mass.

The small q-behaviour $\sim \int \frac{d^4q'}{\mu^2\,(2p\cdot q')\,(2p'\cdot q')} < \infty$.

RECAP:

The onshell matrix element

$\bar{u}(p')\,\Gamma^\mu(p,p')\,u(p)$ has IR divergences (loop momenta of virtual photon $q'\to 0$) $\sim \int \frac{d^4q'}{q'^2\,(p\cdot q')\,(p'\cdot q')} \to \infty$.

Regulate the IR divergence by replacing the photon propagator in the loop by

$$\frac{1}{q'^2} \to \frac{1}{q'^2-\mu^2}$$

Alternatively, again dimensionally regularize $D = 4 + \varepsilon_{IR}$ $(\varepsilon_{IR} > 0)$ as opposed to $d = 4 - \varepsilon_{UV}$)

[Recall, this divergent part comes from the UV finite parts].

$$\int_0 \frac{d^Dq'}{q'^2\,(p\cdot q')\,(p'\cdot q')}$$ would be finite.

⑯

19.09.2014

The IR-divergent piece of $\bar{u}(p')\Gamma^\mu(p,p')u(p)$ is

$$\propto \frac{\alpha}{2\pi}\gamma^\mu \int_0^1 dx \int_0^{1-x} dy \; \frac{-m^2\left[(x+y)^2-2(x+y)+2\right]+q^2(1-x)(1-y)}{\underbrace{m^2(x+y)^2-q^2xy+\mu^2(1-x-y)}_{\text{Modified }\Delta}}$$

The divergence was coming from $(x,y)\to 0$. In this region, the integrand

$$\sim \frac{\alpha\gamma^\mu}{2\pi}\int_0 dx \int_0 dy \; \frac{(-2m^2+q^2)}{m^2(x+y)^2-q^2xy+\mu^2}$$

To focus on the small (x,y) region, change variables

$$x=(1-\xi)\,\omega \qquad \left(x+y=\omega,\quad x/y=\frac{1-\xi}{\xi}\right)$$
$$y=\xi\,\omega$$

$$\text{Integral} \sim \frac{\alpha}{2\pi}\gamma^\mu \int_0 d\xi \int_0 \omega\, d\omega \; \frac{(-2m^2+q^2)}{\left[m^2-q^2\xi(1-\xi)\right]\omega^2+\mu^2}$$

$$\sim \frac{\alpha}{2\pi}\gamma^\mu (q^2-2m^2)\int_0^1 d\xi \;\ln\left[\frac{m^2-q^2\xi(1-\xi)}{\mu^2}\right]$$

It is log divergence in terms of the cut-off μ. In the limit $q^2 \gg$ ~~μ~~ m^2, it is easy to evaluate the ξ integral

$$\cancel{\ln\left[\frac{-q^2\xi(1-\xi)}{\mu^2}\left(1-\frac{m^2}{q^2\xi(1-\xi)}\right)\right]}$$

$$\overset{\text{ibp}}{\sim} -\frac{\alpha}{4\pi}\int_0^1 \frac{d\xi\,(-q^2)}{(m^2-q^2\xi(1-\xi))}\;\ln(-q^2/\mu^2)$$

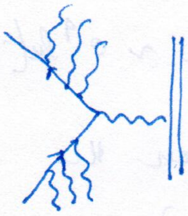
$$\approx -\frac{\alpha}{2\pi}\underbrace{\ln(-q^2/\mu^2)\,\ln(-q^2/m^2)}_{\text{double log form}}.$$

25.09.2014

are corrections to the scattering of a charged particle by a source.

$\swarrow$ Electric FF

There is an IR divergence in $F_1(q^2)$ (term $\propto \gamma^\mu$ in $\bar{u}(p')\Gamma^\mu(p,p')u(p)$)



There is a classical $\overset{IR}{\wedge}$ divergence arising from the emission of "soft" photons in the scattering of an e^- off a source.

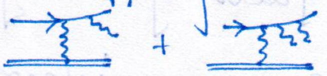
The "tree" level amplitude for (e^- + source)

$\longrightarrow$ (e^- + source) is also accompanied by processes (e^- + source)

$\longrightarrow$ (e^- + source) + $n\gamma$

$\hookrightarrow$ soft-photons

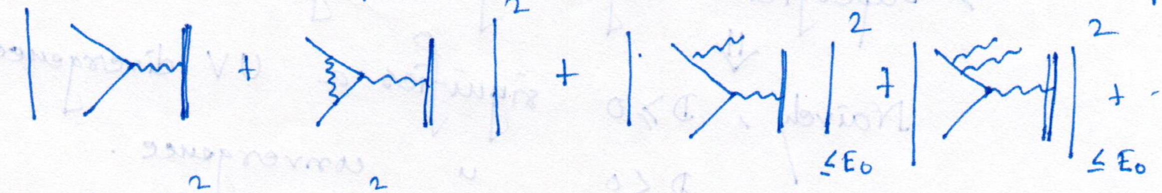
"not measurable"

Every measurement detector for this process will have a cut off $E_0 \neq 0$ which will be the min energy of γ it can detect. The total amplitude (sum of all  + ...) $< E_0$ is also $\overset{\text{divergent}}{\wedge}$ amplitude. But if we put a IR cut off μ on the photon energy, we can regularize this. The regularize amplitude

$\sim \frac{\alpha}{2\pi} \ln(E_0^2/\mu^2)$

$$\Big(\underset{\substack{\Downarrow \\ \text{diverge}}}{0 \leq E(\gamma) \leq E_0} \xrightarrow[\text{off}]{\text{cut}} \underset{\substack{\Downarrow \\ \text{finite}}}{\mu \leq E(\gamma) \leq E_0} \Big).$$

QM cally we've to compute the total $|\text{amplitude}|^2$ for scattering

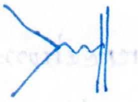


$O(e)$

$\xrightarrow{} \quad |e(1+e^2)|^2 + |e^2|^2$

$\longrightarrow$ 

(17) 23·09·2014

Therefore, the cross term  ·  is IR diverg.

$\propto e^4 \rightarrow e^4 \ln(-q^2/\mu^2)$.

The classical IR divergent contri. $\Big|$  $\Big|^2$

$\sim e^4 \ln\left(E_0^2/\mu^2\right)$ w/ reight of s.t. the sum $\sim e^4 \ln(-q^2/E_0^2)$.

The sum is measurable and has no dependence on the arbitrary cut-off μ, (but of course depends on E_0).

⊠ ULTRAVIOLET DIVERGENCES AND FEYNMAN DIAGRAMS :→

Naive estimate of the divergence of general Feynman diagram in QED :

A general diagram has (N_e, N_γ) ext lines of (e^-, γ) and (P_e, P_γ) internal lines, V vertices (), L-loops (# of indep. closed faces of the graph = # ind loop momenta in the diagram).

The Feynman integral

$$\sim \prod_i \int d^4 k_i \, (S(k))^{P_e} \, (D(k))^{P_\gamma}$$

will have a naive UV divergence if the powers of mom in the numerator $\geqslant$ that of denom.

Define $D \equiv$ (# of powers of k in num) − (# that of denom)

$= 4L - P_e - 2P_\gamma$

↳ Superficial degree of divergence

⇓

Naively, $D \geqslant 0$ signifies a UV divergence.

$D < 0$ " convergence.

Simplify D ⇒ (i) for a general graph (possibly a general rel^n by Euler)

$$\boxed{L = P - (V-1)}$$

$P \to P_e + P_\gamma$

The # of diff internal momenta in the graph $= P$

Constraints on internal momenta $= V-1$

$\swarrow$ is a constraint on the external mom → mom conser (overall).

(ii) At each vertex, there are $2e^-$ lines and $1\,\gamma$ line.

⇒ Count V- photon lines. But there is over counting.

$$V = 2P_\gamma + \cancel{N_e}\, N_\gamma$$

for e^-

$$2V = 2P_e + N_e$$

$$\Rightarrow \quad P_\gamma = \frac{V-N_\gamma}{2}, \qquad P_e = \frac{2V-N_e}{2}$$

Now, $L = P_e + P_\gamma - V + 1$

$$D = 4L - \cancel{2}P_e - 2P_\gamma$$

$$= 4(P_e + P_\gamma - V + 1) - P_e - 2P_\gamma$$

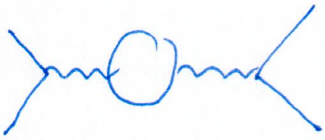
$$= 3P_e + 2P_\gamma - 4V + 4$$

$$= 3\left(V - \frac{N_e}{2}\right) + \cancel{2}\,\frac{V-N_\gamma}{\cancel{2}} - 4V + 4$$

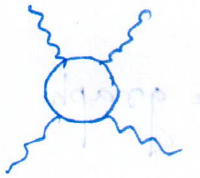
$$\Rightarrow \boxed{D = 4 - \frac{3N_e}{2} - N_\gamma}$$

→ only depends on the external legs of the graph (indep. of internal details).

for  $D = 2$, but actually $D = 0$ i.e. log diver.



$D = 4 - 6 = -2$, but actually divergent.



$D = 0$, but actually convergent.

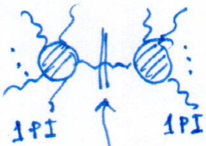
WEINBERG'S THEOREM :-

~~A graph w/ $D \geq 0$ will diverge if $D \geq 0$ for the graph, and all its sub.~~

A graph w/ $D < 0$ and for which all its subgraphs also have $D < 0$ will converge.

If we want to isolate the potential divergences, we only have to look at the ~~subgraphs~~ subclass of primitively divergent graphs. for which $D \geq 0$.

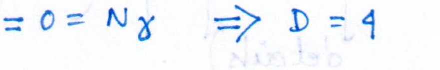
We only need to consider 1PI graphs w/ $D \geq 0$ ($\because$ A general graph which has $D \geq 0$ must have its divergence coming from one or more of its 1PI parts w/ $D \geq 0$).



this prop should be expressed in term of only external momenta.

Coming back to QED, we can see that there are potentially at most 7 - classes of primitively divergent graphs $\Rightarrow D \geq 0$:

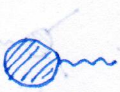
ⓐ $N_e = 0 = N_\gamma \Rightarrow D = 4$



~ vacuum diagrams.

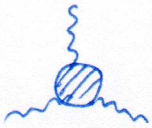
~ these contribute to vacuum energy. But these are set to zero by infinite shift (except gravity → couples to energy directly, not the difference of energy.)

ⓑ $N_\gamma = 1$, $N_e = 0 \Rightarrow D = 3$



$= \langle \Omega | A_\mu(x) | \Omega \rangle = 0$ by Lorentz inv. (vacuum should have directionality). Also, by using charge conjug. one can prove this.

ⓒ $N_\gamma = 3$, $N_e = 0 \Rightarrow D = 1$



$= \langle \Omega | A_\mu A_\nu A_\rho | \Omega \rangle = 0$

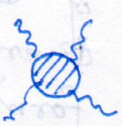
(Furry's thm).

ⓓ $N_\gamma = 2$, $N_e = 0 \Rightarrow D = 2$

Actually log-divergent. (due to gauge inv.)

$\sim \hookrightarrow (q^2 g^{\mu\nu} - q^\mu q^\nu)\, \Pi_2(q^2) \longrightarrow$ log-diver.

ⓔ $N_\gamma = 4$, $N_e = 0 \Rightarrow D = 0$

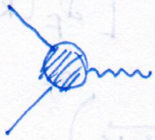


Actually convergent (due to gauge inv.).

ⓕ $N_e = 2$, $N_\gamma = 0 \Rightarrow D = 1$

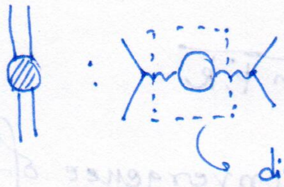
Actually log-divergent (due to chiral symm.)

ⓖ $N_e = 2$, $N_\gamma = 1 \Rightarrow D = 0$



Indeed log-divergent.

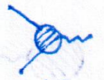
y.



: $\Rightarrow$ div.

$\hookrightarrow$ div. (part of primitive).

To look at div. it's sufficient to look at only primitive div. diagrams.

General lesson: There are no new divergences in higher point functions than the ones in the 3-classes of primitively divergent subgraphs ( ,  , ). If we understand these divergences we are essentially done. No new renormalizations need to be done.

Recall,

 $\Rightarrow$ 1-div. ~ A_μ

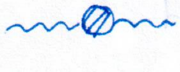
 $\Rightarrow$ 2-div ~ $\not{k}$, m_e.

 $\Rightarrow$ 1-div ~ e.

RECAP:

For QED: $D = 4 - \frac{3}{2}N_e - N_\gamma$

$D \geq 0$ for only 3-non-trivial classes of 1PI graphs. Each such class contains an infinite # of potentially divergent diagrams (from each loop).



$D = 2$

$D = 1$

$D = 0$

We have not yet characterized the nature of divergence for each of these classes. We'll argue that the divergent terms are analytic in ext. momenta and of the form we saw at one-loop. Consider the e^- self-energy $\Sigma(p)$

Take derivatives of $\Sigma(p)$ wrt p (external mom.)

On the one hand when we take this derivatives, they will act on the propagators inside the integral

$$\frac{\partial}{\partial p^\mu}\Sigma(p) = \frac{\partial}{\partial p^\mu}\int\prod_i \frac{d^4k_i}{(2\pi)^4}\left[S_F(k_i,p)\right]^{P_e}\left[D_F(k_i,p)\right]^{P_\gamma}$$

For S_F : $\frac{\partial}{\partial p^\mu}\left(\frac{1}{\text{linear in } \not{p}, \not{k}}\right) \sim \left(\frac{1}{\text{linear in } \not{p}, \not{k}}\right)^2$

For D_F : $\frac{\partial}{\partial p^\mu}\left(\frac{1}{\text{quadratic in } p, k}\right) \sim \frac{1}{\text{cubic in } p, k}$

Taking derivatives wrt p improves the convergence of the integral by increasing the powers of k in the denominator. Each derivative wrt p reduces D by one.

Taking sufficiently many derivatives makes $D<0$. For $\Sigma(p)$, if we take two derivatives $\Rightarrow D<0$.

$\Sigma(p)$ in a Taylor expansion (about say $p=0$) is determined by its derivatives (at $p=0$)

$$\Sigma(p) = \underbrace{\Sigma(0)}_{D=1} + \underbrace{\frac{\partial\Sigma(p)}{\partial \not p}\bigg|_{\not p=0}}_{D=0} \not p + \underbrace{\frac{\partial^2\Sigma(p)}{\partial \not p^2}\bigg|_{\not p=0} \not p^2}_{D<0} + \ldots$$

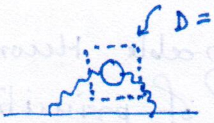
~~(for other subgroups)~~

$\downarrow$ Constant piece (indep. of p) which is divergent

$\frac{1}{\not p - m_0 - \Sigma(p)}$ $\rightarrow$ Mass renormalization.

linear in p which is also divergent

$\rightarrow \psi$ - renormalization.



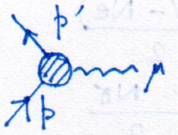
can be made convergent iteratively / recursively by renormalizing the sub-graphs (sub-divergences).

Thus $\Sigma(p) = A_0 + A_1 \not p + A_2 \not p^2 + \ldots$

w/ A_0, A_1 divergent [and $A_3, A_4, \ldots$ finite provided one was recursively renormalized all divergences at lower loop order]. These A_0, A_1 are precisely the terms which can be absorbed in ~~terms of~~ mass & ψ-field renorm. [Position + residue of pole].

δm $\qquad$ Z_2

Similarly,



for $\Gamma^\mu(p, p')$ we can diff wrt p or $\not p$ and D becomes $D<0$.

$D=0$

$$\Gamma^\mu_{\alpha\beta}(p,p') = \Gamma^\mu_{\alpha\beta}(0,0) + \underbrace{\frac{\partial \Gamma^\mu_{\alpha\beta}(p,p')}{\partial p^\mu}\bigg|_{p=0} p^\mu + \ldots}_{D<0,\ \text{lead to finite answer}}$$

$D=0$, div. piece

$\propto \gamma^\mu$

$\Downarrow$

can be absorbed into coupling cons renormalization (Z_1).

For , $\Pi^{\mu\nu}(q^2) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi_2(q^2)$

$$\Pi_2(q^2) = \Pi_2(0) + \frac{\partial \Pi_2(q)}{\partial q^\mu}\Big|_{q=0} q^\mu + \cdots$$

$\Pi_2^{\mu\nu}(q^2)$ starts off quadratically ~~#~~ w/ q. The cons $(D=2)$ and 1st derivatives $(D=1)$ terms in q are absent. Only 2nd deriv. $(D=0)$ and higher are non-zero $\Rightarrow \Pi_2(q^2)$ has $D=0$!

Then $\Pi_2(0) \to D=0$

$$\frac{\partial \Pi_2(q)}{\partial q^\mu}\Big|_{q=0} \sim D=-1 < 0 .$$

So, only $\Pi_2(0)$ is divergent. Again indep. of q. Can be absorbed into A_μ renorm (residue of the photon prop at the pole $q^2=0$).

* QED in 4D is an example of a renormalizable theory.
— there are only a finite # of classes of primitively divergent graphs (in this case 3).
→ a finite # of types of divergences (could be absorbed into the redefinition of the couplings/parameters and fields of the theory).

Consider QED in d-dim

$$D = Ld - P_e - 2P_\gamma$$

Now, $L = (P_e + P_\gamma) - (V-1)$

$$\left.\begin{array}{l} 2V = N_e + 2P_e \\ V = N_\gamma + 2P_\gamma \end{array}\right] \Rightarrow \begin{array}{l} P_e = \frac{(2V-N_e)}{2} \\ P_\gamma = \frac{V-N_\gamma}{2} \end{array}$$

So, $D = (P_e+P_\gamma)d - Vd + d - \frac{1}{2}(2V-N_e) - (V-N_\gamma)$

$$= \left(\frac{2V-N_e}{2} + \frac{V-N_\gamma}{2}\right)d - V(d+2) + d + \frac{N_e}{2} + N_\gamma$$

$$= d - \frac{d-1}{2}N_e - \frac{d-2}{2}N_\gamma + \frac{d-4}{2}V$$

26.09.2014

Three cases :

ⓐ $2 \leq d < 4 \Rightarrow D \geq 0$ only for a finite # of (N_e, N_γ, V).
So, only a finite num of diagrams (:involvement of v i.e. vertices) will be primitively div.
Divergent only up to a max num of V i.e. loops. (in contrast to earlier - finite num of classes)
Such theories are called super-renormalizable.

ⓑ $d = 4 \Rightarrow$ Only a finite classes of primitive div. graphs (finite num of (N_e, N_γ)).
— Renormalizable.

ⓒ $d > 4 \Rightarrow$ for any (N_e, N_γ) there will always be diagrams (an infinite # of them) w/ $D \geq 0$. So, all Green's fun. are primitively div. D can be made arbitrarily large by increasing V. No finite set of couplings which can absorb all the divergences.
— Non-renormalizable.

* RECAP:

ⓐ Super-renormalizable (QED in $d < 4$) $\Rightarrow$ finite # of diagrams which are primitively divergent.

ⓑ Renormalizable (QED in $d = 4$) $\Rightarrow$ finite # of classes of diagrams which are prim. divergent.

ⓒ Non-renormalizable (QED in $d > 4$) $\Rightarrow$ infinite # of classes of diagrams which are prim. divergent.

* SCALAR FIELD THEORY IN 4-DIM :-

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} M^2 \phi^2 - V(\phi) \quad , \quad V(\phi) = \sum_{n=3}^{m} \frac{\lambda_n}{n!} \phi^n$$

Denote the # of vertices of order $n = V_n \; (3 \leq n \leq m)$.

# of propagators $= P$ ('internal lines').

# of external lines $= N$

# of loops $= L$

$D = 4L - 2P$

Euler rel^n : $L = P - (V-1) = P - \sum_n V_n + 1$

Also, $\underbrace{\sum_{n=3}^{m} n V_n}_{\text{\# of lines coming out of vertices}} = 2P + N \Rightarrow P = \frac{1}{2}\left(\sum n V_n - N\right)$

So, $D = 4L - 2P$

$= 4\left(P - \sum V_n + 1\right) - 2P$

$= 2P - 4\sum V_n + 4$

$= \sum n V_n - N - 4\sum V_n \cancel{+2N} + 4$

$$\boxed{D = 4 - \sum_{n=3}^{m} (4-n) V_n - N}$$

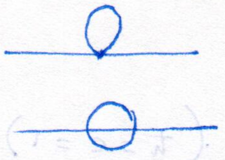
For $m \leq 4$ i.e. $n = 3, 4$

$D = 4 - V_3 - N \Rightarrow$ There are only finite # of classes of primi. div. diagrams.

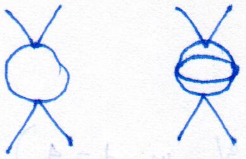
$\Rightarrow$ Renormalizable theory.

$D \geq 0 \Rightarrow 4 \geq N + V_3$

If $V_3 = 0$ then $N \leq 4$ i.e. 2, 3, 4 pt. functions are pot. divergent (1-pt fun we put zero).

N = 2  . . .   D = 2

N = $\cancel{3}$ ∵ $V_3 = 0$ → only quartic vertices

N = 4  . . . $D = 0$,

In the theory $V_3 = 0$ ($\lambda_3 = 0$), only N = 2, 4 are primi. divergent

$\boxed{N = 2}$.

$$\underset{D=2}{\Sigma(p)} = \underset{D=2}{A_0} + \underset{D=0}{A_1(p^2)} + \underbrace{\overset{D=-2}{A_2(p_2)^2} + \dots}_{D<0}$$

A_0 ↓ Quad. div. ↓ mass renor.

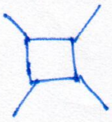
A_1 ↓ log-div. ↓ ϕ-renor.

$$\frac{1}{p^2 - m^2 - \Sigma(p)}$$

$\boxed{N = 4}$ $D = 0$ → log-divergent ('indep. of ext. momenta)

→ Renormalization of λ_4.

If we had only ϕ^3 coupling, then actually one will generate ϕ^4 coupling as well.

 ~ Lead to quartic coupling. (finite).

— super renormalizable.

Any potential w/ $n > 4$ in 4Dimensions is non-renormalizable.

__Ex__: Carry out the analysis for general d-dim & classify theories in $2 \leq d \leq 6$.

⧆ DIMENSIONS (CANONICAL) OF FIELDS :

The action of any QFT is dimensionless ($\hbar = c = 1$).

$\Rightarrow$ $\mathcal{L}$ has dim (mass) $\doteq d = \#$ ST dim.

In QED, $\bar{\psi} \underbrace{(i\not{\partial} - m)}_{1} \psi \subset \mathcal{L}_{QED}$

$\Rightarrow$ So, $\boxed{d[\psi] = \frac{d-1}{2}} \equiv [\psi]$ $\left(= \frac{3}{2} \text{ in } d = 4\right)$.

Also, $\int F_{\mu\nu}^2 \sim (\partial A)^2$

$\Rightarrow$ $\boxed{d[A] = \frac{d-2}{2}} \equiv [A]$ $(= 1 \text{ in } d = 4)$

Also, $e \bar{\psi} \gamma^\mu \psi A_\mu \subset \mathcal{L}$

$$[e] + (d-1) + \left(\frac{d-2}{2}\right) = d$$

$$\Rightarrow \boxed{[e] = \frac{4-d}{2}} \ : \begin{cases} > 0 & \text{for } d < 4 \\ = 0 & \text{for } d = 4 \\ < 0 & \text{for } d > 4 \end{cases}$$

So; $[e] < 0$ $\longleftrightarrow$ Non-renor.

$[e] = 0$ $\longleftrightarrow$ Reno.

$[e] > 0$ $\longleftrightarrow$ Super renor.

In scalar theories :

$\boxed{[\phi] = \frac{d-2}{2}}$ $(= 1 \text{ in } d = 4)$.

Now. $\lambda_n \phi^n \subset \mathcal{L}$

So, $[\lambda_n] + n \frac{d-2}{2} = d$

$$\Rightarrow \boxed{[\lambda_n] = d\left(1 - n/2\right) + n}$$

In $d = 4$, $[\lambda_3], [\lambda_4] \geq 0$

General lesson :

Renormalizable (+ super-reno.) interactions are associated w/ couplings w/ mass dim $= 0$ (> 0).

Non-renormalizable interactions $\leftrightarrow$ couplings w/ mass dim < 0

Recall,

$$D_{QED} = d - \underbrace{\frac{4-d}{2}}_{[e]} V - \underbrace{\frac{d-1}{2}}_{[\psi]} Ne - \underbrace{\frac{d-2}{2}}_{[A]} N\gamma$$

* METHOD OF COUNTER TERMS :

Reorganise perturbation theory s.t. we only need to talk in terms of (e, m), (ψ, A_μ) ~ physical parameters & physical fields , not in terms of (e_0, m_0), $(\psi_0, A_{\mu,0})$ ~ bare quantities.

$$\mathcal{L} = -\frac{1}{4}(F^0_{\mu\nu})^2 + \bar\psi_0 (i\partial\!\!\!/ - m_0)\psi_0 - e_0(\bar\psi_0 \gamma^\mu \psi_0) A^0_\mu$$

Now, $\begin{cases} A_\mu = Z_3^{-1/2} A^0_\mu \\ \psi = Z_2^{-1/2} \psi_0 \end{cases}$

$$\mathcal{L} = -\frac{1}{4} Z_3 (F_{\mu\nu})^2 + Z_2 \bar\psi (i\partial\!\!\!/ - m_0)\psi - e_0 Z_2 Z_3^{1/2} (\bar\psi \gamma^\mu \psi) A_\mu$$

$$= \left[-\frac{1}{4} F_{\mu\nu}^2 + \bar\psi (i\partial\!\!\!/ - m)\psi - e(\bar\psi \gamma^\mu \psi) A_\mu \right]$$

$$+ \left[-\frac{1}{4} \delta_3 F_{\mu\nu}^2 + \bar\psi (i\delta_2 \partial\!\!\!/ - \delta m)\psi - e\delta_1 (\bar\psi \gamma^\mu \psi) A_\mu \right]$$

where, $\delta_3 = Z_3 - 1$, $\delta_2 = Z_2 - 1$, $\delta m = Z_2 m_0 - m$

$\delta_1 \equiv (Z_1 - 1)$, $e \dfrac{Z_1}{Z_2 Z_3^{1/2}} = e_0$.

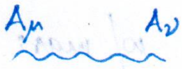
Do perturbation theory about the non-inter. Lag

$$-\frac{1}{4} F_{\mu\nu}^2 + \bar\psi (i\partial\!\!\!/ - m)\psi .$$

Compute Green's fun of (A_μ, ψ) ~ physical fields.

Treat all the other terms as perturbation.

Modified Feynman rules:



$$\frac{-i g_{\mu\nu}}{q^2+i\varepsilon}$$

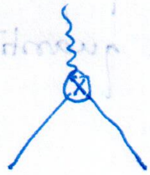
$$\frac{i}{\not p - m}$$

$$-ie\gamma^\mu$$

|| δ's are $O(e^2)+\dots$

$$-i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\delta_3$$

$$i(\not p\,\delta_2 - \delta m)$$



$$-ie\gamma^\mu\delta_1$$

We adjust the counterterms couplings s.t. they cancel divergences order by order in pert. theory.

① 2-pt. fun $\langle\bar\psi\psi\rangle$

$\Sigma_2(p) =$  $+$  $=$ finite.

$$-\frac{e^2}{8\pi^2\varepsilon}(\not p - 4m) + (\not p\,\delta_2 - \delta m)$$

+ finite.

DEMAND FINITE

Choose $\Rightarrow$ $\delta_2 = \frac{e^2}{8\pi^2\varepsilon}$, $\delta m = +\frac{4me^2}{8\pi^2\varepsilon}$

So, $\langle\bar\psi\psi\rangle = \frac{i}{\not p - m - \Sigma_{fin}(p)}$

② 2-pt. fun of A_μ. $= \dfrac{-ig_{\mu\nu}}{q^2(1-\Pi_2^{\text{fin}}(q^2))}$

$\Pi_2 =$  $+$ 

$\downarrow$

$-e^2/6\pi^2\varepsilon \quad + \quad \cancel{\delta m}\ \delta_3$

$+ \text{fin}$

$\Rightarrow \delta_3 = e^2/6\pi^2\varepsilon$

③ 3-pt. fun $\langle \bar{\psi}\psi A \rangle$

$-ie\Gamma^\mu_{(2)} = -e\left(e^2/8\pi^2\varepsilon\right)\gamma^\mu + \delta_1(-ie\gamma^\mu)$

$+ \text{fin}.$

$\Rightarrow \delta_1 = e^2/8\pi^2\varepsilon.$

16.10.14

* $\langle \Omega_H | T\{ \hat{\phi}_H(x_1) \dots \hat{\phi}_H(x_n) \} | \Omega_H \rangle \equiv G(x_1, \dots, x_n)$

$$= \frac{\int [\mathcal{D}\phi]\, \phi(x_1) \cdots \phi(x_n)\, e^{i\int \mathcal{L}[\phi]\, d^4x}}{\int [\mathcal{D}\phi]\, e^{i\int \mathcal{L}[\phi]\, d^4x}}$$

LHS ~ operators $\hat{\phi}$

RHS ~ integration variable (dummy) ϕ

* $Z[J] = \int [\mathcal{D}\phi]\, e^{i\int d^4x\, [\mathcal{L} + J(x)\phi(x)]}$

$$\frac{Z[J]}{Z[0]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \prod_{i=1}^{n} d^4x_i\, G(x_1, \dots, x_n)\, J(x_1) \cdots J(x_n)$$

$$\frac{Z_0[J]}{Z_0[0]} = \exp\left[-\frac{1}{2} \int d^4x\, d^4y\, J(x)\, D_F(x-y)\, J(y) \right]$$

free scalar theory

In the free theory

$$\frac{1}{Z_0[0]} \left(-i \frac{\delta}{\delta J(x_1)} \right) \left(-i \frac{\delta}{\delta J(x_2)} \right) Z_0[J] \Big|_{J=0} = \langle 0 | T\{ \hat{\phi}_H(x_1) \hat{\phi}_H(x_2) \} | 0 \rangle$$

$$= (-i) \left(-i \frac{\delta}{\delta J(x_1)} \right) \Big[-\frac{1}{2} \int d^4x\, d^4y\, \delta^4(x - x_2)\, D_F(x-y)\, J(y)$$
$$- \frac{1}{2} \int d^4x\, d^4y\, \delta^4(y - x_2)\, D_F(x-y)\, J(x) \Big] \exp\left[-\frac{1}{2} \int d^4x\, d^4y\, J D J \right] \Big|_{J=0}$$

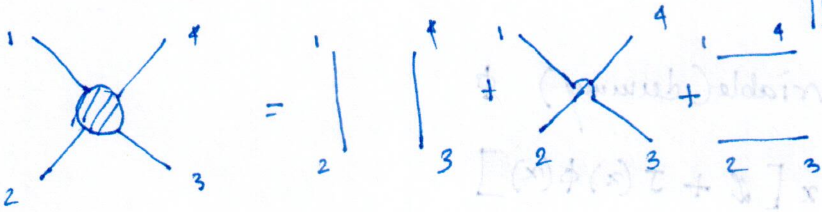
$\downarrow$ Only non-zero at $J=0$ (after performing differentiation)

$$= \frac{1}{2} D_F(x_2 - x_1) + \frac{1}{2} D_F(x_1 - x_2)$$
$$= D_F(x_1 - x_2).$$

$$\langle 0| T\{\hat{\phi}_H(x_1)\cdots\hat{\phi}_H(x_4)\}|0\rangle$$

$$= \left(-i\frac{\delta}{\delta J_1}\right)\cdots\left(-i\frac{\delta}{\delta J_4}\right) e^{-\frac{1}{2}\int JDJ}\Bigg|_{J=0}$$

$$= D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23} \qquad \Big\| \quad D_{ij} = D(x_i - x_j)$$



Similarly convince yourself that all $2n$-point fun are given by the different Wick contractions of the free theory.

Note that for $n>1$, the $2n$ pt contributions are all given by disconnected diagrams. All of them are expressible in terms of the 2-pt. fun $D_F(x-y)$. If we define

$$\frac{Z_0[J]}{Z_0[0]} = \exp\left[W_0[J]\right]$$

then $W_0[J] = -\frac{1}{2}\int d^4x\, d^4y\; J(x)\, D_F(x-y)\, J(y)$;

This generates the connected Green's functions.

$$\prod_{i=1}^{n}\left(-i\frac{\delta}{\delta J(x_i)}\right) W_0[J]\Big|_{J=0} = G_c(x_1,\ldots,x_n) = \langle 0|T\{\hat{\phi}_H(x_1)\cdots\hat{\phi}_H(x_n)\}|0\rangle_{\text{conn}}$$

$$G_c(x_1,\ldots,x_n) = 0 \qquad \text{for } n>2$$

$$G_c(x_1,x_2) = D_F(x_1-x_2)$$

This agrees with the canonical answer for the connected Green's fun.

For a general theory we can define $W[J] = \ln\left(\frac{Z[J]}{Z[0]}\right)$,

then $W[J]$ is again the generating fun for the connected Green's fun of the theory.

(Exponentiation of connected Green's fun $\rightsquigarrow$ all Green's fun including disconnected).

Now see how the Feynman diagram expansion ~~of~~ in perturbation theory in Canonical approach arises from the functional integral approach.

Consider $\int [\mathcal{D}\phi]\, e^{i\int \mathcal{L}\, d^4x}$

$$\mathcal{L}[\phi] = \underset{\substack{\downarrow \\ \text{free}}}{\mathcal{L}_0[\phi]} + \underset{\hookrightarrow \text{ treat perturbatively}}{\delta\mathcal{L}[\phi]}$$

If $\delta\mathcal{L}[\phi]$ is not explicitly dependent on $\partial_0\phi$ or time,

then $\delta\mathcal{L}[\phi] = -\delta\mathcal{H}[\phi]$ $\left(\cancel{\mathcal{L}}\ \mathcal{H}[\phi] = \mathcal{H}_0[\phi] + \delta\mathcal{H}[\phi] \right)$.

So,

$$Z[J=0] = \int [\mathcal{D}\phi]\, e^{i\int \mathcal{L}_0[\phi]\, d^4x}\, e^{-i\int d^4x\, \delta\mathcal{H}}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int [\mathcal{D}\phi]\, e^{i\int d^4x\, \mathcal{L}_0[\phi]} \left(\int d^4x\, \delta\mathcal{H}\right)^n$$

$$= Z_0[J=0] \times \left(\sum \frac{(-i)^n}{n!} \int \prod_{i=1}^{n} d^4x_i\, \langle 0| T\{ \delta\hat{\mathcal{H}}(x_1) \cdots \delta\hat{\mathcal{H}}(x_n) \} |0\rangle \right.$$

$$= Z_0[J=0] \times \langle 0| T\{ e^{-i\int d^4x\, \delta\hat{\mathcal{H}}(x} \} |0\rangle$$

$$\Rightarrow \frac{Z[J=0]}{Z_0[J=0]} = \langle 0| T\{ e^{-i\int d^4x\, \delta\hat{\mathcal{H}}(x)} \} |0\rangle$$

$\hookrightarrow$ correlator in free theory

For a two pt. fun

$$\frac{1}{Z[J=0]} \int [\mathcal{D}\phi]\, \phi(y)\, \phi(z)\, e^{i\int [\mathcal{L}_0 - \delta\mathcal{H}]\, d^4x} = \langle \Omega_H | T\{ \hat{\phi}_H(y)\, \hat{\phi}_H(z) \} | \Omega_H \rangle$$

$$= \frac{1}{Z[J=0]} \times Z_0[J=0] \times \left(\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \prod d^4x_i \, \langle 0 | T\{ \hat{\phi}_H(y) \hat{\phi}_H(z) \, \delta\hat{H}(x_1) \cdots \delta\hat{H}(x_n) \} | 0 \rangle \right.$$

$$= \frac{\langle 0 | T\{ \hat{\phi}_H(x_1) \hat{\phi}_H(x_2) \, e^{-i \int d^4x \, \delta\hat{H}(x)} \} | 0 \rangle}{\langle 0 | T\{ e^{-i\int d^4x \, H(x)} \} | 0 \rangle}$$

* QUANTIZING GAUGE FIELDS :-

There are couple of interrelated problems when one naively tries to quantize the free Maxwell Lagrangian for ED.

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu}^2$$

$$S_M = -\frac{1}{4} \int d^4x \, (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\frac{1}{2} \int d^4x \, (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu)$$

$$= +\frac{1}{2} \int d^4x \, A_\mu \underbrace{(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu)}_{K^{\mu\nu}} A_\nu \quad [\text{IBP}]$$

The propagator (naively) would be given by the inverse of the quadratic term. In other words if we try to find $D_{\alpha\beta}(x)$ s.t.

$$K^{\mu\alpha} D_{\alpha\beta} = \delta^\mu_\beta \, \delta^4(x)$$

There is no such $D_{\alpha\beta}(x)$, because the operator $K^{\mu\nu}$ is not invertible — it has a zero eigenvector

$$(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) \underbrace{(\partial_\nu \Lambda)}_{\text{Gauge DOF}} = (\partial^2 \partial^\mu \Lambda - \partial^\mu \partial^2 \Lambda) = 0$$

Thus the gauge inv. of $\mathcal{L}_M$ ⇒ a zero e.v. for $K^{\mu\nu}$.

17·10·14

Problems :

1. The quadratic form in the Maxwell action is not invertible → propagator not well defined.

$$S_M = \frac{1}{2}\int d^4x \; A_\mu K^{\mu\nu} A_\nu$$

$$K^{\mu\nu} = (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu)$$

No $D_{\alpha\beta}$ s.t $K^{\mu\nu} D_{\nu\beta} = \delta^\mu{}_\beta \, \delta^{(4)}(x)$ as $K^{\mu\nu}$ has zero eigen value and eigen vector $\propto \partial_\mu \Lambda(x)$.

Classically this is fixed by gauge fixing. QM'cally ?

2. The time like component of A_μ has no dynamics — no time derivatives of A_0 enters into the Maxwell Lagrangian. (It is a Lagrangian multiplier).

The Euler-Lagrange EqM

$$\partial_0 \cancel{\partial_\mu} \left(\frac{\delta \mathcal{L}}{\delta (\partial_0 A_\mu)} \right) = \frac{\partial \mathcal{L}}{\delta A_\mu}$$

$$\Rightarrow \boxed{\frac{\delta \mathcal{L}}{\delta A_0} = 0} \rightsquigarrow \text{constraint on the classical configuration space.}$$

There is no canonical momentum for A_0 ~ unclear how to implement quantization.

The reason for all these problems can be traced to our attempt to describe a photon (w/ 2 physical DOF) in terms of a field A_μ (w/ 4 DOF). So, all the components can't be physical. Gauge invariance & contraints are the reflections of redundancy in the description in terms of A_μ.

We'll quantize the theory keeping track of gauge inv. + constraint rather than giving up Lorentz invariance/locality

* A QM SYSTEM WITH CONSTRAINTS + GAUGE INV

Consider a Lagrangian with dof (x_1, x_2, A):

$$L = \frac{1}{2}m(\dot{x}_1 + eA)^2 + \frac{1}{2}m(\dot{x}_2 + eA)^2 - V(x_1 - x_2)$$

Has a gauge invariance under

$$x_i(t) \to x_i(t) + \lambda(t) \quad \text{where } \lambda(t) \text{ is some arbit fn.}$$

$$A(t) \to A(t) - \frac{1}{e}\dot{\lambda}(t) \quad .$$

AND A has no time derivative → Lagrange multiplier → its eqn of motn gives constraint → $\frac{\partial L}{\partial A} = 0$

EqM : ⓐ $m\frac{d}{dt}(\dot{x}_i + eA) = -\frac{\partial V}{\partial x_i}$

$$\Rightarrow \frac{dp_i}{dt} = -\frac{\partial V}{\partial x_i} \qquad p_i = m(\dot{x}_i + eA)$$

ⓑ $m(\dot{x}_1 + eA) + m(\dot{x}_2 + eA) = 0$

$\Rightarrow p_1 + p_2 = 0 \rightsquigarrow$ constraint $\rightsquigarrow$ canonical mom are not 'independent' ~ can't impose 'indep.' commutation reln.

NOTE :

The constraint is the generator of the (infinitesimal) gauge invariance.

$$\{(p_1 + p_2)\lambda(t), x_i\} = \lambda(t)$$

The classical phase space of (x_1, x_2, p_1, p_2) has a constraint imposed on it by the Lag multiplier condn $p_1 + p_2 = 0$.

How to quantize this system keeping in mind the existence of constraint (+ gauge inv.) ?

→ CANONICALLY (HAMILTONIAN)

→ PATH INTEGRAL (LAGRANGIAN) .

$$H = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + V(\hat{x}_1 - \hat{x}_2) = \frac{1}{8M}\hat{P}_{cm}^2 + \frac{1}{8M}\hat{P}_{rel}^2 + V(\hat{x}_{rel})$$

$$[\hat{p}_i, \hat{p}_j] = 0$$

$$[\hat{p}_i, \hat{x}_j] = -i\hbar\,\delta_{ij}$$

|| where, $x_{cm} = \frac{x_1 + x_2}{2}$, $x_{rel} = \frac{x_1 - x_2}{2}$

$P_{cm} = \frac{1}{2}(P_1 + P_2)$, $P_{rel} = P_1 - P_2$

What happens to the constraint $P_1 + P_2 = 0$? Can't impose the operator eqn $\hat{P}_1 + \hat{P}_2 = 0$. This is inconsistent with $[\hat{x}, \hat{P}_1 + \hat{P}_2] = i\hbar$.

We'll instead a physical subspace of the full Hilbert space – $|\Psi_{phys}\rangle$ on which

$$(\hat{P}_1 + \hat{P}_2)|\Psi_{phy}\rangle = 0 = \hat{P}_{cm}|\Psi_{phy}\rangle$$

The full Hilbert space $= \{ |\Psi(x_{cm})\rangle \otimes |\tilde{\Psi}(x_{rel})\rangle \}$

The eigenfunctions of $\hat{H}$ are of the form

$$e^{iK_{cm}x_{cm}}\,\Psi_n(x_{rel})$$

↳ eigenfun of $\frac{\hat{P}_{rel}^2}{2m} + V(\hat{x}_{rel})$.

The physical wave fun which obey $\hat{P}_{cm}|\Psi_{phy}\rangle = 0$ are of the form

$$|\Psi_{phy}\rangle = |\Psi_n(x_{rel})\rangle \quad (\text{i.e. } k_{cm} = 0) .$$

NOTE: ① Imposing $(\hat{P}_1 + \hat{P}_2) = 0$ means specifying in full Hilbert space. This means operating on any state gives "0". We're imposing this condn only on a sub-space.

Ⓘⓘ $[\hat{P}_{cm}, \hat{H}] = 0$

The norm in the physical Hilbert space diverges if one naively treats it

$$\text{Norm} = \int dx_{rel}\, dx_{CM}\, |\psi(x_{rel}, x_{CM})|^2$$

for the physical wave fun

$$\int dx_{cm}\, dx_{rel}\, |\psi_{phy}(x_{rel})|^2$$

$$= \int dx_{cm}$$

$$= \text{vol}(\text{COM}) .$$

Define the norm for physical states after dividing it by $\frac{1}{\text{vol}(\text{COM})}$.

Gauge dir^{n} because under a gauge transf..

$$x_{CM} \to x_{CM} + \lambda(t) .$$

The inner product of a physical state w/ an unphysical state $= 0$ i.e. $\langle \psi_{phy} | \psi_{unphy} \rangle = 0$ if where $\hat{P}_{CM} | \psi_{unphy} \rangle = K_{CM} | \psi_{unphy} \rangle$ with $K_{CM} \neq 0$.

<u>RECAP</u> :

$$L = \frac{1}{2} m (\dot{x}_1 + eA)^2 + \frac{1}{2} m (\dot{x}_2 + eA)^2 - v(x_1 - x_2)$$

$$x_i(t) \longrightarrow x_i(t) + \lambda(t)$$

$$A(t) \longrightarrow A(t) - \frac{1}{e} \dot{\lambda}(t)$$

A EOM : $p_1 + p_2 = 0$, $p_i = m(\dot{x}_i + eA)$

<u>Canonical approach</u> :

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + v(\hat{x}_1 - \hat{x}_2) - \frac{e}{m} A (\hat{p}_1 + \hat{p}_2) .$$

On the physical Hilbert space $(\hat{p}_1 + \hat{p}_2) | \psi_{phy} \rangle = 0$.

$$\hat{H} \overset{phy}{=} \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + v(\hat{x}_1 - \hat{x}_2)$$

$$= \underset{0}{\cancel{\frac{\hat{P}_{CM}^2}{2m}}} + \frac{\hat{P}_{rel}^2}{8m} + v(\hat{x}_{rel}) \quad , \quad |\psi_{phy}\rangle = |\psi(x_{rel})\rangle$$

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p_i , $\frac{\hat{p}_{rel}^2}{8m}$, $V(\hat{x}_{rel})$ ⇝ Gauge invariant.

Here COM DOF is the gauge DOF.

PATH INTEGRAL APPROACH :-

In the PI approach, the constraint can be imposed by a δ- fun in the measure of the phase space PI:

$$\langle x_i^{(2)}, T \mid x_i^{(1)}, 0\rangle \overset{?}{=} \int [\mathcal{D}x_1][\mathcal{D}x_2][\mathcal{D}p_1][\mathcal{D}p_2] \prod \delta\left(p_1(t)+p_2(t)\right) e^{i\int_0^T (p_1\dot{x}_1 + p_2\dot{x}_2 - H)\,dt}$$

$$= \int [\mathcal{D}x_1][\mathcal{D}x_2][\mathcal{D}p_1][\mathcal{D}\Lambda][\mathcal{D}p_2]\, e^{i\int_0^T [p_1\dot{x}_1 + p_2\dot{x}_2 - H - \frac{e}{m}\Lambda(p_1+p_2)]\,dt}$$

$$H \;\swarrow\; \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1 - x_2)$$

$$= \int [\mathcal{D}x_{CM}][\mathcal{D}x_{rel}][\mathcal{D}p_{CM}][\mathcal{D}p_{rel}]\; \delta\left(p_{CM}(t)\right) e^{i\int_0^T \left\{\frac{1}{2}p_{rel}\dot{x}_{rel} + \frac{1}{2}p_{CM}\dot{x}_{CM} - \frac{p_{rel}^2}{8M} - \frac{p_{CM}^2}{8M} + V(x_{rel})\right\}dt}$$

$$= \int \underbrace{[\mathcal{D}x_{CM}]}_{\downarrow}[\mathcal{D}x_{rel}][\mathcal{D}p_{rel}]\; e^{i\int_0^T \left\{\frac{1}{2}p_{rel}\dot{x}_{rel} - \frac{p_{rel}^2}{8m} + V(x_{rel})\right\}dt}$$

This will give a divergent contribution because of gauge invariance $x_{CM}(t) \to x_{CM}(t) + 2\lambda(t)$.

Gauge fixing can be done by imposing a δ-fun constraint eg. by $\delta(x_{CM} - \overset{\circ}{x}_{CM})$ in the PI.

We're getting infinity because of $[\mathcal{D}x_{CM}]$. But all of these states are basically same, related by gauge tranf. So, we're over counting this ⇝ so infinity.

The finite physical amplitude is

$$\langle x_i^{(2)}, T \mid x_i^{(1)}, 0 \rangle$$

$$= \int [\mathcal{D}x_1][\mathcal{D}x_2][\mathcal{D}p_1][\mathcal{D}p_2] \prod_i \delta(p_1+p_2) \prod_i \delta\left(x_{CM} - x_{CM}^{(0)}\right) e^{i\int (p_i \dot{x}_i - H)\, dt}$$

$$= \int [\mathcal{D}x_{rel}][\mathcal{D}p_{rel}]\, e^{i\int \left(\frac{1}{2} p_{rel}\dot{x}_{rel} - \left(\frac{p_{rel}^2}{8m} + V(x_{rel})\right)\right) dt}$$

Gauge fixing ⇒ choosing $\underrightarrow{\text{one}}$ representative among all the configurations all of which are basically same, related by gauge transf.

$\delta(p_1+p_2) \rightsquigarrow$ constraint

$\delta(x_{CM} - x_{CM}^{0}) \rightsquigarrow$ gauge fixing (to avoid over counting).

$$= \int [\mathcal{D}x_{rel}]\, e^{i\int_0^T \left(\frac{1}{2} m \dot{x}_{rel}^2 - V(x_{rel})\right) dt}$$

$\rightsquigarrow$ in terms of physical DOF.

Write

$$\prod_i \delta(p_1(t) + p_2(t)) = \int [\mathcal{D}A]\, e^{-ie/m \int_0^T A(p_1+p_2)\, dt}$$

Then the amplitude is

$$\langle x_i^{(2)}, T \mid x_i^{(1)}, 0 \rangle = \int [\mathcal{D}x_i][\mathcal{D}p_i][\mathcal{D}A]\, e^{i\int dt \left(p_i \dot{x}_i - \left(\frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1 - x_2) - \frac{e}{m} A(p_1+p_2)\right)\right)}$$

$$\prod \delta(x_{cm} - x_{cm}^{0})$$

↳ arbit fun x_{cm}^{0}

— amplitude is indep of it.

We can therefore introduce (result being indep. of $\overset{\circ}{x}_{cm}$)

$$\int [\mathcal{D}\overset{\circ}{x}_{cm}] \, e^{i \frac{1}{\alpha} \int_0^T \overset{\circ}{x}_{cm}^2 dt} = 1$$

So,

$$= \int [\mathcal{D}A] [\mathcal{D}\overset{\circ}{x}_{cm}] \, e^{i \frac{1}{\alpha} \int \overset{\circ}{x}_{cm}^2 dt} \, [\mathcal{D}x_i] [\mathcal{D}p_i] \, e^{i \int (p_i \dot{x}_i - H)} \, \delta(x_{cm} - \overset{\circ}{x}_{cm})$$

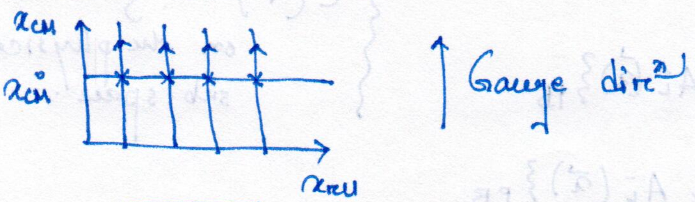
$$= \int [\mathcal{D}A] [\mathcal{D}x_i] [\mathcal{D}p_i] \, e^{i \int ((p_i \dot{x}_i - H) + \frac{1}{\alpha} x_{cm}^2) dt}$$

↳ Highly gauge redundant description — all the DOF including gauge DOF.

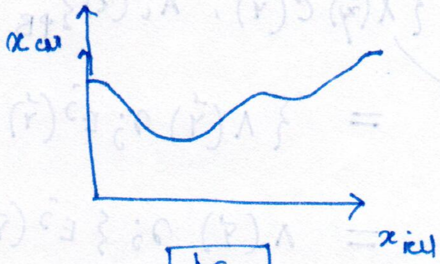
↳ Gauge fixing term in Lagrangian

If we had chosen more complicated gauge condition

$$f(x_{cm}, x_{rel}) = 0$$ [s.t. for any x_{rel}, there is a unique x_{cm}].



Earlier



Now

$$\mathcal{D}x_{cm} \, \delta(x_{cm} - \overset{\circ}{x}_{cm}) = \mathcal{D}x_{cm} \, \delta(f(x_{cm}, x_{rel})) \, |J|$$

where, $|J| = \det \left(\frac{\delta f}{\partial x_{cm}} \right) \Big|_{f=0}$

$$\prod_i dx_i \, \delta(x_i - x_0) = \prod_i dx_i \, \delta(f(x_i)) \, \det \left(\frac{\partial f}{\partial x_i} \right) \Big|_{f_i = 0}$$

With such a gauge choice the PI

$$\langle x_i^{(2)}, T | x_i^{(1)}, 0 \rangle = \int [\mathcal{D}x_i] [\mathcal{D}p_i] \prod \delta(p_1 + p_2) \prod \delta(f(x_{cm}, x_{rel})) \det \left(\frac{\delta f}{\delta x_{cm}} \right) e^{i \int (p_i x_i - H) dt}$$

ELECTROMAGNETIC FIELD :

① Constraint : A_0 is not dynamical

$$\frac{\partial \mathcal{L}_M}{\delta A_0} = 0 = \partial_i E^i \qquad , \quad E^i = \partial^i A^0 - \partial^0 A^i$$

(Gauss law) ;

$$E^i = \frac{\delta \mathcal{L}_M}{\delta(\partial_0 A_i)} = \pi^i \rightsquigarrow \text{Canonically conjug. mom to } A_i .$$

So, $\{A_i(\vec{x},t), \pi^j(\vec{y},t)\}_{PB}$

$$= \delta_i^j \, \delta^{(3)}(\vec{x}-\vec{y})$$

One can't specify π^i arbitrarily, it must satisfy $\partial_i \pi^i = 0 \rightsquigarrow$ constraint. Phase space of (A_i, π^i) has constraint.

The constraint generates 'infinitesimal gauge transformation' on A_i :

Correspondence
$A_0 \to A$
$A_i \to x_1, x_2$

$$\{\Lambda(\vec{y})\, C(\vec{y}) , A_i(\vec{x})\}_{PB}$$

(Arbit fun)

$$= \{\Lambda(\vec{y})\, \partial_j E^j(\vec{y}), A_i(\vec{x})\}_{PB}$$

$$= \Lambda(\vec{y})\, \partial_j \{E^j(\vec{y}), A_i(\vec{x})\}_{PB}$$

$$= -\Lambda(\vec{y})\, \delta^j_i \, \partial_j \, \delta^{(3)}(\vec{x}-\vec{y})$$

$$= \partial_i \Lambda(\vec{y}) \, \delta^{(3)}(\vec{x}-\vec{y})$$

{ $C(\vec{y})$ is zero only on the physical sub-space.

So, $$\left\{ \int \Lambda(\vec{y})\, C(\vec{y})\, d^3y , A_i(x) \right\}_{PB} = \partial_i \Lambda(\vec{x})$$

gauge (infinitesimal) transf on A_i .

$$\mathcal{H}(E^i, A_i) = \pi^i \partial_0 A_i - \mathcal{L}_M(A_i, A_0)$$

$$= \cancel{\pi} \, \pi^i \partial_0 A_i - \frac{1}{2}\left(\pi_i^2 - B_i^2\right) \qquad , \quad B_i = \cancel{\tfrac{1}{2}} \epsilon_{ijk} \partial_j A_k$$

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$$= \pi^i\,(\partial_0 A_i - \partial_i A_0) + \pi^i \partial_i A_0 - \tfrac{1}{2}\pi_i^2 + \tfrac{1}{2} B_i^2$$

$$= \tfrac{1}{2}(\pi^i)^2 + \tfrac{1}{2} B_i^2 + \pi^i \partial_i A_0$$

So, $H = \int d^3x\, \mathcal{H}(x) = \int \left[\frac{1}{2}\left(\pi_i^2 + B_i^2\right) - \underset{\text{Lag. multiplier}}{A_0}\,(\partial_i \pi^i) \right] d^3x$

Lag. multiplier $\cdot$ $\partial_i \pi^i = 0$.

RECAP :

Constraint $C(\vec{x}) = \partial_i \pi^i = \partial_i E^i = 0$

$$\{\pi_i(\vec{x}), A^j(\vec{y})\}_{PB} = -\delta_i^j\, \delta^{(3)}(\vec{x} - \vec{y})$$

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left[\tfrac{1}{2}\pi^{i^2} + \tfrac{1}{2} B_i^2 + A^0 (\partial_i \pi^i) \right]$$

Quantum theory :

① Impose the constraint as a δ-fun.

② 〃 a gauge fixing cond^n as a δ-fun.

} Together with right measure.

③ Do phase space functional integral over the dynamical variables (A_i, π^i).

We can make space dependent gauge transformation $\Lambda(\vec{x})$ under which $A_i(\vec{x}) \longrightarrow A_i(\vec{x}) + \partial_i \Lambda(\vec{x})$.

So, we can choose a gauge where $A_3 = 0$. Then we've functional integral

$$\int [\mathcal{D}A_i][\mathcal{D}\pi^i]\; \delta(\partial_i \pi^i)\; \delta(A_3)\; (\underset{}{\overset{\text{Measure}}{\;\;}})\; e^{i\int(\pi^i \partial_0 A_0 - \mathcal{H})\, d^4x}$$

NOTE : If we've $dx\,dp\; \underset{\text{constraint}}{\delta(f(x,p))}\; \underset{\text{gauge}}{\delta(g(x,p))}$ $\underbrace{\begin{vmatrix} \partial f/\partial x & \partial g/\partial x \\ \partial f/\partial p & \partial g/\partial p \end{vmatrix}}_{\text{fixing}}$

~~$\{X, B\}_P$~~

$\{f, g\}_{PB}$

as $\int df\, dg\; \delta(f)\, \delta(g) = 1$

The invariant measure in the functional integral is

$$\mathcal{D}x\, \mathcal{D}p\; \delta(f(x,p))\, \delta(g(x,p))\; \det\{f,g\}_{PB}$$

Measure factor in this gauge fixing $A_3 = 0$ is

$$\det\{\partial_i \pi^i, A_3\}_{PB} = \det(\partial_3)$$

- in this case it is field independent and can be taken outside the functional integral - where they give a normalization factor (cancel in ratios).

If we had a gauge fixing cond^{n} $G(A_i) = 0$. Then the measure term would have been

$$\det \{\partial_i \pi^i, G(A_j)\}_{PB} = \det\left(\partial_i \frac{\delta G(A_j)}{\delta A_i}\right)$$

for Coulomb gauge $G = \partial_i A^j = 0$.

$$\det(\partial_i \partial_j \delta^j_i) = \det\underbrace{(\partial_i \partial^i)}_{\nabla^2} \longleftarrow \text{field independent}$$

For any linear gauge fixing condition - this will not play any role.

If we did the integral over A_3, we would be left with $(A_1, A_2) \to$ two physical components of the gauge field. If we solved for π_3 from the constraint, we would get a non-local action involving (π_1, π_2, A_1, A_2) $\left[\because \partial_3 \pi^3 = -(\partial_1 \pi^1 + \partial_2 \pi^2)\right]$.

$$\left[\Rightarrow \pi^3 = -\int d^3x\, (\partial_1 \pi^1 + \partial_2 \pi^2)\right]$$

Instead, we'll add in all the unphysical dof (non-dyn. var) and keep a local (and w/ a covariant gauge cond^{n}, a LI) action.

$$\int [\mathcal{D}A_0][\mathcal{D}A_i][\mathcal{D}\pi^i]\; \delta\left(G(A)\right)\; \det\left(\partial_i \frac{\delta G(A_i)}{\delta A_i}\right)$$

$$\exp\left[i\int\left(\underbrace{A^0\partial_i\pi^i + \pi^i\partial_0 A_i}_{\pi^i(\partial_0 A_i - \partial_i A_0)} - \frac{1}{2}\left(\pi^{i\,2} + (\vec{\nabla}\times\vec{A})^2\right)\right) d^4x\right]$$

$$= \int [\mathcal{D}A_0][\mathcal{D}A_i]\; \underbrace{\delta\left(G(A)\right)\; \det\left(\partial_i \frac{\delta G(A)}{\delta A_i}\right)}_{\text{Extra term / factor}}$$

$$\exp\left[i\int d^4x\; \underbrace{\left(\frac{1}{2}(\partial_0 A_i - \partial_i A_0) - (\vec{\nabla}\times\vec{A})^2\right)}_{\mathcal{L}_M}\right]$$

$$= \int [\mathcal{D}A_\mu]\; \delta\left(G(A)\right)\; \det\left(\partial_i \frac{\delta G}{\delta A_i}\right)\; e^{i\int d^4x\, \mathcal{L}_M[A_\mu]}$$

We can make this manifestly Lorentz invariant by choosing eg. Lorentz gauge ($\partial_\mu A^\mu - C(x) = 0$, where $C(x)$ is an arbit fun of $(\vec{x}, t)$). Since this is linear in A_μ, the det factor is indep of A and can be taken outside. We then have a manifestly LI functional integral.

$$\int [\mathcal{D}A_\mu]\; \delta(\partial^\mu A_\mu - c(x))\; e^{i\int d^4x\, \mathcal{L}_M}$$

Since the functional integral is indep of $c(x)$, we can insert $1 = \int [\mathcal{D}c]\; e^{-\frac{i}{2\xi}\int d^4x\; c^2(x)}$

So, $$\int [\mathcal{D}C][\mathcal{D}A_\mu]\; \delta(\partial_\mu A^\mu - C(x))\; e^{-\frac{i}{2\xi}\int d^4x\; c^2 x + i\int d^4x\, \mathcal{L}_M}$$

$$= \int [\mathcal{D}A_\mu]\; e^{i\int d^4x\, \underbrace{\left[\mathcal{L}_M - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]}_{\text{Modified Lagrangian. not GI.}}}$$

In this gauge fixed/modified Lagrangian the kinetic term (quadratic term) for the gauge field A_μ is invertible.

The whole Lagrangian can be written as

$$\frac{1}{2}\int d^4x\, A_\mu(x)\underbrace{\left[\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu + \frac{1}{\xi}\partial^\mu\partial^\nu\right]}_{\eta^{\mu\nu}\partial^2 - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu} A_\nu(x)$$

[Ex: Show that for general ξ

$$D_{\mu\nu}(q) = \frac{i}{q^2+i\epsilon}\left[\eta_{\mu\nu} - \underbrace{(1-\xi)\frac{q_\mu q_\nu}{q^2}}_{\text{will not contribute in any physical process since } \xi \text{ is arbit.}}\right]$$

$\xi = 1 \rightsquigarrow$ Feynman gauge.

$\xi = 0 \rightsquigarrow$ Landau gauge.

* FADEEV - POPOV METHOD

- Start with a Lagrangian fun. integral

$$\int [\mathcal{D}A_\mu]\, e^{i\int \mathcal{L}_M d^4x}$$

This can't be the correct answer since one is over-counting equivalent gauge config $A_\mu \sim A_\mu + \partial_\mu\Lambda$.

$I = \int d^n x\, f(x_i)$ → If $f(\{x_i\})$ is indep of one of the coordinates say x_1 then $f(x_1+a) = f(x_1)$

So, $I = (\text{Vol } x_1)\int d^{n-1}x\ f(\{x_i\})$

Need to divide by (vol x_1) to get a final answer. Or, equivalently insert $\delta(x_1 - x_1^0)$ in the integral

$$\int d^n x \; f(\{x_i\}) \, \delta(x_1 - x_1^0) = \int d^{n-1} x \; f(\{x_i\}) .$$

We fix the gauge invariance by inserting

$$1 = \int dx_1^0 \, \delta(x_1 - x_1^0)$$

and $\int d^n x \, dx_1^0 \, \delta(x_1 - x_1^0) \, f(\{x_i\})$

$$= \underbrace{\int dx_1^0}_{\text{vol. of gauge freedom}} \int d^{n-1} x \; f(\{x_i\})$$

$$1 = \int [\mathcal{D}\Lambda] \, \delta(G(A^{\Lambda})) \det\left(\frac{\delta G(A^{\Lambda})}{\delta \Lambda}\right)\Bigg|_{\Lambda=0} \qquad \Bigg\| \quad \hat{A}_\mu = A_\mu + \partial_\mu \Lambda$$

$$\Bigg[\text{ Analogue of } \quad 1 = \int dx \, \delta(g(x,y)) \; \frac{\delta g(x,y)}{\partial x}\Bigg|_{g=0}$$

Then $\displaystyle\int [\mathcal{D}A_\mu][\mathcal{D}\Lambda] \, e^{i\int \cancel{d^4x}\, d^4x \, \mathcal{L}_M[A]} \, \delta(G(A^{\Lambda})) \det\left(\frac{\delta G(A^{\Lambda})}{\delta \Lambda}\right)$

$$= \int [\mathcal{D}\hat{A}_\mu][\mathcal{D}\Lambda] \, e^{i\int \mathcal{L}_M[A^{\Lambda}] \, d^4x} \, \delta(G(A^{\Lambda})) \det\left(\frac{\delta G(A^{\Lambda})}{\delta \Lambda}\right)$$

For linear gauge fixing condition det( ) is indep. of A and can be taken out.

So, $\displaystyle \det(\,) \int \cancel{[\mathcal{D}A_\mu]} [\mathcal{D}\Lambda] \underbrace{[\mathcal{D}A] \, e^{i\int \mathcal{L}[A]} \, \delta(G(A))}_{\text{indep. of } \Lambda}$

$$= \det(\,) \underbrace{\left[\int [\mathcal{D}\Lambda]\right]}_{\text{Infinite vol. of gauge group}} \int [\mathcal{D}A] \, e^{i\int \mathcal{L}[A]} \, \delta(G(A))$$

In any case, these infinite over all factors drop out.

So, we can just consider $\displaystyle\int [\mathcal{D}A] \, e^{i\int \mathcal{L}[A]} \, \delta(G(A))$

㉗ 29.10.2014

$$\mathcal{Z} = ()\int [\mathcal{D}A]\, e^{i\int \mathcal{L}_{GF}\, d^4x}$$

$$\mathcal{L}_{GF} = \mathcal{L}_M - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$ ⇝ Leads to an invertible kinetic term → propagator

$$\langle\Omega|T\{O_1(\hat{A}_H)\dots O_n(\hat{A}_H)\}|\Omega\rangle$$

↓ Built from A_μ and are gauge inv.

⇝ All physics is encoded in such gauge invariant correlators.

$$= \frac{\int[\mathcal{D}A]\, O_1(A)\dots O_n(A)\, e^{i\int \mathcal{L}_{GF}\, d^4x}}{\int[\mathcal{D}A]\, e^{i\int \mathcal{L}_{GF}\, d^4x}}$$ } Prescription to compute correlators.

* FUNCTIONAL INTEGRALS FOR SPINORS :

A classical spinor field in term of c-number quantities does not exist (in conflict with $\{\psi(x), \psi(y)\} = 0$). To formulate functional integral for spinors, we need to introduce a generalization of c-numbers – grassmann numbers (a-numbers).

These obey the properties :

① $\theta\eta = -\eta\theta$

$\Rightarrow \{\theta, \eta\} = 0$ for any two Grassmann numbers

eg. $\{\theta, \theta\} = 0 \Rightarrow \theta^2 = 0$

② Product of 2 Grassmann numbers is a c-number.

$$(\theta\eta)(\theta'\eta') = -\theta\theta'\eta\eta' = \theta\theta'\eta'\eta = -\theta'\theta\eta'\eta = (\theta'\eta')(\theta\eta)$$

$\theta\eta$ also commutes w/ any Grassmann number :

$$(\theta\eta)\,\eta' = \eta'(\theta\eta)$$

③ They have all the properties of a vector space (over $\mathbb{R}$ or $\mathbb{C}$)

$$a(\theta_1+\theta_2) = a\theta_1 + a\theta_2 \qquad (a \in \mathbb{R}/\mathbb{C})$$

$$\theta + (-\theta) = 0$$

④ $f(\theta) = a + b\theta \qquad a, b \in \mathbb{C}/\mathbb{R}$

$\because \theta^2 = 0$.

for n variables $\theta_1, \dots, \theta_n$

$$f(\theta_1, \dots, \theta_n) = a_0 + a_i\theta_i + a_{i_1 i_2}\theta_1\theta_2 + \dots + \underbrace{a_{i_1 \dots i_n}}_{\text{anti-symmetric in indices}} \theta_1 \dots \theta_n$$

.

⑤ Define integration s.t.

$$\int d\theta \, f(\theta + \underbrace{\eta}_{\text{cons. shift}}) = \int d\theta \, f(\theta)$$

$$\Rightarrow \int d\theta \, (a + b\theta + b\eta) = \int d\theta \, (a + b\theta)$$

$$\Rightarrow b \int d\theta \, \eta = 0 = -b\eta \int d\theta$$

$$\Rightarrow \int d\theta \cdot 1 = 0$$

So, $\int d\theta (a + b\theta) = b \int d\theta \, \theta \equiv b$ i.e. $\int d\theta \, \theta \equiv 1$.

For many variables we similarly define

$$\int d\theta d\eta \, \eta\theta \equiv 1 = \int d\theta\theta \, d\eta\eta = -\int d\theta d\eta \, \theta\eta$$

⑥ Derivative $\frac{d}{d\theta}\theta = 1 = -\theta\frac{\overleftarrow{d}}{d\theta}$

$$\frac{d}{d\theta}(\eta\theta) = -\frac{d}{d\theta}\theta\eta = -\eta$$

⑦ Define complex Grassmann nos.

$$\frac{\theta_1 + i\theta_2}{\sqrt{2}} = \theta, \quad \frac{\theta_1 - i\theta_2}{\sqrt{2}} = \theta^*$$

$$\int d\theta_1 d\theta_2 = \int d\theta^* d\theta$$

$$(\theta\eta)^* = \eta^*\theta^* = -\theta^*\eta^*$$ (check!)

The Gaussian Grassmann integral is simple.

$$\int d\theta^* d\theta\, e^{-\theta^* b\theta} \qquad b \in \mathbb{R}/\mathbb{C}.$$

$$= \int d\theta^* d\theta\, (1 - \theta^* b\theta)$$

$$= -b\int d\theta^* d\theta\, \theta^*\theta$$

$$= b$$

$$\int d\theta^* d\theta\, (\theta\theta^*)\, e^{-\theta^* b\theta}$$

$$= \int d\theta^* d\theta\, \theta\theta^*$$

$$= 1 = b \times 1/b$$

$$\int \prod_{i=1}^{n} d\theta_i^* d\theta_i\, e^{-\sum_{ij}\theta_i^* B_{ij}\theta_j}$$

unitary transf $\vec{e}$igenvalues of $B_{ij} \sim b_i$

$$= \int \prod_{i=1}^{n} d\tilde{\theta}_i^* d\tilde{\theta}_i\, e^{-\sum_i \tilde{\theta}_i^* b_i \theta_i}$$

$$= (-1)^n \prod_i b_i \int \prod_{i=1}^{n} d\theta_i^* d\theta_i\, \tilde{\theta}_i^* \tilde{\theta}_i$$

$$= \prod_i b_i = (\det B)$$

Note that for c. numbers

$$\int \prod_i dz_i^* \, dz_i \quad e^{-\sum z_i^* B_{ij} z_j}$$

$$= \frac{(2\pi)^n}{(\det B)} .$$

Ex: $\int \prod_{i=1}^{n} d\theta_i^* \, d\theta_i \quad \theta_K \theta_L^* \quad e^{-\sum_{ij} \theta_i^* b_{ij} \theta_j} = (\det B) \, (B^{-1})_{KL}$

for c-nos. $\int (\;) \, z_K z_L^* \; e^{-\sum z_i^* B_{ij} z_j} \sim \frac{1}{(\det B)} (B^{-1})_{kl} .$

* Use Grassmann nos. to define a functional integral for Dirac fields :

∵ The Dirac field is a complex spinor, we define a complex Grassmann field.

$$\psi(x) = \sum_i \theta_i U_i(x)$$

θ_i : Grassmann variables

$U_i(x)$: a complete set of spinor wave fun

eg. $U_\alpha(p) e^{ip \cdot x}$

$v_\alpha(p) e^{ip \cdot x}$.

[Analogue of $\phi(x) = \sum c_i \phi_i(x)$.]

These "classical" spinor fields $\psi(x)$ obey $\{\psi(x), \psi(y)\} = 0$

∵ $\{\theta_i , \theta_j\} = 0$

We can then consider a functional integral over these spinor fields $\psi(x)$ (Essentially an integral over θ_i) .

A Lagrangian functional integral for $\psi(x)$ will reproduce results for a free Dirac field.

$$Z = \int [\mathcal{D}\bar{\psi}] [\mathcal{D}\psi] \, e^{i \int \mathcal{L}_D \, d^4x} \qquad \mathcal{L}_D = \bar{\psi} (i\not{\partial} - m) \psi$$

Now, $\prod_x d\bar{\psi}(x) = \prod_i d\theta_i^*$

and $\prod_x d\psi(x) = \prod_i d\theta_i$

$$\langle 0 | T \{ \hat{\psi}_H(x_1) \hat{\bar{\psi}}_H(x_2) \} | 0 \rangle$$

$$= \frac{1}{Z} \int [\mathcal{D}\bar{\psi}] [\mathcal{D}\psi] \, \psi(x_1) \bar{\psi}(x_2) \, e^{i \int \mathcal{L}_D}$$

$$\rightsquigarrow S_F(x_1 - x_2) = \frac{1}{Z} \det(i\not{\partial} - m) \, (i\not{\partial} - m)^{-1}_{x_1 x_2}$$