UNRUH EFFECT (1976. Unruh)

Talk on 27th March 2012

* Topics of discussion:

A. Overview of Unruh Effect (Physically)

B. Rindler ST

- → Uniformly moving observer
- → Uniformly accelerated Observer

C. Classical field in Rindler ST & thermal spectrum

D. Quantum " " " "

- → Wave Equⁿ
- → Quantization
- → Lightcone Mode Expansion
- → The Bogolyubov Transformation
- → Density of Particles
- → The Unruh Temperature

E. Is it experimentally detectable ?

☒ (A) Overview of Unruh Effect :

* Unruh asked a question ⇒ How does a Minkowski vacuum field look to an accelerated observer ? What would happen if a detector is accelerated through a vacuum field ?

⇓ Answer

When a detector, coupled to a relativistic quantum field in its vacuum state, is uniformly accelerated through Minkowski ST, with proper acceleration a, it registers a thermal BB radia at temp $T = \dfrac{\hbar a}{2\pi c k_B} \sim 10^{-19} a$. In other words, it perceiv a thermal bath of particles in equili. The ground state for an inertial observer is seen as in TDE with a non-zero temp by the uniformly accelerated observer.

* Is it surprising ?

- If you think the vacuum is an empty space, you would be surprised, because how can a detector, accelerated or not, see particles if there is not anyth
- But we know vacuum is not "empty space", but the ground state of some quantum field, and one should expect the detector to react to the presence of the field when it's accelerated through space.

But what is still surprising is the claim that it "perceives" a thermal distribution of radiation.

* GOAL : Our main aim is to derive the above claim mathematically.

B. Rindler ST

C. Classical . . .

See AGTR_P

D. $\boxed{\text{Quantum fields in the Rindler ST}}$:

i) Wave Equ^{n}:

* Goal : Quantize a scalar field in the proper reference frame of a uniformly accelerated observer.

* Simplification: Massless scalar field in 1+1 dimensional ST.

* Action :

$$S[\phi] = -\frac{1}{2}\int g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi\,\sqrt{-g}\,d^2x \qquad [\text{Minimally coupled}]$$

$$x^\mu \equiv (t, x)$$

* Action is conformally invariant : $g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = \Omega^2(t,x)\, g_{\alpha\beta}$

Then, $\sqrt{-g} \to \Omega^2\sqrt{-g}$

$g^{\alpha\beta} \to \Omega^{-2} g^{\alpha\beta}$ ← $\left[\; g_{\alpha\beta}\, g^{\beta\gamma} = \delta^\gamma_\alpha \quad \tilde{g}_{\alpha\beta}\,\tilde{g}^{\beta\gamma} = \delta^\gamma_\alpha \quad \Omega^2 g_{\alpha\beta} \;\therefore\; \tilde{g}^{\beta\gamma} = \Omega^{-2}\ldots \right.$

So, $\sqrt{-g}\, g^{\alpha\beta} \to \sqrt{-g}\, g^{\alpha\beta}$

⊛ $\cancel{\ln(\det M) = \mathrm{Tr}(\ln M)}$

$\Rightarrow \cancel{\frac{\delta(\det M)}{\det M} = \mathrm{Tr}(M^{-1}\delta M)}$

$\Rightarrow \cancel{\delta g = g\,(g^{\mu\nu}\delta g_{\mu\nu})}$

In D dim ST the determinant of the metric g transforms as $\sqrt{-\tilde{g}} = \Omega^D\sqrt{-g}$ (See Back page)

$\Rightarrow \quad S[\phi] \to S[\phi]$

Hence, the minimally coupled massless scalar field in 1+1 dimensional Minkowski ST is conformally invariant. This conformal invariance causes a significant simplification of the theory in 1+1 dimensions.

* In Laboratory coordinates (t, x), the action is

$$S[\phi] = \frac{1}{2}\int\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right]\cancel{\sqrt{-g}\,d^2x}\; dt\, dx$$

* In (t', x') the metric is equal to the flat Minkowski metric times a conformal factor $\Omega^2(t', x') \equiv e^{2ax'}$. Hence, due to conformal invariance, the action has the same form in (t', x'):

$$S[\phi] = \frac{1}{2}\int\left[(\partial_{t'}\phi)^2 - (\partial_{x'}\phi)^2\right]\cancel{\sqrt{-g}}\; dt'\, dx'$$

* Wave equ^{n} in Lab :

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0$$

* In accelerated frame : $\dfrac{\partial^2\phi(t', x')}{\partial t'^2} - \dfrac{\partial^2\phi(t', x')}{\partial x'^2} = 0$

* $\boxed{\sqrt{-\tilde{g}} = \Omega^D \sqrt{-g}}$

D dimension
$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \cdots & \tilde{g}_{1D} \\ \tilde{g}_{21} & \tilde{g}_{22} & \cdots & \tilde{g}_{2D} \\ \vdots & & & \\ \tilde{g}_{D1} & \cdots & \cdots & \tilde{g}_{DD} \end{pmatrix} = \begin{pmatrix} \Omega^2 g_{11} & \Omega^2 g_{12} & \cdots & \Omega^2 g_{1D} \\ \vdots & & & \\ \vdots & & & \end{pmatrix}$$

$\Downarrow$ ~~take dt~~

deter: $\tilde{g} = \Omega^{2D} g$

$$\Rightarrow \quad -\tilde{g} = \Omega^{2D}(-g)$$

$$\Rightarrow \boxed{\sqrt{-\tilde{g}} = \Omega^D \sqrt{-g}}$$

* General sol^{n}:

$$\phi(t,x) = A(t-x) + B(t+x)$$
$$\phi(t',x') = P(t'-x') + Q(t'+x').$$

} A, B, P, Q are arbit smooth functions.

Note that a sol^{n} $\phi(t,x)$ representing a certain state of the field will be a very different function of t', x'.

ii) Quantization:

* Goal: Quantize $\phi(t,x)$ as well as $\phi(t',x')$ and compare the vacuum states in Lab & in acc frame.

* Procedure is same in both coordinate system (t,x) & (t',x')

Lab: Mode expansion (with $\omega = |k|$).

$$\hat{\phi}(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{2|k|}} \left[e^{-i|k|t + ikx} \hat{a}_k^- + e^{i|k|t - ikx} \hat{a}_k^+ \right]$$

where, $\hat{a}_k^\pm$: creation & annihilation operators satisfying

$$[\hat{a}_k^-, \hat{a}_{k'}^+] = 2\pi\, \delta(k-k') \;;\quad \hat{a}_k^- \equiv \frac{1}{\sqrt{2|k|}} \left[\omega(k)\, \hat{\phi}(k) + i\, \hat{\pi}(k) \right]$$

Vacuum: In Lab frame (Minkowski vacuum) $\Rightarrow$

$$\hat{a}_k^- |0_M\rangle = 0 \quad \forall\, k$$

* Mode expansion in accelerated frame:

$$\hat{\phi}(t',x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{2|k|}} \left[e^{-i|k|t' + ikx'} \hat{b}_k^- + e^{i|k|t' - ikx'} \hat{b}_k^+ \right]$$

with $[\hat{b}_k^-, \hat{b}_{k'}^+] = 2\pi\, \delta(k-k')$.

Vacuum: In Rindler frame

$$\hat{b}_k^- |0_R\rangle = 0 \quad \forall\, k.$$

* Since the operators $\hat{b}_k$ differ from $\hat{a}_k$, the Rindler vacuum $|0_R\rangle$ & the Minkowski vacuum are two different quantum states of the field $\hat{\phi}$.

iii) Lightcone Mode Expansion:

* Goal: Express everything in terms of lightcone coordinates.

* $$\left.\begin{aligned} U &\equiv t - x \\ V &\equiv t + x \end{aligned}\right\} \quad \& \quad \left.\begin{aligned} u' &\equiv t' - x' \\ v' &\equiv t' + x' \end{aligned}\right\}.$$

* We've already seen the rel^n bet^n two:

$$\left.\begin{aligned} U &= -\frac{1}{a} e^{-au'} \\ V &= \frac{1}{a} e^{av'} \end{aligned}\right\}.$$

* $$ds^2 = -dU \cdot dV = -e^{a(v'-u')} du' dv'$$

* field equ^n and sol^{ns}:

$$\begin{cases} \dfrac{\partial^2}{\partial U \partial V} \phi(U,V) = 0 \Rightarrow \phi(U,V) = A(U) + B(V) \\ \dfrac{\partial^2}{\partial u' \partial v'} \phi(u',v') = 0 \Rightarrow \phi(u',v') = P(u') + Q(v') \end{cases}$$

* Mode expansion: by splitting the integration into the ranges of +ve & -ve k:

$$\text{Lab} \Rightarrow \hat{\phi}(t,x) = \int_{-\infty}^{0} \frac{dk}{2\pi} \frac{1}{\sqrt{2|k|}} \left[e^{-i|k|t + ikx} \hat{a}_k^- + e^{+i|k|t - ikx} \hat{a}_k^+ \right]$$

$$+ \int_{0}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{2|k|}} \left[e^{-i|k|t + ikx} \hat{a}_k^- + e^{i|k|t - ikx} \hat{a}_k^+ \right]$$

$$= \int_{-\infty}^{0} \frac{dk}{2\pi} \frac{1}{\sqrt{2|k|}} \left[e^{ikt + ikx} \hat{a}_k^- + e^{-ikt - ikx} \hat{a}_k^+ \right]$$

$$+ \int_{0}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{2k}} \left[e^{-ikt + ikx} \hat{a}_k^- + e^{ikt - ikx} \hat{a}_k^+ \right]$$

Introduce $\omega = |k|$ with $0 < \omega < +\infty$

$$\Rightarrow = \int_{\infty}^{0} \frac{-d\omega}{2\pi} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega t - i\omega x} \hat{a}_{-\omega}^- + e^{i\omega t + i\omega x} \hat{a}_{-\omega}^+ \right]$$

$$+ \int^{\infty} \frac{d\omega}{\ } \frac{1}{\ } \left[e^{-i\omega t + i\omega x} \hat{a}_\omega^- + e^{i\omega t - i\omega x} \hat{a}_\omega^+ \right]$$

* A Rindler observer at constant spatial coordinate x' undergoes constant acceleration with magnitude $a e^{-ax'}$, and the observer's proper time coincides with the coordinate t'.

* Inertial & Rindler Bases :

One basis for the space of sol^{ns} to the scalar wave equ^{n} in the global inertial coordinates are the functions

$$f_\omega = \frac{1}{\sqrt{2\omega}} e^{-i\omega u} \longrightarrow \text{+ve freq. } \text{out}\cancel{\text{in}}\text{ward propagating modes.}$$

$$\bar{f}_\omega = \frac{1}{\sqrt{2\omega}} e^{i\omega u}$$

$(\omega > 0)$

$$g_\omega = \frac{1}{\sqrt{2\omega}} e^{-i\omega v} \longrightarrow \text{" " inward " " .}$$

$$\bar{g}_\omega = \frac{1}{\sqrt{2\omega}} e^{i\omega v}$$

Modes are normalized wrt defined inner product.

$$\hat{\phi}(u,v) = \int_0^\infty \frac{d\omega}{2\pi}\left[f_\omega \hat{a}^-_\omega + \bar{f}_\omega \hat{a}^+_\omega + g_\omega \hat{a}^-_{-\omega} + \bar{g}_\omega \hat{a}^+_{-\omega} \right]$$

$$\left.\begin{aligned} a^-_\omega |0\rangle &= 0 \\ a^-_{-\omega} |0\rangle &= 0 \end{aligned}\right\} \text{Annihilating Global Inertial vacuum.}$$

We'll take $|0\rangle$ to be the quantum state of the scalar field.

For Rindler Observer :

$$\left.\begin{aligned} p_\Omega &= \frac{1}{\sqrt{2\Omega}} e^{-i\omega u'} \longrightarrow \text{outward} \\ \bar{p}_\Omega &= \frac{1}{\sqrt{2\Omega}} e^{i\omega u'} \longrightarrow \text{"} \\ q_\Omega &= \frac{1}{\sqrt{2\Omega}} e^{-i\omega v'} \longrightarrow \text{inward} \\ \bar{q}_\Omega &= \frac{1}{\sqrt{2\Omega}} e^{i\omega v'} \longrightarrow \text{"} \end{aligned}\right\}$$

These are only defined in the Rindler wedge.

$$\hat{\phi}(u',v') = \int_0^\infty d\Omega \left[p_\Omega \hat{b}^-_\Omega + \bar{p}_\Omega \hat{b}^+_\Omega + q_\Omega \hat{b}^-_{-\Omega} + \bar{q}_\Omega \hat{b}^+_{-\Omega} \right]$$

So, $$\hat{\phi}(U,V) = \int_0^\infty \frac{d\omega}{2\pi}\frac{1}{\sqrt{2\omega}}\Big[\underbrace{e^{-i\omega U}\hat{a}_\omega^- + e^{i\omega U}\hat{a}_\omega^+}_{\text{fun. of } U} + \underbrace{e^{-i\omega V}\hat{a}_{-\omega}^- + e^{i\omega V}\hat{a}_{-\omega}^+}_{\text{fun. of } V}$$

Lightcone mode expansions explicitly decompose the field $\hat{\phi}(U,V)$ into a sum of functions of U & V. This is expected from the form of the wave eqn.

- Hence, $\hat{\phi}(U,V) = \hat{A}(U) + \hat{B}(V)$

$$\hat{A}(U) = \int_0^\infty \frac{d\omega}{2\pi}\frac{1}{\sqrt{2\omega}}\left[e^{-i\omega U}\hat{a}_\omega^- + e^{i\omega U}\hat{a}_\omega^+\right]$$

$$\hat{B}(V) = \int_0^\infty \frac{d\omega}{2\pi}\frac{1}{\sqrt{2\omega}}\left[e^{-i\omega V}\hat{a}_{-\omega}^- + e^{i\omega V}\hat{a}_{-\omega}^+\right].$$

* Mode expansion in Rindler :

$$\hat{\phi}(u',v') = \hat{P}(u') + \hat{Q}(v').$$

$$= \int_0^\infty \frac{d\Omega}{2\pi}\frac{1}{\sqrt{2\Omega}}\Big[e^{-i\Omega u'}\hat{b}_\Omega^- + e^{i\Omega u'}\hat{b}_\Omega^+ + e^{-i\Omega v'}\hat{b}_{-\Omega}^- + e^{i\Omega v'}\hat{b}_{-\Omega}^+$$

* Note: • $\hat{A}(U)$ [& $\hat{P}(u')$] is expanded into operators $\hat{a}_\omega^\pm$ [& $\hat{b}_\Omega^\pm$] with +ve momenta ω [& Ω] and $\hat{B}(V)$ [$\hat{Q}$ into operators $\hat{a}_{-\omega}^\pm$ ($\hat{b}_{-\Omega}^\pm$) with negative momenta $-\omega$ (-

- ω, Ω can take only +ve values.
- Rindler mode expansion is valid only within the domain $x > |t|$ covered by the Rindler frame; it's only within this domain we can compare the two mode expansions.

The Rindler mode operators $b^{\dagger}_{\Omega}$, $b^{\dagger}_{-\Omega}$ are creation operators for ~~inward &~~ outward & inward propagating Rindler particles resp

The # particles that the accelerating observers measures near $\mathcal{I}_+$ is given by

$$\langle 0 | N_{\Omega} | 0 \rangle = \int_0^{\infty} \frac{d\omega}{2\pi} \left| \beta_{\Omega\omega} \right|^2$$

Minkowski vacuum

Rindler

To compute the flux of Rindler particles across $\mathcal{I}^+$ we only need the Bogoliubov coef :

iv) ## The Bogolyubov Transformations:

* <u>Goal</u>: Rel^{n} bet^{n} the operators $\hat{a}^{\pm}_{\pm\omega}$ & $\hat{b}^{\pm}_{\pm\Omega}$.

* $$\hat{\phi}(u', v) = \hat{P}(u') + \hat{Q}(v')$$
$$= \hat{P}(u'(u)) + \hat{Q}(v'(V)).$$

Since the coordinate transformation does not mix u' & v', we get

$$\left.\begin{aligned} \hat{A}(v(u')) &= \hat{P}(u') \\ \hat{B}(V(v')) &= \hat{Q}(v') \end{aligned}\right\}$$

So,
$$\hat{A}(U) = \int_0^\infty \frac{d\omega}{2\pi} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega u} \hat{a}^-_\omega + e^{i\omega u} \hat{a}^+_\omega \right]$$

$$= \hat{P}(u') = \int_0^\infty \frac{d\Omega}{2\pi} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u'} \hat{b}^-_\Omega + e^{i\Omega u'} \hat{b}^+_\Omega \right]$$

Here u is understood to be the function of u' given by our previous established rel^{ns}. Both sides of the above equ^{ns} are equal as a function of u'.

* So, we've found that the operators $\hat{a}^{\pm}_\omega$ with positive momenta are expressed through $\hat{b}^{\pm}_\Omega$ with positive momenta Ω. and similarly for -ve momenta.

* Now, we'll express positive-momentum operators $\hat{a}^{\pm}_u$ as explicit linear combinations of $\hat{b}^{\pm}_\Omega$. To do this we'll take help of F.T. $\Rightarrow$

F.T. of $\hat{p}(u') \Rightarrow$

$$\int_{-\infty}^{\infty} du' \, e^{i\Omega u'} \hat{p}(u')$$

$$= \int_{-\infty}^{\infty} du' \, e^{i\Omega u'} \int_0^{\infty} \frac{d\Omega'}{2\pi} \frac{1}{\sqrt{2\Omega'}} \left[e^{-i\Omega' u'} \hat{b}^-_{\Omega'} + e^{i\Omega' u'} \hat{b}^+_{\Omega'} \right]$$

$$= \int_0^{\infty} \frac{d\Omega'}{2\pi} \frac{1}{\sqrt{2\Omega'}} \int_{-\infty}^{\infty} du' \Big[\underbrace{e^{i(\Omega-\Omega')u'} \hat{b}^-_{\Omega'}}_{2\pi \cdot \delta(\Omega-\Omega') \hat{b}^-_{\Omega'}} + \underbrace{e^{i(\Omega+\Omega')u'} \hat{b}^+_{\Omega'}}_{2\pi \, \delta(\Omega+\Omega') \hat{b}^+_{\Omega'}} \Big] .$$

$$= \begin{cases} \dfrac{1}{\sqrt{2|\Omega|}} \hat{b}^-_{\Omega} , & \Omega > 0 \\[2ex] \dfrac{1}{\sqrt{2|\Omega|}} \hat{b}^+_{|\Omega|} , & \Omega < 0 . \end{cases}$$

F.T. of $\hat{A}(u) \Rightarrow$

$$\int_{-\infty}^{\infty} du' \, e^{i\Omega u'} \hat{A}(u(u'))$$

$$= \int_0^{\infty} \frac{d\omega}{\sqrt{2\omega}} \int_{-\infty}^{\infty} \frac{du'}{2\pi} \left[e^{i\Omega u' - i\omega u} \hat{a}^-_{\omega} + e^{i\Omega u' + i\omega u} \hat{a}^+_{\omega} \right]$$

$$\equiv \int_0^{\infty} \frac{d\omega}{\sqrt{2\omega}} \left[F(\omega, \Omega) \, \hat{a}^-_{\omega} + F(-\omega, \Omega) \, \hat{a}^+_{\omega} \right] .$$

where, we've introduced auxiliary function:

$$F(\omega, \Omega) \equiv \int_{-\infty}^{\infty} \frac{du'}{2\pi} e^{i\Omega u' - i\omega u}$$

$$= \int_{-\infty}^{\infty} \frac{du'}{2\pi} \exp\left[i\Omega u' + i\omega \frac{1}{a} e^{-au'} \right]$$

* Comparing the two F.T. restricting to +ve Ω $\Rightarrow$

$$\hat{a}_\omega^{\pm} \longleftrightarrow \hat{b}_\Omega^{-}$$

$\omega, \Omega > 0 \Rightarrow$ $$\frac{1}{\sqrt{2\Omega}} \hat{b}_\Omega^{-} = \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \left[F(\omega, \Omega)\, \hat{a}_\omega^{-} + F(-\omega, \Omega)\, \hat{a}_\omega^{+} \right]$$

$$\Rightarrow \boxed{\hat{b}_\Omega^{-} = \int_0^\infty d\omega \left[\alpha_{\omega\Omega}\, \hat{a}_\omega^{-} + \beta_{\omega\Omega}\, \hat{a}_\omega^{+} \right]}$$ Bogolyubov Transformation

where, the coef

$$\left.\begin{aligned} \alpha_{\omega\Omega} &\equiv \sqrt{\frac{\Omega}{\omega}}\, F(\omega, \Omega) \\ \beta_{\omega\Omega} &\equiv \sqrt{\frac{\Omega}{\omega}}\, F(-\omega, \Omega) \end{aligned}\right\} \text{Bogolyubov coef.}$$

* Similarly, $\hat{b}_\Omega^{+}$ can be expressed through $\hat{a}_\omega^{\pm}$ using the Hermitian conjugate of the above eqn and the identity

$$F^*(\omega, \Omega) = F(-\omega, -\Omega).$$

* The Bogolyubov transformation mixes creation & annihilation operators with different momenta $\omega \neq \Omega$.

* Similarly, the rel^n bet^n -ve-momenta operators $\hat{a}_{-\omega}^{\pm}$ & $\hat{b}_{-\Omega}^{\pm}$ is obtained from $\hat{B}(v) = \hat{Q}(v')$.

* Pros of Bogolyubov transformations & coef :

- Quantization condn in terms of creation & annihilation operators :

$$\begin{cases} [\hat{a}^-_\omega , \hat{a}^+_{\omega'}] = 2\pi \, \delta(\omega - \omega') \\ [\hat{b}^-_\Omega , \hat{b}^+_{\Omega'}] = 2\pi \, \delta(\cancel{\omega-\omega'} \, \Omega - \Omega') . \end{cases}$$

- Normalization condn on the coef of Bogolyubov transf :

Take, $2\pi\,\delta(\Omega-\Omega') = [\hat{b}^-_\Omega , \hat{b}^+_{\Omega'}]$

$$= \left[\int_{-\infty}^{\infty} d\omega \left(\alpha_{\omega\Omega} \hat{a}^-_\omega + \beta_{\omega\Omega} \hat{a}^+_\omega \right) , \int_{-\infty}^{\infty} d\omega' \left(\alpha^*_{\omega'\Omega'} \hat{a}^+_{\omega'} + \beta^*_{\omega'\Omega'} \hat{a}^- \right.\right.$$

where we've used the range of integration from $-\infty$ to $+\infty$ which is more general Bogolyubov transformation. For our case this is justified because the only non-zero Bogolyubov coef are those relating the momenta ω, Ω of equal sign, i.e. $\alpha_{-\omega\Omega} = 0$ & $\beta_{-\omega\Omega} = 0$.

$$= \int d\omega \, d\omega' \left(\alpha_{\omega\Omega} \alpha^*_{\omega'\Omega'} \, 2\pi\,\delta(\omega-\omega') - \beta_{\omega\Omega} \beta^*_{\omega'\Omega'} \, 2\pi\,\delta(\omega-\omega') \right).$$

$$\Rightarrow 2\pi\,\delta(\Omega-\Omega') = 2\pi \int d\omega \left[\alpha_{\omega\Omega} \alpha^*_{\omega\Omega'} - \beta_{\omega\Omega} \beta^*_{\omega\Omega'} \right].$$

$$\Rightarrow \boxed{\delta(\Omega-\Omega') = \int_{-\infty}^{\infty} d\omega \left(\alpha_{\omega\Omega} \alpha^*_{\omega\Omega'} - \beta_{\omega\Omega} \beta^*_{\omega\Omega'} \right)}.$$

- If we use $[\hat{b}^-_\Omega , \hat{b}^-_{\Omega'}] = 0$, we'll get

$$\boxed{\int_{-\infty}^{\infty} d\omega \left(\alpha_{\omega\Omega} \beta_{\omega\Omega'} - \alpha_{\omega\Omega'} \beta_{\omega\Omega} \right) = 0}.$$

- From above two relns we can derive inverse Bolodyubov transf :

$$\hat{a}^-_\omega = \int_{-\infty}^{\infty} d\Omega \left(\alpha^*_{\omega\Omega} \hat{b}^-_\Omega - \beta_{\omega\Omega} \hat{b}^+_\Omega \right).$$

v) Density of Particles:

Goal: * Since the vacua $|0_M\rangle$ & $|0_R\rangle$ corresponding to the operators $\hat{a}_\omega^-$ & $\hat{b}_\Omega^-$ are different, the M-vacuum is a state with R-particles & vice-versa. We'll now compute the density of R-particles in the M-vacuum state.

* R-number operator, $\hat{N}_\Omega \equiv \hat{b}_\Omega^+ \hat{b}_\Omega^-$. So, the avg. R-parti[cle] number in the M-vacuum $|0_M\rangle$ is equal to the expectation value of $\hat{N}_\Omega$:

$$\langle \hat{N}_\Omega \rangle \equiv \langle 0_M | \hat{b}_\Omega^+ \hat{b}_\Omega^- | 0_M \rangle$$

$$= \langle 0_M | \int d\omega \left[\alpha^*_{\omega\Omega} \hat{a}_\omega^+ + \beta^*_{\omega\Omega} \hat{a}_\omega^- \right] \int d\omega' \left[\alpha_{\omega'\Omega} \hat{a}_{\omega'}^- + \beta_{\omega'\Omega} \right. \ldots | 0_M \rangle$$

$$= \int_0^\infty d\omega \, |\beta_{\omega\Omega}|^2$$

This is the mean number of particles observed in the acceler[ated] frame. We need to calculate this quantity.

* One identity which will be needed:

$$F(\omega, \Omega) = F(-\omega, \Omega) \exp\left(\frac{\pi\Omega}{a}\right) \qquad \text{where, } \omega > 0,\ a > 0$$

and
$$F(\omega, \Omega) \equiv \int_{-\infty}^{\infty} \frac{du'}{2\pi} e^{i\Omega u' - i\omega u} = \int_{-\infty}^{\infty} \frac{du'}{2\pi} \exp\left[i\Omega u' + i\frac{\omega}{a} e^{-au} \right. \ldots$$

* Normalization Cond^{n} for cof :

$$\delta(\Omega-\Omega') = \int_{-\infty}^{\infty} d\omega \left(\alpha_{\omega\Omega}\, \alpha^*_{\omega\Omega'} - \beta_{\omega\Omega}\, \beta^*_{\omega\Omega'} \right)$$

Recall,

$$\left.\begin{aligned} \alpha_{\omega\Omega} &\equiv \sqrt{\frac{\Omega}{\omega}}\, F(\omega,\Omega) \\ \beta_{\omega\Omega} &\equiv \sqrt{\frac{\Omega}{\omega}}\, F(-\omega,\Omega) \end{aligned}\right\} \quad \omega>0\,,\ \Omega>0\,.$$

So,

$$\delta(\Omega-\Omega') = \int_0^{\infty} d\omega\, \frac{\sqrt{\Omega\Omega'}}{\omega} \left[F(\omega,\Omega)\, F^*(\omega,\Omega') - F(-\omega,\Omega)\, F^*(-\omega,\Omega') \right]$$

Using previous identity

$$= \int_0^{\infty} d\omega\, \frac{\sqrt{\Omega\Omega'}}{\omega}\, F(-\omega,\Omega)\, F^*(-\omega,\Omega') \left[\exp\left(\frac{\pi\Omega + \pi\Omega'}{a} \right) - 1 \right]$$

$$\Rightarrow \int_0^{\infty} d\omega\, \frac{\sqrt{\Omega\Omega'}}{\omega}\, F(-\omega,\Omega)\, F^*(-\omega,\Omega') = \left[\exp\left(\frac{2\pi\Omega}{a} \right) - 1 \right]^{-1} \delta(\Omega-\Omega')$$

* Now,

$$\langle \hat{N}_\Omega \rangle = \int_0^{\infty} d\omega\, |\beta_{\omega\Omega}|^2 = \int_0^{\infty} d\omega\, \frac{\Omega}{\omega}\, F(-\omega,\Omega)\, F^*(-\omega,\Omega)$$

$$= \left[\exp\left(\frac{2\pi\Omega}{a} \right) - 1 \right]^{-1} \delta(0)$$

As usual we expect $\langle \hat{N}_\Omega \rangle$ to be divergent since it's the total number of particles in the entire space. The divergent volume factor $\delta(0)$ represents the vol. of space, & the remaining fact is the density n_Ω of R-particles with momentum Ω :

$$\int^{\infty} d\omega\, |\beta_{\omega\Omega}|^2 \equiv n_\Omega\, \delta(0)$$

Zero-point Energy:

$$E_0 = \langle 0|\hat{H}|0\rangle = \frac{1}{2}\,\delta^{(3)}(0)\int d^3k\, \omega_{\vec{k}}$$

This expression diverges for two reasons

i) $\delta^{(3)}(0)$ diverges

ii) $\int d^3k\, \omega_{\vec{k}} = \int_0^\infty dk\, 4\pi k^2 \sqrt{m^2+k^2}$

diverges at the upper limit.

Now, ① explanation of the term $\delta^{(3)}(0)$:

It's the infinite volume of the entire space because $\Rightarrow$

$$(2\pi)^3\, \delta^{(3)}(\vec{k}-\vec{k}') = \int d^3x\, e^{i(\vec{k}\cdot\vec{x} - \vec{k}'\cdot\vec{x})}$$

dimension $\Rightarrow$ 3-volume.

for a field quantization in a finite box of vol. V,

$$\cancel{(2\pi)^3\, \delta^{(3)}(\vec{k}-\vec{k}')} \quad \int d^3x \cdot = V \quad (\vec{k}=\vec{k}').$$

$$\therefore \boxed{\delta^{(3)}(\vec{0}) = \lim_{V\to\infty} \frac{V}{(2\pi)^3}}$$

Now, dividing the energy E_0 by the volume V and taking the limit $V\to\infty$, we obtain zero point energy density

$$\lim_{V\to\infty} \frac{E_0}{V} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, \omega_{\vec{k}}$$

Renormalizing the zero-point energy:

The energy density is infinite because the integral $\int d^3k\, \omega_{\vec{k}}$ diverges at $|\vec{k}|\to\infty$. This is called ultraviolet divergence because large value of $|\vec{k}|$ corresponds high energy.

$\rightsquigarrow$ Normal ordered.

* ∴ Mean density of particles in the mode with momentum Ω is

$$\boxed{n_\Omega = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1}}$$

This is only for +ve-momentum modes ($\Omega > 0$).

The result for negative-momentum modes is obtained by replacing Ω by $|\Omega|$ in the above result.

VI) The Unruh Temperature:

* A massless particle with momentum Ω has energy $E = |\Omega|$, so the above formula is equivalent to the BE distribution

$$n(E) = \frac{1}{\exp\left(\frac{E}{T}\right) - 1}$$

where, $T \equiv \frac{a}{2\pi}$ is the Unruh Temperature.

* So we've found that an accelerated observer detects particles when the field $\hat{\phi}$ is in the Minkowski vacuum state $|0_M\rangle$. The particle distribution is characteristic of the thermal BB radiation with temp $T = \frac{a}{2\pi}$, where a is the magnitude of the proper acceleration (in Planck units). An accelerated detector behaves as though it were placed in a thermal bath with temp T. This is the Unruh Effect.

* Physical Interpretation: of the UE as seen in the lab frame is the following ⇒ the accelerated detector is coupled to the quantum fields & perturbs their quantum state around its trajectory. This perturbation is small but nevertheless due to this

perturbation the detector registers particles, although the fields were previously in the vacuum state. The detected particles are real & the energy for these particles comes from the agent that accelerates the detector.

$$\boxed{\begin{aligned} &* \quad \Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x} \\ &* \quad \Gamma(ix)\,\Gamma(-ix) = |\Gamma(ix)|^2 = \frac{\pi}{x \sinh(\pi x)} \end{aligned}}$$

$$* \quad \int_0^{+\infty} dx\, e^{-bx} x^{s-1} = b^{-s}\,\Gamma(s) \qquad (\$)$$

Complication arises when s & b are complex nos., because of the ambiguity of the phase of b. for eg. i^i is an inherently ambiguous expression since one may write

$$i^i = \left[\exp\left(\frac{i\pi}{2} + 2\pi i n\right)\right]^i = \exp\left[-\pi/2 - 2\pi n\right] \quad ; \; n \in \mathbb{Z}$$

(1) $\Rightarrow$ consider complex b s.t. $\mathrm{Re}\, b > 0$ [The integral diverges if $\mathrm{Re}\, b < 0$,

$$x^{s-1} e^{-bx} = \exp\left[-bx + (s-1)\ln x\right]$$

Now $\boxed{\ln(A+iB) \equiv \ln|A+iB| + i\,(\mathrm{sign}\, B)\tan^{-1}\frac{|B|}{A} \quad , \; A > 0}$

$$\boxed{*} \quad \boxed{\int_0^{\infty} x^{s-1} e^{-bx}\, dx = \exp(-s \ln b)\,\Gamma(s) \quad ; \; \mathrm{Re}\, b > 0,\ \mathrm{Re}\, s > 0} \qquad (1)$$

$$f(\omega, \Omega) \equiv \int_{-\infty}^{\infty} \frac{du'}{2\pi} \exp\left(i\Omega u' + i\frac{\omega}{a} e^{-au'}\right).$$

Let, $x \equiv e^{-au}$ Then $f(\omega,\Omega) = \frac{1}{2\pi a}\int_0^{\infty} dx\, x^{-\frac{i\Omega}{a}-1} e^{\frac{i\omega}{a}x}$

This has the same form as (1) with $b = -\frac{i\omega}{a}$, $s = -\frac{i\Omega}{a}$

$\because$ $\mathrm{Re}\, s = 0 \rightarrow$ integral diverges. Choose

$$b = -\frac{i\omega}{a} + \epsilon \quad , \quad s = -\frac{i\Omega}{a} + \epsilon \quad ; \quad \epsilon > 0.$$

and take limit $\epsilon \to 0^+$.

. Then $\ln b = \lim_{\epsilon \to +0} \ln\left(-\frac{i\omega}{a} + \epsilon\right) = \ln\left|\frac{\omega}{a}\right| - i\frac{\pi}{2}\,\mathrm{sign}\left(\omega/a\right)$

Hence,

$$F(\omega,\Omega) = \frac{1}{2\pi a}\int_0^\infty dx\; x^{-\frac{i\Omega}{a}-1}\; e^{\frac{i\omega}{a}x}$$

$$= \frac{1}{2\pi a}\exp\left[\cdot\frac{i\Omega}{a}\ln\left|\frac{\omega}{a}\right| + \frac{\pi\Omega}{2a}\,\mathrm{sign}\left(\frac{\omega}{a}\right)\right]\Gamma\left(-\frac{i\Omega}{a}\right).$$

$$F(-\omega,\Omega) = \frac{1}{2\pi a}\left[\exp\left\{\frac{i\Omega}{a}\ln\left|\frac{\omega}{a}\right| - \frac{\pi\Omega}{2a}\,\mathrm{sign}(\omega/a)\right\}\right]\Gamma\left(-\frac{i\Omega}{a}\right).$$

So,

$$\frac{F(\omega,\Omega)}{F(-\omega,\Omega)} = \exp\left(\frac{\pi\Omega}{a}\right) \quad ; \omega>0,\ \Omega>0$$

$$\Rightarrow \boxed{F(\omega,\Omega) = F(-\omega,\Omega)\exp\left(\frac{\pi\Omega}{a}\right)} \quad ; \omega>0,\ \underline{\underline{\Omega>0}}.$$

$$* \ |\beta_{\omega\Omega}|^2 = \frac{\Omega}{\omega}|F(-\omega,\Omega)|^2$$

$$= \frac{\Omega}{\omega}\cdot\frac{1}{4\pi^2a^2}\cdot\exp\left(-\frac{\pi\Omega}{a}\right)\left|\Gamma\left(-\frac{i\Omega}{a}\right)\right|^2$$

$$= \frac{1}{2\pi\omega a}\left[\frac{1}{\exp\left(\frac{2\pi\Omega}{a}\right)-1}\right]$$

$$\Gamma(ix)\Gamma(-ix) = |\Gamma(ix)|^2 = \frac{\pi}{x\sinh(\pi x)}$$