Two-Loop QCD Correction to Massive Spin-2 Resonance $\rightarrow$ 3 gluons

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in collaboration with

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Prologue : Why Quantum Correction?

- Improve the precision of the theoretical predictions and reduce the dependency of the physical observables on unphysical renormalization scale.
- Achieved by higher order quantum corrections in the framework of perturbation theory.

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Prologue : Perturbative Expansion of Cross Section

In perturbative QCD (pQCD) the N-particle scattering cross section :

$$\sigma_N = a_s^{\lambda}(\sigma_N^{(0)} + a_s \sigma_N^{(1)} + a_s^2 \sigma_N^{(2)} + ...), \qquad \lambda = 0, 1, ...$$

- LO :

$$\sigma_N^{(0)} \approx \int |\,|\mathcal{M}_N^{(0)}\rangle\,|^2 \, d\Phi_N$$

- NLO :

$$\sigma_N^{(1)} \approx \int 2\,\mathrm{Re}\,(\,\langle \mathcal{M}_N^{(0)} | \mathcal{M}_N^{(1)} \rangle\,)\, d\Phi_N + \int |\,|\mathcal{M}_{N+1}^{(0)}\rangle\,|^2 \, d\Phi_{N+1}$$

- NNLO :

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Prologue : Perturbative Expansion of Matrix Elements

- Necessary ingredients :
  1. virtual corrections $\rightsquigarrow |\mathcal{M}_N^{(1)}\rangle$, $|\mathcal{M}_N^{(2)}\rangle$, ...
  2. real corrections $\rightsquigarrow |\mathcal{M}_{N+1}^{(0)}\rangle$, $|\mathcal{M}_{N+2}^{(0)}\rangle$, ...
- We address the question : How do we calculate these?
- Therapy : calculable using perturbation theory. In **pQCD**

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Prologue : Goal

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$\downarrow$

Given a Lagrangian, how do we **calculate the loop-amplitude** following modern technique.

Reference

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Outline

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Why NNLO in EDM?

The Effective Action

Calculation of Loop Amplitude
- Feynman Diagrams
- Reduction of Loop-Integrals : IBPs and LIs
- Master Integrals
- Unrenormalizad Results

Renormalization
- UV Renormalization
- IR Factorization
- $h \to ggg$

Conclusions and Remarks

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Why Beyond NLO in Extra Dimensional Model?

1. Well studied at LHC : QCD corrections play an important role to constrain the model parameters.

   ⇝ K-factors for processes like $\gamma\gamma$, ZZ, WW, ll etc. at **NLO are as large as** ~ 1.6 to 1.8.

2. QCD has a very rich infrared divergence structure! UV renormalized QCD amplitude is not divergence free. In dimensional regularization ($d = 4 + \epsilon$)

   $$\text{UV renorm 1-loop QCD amplitude} \quad : \quad \frac{a_2^{(1)}}{\epsilon^2} + \frac{a_1^{(1)}}{\epsilon} + \text{finite}$$

   $$\text{UV renorm 2-loop QCD amplitude} \quad : \quad \frac{a_4^{(2)}}{\epsilon^4} + \frac{a_3^{(2)}}{\epsilon^3} + \frac{a_2^{(2)}}{\epsilon^2} + \frac{a_1^{(2)}}{\epsilon} + \text{finite}$$

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The Effective Action

- We consider the SM with an additional **massive** spin-2 field $h^{\mu\nu}$.
- Assumption : Spin-2 field couples to SM through minimal gravitational coupling i.e.,

$$\int d^4x \mathcal{L}(\eta, \mathrm{S}, \mathrm{V}, \mathrm{F}) \to \int d^4x \sqrt{|\hat{g}|} \mathcal{L}(\hat{g}, \mathrm{S}, \mathrm{V}, \mathrm{F}) \tag{1}$$

where, S = scalar fields, V = vector fields & F = fermionic fields and $\hat{g}^{\mu\nu}$ = induced metric in 4-dimension $\sim \eta^{\mu\nu} + \kappa h^{\mu\nu}$.

- $$\mathcal{S} = \mathcal{S}_{SM} + \mathcal{S}_h - \frac{\kappa}{2} \int d^4x \; T^{SM}_{\mu\nu}(x) \; h^{\mu\nu}(x) \tag{2}$$

$\kappa \sim M_P^{-1}$ is the strength of interaction.

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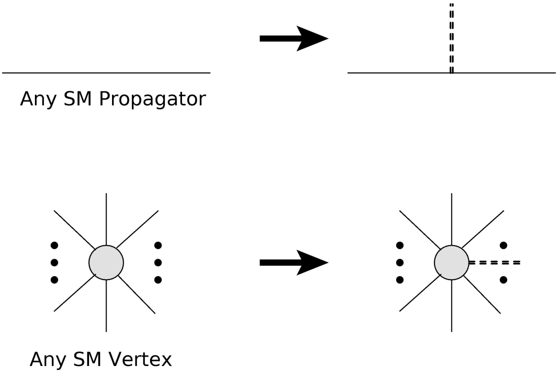

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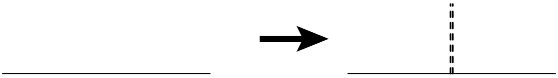
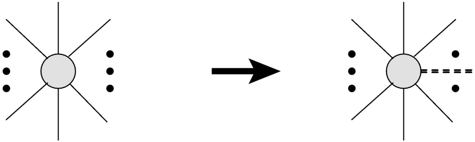


Any SM Propagator



Any SM Vertex

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- $T^{QCD}_{\mu\nu}(x)$: A big expression containing
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Calculation of Loop Amplitude(COLA) : Feynman Diagrams

- **Step 1** : **Generate Feynman diagrams** using QGRAF.

  For our case $h^*(M_h) \to g(p_1) + g(p_2) + g(p_3)$, the no. of diagrams

  Tree : 4

  1-loop : 108

  2-loop : 2362 !

  No spin-2 particle in loop or at intermediate propagator.

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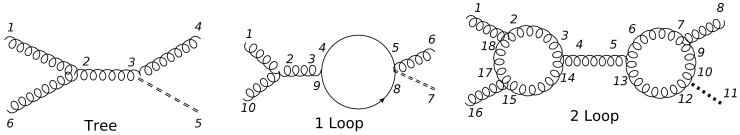

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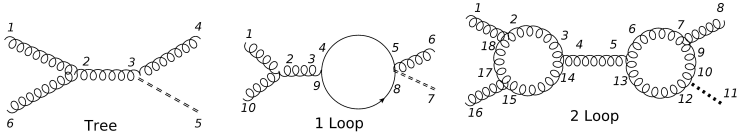

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COLA : Structure

- **Step 2** : In-house form routines convert symbolic raw output of QGRAF to a format usable in **FORM** and **apply Feynman rules**.

- **Step 3** : **Color simplification** using in-house form programs.

- Structure at l-loop for $h^* \to$ g g g :

$$\mathcal{M}_4^{(l)}(p_1, p_2, p_3) = \sum_{f=1}^{\text{no. of FD}} \mathcal{F}_f^{(l)}(p_1, p_2, p_3) \tag{4}$$

with

$$\mathcal{F}_f^{(l)}(p_1, p_2, p_3) = \epsilon^{\mu_1}(p_1)\,\epsilon^{\mu_2}(p_2)\,\epsilon^{\mu_3}(p_3)\,\epsilon^{\nu_1\,\nu_2}(q) \left(\int \prod_{\alpha=1}^{l} \frac{d^d k_\alpha}{(2\pi)^d} \frac{T_{\mu_1\,\mu_2\,\mu_3\,\nu_1\,\nu_2}(\{p_i\},\{k_j\})}{\prod_{\beta=1}^{n_{\text{prop}}} D_\beta}\right) \tag{5}$$

where, n_{prop} = no. of propagators present in the f-th l-loop FD.

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- **Step 4** : To do the polarization sum multiply with

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  OR

  (ii) CC of n-th loop amplitude .

  $n = 0 \Rightarrow \overline{\mathcal{M}_4^{(0)}}\mathcal{M}_4^{(l)}$ i.e. $\overline{\text{Tree amplitude}}$ * l-th loop amplitude.

  $n = 1 \Rightarrow \overline{\mathcal{M}_4^{(1)}}\mathcal{M}_4^{(l)}$ i.e. $\overline{\text{1-loop amplitude}}$ * l-th loop amplitude.

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COLA : Structure in Prescription 2

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\begin{aligned}
&\sum_{\text{spin}} \overline{\mathcal{M}_4^{(n)}} \mathcal{M}_4^{(l)} \\
= \quad & \left(\sum_{\text{spin}} \overline{\epsilon^{\mu_1'}(p_1)} \epsilon^{\mu_1}(p_1) \right) \left(\sum_{\text{spin}} \overline{\epsilon^{\mu_2'}(p_2)} \epsilon^{\mu_2}(p_2) \right) \left(\sum_{\text{spin}} \overline{\epsilon^{\mu_3'}(p_3)} \epsilon^{\mu_3}(p_3) \right) \\
& \left(\sum_{\text{spin}} \overline{\epsilon^{\nu_1' \, \nu_2'}(q)} \epsilon^{\nu_1 \, \nu_2}(q) \right) \\
& \left(\int \prod_{\alpha'=1}^{n} \frac{d^d k'_{\alpha'}}{(2\pi)^d} \overline{\frac{T'_{\mu_1' \, \mu_2' \, \mu_3' \, \nu_1' \, \nu_2'}(\{p_i\}, \{k'_j\})}{\prod_{\beta'=1}^{n_{\text{prop}}} D'_{\beta'}}} \right) \\
& \left(\int \prod_{\alpha=1}^{l} \frac{d^d k_\alpha}{(2\pi)^d} \frac{T_{\mu_1 \, \mu_2 \, \mu_3 \, \nu_1 \, \nu_2}(\{p_i\}, \{k_j\})}{\prod_{\beta=1}^{n_{\text{prop}}} D_\beta} \right) \qquad (6)
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COLA : Polarization Sum

- Choice of gauge :
  - **Axial gauge** for polarization sum of **external gluons**.
  - **Feynman gauge** for **internal gluons** $\Rightarrow$ internal ghost contributions to loop are taken.
- Polarization sum of **gluons** in **axial gauge** in **d-dimensions** :

$$\sum_{\text{spin}} \overline{\epsilon^{\mu}(p_j)}\epsilon^{\nu}(p_j) = -\eta^{\mu\nu} + \frac{p_j^{\mu} r_j^{\nu} + r_j^{\mu} p_j^{\nu}}{p \cdot r}$$

where, r_j^{μ} is reference momentum of corresponding gluon.

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$$\begin{aligned}\sum_{\text{spin}} \overline{\epsilon^{\mu\nu}(q)}\epsilon^{\rho\sigma}(q) \quad = \quad & \left(\eta^{\mu\rho} - \frac{q^{\mu}q^{\rho}}{q.q}\right)\left(\eta^{\nu\sigma} - \frac{q^{\nu}q^{\sigma}}{q.q}\right) + \left(\eta^{\mu\sigma} - \frac{q^{\mu}q^{\sigma}}{q.q}\right) \\ & \left(\eta^{\nu\rho} - \frac{q^{\nu}q^{\rho}}{q.q}\right) - \frac{2}{d-1}\left(\eta^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q.q}\right)\left(\eta^{\rho\sigma} - \frac{q^{\rho}q^{\sigma}}{q.q}\right)\end{aligned}$$

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  $\rightarrow$ Not a brilliant idea!

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  $\mathrm{n}_{SP} = l \cdot \mathrm{n}_{ileg} + \binom{l}{2} + l = l \cdot \mathrm{n}_{ileg} + \frac{l(l+1)}{2}$.

  For $l = 2$ & $\mathrm{n}_{ileg} = 3$, $\mathrm{n}_{SP} = 9$

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- **Step 5** : Classify the integrals into set of **independent integrals**
  $\Rightarrow$ minimize the number of integrals to be computed.

  - Express every integral in terms of ONLY propagators

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    since there are $n_{SP} = 9$ independent SP, we need the same no. of different propagators (#9) for this representation.

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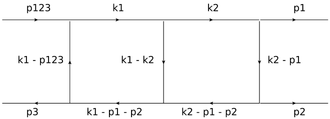

$$
\begin{aligned}
I \quad = \quad & \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{1}{D_1 D_2 D_3 D_4 D_5 D_6 D_7} \\
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\end{aligned}
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Introduce D_8 & D_9 s.t. they form a complete basis $\Rightarrow$ expressible SPs in terms of $D_1, \cdots, D_9$:

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I[a_1,\, a_2,\, \cdots,\, a_9] \equiv \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_9^{a_9}}
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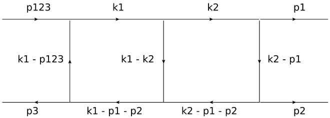

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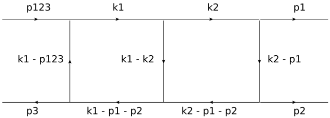

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A possible choice of basis :

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Indeed SPs are expressible in terms of these props and kinematical invariants:

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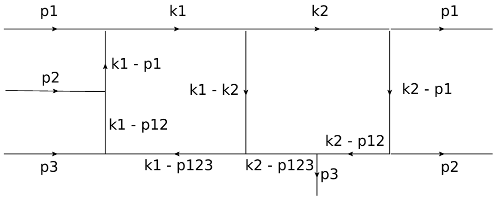
COLA : Basis - Topology

Basis - topology : a diagram containing **ALL** the propagators of a basis. For the above example :



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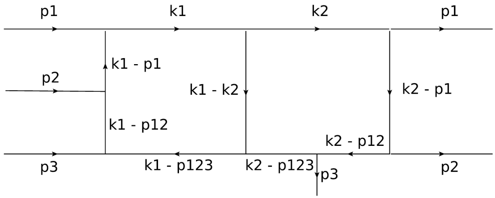

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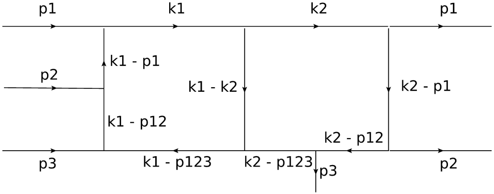

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- **Integrals are invariant w.r.t. shifts in loop momenta**

  $\rightsquigarrow$ all the 2362 2-loop integrals can be cast to belong to only TWO basis-topologies !

Example : Suppose a 2-loop diagram contains

$\{\ k_1,\ k_2,\ k_1+k_2,\ k_2+p_1+p_2,\ k_1+k_2+p_1+p_2\ \}$

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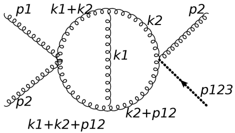

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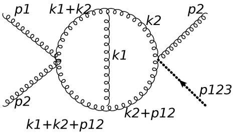

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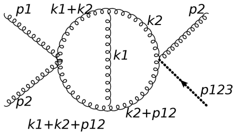
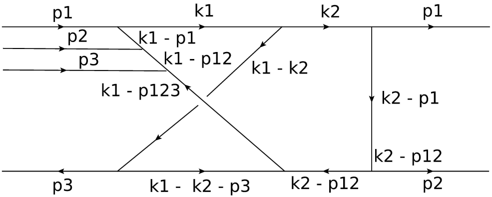


COLA : Basis - Topology

- The other basis-topology:



$\rightsquigarrow$ **non-planar**

$\rightsquigarrow$ k_1, k_2, $(k_1 - k_2)$, $(k_1 - p_1)$, $(k_2 - p_1)$, $(k_1 - k_2 - p_3)$, $(k_1 - p_1 - p_2)$, $(k_2 - p_1 - p_2)$, $(k_1 - p_1 - p_2 - p_3)$.

COLA : Full Set of Basis(2-loop)

- Full basis for **2-loop** 4-leg processes [1 massive] :

1. k_1, k_2, $(k_1 - k_2)$, $(k_1 - p_1)$, $(k_2 - p_1)$, $(k_1 - p_1 - p_2)$, $(k_2 - p_1 - p_2)$, $(k_1 - p_1 - p_2 - p_3)$, $(k_2 - p_1 - p_2 - p_3)$.
2. $\{p_1 \to p_2, p_2 \to p_3, p_3 \to p_1\}$
3. $\{p_1 \to p_3, p_2 \to p_1, p_3 \to p_2\}$

$\rightsquigarrow$ **Planar / basis-topology 1** (2-loop).

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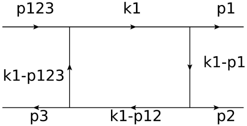

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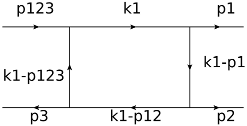

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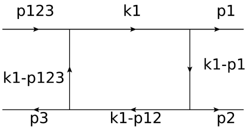

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- **Mini Summary**
  1. Computation of multiloop amplitude is reduced to computation of **scalar integrals**.
  2. All the scalar integrals (diagrams) are organized to belong to any of the above set of basis.
- Still large number of scalar integrals with different set of indices $\rightsquigarrow$ impractical to solve all!
- Identities for each topology $\rightsquigarrow$ further reduction.
  1. **Integraltion-by-parts (IBPs)** identities [*Chetyrkin-Tkachov*]
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  - only 21 types of 2-loop integrals!

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Outline

COLA : Integration-by-parts identities (IBPs)

- Generalization of **Gauss' divergence theorem** in d-dimensions.
- Any d-dimensional integral is **convergent**.
- Necessary condition for convergence : **surface terms vanish**

$$\boxed{\int \prod_{\alpha=1}^{l} \frac{d^d k_\alpha}{(2\pi)^d} \frac{\partial}{\partial k_{j,\mu}} \left(v^\mu f_{scalar}\right) = 0} \tag{7}$$

where,

$f_{scalar} = \left(\frac{1}{\prod_{i=1}^{n_{sp}} D_i^{a_i}}\right)$ and v^μ is loop or external momenta.

- Integrals remain within the same topology and its sub-topologies* due to differentiation

  $\rightsquigarrow$ **IBP of a topology relate integrals belonging to that and/or its sub-topologies**.

- With n_{ext} independent external momenta, there are $l(l+n_{ext})$ IBP identities

  for each set of indices $\{a_1, a_2, \cdots\}$.

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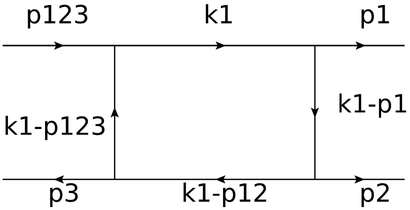
- **Example:** Topology $\rightsquigarrow$ 1-loop box (1-loop basis-topology)



$$\rightsquigarrow \int \frac{d^d k}{(2\pi)^d} \frac{1}{D_1^{a_1}\, D_2^{a_2}\, D_3^{a_3}\, D_4^{a_4}} \equiv I[a_1, a_2, a_3, a_4]$$

with $D_1 \equiv k_1$, $D_2 \equiv (k_1 - p_1)$, $D_3 \equiv (k_1 - p_1 - p_2)$, $D_4 \equiv (k_1 - p_1 - p_2 - p_3)$

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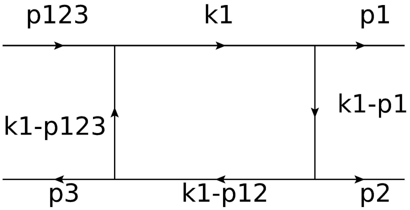

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■ For $v^\mu = p_{1\mu}$, IBP $\rightsquigarrow$

$$0 = \int \frac{d^d k}{(2\pi)^d} \quad \Big[\quad a_1(-1 + \frac{D_2}{D_1}) + a_2(1 - \frac{D_1}{D_2}) - a_3(\frac{D_1}{D_3} - \frac{D_2}{D_3} - \frac{s}{D_3}) - a_4(\frac{D_1}{D_4} - \frac{D_2}{D_4} - \frac{s}{D_4} - \frac{u}{D_4}) \quad \Big] \frac{1}{D_1^{a_1}\, D_2^{a_2}\, D_3^{a_3}\, D_4^{a_4}}$$

$$\Rightarrow \qquad a_1(-1 + 1^+2^-) + a_2(1 - 2^+1^-) - a_3(3^+1^- - 3^+2^- - s\,3^+) - a_4(4^+1^- - 4^+2^- - s\,4^+ - u\,4^+) = 0$$

Convention: $1^+2^- I[a_1, a_2, a_3, a_4] = I[a_1 + 1, a_2 - 1, a_3, a_4]$

Similarly for $p_{2\mu}, p_{3\mu}, k_\mu$.

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■ **Clearly IBP gives recursion relations among the integrals of a topology and/or its sub-topologies.**

⇒ Only 3 independent integrals!, called master integrals (MIs).

$$I[1,0,1,0] \to \{k_1, k_1 - p_1 - p_2\}$$

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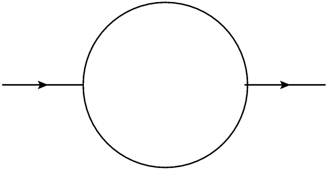
■ First two belong to a sub-topology (bubble)* :



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■ Effectively computation of 2 integrals is needed since first two are similar.

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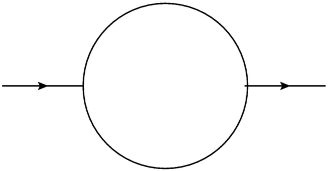

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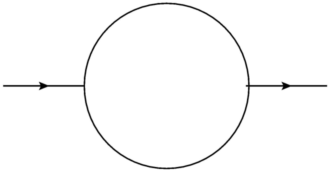

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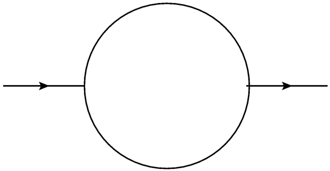

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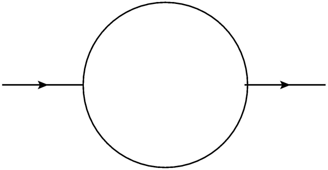

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COLA : Lorentz invariance identities (LIs)

- Scalar loop integrals are **Lorentz scalars** :

$$p_i^\mu \to p_i^\mu + \delta p_i^\mu = p_i^\mu + \omega^{\mu\nu} p_\nu \,, \quad \text{with} \quad \omega^{\mu\nu} = -\omega^{\nu\mu}$$

$$I(p_i + \delta p_i) = I(p_i) + \omega^{\mu\nu} \sum_j p_{j,\nu} \frac{\partial}{\partial p_j^\mu} I(p_i) = I(p_i)$$

- Anti-symmetry of $\omega^{\mu\nu}$ gives

$$\sum_i p_{i,[\mu} \frac{\partial}{\partial p_i^{\nu]}} I(p_i) = 0$$

- Multiply anti-symmetric combination of $p_j^\mu p_k^\nu$ to get scalar relations among the scalar integrals:

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- **Example : 4-pt functions**

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COLA : Implementation of Identities in Computer

- Implemented IBPs and LIs in computer following some algorith :

  AIR

  FIRE

  REDUZE

  LiteRed* - *we have used this.*

COLA : IBPs + LIs

For $h^* \to g(p_1)\ g(p_2)\ g(p_3)$

- 1-loop :

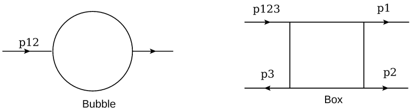
  - No. of **diag.** 108

  - No. of **zero diag. due to color algebra & momentum dependence:** 19

    $\rightsquigarrow$ non-zero 89.

  - **All** $\in$ **planar basis-topology (1-loop box).**

  - IBPs + LIs $\Rightarrow$ **ONLY 2 planar topologies of master integrals!**



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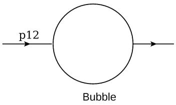

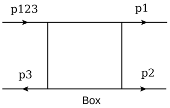

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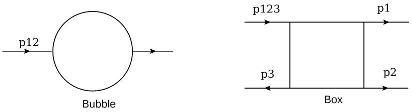

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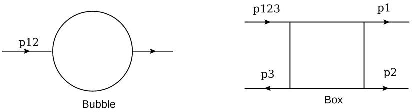

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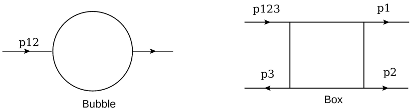

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  • **1863** $\in$ **planar basis-topology (basis-topology 1)**
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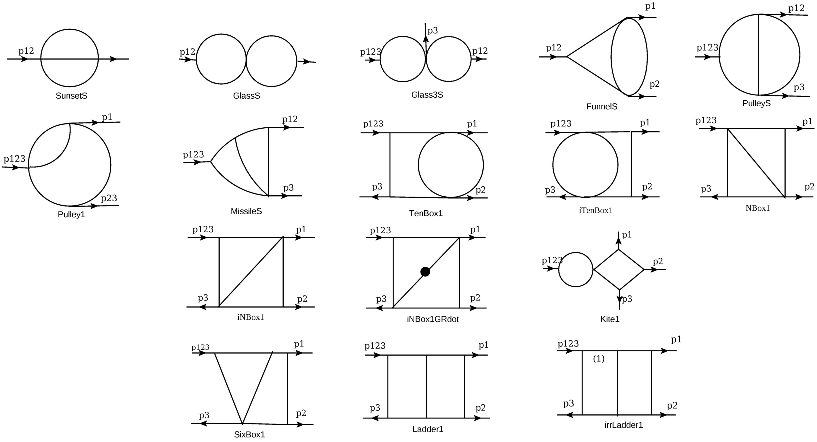
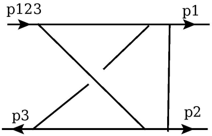
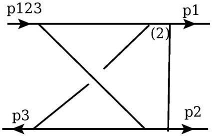


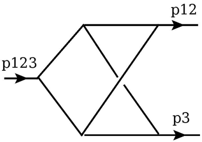
COLA : Non-planar Master-Topologies for h* $\rightarrow$ ggg (2-loop)



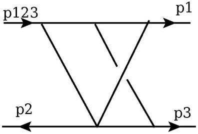
XBox1



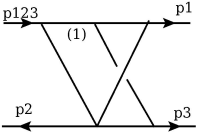
irrXBox1



Rocket3S



VXBox1p



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- The results of these MIs are taken from a paper by *T. Gehrmann and E. Remiddi* which are computed using differential equation method.
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- The loop integrals encounter **singularities** in **4-dimension** :

1. Singularities arising from the **high-momentum** limit of the loop integrals
   $\rightsquigarrow$ **UV divergences**.

2. Singularities arising from
   - the **zero-momentum** limit of the loop integrals (**soft**) :

     $\oplus$ associated with **massless vector bosons** : arise in gauge theories only.

     $\oplus$ present also when matter particles are **massive**.

   - the **collinearity** of loop momenta to one of the massless external particles(**collinear**):

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How do we remove these singularities?

- **Regularization**

  Before renormalizing the theory, regulate the integrals in some regularization scheme (for both UV and IR) :

  1. Momentum cutoff regularization,
  2. Pauli-Villars regularization,
  3. Dimensional regularization (DR).

  $\rightsquigarrow$ In **DR** change the dimension from 4 to $\mathbf{d = 4 + \epsilon}$.

  $\rightsquigarrow$ The regularization is of course removed after the cancellation of divergences.

- The singularities of loop integrals will show up as **poles** in ϵ.

- **Removing UV divergence**

  redefine all the bare fields and couplings of the Lagrangian $\rightarrow$ generate precisely those singular terms required to render the theory UV finite.

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- **Removing Soft divergence**

  1. a quark with **virtual** gluons
  2. a quark accompanied by an arbitrary no. of **real soft** gluons

  } **indistinguishable** $\in$ same energy eigenstate as that of quark.

$\rightsquigarrow$ **degenerate states (soft)**

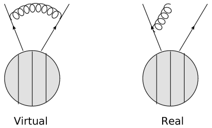
$\rightsquigarrow$ this degeneracy is the origin of the soft divergence in transition matrix element.

$\rightsquigarrow$ **summation over these degenerate states eliminate** the soft - divergence (KLN theorem).

$\rightsquigarrow$

$$\sum_{Degn.st. \in final} (\text{virtual} + \text{real}) \rightsquigarrow \text{soft free}$$

For eg.



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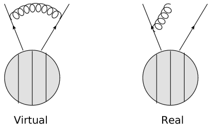

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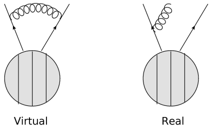

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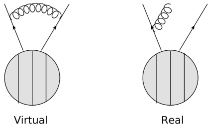

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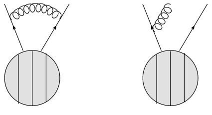

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⇝ **Reason** : no summation is done over initial degenerate states at partonic cross-section level.

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  $\rightsquigarrow$ **Reason** : no summation is done over initial degenerate states at partonic cross-section level.

  For $a + b \to c + d$, we sum over only the states of final particles i.e. c, d. No sum over states of initial particles at the partonic cross-section level.

Renormalization Procedure : Collinear

- **Removing Collinear divergence**

1. a massless quark with **virtual** gluons
2. a massless quark accompanied by an arbitrary no. of **real collinear** gluons

} **indistinguishable**

$\rightsquigarrow$ **degenerate states (collinear)**

$\rightsquigarrow$ this degeneracy is the origin of the collinear divergence in transition matrix element.

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⇝ **Unlike soft** in the collinear, external massless particles are involved : collinearity of the external massless partons with massless real or virtual partons.

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$$\sum_{Deg.sta. \in ini, final} (\text{virtual} + \text{real}) \rightsquigarrow \text{collinear free}$$

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Outline

Renormalization for $h \to ggg$

- For $h \to ggg$

  - $$|\mathcal{M}\rangle = \left(\frac{\hat{a}_s}{\mu_0^\epsilon} S_\epsilon\right)^{\frac{1}{2}} |\hat{\mathcal{M}}^{(0)}\rangle + \left(\frac{\hat{a}_s}{\mu_0^\epsilon} S_\epsilon\right)^{\frac{3}{2}} |\hat{\mathcal{M}}^{(1)}\rangle + \left(\frac{\hat{a}_s}{\mu_0^\epsilon} S_\epsilon\right)^{\frac{5}{2}} |\hat{\mathcal{M}}^{(2)}\rangle + \mathcal{O}(\hat{a}_s^3)\,, \tag{8}$$

  with

  $S_\epsilon = \exp[\frac{\epsilon}{2}(\gamma_E - \ln 4\pi)]$ with Euler constant $\gamma_E = 0.5772\ldots$

  $|\hat{\mathcal{M}}^{(i)}\rangle$: unrenormalized matrix element representing the i^{th} loop amplitude.

  μ_0 : introduced to keep $\hat{a}_s$ dimensionless in d-dimension.

  - UV renormalization in $\overline{MS}$

$$\begin{aligned} \frac{\hat{a}_s}{\mu_0^\epsilon} S_\epsilon &= \frac{a_s}{\mu_R^\epsilon} Z(\mu_R^2) \\ &= \frac{a_s}{\mu_R^\epsilon}\left[1 + a_s \frac{2\beta_0}{\epsilon} + a_s^2\left(\frac{4\beta_0^2}{\epsilon^2} + \frac{\beta_1}{\epsilon}\right) + \mathcal{O}(a_s^3)\right] \end{aligned} \tag{9}$$

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Compairing (8) and (10) we get **UV renormalized** matrix elements :

$$\begin{aligned}
|\mathcal{M}^{(0)}\rangle &= \left(\frac{1}{\mu_R^\epsilon}\right)^{\frac{1}{2}} |\hat{\mathcal{M}}^{(0)}\rangle \,, \\
|\mathcal{M}^{(1)}\rangle &= \left(\frac{1}{\mu_R^\epsilon}\right)^{\frac{3}{2}} \left[|\hat{\mathcal{M}}^{(1)}\rangle + \mu_R^\epsilon \frac{r_1}{2} |\hat{\mathcal{M}}^{(0)}\rangle \right] \,, \\
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IR Structure of $h \to ggg$

• The UV renormalized matrix elements contain divergences coming from **IR** region of massless QCD (**soft + collinear**).

IR divergence in QCD has universal structure! $\rightsquigarrow$ depends only on the external partons.

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\begin{aligned}
|\mathcal{M}^{(1)}\rangle &= 2\,\mathbf{I}_g^{(1)}(\epsilon)\,|\mathcal{M}^{(0)}\rangle + |\mathcal{M}^{(1)fin}\rangle \\
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Catani predicted all $a^{(1)}$ and $a^{(2)}$ except $a_1^{(2)}$, later verified by *Sterman & Tejeda*. *Becher and Neubert* derived the same including $a_1^{(2)}$.

$\rightsquigarrow$ These poles serve as the most **crucial check** of any calculation.

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IR Structure of $h \to ggg$

- Our results agree with these poles including the single pole!
- Even in the presence of **spin-2** particle, QCD amplitude **factorizes** into soft-collinear and hard parts in a **universal** way $\rightsquigarrow$ **something new!**

Conclusions and Remarks

- We have seen the power of IBPs and LIs : thousands of feynman integrals are reducible to only few MIs!
- **Crucial check :** Gauge invarince and universal IR pole structure.
- Explicitly verified the universality of QCD amplitude factorization even with the presence of spin-2 graviton.
- Renormalized finite part can't be shown here...50 pages long!
- This finite part can be analytically continued to get the result for production of massive spin-2 graviton with one jet in gluon gluon fusion.
- We have computed one of the most difficult parts $\rightsquigarrow$ 2-loop contribution to graviton + jet production $\rightsquigarrow$ important ingradients of the full NNLO computation.
- Remaining parts involve real emisiion & phase space integration $\rightsquigarrow$ new developments are underway (phase space slicing, antenna subtraction etc) .

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Harmonic Polylogarithms (HPLs)

- Logarithms, polylogarithms ($\mathrm{Li}_n(x)$) and Nielsen's polylogarithm ($\mathrm{S}_{n,p}(x)$) appear naturally in the analytical expressions of radiative correction in pQCD.

  - $$\ln(x) = \int_1^x \frac{dt}{t}$$
  - $$\mathrm{Li}_n(x) \equiv \sum_{k=1}^{\infty} \frac{x^k}{k^n} = \int_0^x \frac{dt}{t}\mathrm{Li}_{n-1}(t) \qquad \textbf{e.g.} \quad \mathrm{Li}_1(x) = -\ln(1-x)$$
  - $$S_{n,p}(x) \equiv \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{dt}{t}[\ln(t)]^{n-1}[\ln(1-xt)]^p$$

    e.g. $S_{n-1,1}(x) = \mathrm{Li}_n(x)$

- But, for higher order (2-loops and beyond) these functions are not sufficient to evaluate all the loop integrals appearing in the Feynman graphs .
- Overcome by introducing new set of functions : **Harmonic Polylogarithms (HPLs)** $\rightsquigarrow$ generalization of NP.

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HPLs

- **1D HPLs**

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\begin{aligned}
H(0;x) &\equiv \ln(x) \\
H(1;x) &\equiv \int_0^x \frac{dt}{1-t} = -\ln(1-x) \\
H(-1;x) &\equiv \int_0^x \frac{dt}{1+t} = \ln(1+x)
\end{aligned}
$$

Consequently,

$$
\frac{d}{dx} H(a;x) = f(a;x) \qquad a \in \{-1,0,1\}
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- HPLs of **higher weight** are **defined recursively**.
  - Introduce $\vec{m}_\omega \equiv (a, \vec{m}_{\omega-1})$. Each component $\in \{-1, 0, 1\}$.

  e.g. $\vec{0}_\omega = (0, 0, \cdots, 0) \rightsquigarrow \omega$ no. of 0.
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