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Advanced QFT Course by Prof. Ravindran

1. $S = + \frac{1}{2} \int d^4x \left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2 \right)$

(a) $\phi = \chi + \frac{1}{2} \lambda \chi^2$

$$S = \frac{1}{2} \int d^4x \left[\partial_\mu \left(\chi + \frac{1}{2}\lambda \chi^2\right) \partial^\mu \left(\chi + \frac{1}{2}\lambda\chi^2\right) - m^2 \left(\chi + \frac{1}{2}\lambda\chi^2\right)^2 \right]$$

$$= \frac{1}{2} \int d^4x \left[\left(\partial_\mu \chi \, \partial^\mu \chi - m^2 \chi^2 \right) + \lambda \left(2\chi \partial_\mu \chi \, \partial^\mu \chi - m^2 \chi^3 \right) + \lambda^2 \left(\chi^2 \partial_\mu \chi \, \partial^\mu \chi - \frac{1}{4} m^2 \chi^4 \right) \right]$$

$$\equiv S_{free} + S_{int}(\lambda) + S_{int}(\lambda^2)$$

(b) Define, $\tilde{\chi}(k) = \int d^4x \, e^{ik\cdot x} \chi(x)$

$$\Rightarrow \chi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{\chi}(k)$$

$$S_{free} = \frac{1}{2} \int d^4x \left(\partial_\mu \chi \, \partial^\mu \chi - m^2 \chi^2 \right)$$

$$= \frac{1}{2} \int d^4x \left(\partial_\mu \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{\chi}(k) \, \partial^\mu \int \frac{d^4k'}{(2\pi)^4} e^{-ik'\cdot x} \tilde{\chi}(k') - m^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} e^{-i(k+k')\cdot x} \tilde{\chi}(k) \tilde{\chi}(k') \right)$$

$$= \frac{1}{2} \int d^4x \left[\int \frac{d^4k}{(2\pi)^4} (-ik_\mu) e^{-ik\cdot x} \tilde{\chi}(k) \int \frac{d^4k'}{(2\pi)^4} (-ik'^\mu) e^{-ik'\cdot x} \tilde{\chi}(k') - m^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} e^{-i(k+k')\cdot x} \tilde{\chi}(k) \tilde{\chi}(k') \right.$$

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k+k') \Big[(-k\cdot k')\, \tilde{\chi}(k)\, \tilde{\chi}(k') - m^2\, \tilde{\chi}(k)\, \tilde{\chi}(k') \Big]$$

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4}\, \tilde{\chi}(-k)\, (k^2 - m^2)\, \tilde{\chi}(k)$$

So, $$\boxed{S_{free} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4}\, \tilde{\chi}(-k)\, (k^2 - m^2)\, \tilde{\chi}(k)}$$

$$S_{int}(\lambda) = -\lambda/2 \int d^4x \left(2\chi\, \partial_\mu \chi\, \partial^\mu \chi - m^2 \chi^3 \right)$$

$$= -\lambda/2 \int \prod_{i=1}^{3} \left[\frac{d^4k_i}{(2\pi)^4} \tilde{\chi}(k_i) \right] \left(-2k_2\cdot k_3 - m^2 \right) (2\pi)^4 \delta^{(4)}\left(\Sigma k_i\right)$$

$$= -\lambda/2 \int \prod_{i=1}^{3} \left[\frac{d^4k_i}{(2\pi)^4} \tilde{\chi}(k_i) \right] (2\pi)^4 \delta^{(4)}\left(\Sigma k_i\right) \left\{ -\frac{2}{3} \left(k_1\cdot k_2 + k_2\cdot k_3 + k_3\cdot k_1 \right) - m^2 \right\}$$

$$\Rightarrow \boxed{S_{int}(\lambda) = \frac{\lambda}{3!} \int \prod_{i=1}^{3} \left[\frac{d^4k_i}{(2\pi)^4} \tilde{\chi}(k_i) \right] (2\pi)^4 \delta^{(4)}\left(\Sigma k_i\right) \Big[2\left(k_1\cdot k_2 + k_2\cdot k_3 + k_3\cdot k_1 \right) + 3m^2 \Big]}$$

and,

$$S_{int}(\lambda^2) = \lambda^2/2 \int d^4x \left(\chi^2\, \partial_\mu \chi\, \partial^\mu \chi - \frac{m^2}{4} \chi^4 \right)$$

$$= \lambda^2/2 \int \left[\prod_{i=1}^{4} \frac{d^4k_i}{(2\pi)^4} \tilde{\chi}(k_i) \right] (2\pi)^4 \delta^{(4)}\left(\Sigma k_i\right) \left\{ -k_3\cdot k_4 - m^2/4 \right\}$$

$$= \lambda^2/2 \int \left[\prod_{i=1}^{4} \frac{d^4k_i}{(2\pi)^4} \tilde{\chi}(k_i) \right] (2\pi)^4 \delta^{(4)}\left(\Sigma k_i\right) \left\{ -\frac{1}{6} \left(k_1\cdot k_2 + k_2\cdot k_3 + k_3\cdot k_4 + k_1\cdot k_3 + k_1\cdot k_4 + k_2\cdot k_4 \right) - m^2/4 \right\}$$

$$= -\frac{\lambda^2}{4!} \int \left[\prod_{i=1}^{4} \frac{d^4k_i}{(2\pi)^4} \tilde{\chi}(k_i) \right] (2\pi)^4 \delta^{(4)}\left(\Sigma k_i\right) \left\{ 2\left(k_1\cdot k_2 + k_1\cdot k_3 + k_1\cdot k_4 + k_2\cdot k_3 + k_3\cdot k_4 + k_2\cdot k_4 \right) + 3m^2 \right\}$$

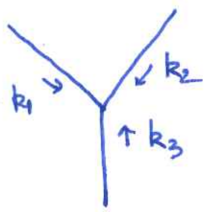
$$\boxed{S_{int}(\lambda^2) = -\frac{\lambda^2}{4!}\int\left[\prod_{i=1}^{4}\frac{d^4k_i}{(2\pi)^4}\,\tilde{\chi}(k_i)\right](2\pi)^4\,\delta^{(4)}\Big(\sum k_i\Big)\left\{2\sum_{\substack{j<k\\ \cancel{j\neq k}}} k_j\cdot k_k + 3m^2\right\}}$$

Momentum space Feynman rules

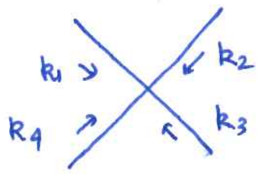
$\boxed{\text{Propagator}} \Rightarrow$

$$\Longleftrightarrow \quad (2\pi)^4\,\delta^4(k_1+k_2)\,\frac{i}{k_1^2-m^2+i\epsilon}$$

$\boxed{\text{Vertices}}$



$$\Longleftrightarrow \quad \frac{i\lambda}{3!}(2\pi)^4\,\delta^{(4)}\Big(\sum_{i=1}^{3}k_i\Big)\left\{2\sum_{\substack{i<j\\ \cancel{i\neq j}\\ i,j=1}}^{3} k_i\cdot k_j + 3m^2\right\}$$

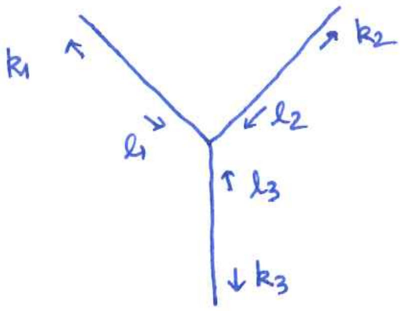


$$\Longleftrightarrow \quad -\frac{i\lambda^2}{4!}(2\pi)^4\,\delta^{(4)}\Big(\sum_{i=1}^{4}k_i\Big)\left\{2\sum_{\substack{i<j\\ i,j=1}}^{4} k_i\cdot k_j + 3m^2\right\}.$$

(c) $\langle \tilde{x}(k_1)\,\tilde{x}(k_2)\,\tilde{x}(k_3)\rangle_c$

<u>At $O(\lambda^0)$</u> there is no contribution.

<u>At $O(\lambda)$</u>



vertex factor symmetry factor

$$\frac{i\lambda}{3!}\int\prod_{i=1}^{3}\frac{d^4 l_i}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}\Big(\sum_{i=1}^{3} l_i\Big)\Big[2\sum_{\substack{i<j\\ i,j=1}}^{3} l_i\cdot l_j+3m^2\Big]\times 3!$$

$$\underbrace{\times\ i(2\pi)^4\delta^4(k_1+l_1)\frac{1}{k_1^2-m^2+i\epsilon}\times i(2\pi)^4\delta^4(k_2+l_2)\frac{1}{k_2^2-m^2+i\epsilon}\times i(2\pi)^4\delta^4(k_3+l_3)\frac{1}{k_3^2-m^2+i\epsilon}}_{\text{propagator factor}}$$

$$=\frac{i\lambda}{3!}\times 3!\,(2\pi)^4\delta^4(k_1+k_2+k_3)\Big[2\sum_{\substack{i<j\\ i,j=1}}^{3}\cancel{l_i\cdot l_j}\ k_i\cdot k_j+3m^2\Big]\prod_{i=1}^{3}\frac{i}{k_i^2-m^2+i\epsilon}$$

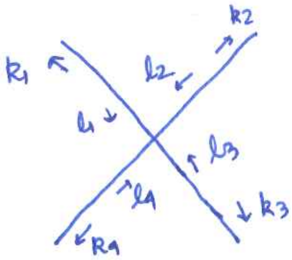
So,

$$\boxed{\langle \tilde{x}(k_1)\,\tilde{x}(k_2)\,\tilde{x}(k_3)\rangle_c = i\lambda\,(2\pi)^4\,\delta^4\Big(\sum k_i\Big)\Big[2\sum_{\substack{i<j\\ i,j=1}}^{3}k_i\cdot k_j+3m^2\Big]\prod_{i=1}^{3}\frac{i}{k_i^2-m^2+i\epsilon}}$$

PS1_ 1.5

(d) $\langle \tilde{x}(k_1)\,\tilde{x}(k_2)\,\tilde{x}(k_3)\,\tilde{x}(k_4)\rangle_c$

Diag 1 :-



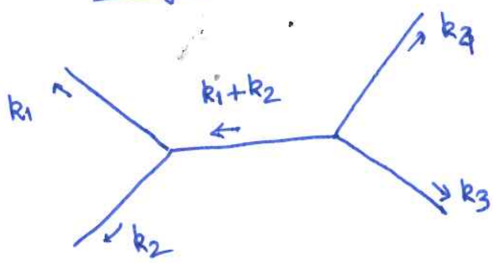
$$= -\frac{i\lambda^2}{4!}\int\left[\prod_{i=1}^{4}\frac{d^4 l_i}{(2\pi)^4}\right](2\pi)^4\,\delta^4\left(\sum l_i\right)\left[2\sum_{\substack{i<j\\ i,j=1}}^{4} l_j\cdot l_i + 3m^2\right]\times 4!$$

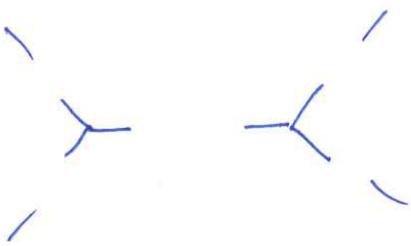
$$\times\prod_{j=1}^{4}\left[i\,(2\pi)^4\,\delta^4(k_j+l_j)\,\frac{1}{k_j^2-m^2+i\epsilon}\right]$$

$$= -i\lambda^2\,(2\pi)^4\,\delta^4\left(\sum_{i=1}^{4}k_i\right)\left[\prod_{j=1}^{4}\frac{i}{k_j^2-m^2+i\epsilon}\right]\left\{2\sum_{i<j}k_i\cdot k_j+3m^2\right\}.$$

.... (1)

Diag 2 :-





$$= \left(\frac{i\lambda}{3!}\right)^2(2\pi)^4\,\delta^4\left(\sum k_i\right)\cancel{\left[2\{k_1\cdot k_2+(2k_2)\right.}$$

$$\left[2\{(-k_1)\cdot(-k_2)+(-k_2)\cdot(k_1+k_2)+(-k_1)\cdot(k_1+k_2)\}+3m^2\right]$$

$$\left[2\{(-k_3)\cdot(-k_4)+(-k_4)\cdot(-k_1-k_2)+(-k_3)\cdot(-k_1-k_2)\}+3m^2\right]$$

$$\left[\prod_{j=1}^{4}\frac{i}{k_j^2-m^2+i\epsilon}\right]\left[\frac{i}{(k_1+k_2)^2-m^2+i\epsilon}\right]\times\underbrace{(6\times2\times3\times2)}_{\text{symmetry factor}}\times\underbrace{\frac{1}{2!}}_{\substack{\text{vertex}\\ \text{symmetry}\\ \text{factor}}}$$

$$= \frac{i^7 \lambda^2}{(3!)^2} (2\pi)^4 \delta^4 (\Sigma k_i) \left[2 \left\{ \cancel{k_1 k_2} - \cancel{k_1 k_2} - k_2^2 - k_1^2 - k_1 \cdot k_2 \right\} + 3m^2 \right]$$

$$\left[2 \left\{ \cancel{k_3 k_4} - \cancel{k_3 k_4} - k_4^2 - k_3^2 - k_3 \cdot k_4 \right\} + 3m^2 \right]$$

$$\left[\prod_{j=1}^{4} \frac{1}{k_j^2 - m^2 + i\epsilon} \right] \frac{1}{(k_1+k_2)^2 - m^2 + i\epsilon} \times 36$$

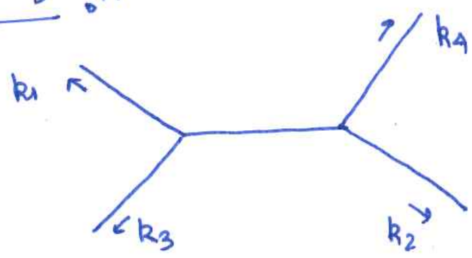
$$= -i \lambda^2 (2\pi)^4 \delta^4 (\Sigma k_i) \left[+2 \left(-2m^2 - k_1 \cdot k_2 \right) + 3m^2 \right]$$

$$\left[2 \left(-2m^2 - k_3 \cdot k_4 \right) + 3m^2 \right] \left[\prod_{j=1}^{4} \frac{1}{k_j^2 - m^2 + i\epsilon} \right] \frac{1}{(k_1+k_2)^2 - m^2 + i\epsilon}$$

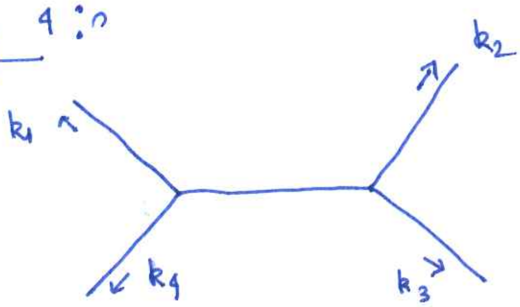
$$= - i\lambda^2 (2\pi)^4 \delta^4 (\Sigma k_i) \left[-m^2 - 2k_1 \cdot k_2 \right] \left[-m^2 - 2k_3 \cdot k_4 \right]$$

$$\left[\prod_{j=1}^{4} \frac{1}{k_j^2 - m^2 + i\epsilon} \right] \frac{1}{(k_1+k_2)^2 - m^2 + i\epsilon} \quad \cdots\cdots \; (2)$$

Diag 3 :-



Diag 4 :-



similar treatment of Diag 2.

Hence,

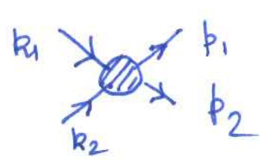
$$\langle \tilde{x}(k_1)\, \tilde{x}(k_2)\, \tilde{x}(k_3)\, \tilde{x}(k_4) \rangle_c$$

$$= \text{Diag } 1 + \text{diag } 2 + \text{diag } 3 + \text{diag } 4 .$$

(d)

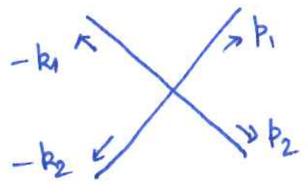
$$s = (k_1+k_2)^2 \qquad\qquad s+t+u = 4m^2$$
$$t = (k_1-p_1)^2$$
$$u = (k_1-p_2)^2$$

AIM: $S(\underbrace{p_1, p_2}_{\text{outgoing}} ; \underbrace{k_1, k_2}_{\text{incoming}}) = \; ?$ to $\mathcal{O}(\lambda^2)$



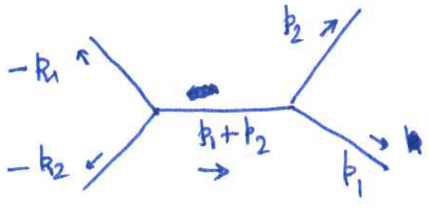
$$S(p_1, p_2 ; k_1, k_2) = z^2 \, \underbrace{\hat{G}^{(4)}}_{\text{truncated Green's function}} (-k_1, -k_2 ; p_1, p_2)$$

contribution 1 (⟺ diag 1 of (c) with $k_1, k_2, k_3, k_4 \to -k_1, -k_2, p_1, p_2$)



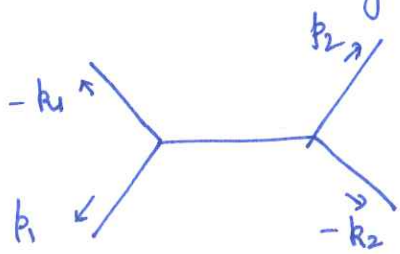
$$\hat{G}_1^{(4)} = -i\lambda^2 (2\pi)^4 \delta^4(p_1+p_2-k_1-k_2) \left\{ 2\left(k_1\cdot k_2 + p_1\cdot p_2 - k_1\cdot p_1 - k_1\cdot p_2 - k_2\cdot p_1 - k_2\cdot p_2\right) + 3m^2 \right\}$$

Contr 2 (⟺ diag 2 of (c))

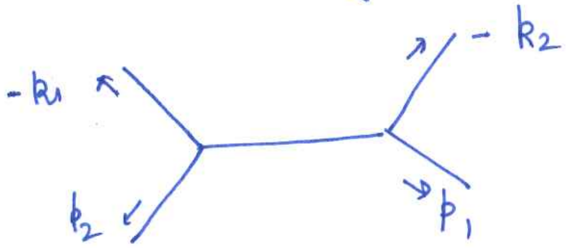


$$\hat{G}_2^{(4)} = -i\lambda^2 (2\pi)^4 \delta^4(p_1+p_2-k_1-k_2) \, (-m^2 - 2k_1\cdot k_2)(-m^2 - 2p_1\cdot p_2) \frac{1}{(k_1+k_2)^2 - m^2 + i\epsilon}$$

Contr 3 (≡ diag ③ of ©).



Contr 4 (≡ diag 4 of ©)



Similar treatment as contribution 2.

$$S(p_1, p_2; k_1, k_2) = \text{contri } 1 + \cdots + \text{contri}. 4$$

use $s + t + u = 4m^2$

~~Litt~~ (BIG algebra)

$$= 0$$

$$\Rightarrow \boxed{S(p_1, p_2; k_1, k_2) = 0}$$

(2)

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \bar\psi\,(i\partial\!\!\!/ - m)\,\psi + \lambda\,\bar\psi\gamma^\mu\psi\,\partial_\mu\phi \right]$$

$$= \int \left[\prod_{i=1}^{2} \frac{d^4k_i}{(2\pi)^4}\right] (2\pi)^4\,\delta^4(k_1+k_2)$$

$$\left[\left\{ \frac{1}{2}(-ik_{1\mu})(-ik_2^\mu) - \frac{1}{2}m^2 \right\} \tilde\phi(k_1)\,\tilde\phi(k_2) \right.$$

$$\left. + \tilde{\bar\psi}(k_1)\,(k\!\!\!/_2 - m)\,\tilde\psi(k_2) \right]$$

$$+ \lambda \int \left[\prod_{i=1}^{3} \frac{d^4k_i}{(2\pi)^4}\right] (2\pi)^4\,\delta^4(k_1+k_2+k_3)$$

$$\tilde{\bar\psi}(k_1)\,\gamma^\mu\,\tilde\psi(k_2)\,(-ik_{3\mu})\,\tilde\phi(k_3)$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[k^2 - m^2\right]\tilde\phi(-k)\,\tilde\phi(k) + \int \frac{d^4k}{(2\pi)^4}\,\tilde{\bar\psi}(-k)\,(k\!\!\!/ - m)\,\tilde\psi(k)$$

$$+ \lambda \int \left[\prod_{i=1}^{3} \frac{d^4k_i}{(2\pi)^4}\right] (2\pi)^4\,\delta^4(k_1+k_2+k_3)\,\tilde{\bar\psi}(k_1)\,\gamma^\mu\,\tilde\psi(k_2)\,(-ik_{3\mu})\,\tilde\phi(k_3)$$

(a) Feynman Rules :-

$$\rightarrow \quad S_{free}[\phi] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\tilde\phi(-k)\,(k^2 - m^2)\,\tilde\phi(k)$$

$$\langle \tilde\phi(k_1)\,\tilde\phi(k_2) \rangle = i\,(2\pi)^4\,\delta^4(k_1+k_2)\,\frac{1}{k^2 - m^2}$$

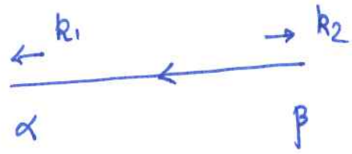
$$\sim S_{free}[\psi] = \int \frac{d^4k}{(2\pi)^4} \tilde{\bar{\psi}}(-k) \cdot (\not{k} - m) \tilde{\psi}(k)$$

$$= \int \frac{d^4k}{(2\pi)^4} \tilde{\bar{\psi}}_\alpha(-k) (\not{k} - m)_{\alpha\beta} \tilde{\psi}_\beta(k)$$

α, β are Dirac indices

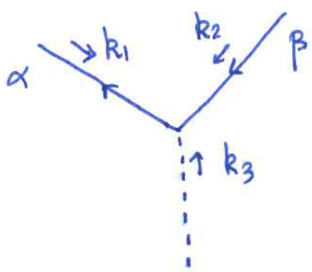
$\not{k} \equiv \gamma^\mu k_\mu$

$$\langle \tilde{\psi}_\alpha(k_1) \tilde{\bar{\psi}}_\beta(k_2) \rangle = i (2\pi)^4 \delta^4(k_1 + k_2) \left(\frac{1}{\not{k}_1 - m} \right)_{\alpha\beta}$$



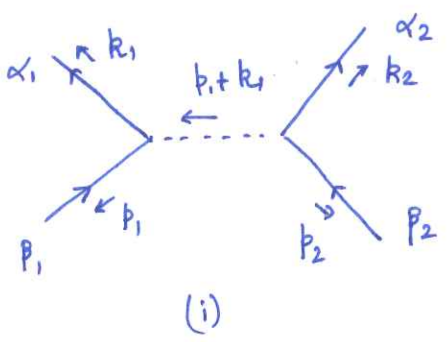
convention of fermionic current flow $\Rightarrow$ arrow towards $\tilde{\psi}$.

$$\sim S_{int} = -i\lambda \int \left[\prod_{i=1}^{3} \frac{d^4k_i}{(2\pi)^4} \right] (2\pi)^4 \delta^4(k_1 + k_2 + k_3) \tilde{\bar{\psi}}_\alpha(k_1) \gamma^\mu_{\alpha\beta} \tilde{\psi}_\beta(k_2) \tilde{\phi}(k_3) k_{3\mu}$$



$$\Longleftrightarrow \lambda (2\pi)^4 \delta^4(k_1 + k_2 + k_3) k_{3\mu} \gamma^\mu_{\alpha\beta}$$

(b) AIM: $\langle \tilde{\psi}_{\alpha_1}(k_1)\,\tilde{\psi}_{\alpha_2}(k_2)\,\tilde{\bar{\psi}}_{\beta_1}(p_1)\,\tilde{\bar{\psi}}_{\beta_2}(p_2)\rangle_c = \,?$ $\qquad O(\lambda^2)$



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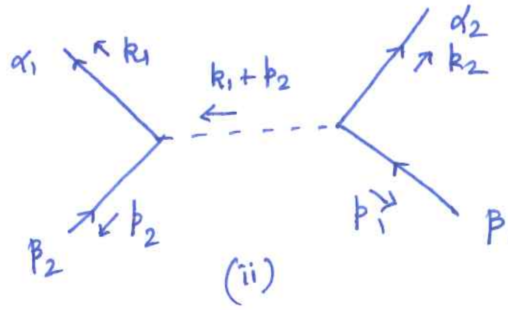


$$= \mathrm{sgn}_1\,(2\pi)^4\,\delta^4(k_1+k_2+p_1+p_2)\left[\frac{i}{\not{k}_1-m}\,\lambda\,(k_1+p_1)_\mu\gamma^\mu\,\frac{i}{-\not{p}_1-m}\right]_{\alpha_1\beta_1}$$

$$\left[\frac{i}{\not{k}_2-m}\,\lambda\,(k_2+p_2)_\mu\gamma^\mu\,\frac{i}{-\not{p}_2-m}\right]_{\alpha_2\beta_2}\frac{i}{(k_1+p_1)^2-m^2+i\epsilon}\times\underbrace{2}_{\text{symm. fac.}}\times\underbrace{\frac{1}{2!}}_{\text{vertex fac.}}$$

$$+\,\mathrm{sgn}_2\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\left[\frac{i}{\not{k}_1-m}\,\lambda\,(k_1+p_2)_\mu\gamma^\mu\,\frac{i}{-\not{p}_2-m}\right]_{\alpha_1\beta_2}$$

$$\left[\frac{i}{\not{k}_2-m}\,\lambda\,(k_2+p_1)_\mu\gamma^\mu\,\frac{i}{-\not{p}_1-m}\right]_{\alpha_2\beta_1}\frac{i}{(k_1+p_2)^2-m^2+i\epsilon}\times 2\times\frac{1}{2!}$$

fixing sign:

(i) $\Rightarrow \langle \tilde{\psi}_{\alpha_1}\tilde{\psi}_{\alpha_2}\tilde{\bar{\psi}}_{\beta_1}\tilde{\bar{\psi}}_{\beta_2}\;\tilde{\bar{\psi}}\tilde{\psi}\phi\;\tilde{\bar{\psi}}\tilde{\psi}\phi\rangle \Rightarrow (-1)^{5+2} = -1 \equiv \mathrm{sgn}_1$

(ii) $\Rightarrow \langle \tilde{\psi}_{\alpha_1}\tilde{\psi}_{\alpha_2}\tilde{\bar{\psi}}_{\beta_1}\tilde{\bar{\psi}}_{\beta_2}\;\tilde{\bar{\psi}}\tilde{\psi}\phi\;\tilde{\bar{\psi}}\tilde{\psi}\phi\rangle \Rightarrow (-1)^{4+2} = 1 \equiv \mathrm{sgn}_2$

General rules $\rightsquigarrow$ sign $= (-1)^{\#\,\text{crossing}\,+\,\#\,\bar{\psi}\psi\text{ pairs}}$

Hence, $\boxed{\langle \tilde{\bar{\psi}}_{\alpha_1}(k_1)\,\tilde{\psi}_{\alpha_2}(k_2)\,\tilde{\bar{\psi}}_{\beta_1}(p_1)\,\tilde{\bar{\psi}}_{\beta_2}(p_2)\rangle_c = \cancel{\mathrm{sgn}}\;(i) + (ii)}$

Do the algebra in your own.

(3) $$S = \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 + \bar{\psi} (i\not{\partial} - m)\psi \right]$$

[a] $$\psi \equiv e^{i\lambda\phi} \chi$$

$$\Rightarrow \partial_\mu \psi = e^{i\lambda\phi} (\partial_\mu \chi + i\lambda \chi \partial_\mu \phi)$$

and, $$\bar{\psi} \equiv \psi^\dagger \gamma^0$$

$$= e^{-i\lambda\phi} \chi^\dagger \gamma^0 \qquad [\phi^* = \phi]$$

$$= e^{-i\lambda\phi} \bar{\chi}$$

So, $$\bar{\psi}\psi = \bar{\chi}\chi$$

$$i\bar{\psi}\not{\partial}\psi = i e^{-i\lambda\phi} \bar{\chi} \gamma^\mu e^{i\lambda\phi} (\partial_\mu \chi + i\lambda \chi \partial_\mu \phi)$$

$$= i \bar{\chi}\gamma^\mu (\partial_\mu \chi + i\lambda \chi \partial_\mu \phi)$$

$$= i \bar{\chi}\not{\partial}\chi - \lambda \bar{\chi}\gamma^\mu \chi \partial_\mu \phi$$

So,

$$\boxed{S = \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 + \bar{\chi}(i\not{\partial} - m)\chi - \lambda \bar{\chi} \gamma^\mu \chi \partial_\mu \phi \right]}$$

PS1 _ 3.2

(b) Consider ϕ, χ as independent fields

λ small parameter

Feynman rules :

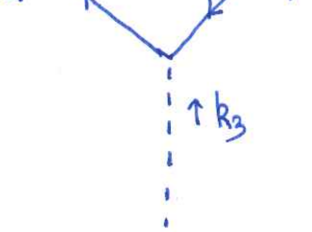
propagators

$$\langle \tilde{\phi}(k_1)\, \tilde{\phi}(k_2) \rangle = (2\pi)^4 \delta^4(k_1+k_2) \frac{i}{k_1^2 - M^2 + i\epsilon}$$

$$\langle \tilde{\psi}_\alpha(k_1)\, \tilde{\bar{\psi}}_\beta(k_2) \rangle = (2\pi)^4 \delta^4(k_1+k_2) \left(\frac{i}{\not{k}_1 - m} \right)_{\alpha\beta}$$

$$S_{int} = -\lambda \int d^4x \; \bar{\chi} \gamma^\mu \chi \, \partial_\mu \phi$$

$$= -\lambda \int \left[\prod_{i=1}^{3} \frac{d^4 k_i}{(2\pi)^4} \right] (2\pi)^4 \delta^4(k_1+k_2+k_3) \; \tilde{\bar{\chi}}_\alpha(k_1)\, \gamma^\mu_{\alpha\beta}\, \tilde{\chi}_\beta(k_2)\, (-i k_{3\mu})\, \tilde{\phi}(k_3)$$



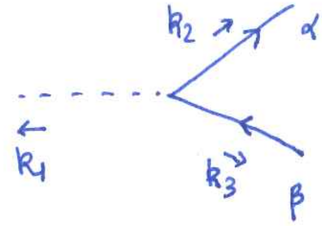
$$= -\lambda (2\pi)^4 \delta^4(k_1+k_2+k_3)\, \gamma^\mu_{\alpha\beta}\, k_{3\mu}$$

S-matrix element $\quad \phi \to \chi \quad \chi$

$$\Rightarrow \left\langle \tilde{\phi}(-k_1)\, \tilde{\chi}_\alpha(k_2)\, \tilde{\bar{\chi}}_\beta(k_3) \right\rangle_c$$

Recall,

$$\left\langle \tilde{\phi}(k_1)\, \tilde{\chi}_\alpha(k_2)\, \tilde{\bar{\chi}}_\beta(k_3) \right\rangle_c$$

= 

$$= (2\pi)^4 \delta^4(k_1+k_2+k_3) \left[\frac{i}{\not{k}_2 - m} (-\lambda) (-k_{1\mu}) \gamma^\mu \frac{i}{-\not{k}_3 - m} \right]_{\alpha\beta} \frac{i}{k_1^2 - M^2 + i\epsilon}$$

So, $\left\langle \tilde{\phi}(-k_1)\, \tilde{\chi}_\alpha(k_2)\, \tilde{\bar{\chi}}_\beta(k_3) \right\rangle_c$

$$= (2\pi)^4 \delta^4(-k_1+k_2+k_3) \left[\frac{i}{\not{k}_2 - m} (-\lambda) k_{1\mu} \gamma^\mu \frac{i}{-\not{k}_3 - m} \right]_{\alpha\beta} \frac{i}{k_1^2 - M^2 + i\epsilon}$$

LSZ - reduction $\Rightarrow$ S-matrix element would be proportional to

$$S(k_2, k_3; +k_1) \propto (2\pi)^4 \delta^4(-k_1+k_2+k_3) \underbrace{\bar{u}(k_2) \not{k}_1 v(k_3)}_{\underbrace{\bar{u}(k_2)(\not{k}_2+\not{k}_3) v(k_3)}_{\underbrace{\bar{u}(k_2)(m-m) v(k_3)}_{0}}}$$

$$\boxed{\therefore \; S(k_2, k_3; k_1) = 0}$$

(4)

$$\theta_i \theta_j = -\theta_j \theta_i \quad \Rightarrow \quad \theta_i^2 = 0$$

(a) $$e^{A\theta_1\theta_2 + B\theta_3\theta_4} = ?$$

$(\because \theta_i^2 = 0)$

$$= 1 + (A\theta_1\theta_2 + B\theta_3\theta_4) + \frac{1}{2}(A\theta_1\theta_2 + B\theta_3\theta_4)^2 + 0.$$

$$= 1 + (A\theta_1\theta_2 + B\theta_3\theta_4) + \frac{1}{2}\Big(\underbrace{A\theta_1\theta_2 A\theta_1\theta_2}_{0} + A\theta_1\theta_2 B\theta_3\theta_4$$
$$+ B\theta_3\theta_4 A\theta_1\theta_2 + \underbrace{B\theta_3\theta_4 B\theta_3\theta_4}_{0}\Big)$$

$$= 1 + (A\theta_1\theta_2 + B\theta_3\theta_4) + AB\,\theta_1\theta_2\theta_3\theta_4$$

(b) $$\int d\theta_i \equiv 0 \qquad \int d\theta_i\,\theta_i \equiv 1$$

$$d\theta_i\, d\theta_j = -d\theta_j\, d\theta_i \qquad d\theta_i\,\theta_j = -\theta_j\, d\theta_i$$

$$\int d\theta_1\, d\theta_2\; e^{A\theta_1\theta_2}$$

$$= \int d\theta_1\, d\theta_2\, (1 + A\theta_1\theta_2)$$

$$= A\int d\theta_1\, d\theta_2\, \theta_1\theta_2$$

$$= -A\int d\theta_1\,\theta_1 \int d\theta_2\,\theta_2$$

$$= -A.$$

(c)

(c)

$$\theta \equiv \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2)$$

$$\theta^* \equiv \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2)$$

$$\eta \equiv \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2)$$

$$\eta^* = \frac{1}{\sqrt{2}}(\eta_1 - i\eta_2)$$

$$(\theta\eta)^* \equiv \eta^* \theta^*$$

$$= \frac{1}{2}(\eta_1 - i\eta_2)(\theta_1 - i\theta_2)$$

$$= \frac{1}{2}(\eta_1\theta_1 - i\eta_2\theta_1 - i\eta_1\theta_2 - \eta_2\theta_2)$$

$$= \frac{1}{2}.(-\theta_1\eta_1 + i\theta_1\eta_2 + i\theta_2\eta_1 + \theta_2\eta_2)$$

$$= \frac{1}{2}\left[-\theta_1(\eta_1 - i\eta_2) + i\theta_2(\eta_1 - i\eta_2)\right]$$

$$= -\theta^*\eta^*$$

(d) $\int d\theta^* d\theta \, e^{-\theta^* B\theta}$

$$= \int d\theta^* d\theta \, (1 - \theta^* B\theta)$$

$$= B$$

Ordinary Gaussian integral results in $2\pi/B$.

(e) $\frac{\partial}{\partial \theta_i} \theta_i = 1$, $\frac{\partial}{\partial \theta_i} 1 = 0$

$$\frac{\partial}{\partial \theta_i}\{f(\theta_1,\dots,\theta_n)\, g(\theta_1,\dots,\theta_n)\} \stackrel{?}{=} \frac{\partial f}{\partial \theta_i} g + f \frac{\partial g}{\partial \theta_i}$$

~~Let, f(θ₁,…,θₙ)~~

Any f and g can be written as

$$f = A + \theta_i B$$
$$g = C + \theta_i D$$

A, B, C, D contain all the other Grassman variables

eg. $A = A(\theta_1, \theta_2, \dots, \cancel{\theta_i}, \dots, \theta_n)$

So, $\frac{\partial}{\partial \theta_i}(fg)$

$$= \frac{\partial}{\partial \theta_i}\Big(AC + \theta_i BC + \underset{(-1)^A}{\theta_i AD} + \underset{0}{\cancel{\theta_i^2}} BD\Big)$$

$$= BC + (-1)^A AD$$

$$\begin{cases} (-1)^A = 1 & \text{if } A \text{ is even fun.} \\ \quad\;\; = -1 & \text{if } A \text{ is odd fun.} \end{cases}$$

If f is even $\Rightarrow$ A is even & B is odd

$\Rightarrow (-1)^f = (-1)^A$

So,

$$\boxed{\frac{\partial}{\partial \theta_i}(fg) = \frac{\partial f}{\partial \theta_i} g + (-1)^f f \frac{\partial g}{\partial \theta_i}}$$

Consider, $\frac{\partial f}{\partial \theta_i} g + (-1)^f f \frac{\partial g}{\partial \theta_i}$

$$= B(C + \theta_i D) + (-1)^A (A + \theta_i B) D$$
$$= BC + (-1)^A AD + B\theta_i D + (-1)^A \theta_i BD$$
$$= BC + (-1)^A AD + \theta_i BD(-1)^B + (-1)^A \theta_i BD$$
$$= BC + (-1)^A AD + \underbrace{\left[(-1)^B + (-1)^A\right]}_{0} \theta_i BD \;=\; BC + (-1)^A AD$$

(5) Text book problem ⇒ SOLVE YOURSELF.