Dirac & Majorana Neutrino
PP Talk 3
11th Oct, 2012

1. Introduction:

When Dirac first wrote down his relativistic equation for a fermionic field, he had electrons in mind. It doesn't require much mind-reading to deduce this conclusion, because his first article on this issue was entitled "The Quantum Theory of the Electron". Electrons have mass & charge. In his solutions, Dirac found ~~the~~ antiparticles having the same mass as electron but with opposite charge.

Dirac's paper was published in 1928. The very next year, Weyl showed that for massless fermions, a simpler eqn would suffice, involving two component fields as opposed to the four component field that Dirac had obtained. We'll see:

massless fermion $\Rightarrow$ Weyl fermion.
massive " $\Rightarrow$ Dirac/Majorana.
massive neutral $\Rightarrow$ Majorana.

where various fermions, namely Weyl, Dirac & Majorana fermions correspond to various representations of the gamma matrices

And then, in 1930, Pauli proposed the neutrinos to explain the continuous energy spectrum of electrons coming out in beta decay. The neutrinos had to be uncharged because of conservation of electric charge, & they seemed to have vanishing mass from the analysis of β-decay data. It was therefore conjectured that the neutrinos are massless. Naturally, it was assumed that the neutrinos are therefore Weyl fermions, i.e. their properties are described by Weyl's theory.

There was also the possibility that neutrinos are the antiparticles of themselves, since they are uncharged. Description of such fermion fields was pioneered by Majorana in 1937. The question was not taken seriously because, at that time, everybody was convinced that neutrinos are Weyl fermions. Because later we'll see a fermion can't be simultaneously Majorana as well as Weyl.

The question became important much later, beginning in 1960s, when people started examining the consequences of small but non-zero neutrino masses, and possibilities of detecting them. If neutrinos have mass, they can't be Weyl fermions. This opened the discussion of whether the neutrinos are Dirac or Majorana fermions.

So, first we'll try to understand the basic framework of Dirac, Weyl & Majorana fermions and then we'll focus on neutrinos. According to SM neutrinos are massless. But they do have non-zero mass. Our AIM will be to see how one can incorporate the non-zero mass of neutrino into our physical theory.

2. The Dirac Equation & its sol^{ns} :

$$(i\not{\partial} - m)\psi(x) = 0$$

$$\mathcal{L} = \bar{\psi}\,(i\not{\partial} - m)\psi$$

$$H = \vec{\alpha}\cdot\vec{p} + \beta m = \gamma^0(\vec{\gamma}\cdot\vec{p} + m)$$

with $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

& $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$

2.A. Real sol^{ns} :

* Is the DE a real equation like the KG eq^{n} ?
  ⇒ Answer depends on what γ^μ's are. If all non-zero elements of all four γ^μ's are purely imaginary, then DE is real.

* Question ⇝ Is it possible to define γ^μ's, satisfying the required conditions, as purely imaginary ?

  ⇒ Majorana representation [Recall we derived the pros of γ^μ matrices demanding the covariance of DE under LT. And Pauli's theorem tells us that there could be infinite # of such represent related by unitary similarity transformation.] .

Majorana : $\gamma_M^0 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}$, $\gamma_M^1 = \begin{bmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{bmatrix}$, $\gamma_M^2 = \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}$, $\gamma_M^3 = \begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{bmatrix}$

Let's also write down the Dirac & Weyl representations.

Dirac : $\gamma_D^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, $\vec{\gamma}_D = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}$, $\gamma_D^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$

Weyl : $\gamma_C^0 = -\gamma_D^5 = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$, $\vec{\gamma}_C = \vec{\gamma}_D$, $\gamma_C^5 = \gamma_D^0$

See for Majorana : $\boxed{\gamma_M^* = -\gamma_M}$ (1)

* DE becomes a real equation in Majorana repres. $\Rightarrow$ Real sol^{n} should exist. i.e.

$$\boxed{\begin{array}{c} \psi_{M\alpha} = \psi^*_{M,\alpha} \\ \Rightarrow \psi_M = \psi^*_M \end{array}}$$ $\rightarrow$ Majorana fermions.

$\alpha = (1,2,3,4)$: 4-spinors

$\Uparrow$

Valid only in Majorana representation.

* Majorana cond^{n} in some general representation:

Pauli's thm $\rightarrow$ $\boxed{\begin{array}{l} \gamma^\mu = U \gamma^\mu_M U^\dagger \\ \& \; \psi = U \psi_M \end{array}}$ where, $\{\gamma^\mu, \psi\}$ in some other represent. (2)

with $UU^\dagger = U^\dagger U = I$

Then, $\psi = U\psi_M$

$\Rightarrow \psi^*_M = (U^\dagger \psi)^*$

Cond^{n}: $\psi_M = \psi_M^*$

$\Rightarrow U^\dagger \psi = (U^\dagger \psi)^*$ $\Rightarrow$ $\boxed{\psi = U U^T \psi^*}$

$\Downarrow$

Now, unitarity of $U \Rightarrow UU^T$ is too unitary.

We define $UU^T \equiv \gamma_0 C$ $\Rightarrow$ $\boxed{\psi = \gamma_0 C \psi^*}$

$\uparrow$

unitary matrix

Define : $\hat{\psi} \equiv \gamma_0 C \psi^*$

Majorana fermion field : $\boxed{\psi = \hat{\psi} \equiv \gamma_0 C \psi^*}$ (3)

2.8 Pros of C: $\quad UU^T \equiv \gamma^0 C \Rightarrow C = \gamma^0 UU^T$

ⓐ $$C^{-1}\gamma^\mu C = (\gamma^0 UU^T)^{-1}\gamma^\mu(\gamma^0 UU^T)$$

$$= U^* U^\dagger \underbrace{\gamma^0\gamma^\mu\gamma^0}_{\gamma^{\mu\dagger}} UU^T$$

$$= U^* \underbrace{\underbrace{U^\dagger \gamma^{\mu\dagger} U}_{(U^\dagger\gamma^\mu U)^\dagger}}_{(\gamma_M^\mu)^\dagger} U^T \qquad [\because \gamma^\mu = U\gamma_M^\mu U^\dagger]$$

$$= U^* \gamma_M^{\mu\dagger} U^T$$

$$= (U\gamma_M^{\mu *} U^\dagger)^T$$

$$= -(U\gamma_M^\mu U^\dagger)^T \qquad [\because \gamma_M^{\mu *} = -\gamma_M^\mu]$$

$$= -(\gamma^\mu)^T$$

$$\Rightarrow \boxed{C^{-1}\gamma^\mu C = -(\gamma^\mu)^T} \quad ④$$

ⓑ $$UU^T \underbrace{\underbrace{(UU^T)^\dagger}_{U^* U^\dagger}}_{(UU^T)^*} = \mathbb{1} \quad \longleftarrow \text{unitarity of } UU^T$$

$$\Rightarrow (UU^T)(UU^T)^* = \mathbb{1}$$

$$\Rightarrow (\gamma^0 C)(\gamma^0 C)^* = \mathbb{1}$$

$$\Rightarrow \gamma^0 C \gamma^{0*} C^* = \mathbb{1}$$

$$\Rightarrow \gamma^{0*} C^* C = C^{-1}\gamma^0 C = -(\gamma^0)^T \quad [\text{using } ④]$$

$$* \Rightarrow \gamma^0 C C^* = -\gamma^{0\dagger} = -\gamma^0$$

$$\Rightarrow CC^* = -\mathbb{1}$$

$$\Rightarrow C^* = -C^{-1} \Rightarrow \boxed{C^T = -C} \quad ⑤$$

Antisymmetric.

Hence,

(i) $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$

(ii) $C^\dagger = C^{-1}$

(iii) $C^T = -C$ ✗

$\Rightarrow$ we can identify C as charge conjugation operator

$\Rightarrow$ Applying twice gives back the original field.

i.e. $(\hat{\hat{\psi}}) = \psi$.

<u>Proof</u>: ~~Chiral representation~~

It can be shown $C = i\gamma^2\gamma^0$

Then $(\gamma^0 C)(\gamma^0 C)$

$= (\gamma^0 i\gamma^2\gamma^0)(\gamma^0 i\gamma^2\gamma^0)$

$= -\gamma^0 \underbrace{\gamma^2\gamma^2}_{-\mathbb{1}} \gamma^0$

$= \gamma^{0\,2}$

$= \mathbb{1}$.

✗

(iv) $\Rightarrow$ Also, consider $C\bar{\psi}^T = C(\psi^\dagger\gamma^0)^T$

$= C\gamma^{0T}\psi^*$

$= -\gamma^0 C\psi^*$

$= e^{i\pi}\gamma^0 C\psi^*$.

$C^{-1}\gamma^\mu C = -\gamma^{\mu T}$

$\Rightarrow \gamma^\mu C = -C\gamma^{\mu T}$

Under charge conjugation $\psi \to \psi^c = C\bar{\psi}^T = e^{i\pi}\gamma^0 C\psi^*$

$= e^{i\pi}\hat{\psi}$

So, apart from a phase factor, $\boxed{\hat{\psi} = \psi^c}$

So, <u>Majorana condition</u> can be recast as $\boxed{\psi = \psi^c}$

2.BC ~~Fourier expansion:~~

2.BC Fourier expansion:

* ψ_M - real $\Rightarrow$ one should be able to write ψ_M in the following form:

$$\psi_{M,\alpha}(x) = \sum_s \int_p \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[a(\vec{p},s)\, u_{M,\alpha}(\vec{p},s)\, e^{-ip\cdot x} + a^\dagger(\vec{p},s)\, u^*_{M,\alpha}(\vec{p},s)\, e^{ip\cdot x} \right]$$

Rel. normalized

Hermitian conjugate

$\Downarrow$

$\psi^*_{M,\alpha}(x) = \psi_{M,\alpha}(x)$

~~Etc.~~

4- basis spinors $\Rightarrow \begin{cases} u_M(\vec{p},s) & : s = \pm 1/2 \\ u^*_M(\vec{p},s) & : s = \pm 1/2 \end{cases}$

* Fourier expansion in arbitrary representation $\Rightarrow$

Now,

$$\psi = U\psi_M$$

$$= \sum_s \int_p \left[a(\vec{p},s)\, \underbrace{U u_M(\vec{p},s)}_{\text{Define}}\, e^{-ip\cdot x} + a^\dagger(\vec{p},s)\, U u^*_M(\vec{p},s)\, e^{ip\cdot x} \right]$$

$u(\vec{p},s) \equiv U u_M(\vec{p},s)$. Then,

$U u^*_M = U\left(U^\dagger u(\vec{p},s)\right)^* = U U^T u^*(\vec{p},s) = \gamma^0 C\, u^*(\vec{p},s)$

Hence,

$$\boxed{\psi(x) = \sum_s \int_p \left[a(\vec{p},s)\, u(\vec{p},s)\, e^{-ip\cdot x} + a^\dagger(\vec{p},s)\, \underbrace{v(\vec{p},s)}_{\text{Defining}}\, e^{ip\cdot x} \right]} \quad (16)$$

$v(\vec{p},s) \equiv \gamma^0 C\, u^*(\vec{p},s)$

$\Rightarrow u(\vec{p},s) = \gamma^0 C\, v^*(\vec{p},s)$

(•) To prove $u(\vec{p},s) = \gamma^0 C\, v^*(\vec{p},s)$
Take complex conj of $v(\vec{p},s) = \gamma^0 C u^*(\vec{p},s)$ $\Longleftrightarrow$
and then multiply by UU^T.

(4) $\Rightarrow$ (i) $\psi(x)$ is indeed a solution of DE.

(ii) Since same a & $a^\dagger$ appear $\rightsquigarrow$ particle itself is its own antiparticle. This is also confirmed from $\psi = \psi^c$.

(iii) Majorana field can't have a conserved charge because

$j^\mu = \bar{\psi}\gamma^\mu\psi = 0$

Proof:

$$\bar{\psi}\gamma^\mu\psi$$

$$= \bar{\psi^c}\gamma^\mu\psi^c \qquad \psi^c = C\bar{\psi}^T$$

$$= -\psi^T C^\dagger\gamma^\mu C\bar{\psi}^T \qquad \Rightarrow \bar{\psi^c} = \psi^T\bar{C}$$

$$= \bar{\psi}C\gamma^{\mu T}C^\dagger\psi$$

$$= -\bar{\psi}\gamma^\mu\psi$$

$\Rightarrow \bar{\psi}\gamma^\mu\psi = 0$

$\Rightarrow$ Majorana particle must be neutral.

2.D Lorentz Invariance of the Reality Cond^{n}:

* Condition would be physically meaningful if it holds true in any reference frame. i.e. is LI. We'll prove that this cond^{n} is indeed LI.

$\Rightarrow$ Dirac fermionic field $\psi(x)$ under proper orthochronous LT transforms as

$$\boxed{\psi'(x') = \exp\left[-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}_{spinor}\right]\psi(x)} \qquad (8)$$

where, $J^{\mu\nu}_{spinor} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}\sigma^{\mu\nu}$

$$\Rightarrow \psi'^*(x') = \exp\left[\frac{i}{4}\omega_{\mu\nu}\sigma^{*\mu\nu}\right]\psi^*(x)$$

$$\gamma^0 C \Rightarrow \underbrace{\gamma^0 C\psi'^*(x')}_{\hat{\psi}'(x')} = \underbrace{\gamma^0 C \exp\left[\frac{i}{4}\omega_{\mu\nu}\sigma^{*\mu\nu}\right]\underbrace{\psi^*(x)}_{(\gamma^0 C)^{-1}\hat{\psi}(x)}}_{\text{We've to find}} \quad [\because \hat{\psi} \equiv \gamma^0 C\psi^*$$

$$(\gamma^0 C)\sigma^{*\mu\nu}(\gamma^0 C)^{-1} = ?$$

first we'll prove $(\gamma^0 C)\gamma^{\mu *}(\gamma^0 C)^{-1} = -\gamma^\mu$

$$\Rightarrow (\gamma^0 C)\sigma^{*\mu\nu}(\gamma^0 C)^{-1} = -\sigma^{\mu\nu}.$$

Proof: $\gamma^{\mu *} = (\gamma^{\mu\dagger})^T = (\gamma^0\gamma^\mu\gamma^0)^T = \gamma^{0T}\gamma^{\mu T}\gamma^{0T}$

$$= -\gamma^{0T} C^{-1}\gamma^\mu C\,\gamma^{0T}$$

$$= -C^{-1}\gamma^0\gamma^\mu\gamma^0 C$$

$$= -(\gamma^0 C)^{-1}\gamma^\mu(\gamma^0 C)$$

$$\Rightarrow (\gamma^0 C)\gamma^{\mu *}(\gamma^0 C)^{-1} = -\gamma^\mu.$$

Hence,

$$\boxed{\hat{\psi}'(x') = \exp\left[-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right]\hat{\psi}(x)} \qquad (9)$$

So, if (8) holds then $\Rightarrow$ (9) also holds true.

$\hat{\psi}$ transforms exactly same way as ψ does under proper Lorentz transformation.

$\Rightarrow$ Reality condn is Lorentz invariant.

3. Left or Right $\Rightarrow$ Helicity & Chirality :

3.A. Helicity :

* A definition of "handedness" that can be applied to any particle has to do with the relative orientation of its momentum & AM. The definition hinges on a property called "helicity", which is defined as:

$$\boxed{h \equiv \frac{2\vec{J}\cdot\vec{p}}{|\vec{p}|}}$$

$\downarrow$ fermion obeying DE

$$\boxed{h = \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}} = \vec{\Sigma}\cdot\hat{p} \qquad \text{where } \vec{\Sigma} \equiv \{\Sigma_1, \Sigma_2, \Sigma_3)$$

$$\text{with } \Sigma_i = \epsilon_{ijk} J^{jk}_{\text{spinor}} = \frac{i}{2}\epsilon_{ijk}\gamma^j\gamma^k$$

* Properties:

(a)

$$\text{Now, } h^2 = \frac{p_i p_j}{p^2}\Sigma_i\Sigma_j$$

$$= \frac{p_i p_j}{p^2}\frac{1}{2}\underbrace{(\Sigma_i\Sigma_j + \Sigma_j\Sigma_i)}_{\underbrace{\{\Sigma_i, \Sigma_j\}}_{2\delta_{ij} I_4}}$$

$$= I_4$$

Eigenvalues of h are ± 1 — Eigenstate with $+1$ $\rightarrow$ Right-helical.

Eigenstate with -1 $\rightarrow$ Left-~~handed~~. helical

(b) $$\boxed{\left[h, H^{\text{free}}_{\text{Dirac}}\right] = 0}$$

Proof: Dirac representation:

$$H = \vec{\alpha}\cdot\vec{p} + \beta m$$
$$= \gamma^0 \vec{\gamma}\cdot\vec{p} + \gamma^0 m$$
$$= \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} + \gamma^0 m$$
$$= \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}$$

$$[h, H_{\text{Dirac}}]$$
$$= \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} - \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}$$

$$= \begin{pmatrix} m\,\vec{\sigma}\cdot\hat{p} & p \\ p & -m\,\vec{\sigma}\cdot\hat{p} \end{pmatrix} - \begin{pmatrix} m\,\vec{\sigma}\cdot\hat{p} & p \\ p & -m\,\vec{\sigma}\cdot\hat{p} \end{pmatrix}$$

$$= 0$$

$\Rightarrow$ for a free Dirac particle, helicity is conserved: it does not change with time.

(c) Rotation in spinor space is unitary $\Rightarrow$ $\vec{\Sigma}\cdot\hat{p}$ (dot product) remains invariant under rot^n in spinor space.

(d) However, helicity is not invariant under boosts in spinor space (remember boosts in spinor space is not unitary, rather hermitian). Instead of proving mathematically we can easily understand this with an example.

Consider a fermion whose spin & momentum are both in the same dir^n (say x-dir^n) $\Rightarrow h = +1$. Now, consider the same particle from the point of view of an observer moving along the x-dir^n with a velocity greater than that of the particle.

for this observer, the particle is moving in the opposite dirn, so the unit vector along the particle momentum is in the negative x-dirn. The spin, however, does not change, since $\Sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma^{jk}$ tells us that the x-component of spin is really the yz component of a rank-2 anti-symmetric tensor, and components perpendicular to the frame velocity remain unaffected in a LT. Hence, wrt the new observer helicity turns out to be -1.

$\Rightarrow$ A <u>massive</u> fermion cannot be exclusively left-helical or right-helical. Helicity depends on the observer who is looking on it

<u>NOTE:</u> For a <u>massless</u> fermion, no observer can move faster than the massless fermion. Hence, helicity is ~~also~~ LI for a massless particle.

3.2 Chirality:

* By definition $\gamma_5 = \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ is called chirality operator.

$\gamma_5^\dagger = \gamma_5$

$(\gamma_5)^2 = \mathbb{1} \Rightarrow \gamma_5 \to$ Eigenstate with ev $+1$ $\to$ Right-chiral.

Eigenstate with ev -1 $\to$ Left-handed.

$\{\gamma_5, \gamma^\mu\} = 0 \quad \forall \mu.$

* Let's denote by ψ_R & ψ_L the fields which are eigenstates of γ_5 with ev $+1$ & -1 respectively:

$$\begin{cases} \gamma^5\psi_R = \psi_R \\ \gamma^5\psi_L = -\psi_L \end{cases}$$

* Since γ_5 is hermitian $\Rightarrow$ eigenstates form a complete set i.e. basis.

Any generic spinor field ψ can be splitted into its Left-chiral & right-chiral components:

$$\psi = \psi_R + \psi_L$$

with $\psi_R \equiv \frac{1+\gamma^5}{2}\psi \equiv P_R \psi$

$\psi_L \equiv \frac{1-\gamma^5}{2}\psi \equiv P_L \psi$

where, $\left.\begin{array}{l} P_R \equiv \frac{1}{2}(1+\gamma_5) \\ P_L \equiv \frac{1}{2}(1-\gamma_5) \end{array}\right\}$ Chirality $\underbrace{\text{projection}}$ operators.

$\Downarrow$ because

~~They~~

$$\left\{\begin{array}{l} P_R P_L = 0 = P_L P_R \\ P_R^2 = P_R \\ P_L^2 = P_L \\ P_R + P_L = \mathbb{1}. \end{array}\right.$$

* Consider Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$$= (\bar{\psi}_R + \bar{\psi}_L)(i\partial\!\!\!/ - m)(\psi_R + \psi_L)$$

Now, $P_R^\dagger = P_R$; $P_L^\dagger = P_L$; $P_R\gamma^0 = \frac{1}{2}(1+\gamma_5)\gamma^0 = \frac{1}{2}\gamma^0(1-\gamma_5)$
$= \gamma^0 P_L$

& $P_L\gamma^0 = \gamma^0 P_R$

So, $\bar{\psi}_R = \overline{(P_R\psi)} = (P_R\psi)^\dagger\gamma^0 = \psi^\dagger P_R\gamma^0 = \psi^\dagger\gamma^0 P_L = \bar{\psi} P_L$

$\bar{\psi}_L = \bar{\psi} P_R$

Hence, $i\bar{\psi}_R \partial\!\!\!/ \psi_L = i\bar{\psi} P_L \gamma^\mu \partial_\mu P_L \psi = i\bar{\psi}\gamma^\mu\partial_\mu P_R P_L \psi = 0$

$i\bar{\psi}_L \partial\!\!\!/ \psi_R = 0$

$$m\bar{\psi}_L\psi_L = m\bar{\psi}P_R P_L\psi = 0$$

$$m\bar{\psi}_R\psi_R = 0$$

$$\therefore \boxed{\mathcal{L} = \bar{\psi}_R i\not{\partial}\psi_R + \bar{\psi}_L i\not{\partial}\psi_L - m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)}$$

One can see that the chiral fields ψ_R & ψ_L have independent kinetic terms but they are coupled by the mass term.

Field eqn[ns] :

$$\left.\boxed{\begin{aligned} i\not{\partial}\psi_R &= m\psi_L \\ i\not{\partial}\psi_L &= m\psi_R \end{aligned}}\right\} \Rightarrow$$

ST evolution of chiral fields are related by mass m.

The fields ψ_R & ψ_L are called Weyl spinors. A Weyl spinor has only two independent components, as we can understand by noting that the decomposition $\psi = \psi_R + \psi_L$ of a four component spinor must split the four independent components equally into two groups, one for each chiral components. One can see this explicitly using a definite representation of the Dirac matrices.

In Weyl representation : $\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

Then, $P_R = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ & $P_L = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$

Writing four component spinor ψ as

$$\psi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix}$$

So, $\psi_R = \begin{pmatrix} \chi_R \\ 0 \end{pmatrix}$, $\psi_L = \begin{pmatrix} 0 \\ \chi_L \end{pmatrix}$

* (a) $[\gamma_5, \sigma^{\mu\nu}] = 0 \quad \forall \ \mu, \nu$

Proof:

$$[\gamma_5, i\gamma^\mu\gamma^\nu]$$

$$= i\underbrace{\{\gamma_5, \gamma^\mu\}}_{0}\gamma^\nu - i\gamma^\mu\underbrace{\{\gamma_5, \gamma^\nu\}}_{0}$$

$$= 0$$

$\Rightarrow$ Chirality remains invariant under proper orthochronous LT. in the spinor space. It can be made Lorentz covariant way.

(b) $[\gamma_5, H^{\text{free}}_{\text{Dirac}}] \neq 0$ for $m \neq 0$.

Proof: In Dirac representation

$$[\gamma_5, H] = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} - \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$= 2m\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$\neq 0 \quad \text{for } m \neq 0.$$

$\Rightarrow$ Chirality is not conserved even for a free massive fermion.

Therefore,

$$\boxed{\begin{array}{lll} \text{Helicity: conserved} & + \ \text{NOT LI} & \\ \text{Chirality: NOT conserved} & + \ \text{LI} & \end{array} \Bigg\} \ m \neq 0}$$

If a particle is left-helical, it will not appear to be so to a suitably boosted observer. On the other hand, if at one time a particle is found to be left-chiral, it won't remain so at a later time.

4. Massless fermion:

* We've seen that the problem with assigning a frame-independent helicity to a fermion disappears if the fermion is massless. The problem with a conserved value of γ^5 also disappears in this limit. This shows that without any ambiguity, one can talk about a +ve or -ve helicity fermion or a left or right chiral fermion when one talks about massless fermions.

$\Rightarrow$ Called Weyl fermions.

* $m=0$, Weyl equations $\Rightarrow$

$$\left.\boxed{\begin{aligned} i\not\partial \psi_R &= 0 \\ i\not\partial \psi_L &= 0 \end{aligned}}\right\} \psi_R \,\&\, \psi_L \text{ decouple.}$$

* Consider a sol^{n} $\psi(x)$ of Weyl equ^{ns}:

$$i\not\partial \psi(x) = 0$$

$$\cancel{p^\mu \psi(x) = p^\mu \psi(x)}$$

In Momentum space $\Rightarrow$ $(\gamma^0|\vec{p}| - \vec{\gamma}\cdot\vec{p})\, w(\vec{p}) = 0$; $w(\vec{p}) = (u(\vec{p}), v(\vec{p}))$

$$\gamma^5 \gamma^0 \Rightarrow \left(1 - \gamma^0\vec{\gamma}\cdot\frac{\vec{p}}{|\vec{p}|}\right) w(\vec{p}) = 0$$

$$\Rightarrow \left(1 - \gamma^5\,\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}\right) w(\vec{p}) = 0 \quad \left[\text{using } \gamma^0\gamma^i = \gamma_5 \Sigma_i\right]$$

$$\Rightarrow \boxed{\underbrace{\gamma_5}_{\text{Chirality}}\, w(\vec{p}) = \underbrace{\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}}_{\text{Helicity}}\, w(\vec{p})}$$

So, $m=0 \Rightarrow$ chirality $\equiv$ helicity

5. Majorana fermions from Weyl fermions :

* One can build Majorana fermion field out of Weyl fields.
* A Majorana fermion has mass. Hence, it must have both left & right components. One needs left chiral as well as right chiral Weyl fermion to obtain a Majorana field. However to construct Majorana fermionic field one has to keep in mind the ~~Majority~~ Majorana reality cond^{n}.

$$\Downarrow$$

How can one arrange to have two Weyl fields of two chiralities such that they satisfy Majorana cond^{n} ?

Take Weyl equ^{n}:

$$P_R \psi_L = 0 \Rightarrow$$

$$\Rightarrow (1+\gamma_5)\psi_L = 0$$

$$\Rightarrow (1+\gamma_5^*)\psi_L^* = 0$$

$$\gamma^0 C \quad \Rightarrow \gamma^0 C(1+\gamma_5^*)\psi_L^* = 0$$

Now, one can show $C^{-1}\gamma_5 C = \gamma_5{}^T$ || using $\gamma_5^* = \gamma_5{}^T$ & $C^{-1}\gamma^\mu C = -\gamma^{\mu T}$

$$\Rightarrow \gamma_5 C = C\gamma_5{}^T$$

$$= C\gamma_5^*$$

So,

$$\gamma^0 C(1+\gamma_5^*)\psi_L^* = 0$$

$$\Rightarrow \gamma^0(1+\gamma_5)C\psi_L^* = 0$$

$$\Rightarrow (1-\gamma_5)\gamma^0 C\psi_L^* = 0$$

$$\Rightarrow (1-\gamma_5)\gamma^0 C\psi_L^* = 0$$

$$\Rightarrow \underbrace{P_L}_{\downarrow}\underbrace{(\gamma^0 C\psi_L^*)}_{\downarrow} = 0$$

$\gamma^0 C\psi_L^*$ must be right-chiral

$$\equiv \hat{\psi}_L$$

Thus, if we define a field by

$$\psi = \psi_L + \gamma^0 C \psi_L^* \equiv \underset{\substack{\uparrow \\ \text{Left}}}{\psi_L} + \underset{\substack{\uparrow \\ \text{right}}}{\hat{\psi}_L}$$

$$\Rightarrow \hat{\psi} = \hat{\psi}_L + \hat{\hat{\psi}}_L = \hat{\psi}_L + \psi_L = \psi$$

→ Majorana condn is satisfied.

* Next question we address ⇒ Weyl fermion is massless whereas a Majorana fermion has mass. Then how do we get mass by adding two massless fermionic terms?

⇒ The point is that we've not really 'generated' a mass, we've only created an arrangement where mass can be allowed.

The mass term in Dirac Lagrangian : $\bar{\psi}\psi$

In chiral represent : $\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$ [Page 13]

⇓ Recall

$$\mathcal{L} = \bar{\psi}_R i\cancel{\partial}\psi_R + \bar{\psi}_L i\cancel{\partial}\psi_L - m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$$

→ no term like $\bar{\psi}_L\psi_L$ or $\bar{\psi}_R\psi_R$ since these are identically zero.

→ For a Weyl fermion which has a specific chirality, the mass term must therefore vanish. In other words, the mass term must contain two different chiralities : a Weyl fermion is unable to meet this demand.

→ But since a Majorana fermion has both components, so it can have ~~is~~ mass~~ive~~. A massive fermion must have a left-chiral as well as a right chiral component.

6. Dirac fermions from Weyl fermions:

* Dirac fermions → massive → require Weyl fermions of both helicities. In general, they do not satisfy Majorana cond^{n}. They can have non-zero charge.

* If we take two independent left-chiral Weyl fields ψ_{L1} & ψ_{L2}, and make the combination

$$\psi(x) = \psi_{L1}(x) + \hat{\psi}_{L2}(x)$$

→ this defines a Dirac field.

* Summary:

Dirac field ⇒ completely unconstrained sol^{n} of DE.

Both Weyl & Majorana fields are simpler sol^{ns}, with some kind of constraints imposed on the sol^{n}. We've seen there are two types of conditions that can be imposed in a Lorentz covariant manner on a sol^{n} of a DE.

Reality cond^{n} ⇓ Majorana

Chirality cond^{n} ⇓ Weyl.

7. Majorana Neutrinos:

It is interesting that the possibility of describing a physical particle with a Weyl spinor was rejected by Pauli in 1933 because it leads to the violation of parity. In fact, space inversion transforms ψ_L into ψ_R and vice versa, implying that parity conservation ~~implies~~ requires the simultaneously existence of both chiral components.

However, the discovery of parity violation in 1956-57 invalidated Pauli's reasoning, reviving the possibility to describe massless particles with Weyl spinor fields. In particular, since there was no indication of the existence of a neutrino mass and it was likely that the neutrino participates in weak interactions through its left chiral component, therefore. Landau, Lee, Yang & Salam proposed to describe the neutrino with a left-handed Weyl spinor. This is the so called two component theory of massless neutrinos, which has been incorporated in the SM, where neutrinos are massless and described by left-handed Weyl spinors.

Our GOAL is to incorporate the non-zero mass of the neutrinos into our physical theory.

8. Majorana Mass term:

* In order to understand the theory of Majorana neutrinos, let us consider a single neutrino type ν. A Majorana mass is generated by a Lagrangian mass term with a chiral fermion field alone. Since neutrinos are left-handed, we use the left-handed chiral field ν_L.

* Is it possible to write down a mass term using ν_L alone?

⟹ In order to answer this question let's consider first a Dirac mass term for a Dirac neutrino field $\nu = \nu_L + \nu_R$,

$$\mathcal{L}^D_{mass} = -m\bar{\nu}\nu = -m(\bar{\nu}_R\nu_L + \bar{\nu}_L\nu_R) = -m\,\bar{\nu}_R\nu_L + h.c.$$

Only the $\bar{\nu}_R\nu_L$ & $\bar{\nu}_L\nu_R$ couplings survive; $\bar{\nu}_R\nu_R \equiv 0 = \bar{\nu}_L\nu_L$.
The Dirac mass term in the above equation is a Lorentz scalar.
Because,

$$\begin{aligned}\nu_L'(x') &= P_L\,\nu'(x') = P_L\,S\,\nu(x) \\ &= S\,P_L\,\nu(x) \qquad \left[\because\ [\gamma_5, \sigma^{\mu\nu}] = 0\right] \\ &= S\,\nu_L(x)\end{aligned}$$

– similarly others.

◆ In order to write a Majorana mass term using ν_L alone, we must find a right-handed function of ν_L which transforms as ν_L under LT. and can be substituted in place of ν_R in the $\mathcal{L}^D_{mass}$.
This function of ν_L is precisely the charge conjugated field

$$\nu_L^C = C\,\bar{\nu}_L^T.$$

Since ν_L^c is right-handed (as we proved earlier; OR $P_L(C\bar{\Psi}_L^T)$ $= C P_L^T \bar{\Psi}_L^T = C(\bar{\Psi}_L P_L)^T = C[\bar{\Psi}\underset{\sim}{P_R\Psi_L}]^T = 0$; where we've used $P_L C = C P_L^T$ & $\bar{\Psi}_L = \bar{\Psi}P_R$).

the coupling $\overline{\nu_L^c}\nu_L$ does not vanish. Furthermore, under a $\mathcal{L}T$ the charge conjugated field $\nu_L^c(x)$ transforms as

$$\nu_L^c(x) = C\,\overline{\nu_L}^T \; \cancel{=} \; C\,(\nu_L^\dagger \gamma^0)^T \xrightarrow{LT} C\,(\nu_L'^\dagger \gamma^0)^T$$

$$\xrightarrow{LT} C\,(\overline{\nu_L} S^{-1})^T$$

$$= C\,(S^{-1})^T\,\overline{\nu_L}^T$$

$$= \underbrace{C\,(S^{-1})^T C^{-1}}\; C\,\overline{\nu_L}^T$$

$$= S\,\nu_L^c$$

$$\left.\begin{array}{lll} \therefore\ \nu_L^c(x) & \text{transforms as} & \nu_L \\ \Rightarrow \overline{\nu_L^c}(x) & \text{"} \quad \text{"} & \overline{\nu_L} \end{array}\right\} \Rightarrow \begin{array}{l} \nu_L^c \to S\,\nu_L^c \\ \overline{\nu_L^c} \to \overline{\nu_L^c}\,S^{-1} \end{array}$$

Therefore, ν_L^c has the correct pro^s to be used in place of ν_R ~~in~~ leading to the Majorana mass term:

$$\mathcal{L}^M_{mass} = -\tfrac{1}{2} m\,\overline{\nu_L^c}\,\nu_L + h.c.$$

Full Majorana Lagrangian:

$$\boxed{\mathcal{L}^M = \frac{1}{2}\left[\overline{\nu_L}\, i\not\partial\, \nu_L + \overline{\nu_L^c}\, i\not\partial\, \nu_L^c - m\left(\overline{\nu_L^c}\,\nu_L + \overline{\nu_L}\,\nu_L^c\right)\right]}$$

Where additional $1/2$ is introduced in order to avoid double counting due to the fact that ν_L^c & $\overline{\nu_L^c}$ are not independent.

Conveniently one defines Majorana field as $\nu = \nu_L + \nu_L^c$ which satisfies $\nu = \nu^c$. Then

$$\boxed{\mathcal{L}^M = \tfrac{1}{2}\,\overline{\nu}\,(i\not\partial - m)\,\nu}$$

9. Conclusion:

Among known elementary fermions only the neutrinos are neutral & they can be Majorana particles. As already noted by Majorana, since a Majorana spinor has only two independent comp., the Majorana theory is simpler and more economical than the Dirac theory. Hence, the Majorana nature of massive neutrinos may be more natural than the Dirac nature. In fact, neutrinos are Majorana particles in most theories beyond the SM.

The Dirac & Majorana descriptions of a neutrino have different phenomenological consequences only if the neutrino is massive. In the massless Dirac theory, the independent left-handed & right-handed chiral components of the neutrino field obey the decoupled Weyl equations. In the massless Majorana theory, the same Weyl equations hold, with the left & right handed chiral fields related by the reality condn. However, only the left-handed chiral component of the neutrino field interacts. If the neutrino is massless, since the left chiral component of the neutrino field obeys the Weyl eqn in both Dirac & Majorana descriptions and the right chiral component is irrelevant for neutrino interactions, the Dirac & Majorana theories are physically equivalent.

From these considerations, it is clear that in practice one can distinguish a Dirac from a Majorana neutrino only by measuring some effect due to the neutrino mass, since otherwise the massless theory applies in an effective way.

$\boxed{*}$ Sol^n of DE:

* $$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{h=\pm1}\left[a(\vec{p},h)\,u(h,\vec{p})\,e^{-ip\cdot x} + b^\dagger(\vec{p},h)\,v(\vec{p},h)\,e^{ip\cdot x}\right]$$

where, $h \sim$ helicity.

$E_p = \sqrt{\vec{p}^2 + m^2}$ to satisfy KG equ^n $(\Box + m^2)\psi = 0$

$\partial_\mu\partial^\mu$

DE $\Rightarrow$

$(\not{p} - m)\,u(\vec{p},h) = 0$

$(\not{p} + m)\,v(\vec{p},h) = 0$

and $\bar{u}(\vec{p},h)(\not{p} - m) = 0$

$\bar{v}(\vec{p},h)(\not{p} + m) = 0$

|| The KG ~~field~~ equ^n must be satisfied by any free field because it's equivalent to the relativistic energy-momentum dispersion rel^n.

Also, $\overline{u(\vec{p},h)}\,v(\vec{p},h') = 0$.

* Helicity pro^s of $u(\vec{p},h)$, $v(\vec{p},h)$:

$$\psi(x) = \sum_{h=\pm1}\psi(x,h)$$

where, $$\psi(x,h) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a(\vec{p},h)u(\vec{p},h)e^{-ip\cdot x} + b^\dagger(\vec{p},h)\,v(\vec{p},h)e^{ip\cdot x}\right]$$

is an eigenfield of the helicity operator with ev h:

$$\hat{h}\,\psi(x,h) = h\,\psi(x,h).$$

$$\hat{h}\,\psi(x,h) = \frac{\vec{\Sigma}\cdot\vec{P}}{|\vec{P}|}\psi(x,h)$$ operator

momentum ev

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a(\vec{p},h)\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}u(\vec{p},h)\,e^{-ip\cdot x} - b^\dagger(\vec{p},h)\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}v(\vec{p},h)e^{ip\cdot x}\right]$$

$\because e^{+ip\cdot x} \overset{\vec{P}}{\Longrightarrow} -\vec{p}$

In order to satisfy $\hat{h}\,\psi(x,h) = h\,\psi(x,h)$, $u(\vec{p},h)$ & $v(\vec{p},h)$ must be eigenfunctions of $\hat{h}$ in momentum space $\vec{\Sigma}\cdot\vec{p}/|\vec{p}|$ with opposite eigenvalues:

$$\begin{cases} \dfrac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\, u(\vec{p},h) = h\, u(\vec{p},h) \\[2ex] \dfrac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\, v(\vec{p},h) = -h\, v(\vec{p},h) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \overline{u(\vec{p},h)}\, \dfrac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|} = h\, \overline{u(\vec{p},h)} \\[2ex] \overline{v(\vec{p},h)}\, \dfrac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|} = -h\, \overline{v(\vec{p},h)} \end{cases}$$

See, $\bar{u}(\vec{p},h)\ \uparrow\ u(\vec{p},h')$

$$= \underbrace{\bar{u}(\vec{p},h)\, \frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}}_{h\,\bar{u}(\vec{p},h)}\ \underbrace{\frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\, u(\vec{p},h')}_{h'\,u(\vec{p},h')}$$

$$= h h'\, \overline{u(\vec{p},h)}\, u(\vec{p},h').$$

cons. $\delta_{hh'}$

$$\Rightarrow \overline{u(\vec{p},h)}\, u(\vec{p},h') = h h'\, \overline{u(\vec{p},h)}\, u(\vec{p},h')$$

$\Rightarrow$ See $h = \pm 1$, $h' = \pm 1$

if $h = -h'$ $\rightsquigarrow$ inconsistent

only choice $h = h'$ $\rightsquigarrow$ consistent

$\therefore u(\vec{p},h)$

$j^0 = \psi^\dagger \psi = 2E$

$$\Downarrow$$

Normalization $\Rightarrow$ (Dirac 31 & 32)

$$\begin{aligned} \overline{u(\vec{p},h)}\, u(\vec{p},h') &= 2m\, \delta_{hh'} \\ \overline{v(\vec{p},h)}\, v(\vec{p},h') &= -2m\, \delta_{hh'} \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} u^\dagger(\vec{p},h)\, u(\vec{p},h') &= 2E\, \delta_{hh'} \\ v^\dagger(-\vec{p},h)\, v(-\vec{p},h') &= 2E\, \delta_{hh'} \\ v^\dagger(-\vec{p},h)\, u(\vec{p},h') &= 0 \end{aligned}$$

* $\overline{u(\vec{p},h)}\,\gamma^\mu\, u(\vec{p},h') = \overline{v(\vec{p},h)}\,\gamma^\mu\, v(\vec{p},h') = 2p^\mu\,\delta_{hh'}$

Proof: $\overline{u(\vec{p},h)}\,\gamma^\mu\, u(\vec{p},h')$

$$= \overline{u(\vec{p},h)}\,\frac{\gamma^\mu \not{p} + \not{p}\gamma^\mu}{2m}\,u(\vec{p},h')$$

$$= \frac{p^\mu}{m}\,\overline{u(\vec{p},h)}\,u(\vec{p},h')$$

$$= 2p^\mu\,\delta_{hh'}$$

$\overline{u(\vec{p},h)}\,u(\vec{p},h') = 2m\,\delta_{hh'}$

$\overline{v(\vec{p},h)}\,v(\vec{p},h') = -2m\,\delta_{hh'}$

$\overline{u(\vec{p},h)}\,\gamma_5\, u(\vec{p},h') = 0$

Proof: $\overline{u(\vec{p},h)}\,\gamma_5\,\frac{\not{p}}{m}\,u(\vec{p},h')$

$$= -\,\overline{u(\vec{p},h)}\,\frac{\not{p}}{m}\,\gamma_5\,u(\vec{p},h')$$

$$= -\,\overline{u(\vec{p},h)}\,\gamma_5\,u(\vec{p},h')$$

$$= 0$$

$u^\dagger(\vec{p},h)\,v(-\vec{p},h') = v^\dagger(\vec{p},h)\,u(-\vec{p},h') = 0.$

* Dirac & Majorana 25 $\Rightarrow$

$$\Lambda_\pm(p) = \frac{1}{2}\left(1 \pm \not{p}/m\right)$$

$\Lambda_+(p)\,u(\vec{p},h) = u(\vec{p},h)$, $\Lambda_+\,v(\vec{p},h) = 0$

$\Lambda_-(p)\,u(\vec{p},h) = 0$, $\Lambda_-(p)\,v(\vec{p},h) = v(\vec{p},h)$.

Completeness,
$$\sum_{h=\pm 1}\left[\frac{u(\vec{p},h)\,\overline{u(\vec{p},h)}}{2m} - \frac{v(\vec{p},h)\,\overline{v(\vec{p},h)}}{2m}\right] = \mathbb{1}$$

Now, our aim is to proof

$$\boxed{\begin{aligned}\Lambda_+(p) &= \sum_{h=\pm 1} \frac{u(\vec{p},h)\,\overline{u(\vec{p},h)}}{2m} \\ \Lambda_-(p) &= -\sum_{h=\pm 1} \frac{v(\vec{p},h)\,\overline{v(\vec{p},h)}}{2m}\end{aligned}}$$

<u>Proof:</u>

$$\Lambda_+(p)\, u(\vec{p},h) = u(\vec{p},h)$$

$$\Rightarrow \Lambda_+(p)\, u(\vec{p},h)\,\overline{u(\vec{p},h)} = u(\vec{p},h)\,\overline{u(\vec{p},h)}$$

$$\Rightarrow \Lambda_+(p) \underbrace{\sum_{h=\pm 1} u(\vec{p},h)\,\overline{u(\vec{p},h)}}_{2m + \sum v(\vec{p},h)\,\overline{v(\vec{p},h)}} = \sum_{h=\pm} u(\vec{p},h)\,\overline{u(\vec{p},h)}$$

and $\Lambda_+(p)\, v(\vec{p},h) = 0$.

$$\Rightarrow \Lambda_+(p) = \sum_{h=\pm 1} \frac{u(\vec{p},h)\,\overline{u(\vec{p},h)}}{2m} .$$ Pvd

* Dirac & Majorana 27 $\Rightarrow$

$$\cancel{P_h = \frac{1}{2}(1+\gamma_5 \not{s}_h)}$$

$$\cancel{P_h\, u(\vec{p},h) = \frac{1}{2}(1+\gamma_5 \not{s}_h)\, u(\vec{p},h)}$$

$$\cancel{P_h}$$

$$P_h(u)\, u(\vec{p},h) = \frac{1}{2}(1 + h\,\hat{p}\cdot\vec{\Sigma})\, u(\vec{p},h)$$

$$= \frac{1}{2}(1+h\,h)\, u(\vec{p},h) = u(\vec{p},h)$$

$$P_h(u)\, v(\vec{p},h) = \frac{1}{2}(1+h\,\hat{p}\cdot\vec{\Sigma})\, v(\vec{p},h)$$

$$\neq \cancel{u(\vec{p},h)} = \frac{1}{2}(1-h^2)\, v(\vec{p},h)$$

$$= 0$$

$$P_h(v)\, v(\vec{p},h) = \frac{1}{2}(1-h\,\hat{p}\cdot\vec{\Sigma})\, v(\vec{p},h)$$

$$= \frac{1}{2}(1+h^2)\, v(\vec{p},h)$$

$$= 2v(\vec{p},h) .$$

$$P_h(u) = \frac{1}{2}\left(1 + h\,\hat{p}\cdot\vec{\Sigma}\right)$$

$$P_h(v) = \frac{1}{2}\left(1 - h\,\hat{p}\cdot\vec{\Sigma}\right).$$

$$P_h = \frac{1}{2}\left(1 \pm h\,\hat{p}\cdot\vec{\Sigma}\right).$$

and $\frac{\vec{p}}{|\vec{p}|}\cdot\vec{\Sigma}\; u(\vec{p},h) = h\, u(\vec{p},h)$

$\hat{p}\cdot\vec{\Sigma}\; v(\vec{p},h) = -h v(\vec{p},h)$.

$$\boxed{\cancel{P_h = \frac{1}{2}\left(1 + \gamma_5 \not{s}_h\right)} \quad \cancel{= \frac{1}{2}\left(1 + \gamma_5\, s\right)}}.$$

We've seen also

$$\hat{p}\cdot\vec{\Sigma}\; u(\vec{p},h) = h\,\gamma_5 \not{s}_h\, u(\vec{p},h)$$

$$\Rightarrow \frac{\hat{p}\cdot\vec{\Sigma}}{h}\, u(\vec{p},h) = \gamma_5 \not{s}_h\, u(\vec{p},h)$$

and $\hat{p}\cdot\vec{\Sigma}\; v(\vec{p},h) = -h\,\gamma_5 \not{s}_h\; v(\vec{p},h)$ $\cancel{\neq -h\, v(\vec{p},h)}$

$$\Rightarrow + \frac{\hat{p}\cdot\vec{\Sigma}}{h}\, v(\vec{p},h) = -\gamma_5 \not{s}_h\; v(\vec{p},h)$$

So, $P_h(u) = \frac{1}{2}\left(1 + h\,\hat{p}\cdot\vec{\Sigma}\right) = \frac{1}{2}\left(1 + \gamma_5 \not{s}_h\right)$.

$P_h(v) = \frac{1}{2}\left(1 - h\,\hat{p}\cdot\vec{\Sigma}\right) = \frac{1}{2}\left(1 + \gamma_5 \not{s}_h\right)$.

So, $P_h = \frac{1}{2}\left(1 + \gamma_5 \not{s}_h\right)$. where

$$s_h^\mu = h\left(\frac{|\vec{p}|}{m}, \hat{p}\,\frac{E}{m}\right).$$

* Dirac & Majorana 27 $\Rightarrow$.

$$\Lambda_\pm^h(p) = \Lambda_\pm(p)\, P_h = P_h\, \Lambda_\pm(p)$$

$$= \frac{1}{2}\left(1 \pm \frac{\not{p}}{m}\right)\frac{1}{2}\left(1 + \gamma_5 \not{s}_h\right)$$

and $\Lambda_+^h(p)\, u(\vec{p},h') = \delta_{hh'}\, u(\vec{p},h')$, $\Lambda_+^h(p)\, v(\vec{p},h') = 0$

$\Lambda_-^h(p)\, u(\vec{p},h') = 0$, $\Lambda_-^h(p)\, v(\vec{p},h') = \delta_{hh'}\, v(\vec{p},h')$

Also, $\sum_{r=\pm}\sum_{h=\pm 1} \Lambda_r^h(p) = \mathbb{1}$, $\Lambda_r^h(p)\,\Lambda_s^{h'}(p) = \Lambda_r^h(p)\,\delta_{rs}\,\delta_{hh'}$

$$\Lambda_-^h(p)\, v(\vec{p}, h') = \delta_{hh'}\, v(\vec{p}, h')$$

$$\Lambda_+^h(p)\, u(\vec{p}, h') = \delta_{hh'}\, u(\vec{p}, h')$$

$$\cancel{\Rightarrow \Lambda_+^h(p) \sum_{h'=\pm 1} u(\vec{p}, h')\, \overline{u(\vec{p}, h')}} = \delta_{hh'} \sum_{h'=\pm 1} \cancel{u(\vec{p}, h')\, \overline{u(\vec{p}, h')}}$$

$$\boxed{\begin{aligned} \Lambda_+^h(p) &= \frac{u(\vec{p}, h)\, \overline{u(\vec{p}, h)}}{2m} \\ \Lambda_-^h(p) &= -\frac{v(\vec{p}, h)\, \overline{v(\vec{p}, h)}}{2m} \end{aligned}}$$

Now, $\Lambda_+^h(p) = \dfrac{u(\vec{p}, h)\, \overline{u(\vec{p}, h)}}{2m}$

$$\Lambda_-^h(p) = -\frac{v(\vec{p}, h)\, \overline{v(\vec{p}, h)}}{2m}$$

So, $u(\vec{p}, h)\, \overline{u(\vec{p}, h)} = 2m\, \Lambda_+^h(p)$

$$= 2m\, \frac{1}{2}\left(1 + \frac{\not{p}}{m}\right) \frac{1}{2}\left(1 + \gamma_5 \not{s}_h\right)$$

$$= m\left(\frac{m + \not{p}}{m}\right) \frac{1}{2}\left(1 + \gamma_5 \not{s}_h\right)$$

$$= (\not{p} + m)\, \frac{1}{2}\left(1 + \gamma_5 \not{s}_h\right).$$

$$\not{s}_h = s_h^\mu \gamma_\mu = \cancel{\gamma_\mu \left(\frac{|\vec{p}|}{m}, \hat{p}\frac{E}{m}\right) h} = \cancel{\gamma^\mu h\left(\frac{|\vec{p}|}{m}, -\hat{p}\frac{E}{m}\right)}$$

$$\cancel{= \gamma^0 \frac{|\vec{p}|}{m}}$$

Massless limit $\Rightarrow$ $\gamma_5 \not{s}_h = \begin{cases} \gamma_5 & \text{particle} \rightarrow u \\ -\gamma_5 & \text{anti} \rightarrow v \end{cases}$

of 4D spinors

* four spinors form a basis → they are also mutually orthogonal.

$$\Downarrow$$

$$u(\vec{p},+),\ u(\vec{p},-),\ v(\vec{p},+),\ v(\vec{p},-)$$

The outer products

$$\left.\begin{array}{l} u(\vec{p},+)\,\overline{u(\vec{p},+)} \\ u(\vec{p},-)\,\overline{u(\vec{p},-)} \\ v(\vec{p},+)\,\overline{v(\vec{p},+)} \\ v(\vec{p},-)\,\overline{v(\vec{p},-)} \end{array}\right\}$$

form a basis of the VS of 4×4 matrices. They satisfy the completeness rel^{n}:

Now,

$$\sum_{h=\pm 1}\left[\frac{u(\vec{p},h)\,\overline{u(\vec{p},h)}}{2m} - \frac{v(\vec{p},h)\,\overline{v(\vec{p},h)}}{2m}\right] = \mathbb{1}$$

* Energy Projection operator (Dirac 22) : Recall ⇒

$$\begin{cases} \frac{\not p}{m}\,u(\vec{p},h) = u(\vec{p},h) \\ \frac{\not p}{m}\,v(\vec{p},h) = -v(\vec{p},h) \end{cases}$$

→ $\Lambda_{\pm}(p) = \frac{1}{2}\left(1 \pm \not p/m\right)$; $\{ + \to u,\ - \to v \}$

→ $\Lambda_{+}(p)\,u(\vec{p},h) = u(\vec{p},h)$

$\Lambda_{+}(p)\,v(\vec{p},h) = 0$

$\Lambda_{-}(p)\,u(\vec{p},h) = 0$

$\Lambda_{-}(p)\,v(\vec{p},h) = v(\vec{p},h)$

$$\left.\begin{array}{l} \Lambda_{+}(p) = \sum_{h=\pm 1} \frac{u(\vec{p},h)\,\overline{u(\vec{p},h)}}{2m} \\ \Lambda_{-}(p) = -\sum_{h=\pm 1} \frac{v(\vec{p},h)\,\overline{v(\vec{p},h)}}{2m} \end{array}\right\}$$ Dirac & Majorana 24.2

* Helicity Projection operator :

→ Recall ⇒

$$\begin{cases} \frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\,u(\vec{p},h) = h\,u(\vec{p},h) \\ \frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\,v(\vec{p},h) = -h\,v(\vec{p},h) \end{cases}$$

$$\frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\,u(\vec{p},+) = u(\vec{p},+)$$

$$\frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\,u(\vec{p},-) = -u(\vec{p},-)$$

$(h^2 = 1)$

$$P_h(u) = \frac{1}{2}\left(1 + \frac{1}{h}\,\hat{p}\cdot\vec{\Sigma}\right) = \frac{1}{2}\left(1 + \frac{h}{h^2}\,\hat{p}\cdot\vec{\Sigma}\right) = \frac{1}{2}\left(1 + h\,\hat{p}\cdot\vec{\Sigma}\right)$$

↳ projection operators on u with definite helicity.

$P_h(v) = \frac{1}{2}\left(1 - h\,\hat{p}\cdot\vec{\Sigma}\right)$ → proj. oper. on v with definite helicity

It's possible to write these projection operators in a unified <u>covariant form</u>.

$$\hat{p}\cdot\vec{\Sigma}\; u(\vec{p},h) = \hat{p}\cdot\vec{\Sigma}\;\frac{\not{p}}{m}u(\vec{p},h)$$

$$= \hat{p}\cdot\gamma^5\gamma^0\vec{\gamma}\;\frac{E\gamma^0 - \vec{p}\cdot\vec{\gamma}}{m}\;u(\vec{p},h)$$

$$\cancel{= \gamma^5 \gamma^0 \gamma^i p^i}\,.$$

$$= \left[-\frac{\gamma^5\,\hat{p}\cdot\vec{\gamma}\,\gamma^0 E\gamma^0}{m} - \frac{\gamma^5\gamma^0}{m|\vec{p}|}\,p^i\gamma^i\,p^j\gamma^j\right]u(\vec{p},h)$$

$$\frac{p^ip^j}{2}\underbrace{(\gamma^i\gamma^j + \gamma^j\gamma^i)}_{\underbrace{2\eta^{ij}}_{-2\delta_{ij}}}$$

$$= \left[-\gamma^5\,\frac{\hat{p}\cdot\vec{\gamma}}{m}E + \frac{\gamma^5\gamma^0}{m|\vec{p}|}|\vec{p}|^2\right]u(\vec{p},h)$$

$$= \gamma^5\left[\frac{|\vec{p}|}{m}\gamma^0 - \frac{E}{m}\,\frac{\vec{\gamma}\cdot\vec{p}}{|\vec{p}|}\right]u(\vec{p},h)$$

$$\Rightarrow \boxed{(\hat{p}\cdot\vec{\Sigma})\,u(\vec{p},h) = h\,\gamma^5\not{s}_h\,u(\vec{p},h)}$$

$s_h^\mu \rightarrow$ polarization 4-vector

where, $s_h^\mu = h\left(\frac{|\vec{p}|}{m}, \frac{E}{m}\frac{\vec{p}}{|\vec{p}|}\right) = h\left(\frac{|\vec{p}|}{m}, \hat{p}\frac{E}{m}\right)$.

with $s_h^2 = -1$ & $s_h\cdot p = 0$

and similarly,

$$\boxed{(\hat{p}\cdot\vec{\Sigma})\,v(\vec{p},h) = -h\,\gamma^5\not{s}_h\,v(\vec{p},h)}$$

→ Recall, $P_h(u) = \frac{1}{2}(1 + h\,\hat{p}\cdot\vec{\Sigma})$

$P_h(v) = \frac{1}{2}(1 - h\,\hat{p}\cdot\vec{\Sigma})$

<u>Helicity</u> Projection operator <u>in covariant form</u> $\Longrightarrow$ $\boxed{P_h = \frac{1}{2}(1 + \gamma^5\not{s}_h)}$

Now, $[\gamma^5 \not{s}_h, \not{p}]$

$$= \gamma^5 \underbrace{\{\not{s}_h, \not{p}\}}_{0} - \underbrace{\{\gamma^5, \not{p}\}}_{0} \not{s}_h$$

$$\cancel{\gamma^5 \not{s}_h \not{p}} - \cancel{\not{p}\gamma^5 \not{s}_h}$$
$$= -\cancel{\gamma^5 \not{s}_h \not{p}} + \cancel{\gamma^5 \not{p} \not{s}_h}$$

$$= 0$$

$\{\gamma^\mu, \gamma^5\} = 0$

$s_h \cdot p = 0$

now, $\not{s}_h \not{p} + \not{p}\not{s}_h = 2 s_h \cdot p = 0$

$= \{\not{s}_h, \not{p}\}$

$= 0$

Therefore, we can define the four projection operators on the components with definite energy & helicity as

$$\Lambda^h_\pm(p) \equiv \Lambda_\pm(p) P_h = P_h \Lambda_\pm(p)$$
$$= \frac{1}{2}\left(1 \pm \frac{\not{p}}{m}\right) \frac{1}{2}\left(1 + \gamma^5 \not{s}_h\right)$$

s.t. $$\sum_{r=\pm} \sum_{h=\pm 1} \Lambda^h_r(p) = 1$$

$$\Lambda^h_r(p) \Lambda^{h'}_s(p) = \Lambda^h_r(p)\, \delta_{rs} \delta_{hh'}$$

$\boxed{*}$

$$\boxed{\begin{aligned} \Lambda_+ u(\vec{p}, h) &= u(\vec{p}, h) \\ \Lambda_- v(\vec{p}, h) &= v(\vec{p}, h) \\ P_h u(\vec{p}, h) &= h\, u(\vec{p}, h) \\ P_h v(\vec{p}, h) &= +h\, v(\vec{p}, h) \end{aligned}}$$

with $\Lambda_\pm(p) = \frac{1}{2}(1 \pm \not{p}/m)$

$P_h = \frac{1}{2}(1 + \gamma^5 \not{s}_h)$

and
$$\left\{\begin{aligned} \Lambda^h_+(p)\, u(\vec{p}, h') &= \delta_{hh'}\, u(\vec{p}, h') \\ \Lambda^h_+(p)\, v(\vec{p}, h') &= 0 \\ \Lambda^h_-(p)\, v(\vec{p}, h') &= \delta_{hh'}\, v(\vec{p}, h') \\ \Lambda^h_-(p)\, u(\vec{p}, h') &= 0 \end{aligned}\right.$$