Problem 1: Meson-antimeson mixing

The $M^0$-$\bar{M}^0$ mixing amplitude $M_{12}$, where $M = K, B_d$ or $B_s$, can be written as

$$M_{12} = A_M \sum_{i,j=\text{u,c,t}} \lambda_i^M \lambda_j^M \tilde{S}(x_i, x_j)$$  \hspace{1cm} (1)

if one neglects QCD corrections. Here $x_i = m_i^2 / M_W^2$ and $\tilde{S}(x_i, x_j) = \tilde{S}(x_j, x_i)$ is obtained from the box diagram:

For $M = K, B_d, B_s$ one has:

<table>
<thead>
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<th>$M = K$</th>
<th>$M = B_d$</th>
<th>$M = B_s$</th>
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</thead>
<tbody>
<tr>
<td>$A_M [\text{GeV}]$</td>
<td>$6.2 \pm 0.8 \cdot 10^{-11}$</td>
<td>$1.3 \pm 0.4 \cdot 10^{-9}$</td>
<td>$2.0 \pm 0.6 \cdot 10^{-9}$</td>
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<tr>
<td>$\lambda_i^M$</td>
<td>$V_{ts}V_{td}^*$</td>
<td>$V_{tb}V_{td}^*$</td>
<td>$V_{tb}V_{ts}^*$</td>
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The uncertainties stem from poorly known hadronic quantities.

a) Use CKM unitarity to eliminate $\lambda_u^M$ in favour of $\lambda_c^M$ and $\lambda_t^M$ to write

$$M_{12} = A_M[(\lambda_t^M)^2 S(x_t) + 2\lambda_c^M \lambda_t^M S(x_c, x_t) + (\lambda_c^M)^2 S(x_c)]$$  \hspace{1cm} (2)

and find the relation between $S$ in (2) and $\tilde{S}$ in (1), setting $x_u = 0$. Verify the GIM mechanism: Which function vanishes for (i) $x_c = x_t$ and (ii) $x_c = 0$?
b) With $S(x_c) = 2.6 \cdot 10^{-4}$, $S(x_c, x_t) = 2.3 \cdot 10^{-3}$ and $S(x_t) = 2.3$ compute the three terms in (2) and assess the relevance of the second and third term for the three meson systems. Mixing-induced CP asymmetries determine $\sin(2 \arg M_{12})$ (assuming the standard CKM phase convention). How big is the relative deviation of $\arg M_{12}(B_d)$ from $2\beta \simeq 46^\circ$ and of $\arg M_{12}(B_s)$ from $-2\beta_s \simeq -2.2^\circ$?

c) The mass difference between the mass eigenstates is given by $\Delta m = 2|M_{12}|$. Compute $\Delta m$ from the formula above and assess the relevance of neglected QCD effects by comparing your results with the experimental numbers

\[
\begin{align*}
\Delta m_K &= (3.483 \pm 0.006) \cdot 10^{-15} \text{ GeV}, \\
\Delta m_{B_d} &= (3.34 \pm 0.03) \cdot 10^{-13} \text{ GeV}, \\
\Delta m_{B_s} &= (1.170 \pm 0.008) \cdot 10^{-11} \text{ GeV}.
\end{align*}
\]

d) The CP-violating quantity $|\varepsilon_K|$ in $K^0 - \bar{K}^0$ mixing,

\[|\varepsilon_K| \approx \frac{1}{2\sqrt{2}} \arg M_{12},\]

has been measured as $|\varepsilon_K| = (2.28 \pm 0.02) \cdot 10^{-3}$. Express the $\lambda^K_c$ and $\lambda^K_t$ in terms of Wolfenstein parameters (to leading non-vanishing order in $\lambda$) and determine the constraint on $(\bar{\rho}, \bar{\eta})$ found from $|\varepsilon_K|$. 

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