Lecture 6: The idea of renormalization, counting divergences in Feynman diagrams

When we discussed perturbative computations in the first two lectures of this semester, we saw that, quite often, we obtain ill-defined expressions. There are different reasons why these ill-defined expressions appear and one requires a two-step procedure to deal with them:

1) **regularization**, which means re-defining original computations in such a way that calculations become formally well-defined. Usually, this requires introduction of some regularization parameter (photon mass, dimensional reg. parameter &c., etc.) which should be considered to be either small or large, to approach an interesting physical regime;

2) understanding of how the dependence of the regularization parameter disappears when physical quantities are computed. In case when the dependence of the regularization parameter is related to behavior of Feynman integrals at very large
momenta, it is called renormalization.

Before we discuss the details of the renormalization procedure, we will try to understand its meaning. To this end, we will consider an example of a massless scalar field, described by the following Lagrangian

$$\mathcal{L} = -\frac{1}{2} \phi \square \phi - \frac{A}{4!} \phi^4$$

The constant $A$ is dimensionless coupling, that is sufficiently small so that the scattering of $\phi$-particles can be treated in perturbation theory.

At tree level we have a simple diagram

$$i M_1 = i \lambda \phi \phi$$

At one loop, there are three diagrams that describe the $4-\phi$ scattering

$$i M_2 = \phi \phi$$

All of these diagrams are similar, so let's focus on the first one.
The expression for $iM_2^{(a)}$ is

$$iM_2^{(a)} = \frac{(-iA)^2}{2} i^2 \int \frac{d^4 k}{(2\pi)^4} \frac{4}{k^2 (p_{12} + k)^2}, \quad (\times)$$

where $p_{12} = p_1 + p_2$. Indeed, and $\frac{1}{2}$ is a symmetry factor. Next, consider the integral

$$iM_2^{(a)}$$

over $k$, in Eq. (\times). The integral does not converge at large values of $k$. Indeed, if $k^n \to \infty$, $k^2 (p_{12} + k)^2 \propto k^4$ and

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 (p_{12} + k)^2} \propto \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4}$$

To understand that last integral, perform the Wick rotation, use the fact that the 4-dim. solid angle is $2\pi^2$ and write

$$\int_0^\infty \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4} = \frac{i}{8\pi^2} \int \frac{dk_E}{k_E}$$

As we see, the integral doesn't exist; it diverges at $k = \infty$. To make sense out of this integral, we need to avoid reaching $k_E = \infty$. We do that by introducing a cut-off $(\Lambda \gg \sqrt{s/12})$

$$\frac{i}{8\pi^2} \int \frac{dk_E}{k_E} \Rightarrow \frac{i}{8\pi^2} \int \frac{dk_E}{\sqrt{s/12}} = \frac{i}{8\pi^2} \ln\left(\frac{\Lambda}{\sqrt{s/12}}\right)$$
For any large, but finite, \( \Lambda \) we can compute the integral. For \( \Lambda \to \infty \), we recover the problem of the original integral. This is an example of the regularization procedure introduced earlier.

With the cut-off regularization, we obtain

\[
i M_2^{(a)} = \frac{A^2}{2} \frac{i}{8 \pi^2} \ln \frac{\Lambda}{\sqrt{t_{12}}} = \frac{i \Lambda^2}{32 \pi^2} \ln \left( \frac{\Lambda^2}{s_{12}} \right)
\]

It is clear that a similar calculation for b) & c) gives similar results up to \( s_{12} \to -t \) for b) & \( s_{12} \to -u \) for c).

Therefore

\[
i M_2 = \frac{i \Lambda^2}{32 \pi^2} \left( \frac{\ln \frac{\Lambda^2}{s}}{s} + \frac{\ln \frac{\Lambda^2}{-t}}{-t} + \frac{\ln \frac{\Lambda^2}{-u}}{-u} \right)
\]

The full amplitude is

\[
i M = i M_1 + i M_2 = -i \Lambda + \frac{i \Lambda^2}{32 \pi^2} \left[ \frac{\ln \frac{\Lambda^2}{s}}{s} + \frac{\ln \frac{\Lambda^2}{-t}}{-t} + \frac{\ln \frac{\Lambda^2}{-u}}{-u} \right]
\]

Let's try to understand what this expression means. First point to note is that if we take \( \Lambda \to \infty \) keeping \( \Lambda \) fixed, \( iM \) flows up to this is not a meaningful full procedure.

Second point is that if we calculate the difference of the two scattering amplitudes at two values of \( s, t \& u \), the result is \( \Lambda \)-independent.
\[ iM(s, t, u) - iM(s, t, u) = - \frac{i \lambda^2}{32\pi^2} \left( \frac{\hbar s}{-t} + \frac{\hbar u}{-u} \right) \]

At this point, we can ask the following question — how do we know what \( \lambda \) is?

True, \( \lambda \) is a parameter in the Lagrangian but for us to actually know what it is, it must be measured. We can only measure it from the scattering of \( \phi \) - particles but then, according to our results, we will never get \( \lambda \), but rather \( \lambda = \frac{\lambda^2}{32\pi^2} \left( 3\hbar \Lambda^2 + \ldots \right) \).

Therefore, the only quantity that we can determine experimentally is a strange combination of \( \lambda \) and \( \hbar \Lambda^2 \) and it is this combination that we would have to call a physical coupling.

To make this explicit, let us declare, that a physical coupling \( \lambda \) is measured at a perfect kinematic point \( (s_0, t_0, u_0) \); i.e.

\[ iM(s_0, t_0, u_0) = -i \lambda_{\text{phys}} \]

According to our calculation

\[ \lambda_{\text{phys}} = \lambda - \frac{\lambda^2}{32\pi^2} \left( \frac{\hbar \Lambda^2}{s_0} + \frac{\hbar \Lambda^2}{t_0} + \frac{\hbar \Lambda^2}{-u_0} \right) \]

It follows that

\[ \lambda = \lambda_{\text{phys}} + \frac{\lambda^2}{32\pi^2} \left( \frac{\hbar \Lambda^2}{s_0} + \frac{\hbar \Lambda^2}{t_0} + \frac{\hbar \Lambda^2}{-u_0} \right) + O(\lambda_{\text{phys}}^3) \]
and, if we use this formula to re-express the scattering amplitude in terms of physical coupling $\lambda_{\text{phys}}$, we find

$$iM = -i \lambda_{\text{phys}} + \frac{i\lambda_{\text{phys}}^2}{32\pi^2} \left( \ln(\frac{S_0}{\tilde{S}}) + \ln(\frac{t_0}{t}) + \ln(\frac{u_0}{u}) \right)$$

The scattering amplitude become independent of $A^2$ and all divergences are absorbed into $\lambda_{\text{phys}}$. Because of our definition, $\lambda_{\text{phys}}$ is directly measurable and we can predict $iM$ in terms of $\lambda_{\text{phys}}$ without experiencing any problems with divergences. This is the program of the renormalization in a nutshell.

The key message is that divergences that are encountered in perturbation theory disappear if observables are written in terms of observables.

There are subtleties associated with the above statement but it conveys the general idea. We will now turn to more systematic studies of divergences in perturbative QFT.
To this end, we consider Quantum Electrodynamics (QED). A typical diagram in QED is characterized by:

1) \( N_e \), the number of external lepton lines;
2) \( N_p \), the number of external photon lines;
3) \( P_e, P_p \), the number of internal (i.e., loop-momentum dependent) propagators for electrons and photons, and
4) \( V \), the number of vertices that involve internal lines and
5) \( L \), the number of loops.

A typical diagram looks like:

\[
\sim \int \frac{1}{k_{1}-m} \frac{1}{k_{j}^{2}} \quad (**) 
\]

We would like to know what happens to this expression if all momenta are scaled by a uniform (large) factor:

\[ k_{i\ell} \rightarrow \lambda k_{i\ell}, \quad \lambda \rightarrow \infty. \]

If we assume that the \( L \)-loop integral in Eq. (**) contains \( P_e \) external electron lines and \( P_p \) external photon lines, the scaling is easily computed:

\[ I^{**} \sim 2^{4L-P_e-2P_p} \]

We call the exponent of \( \lambda \) \( D = 4L-P_e-2P_p \) as the superficial degree of divergence.
The meaning of $D$ is clear. It tells us about the contribution to $I$ from the integration region where all loop momenta become very large. If $D < 0$, the integral over that region converges; if $D = 0$, the integral diverges logarithmically; if $D > 0$, the integral diverges as a power.

It is important to have in mind that even diagrams with $D < 0$ can have divergent subdiagrams whose presence isn’t captured by $D$. This isn’t a problem since $D$ is not supposed to identify all divergences. It is designed to capture particular ones.

Our next step is to express $D = 4L - P_e - 2P_f$. As our first step, note that in QED $L = P_e + P_f - V + 1$

This relation is valid because the number of loop momenta that are independent follows from the number of original # of independent momenta ($P_e + P_f$, i.e. one per propagator) reduced by the number of equations ($V-1$) that force energy-momentum conservation.
The number of vertices can also be constrained. Indeed, each vertex contains 1 photon line and 2 electron lines.

Hence, the # of vertices is

\[ V = 2P_g + N_f \] or \[ V = \frac{1}{2} (2P_e + N_e) \]

We can use these relations to write \( D \) as:

\[ D = 4L - P_e - 2P_g = 4(P_e + P_g - V + 1) - P_e - 2P_g \]

\[ = 3P_e + 2P_g - 4V + 4 = 3(P_e - V) + 2P_g - V + 4 \]

\[ = 3\left(-\frac{N_e}{2}\right) - N_f + 4 = 4 - \frac{3N_e}{2} - N_f \]

\[ D = 4 - \frac{3N_e}{2} - N_f \]

We see that the degree of divergence in QED only depends on the number of external legs in a particular diagram. This allows us to easily find all Green's functions that have \( D > 0 \). This is so, because \( D \) decreases when the number of external legs increases; so there is a countable number of cases. We find

\[ D = \begin{cases} 4 & \text{for } 1 \text{ electron line} \\ 3 & \text{for } 2 \text{ electron lines} \\ 2 & \text{for } 3 \text{ electron lines} \\ 1 & \text{for } 4 \text{ electron lines} \\ 0 & \text{for } 5 \text{ electron lines} \end{cases} \]
and all other Green's functions have $D < 0$. Next, we can argue 3 divergent Green's functions away:

- is the vacuum fluctuations, do not have external legs, do not need to construct scattering amplitudes, \( \Rightarrow \) irrelevant.

In QED all Green's functions with odd number of photons vanish (Furry's theorem).

This leaves us with 4 divergent Green's functions to understand.

Let us generalize these considerations.

Consider QED in $d$, rather than 4, dimensions.

Repeating the above derivation, we find

\[
D = d - L - P_e - 2 P_f \Rightarrow \\
D = d + \left( \frac{d-4}{2} \right) V - \left( \frac{d-2}{2} \right) N_f + \left( \frac{d-1}{2} \right) N_e.
\]

This result differs from $d=4$ one in that $D$ depends on the number of vertices. If $d<4$, $D$ becomes smaller for larger $V$, for fixed $N_f$ & $N_e$. 
Therefore, for \( d < 4 \), the number of divergent diagrams in QED\( d \) (not Green's functions but diagrams) is finite. For \( d = 4 \), the number of divergent Green's functions is finite. For \( d > 4 \), for any \( N_e \) and \( N_f \), there exists an order in perturbation theory where contributions to any Green's function become divergent.

The three different situations that we described are referred to as "super-renormalizable" \((d < 4)\), "renormalizable" \((d = 4)\) and "not renormalizable" \((d > 4)\).

We have seen that large-momentum (ultraviolet) properties of the theory can be changed by changing dimensionality of space-time. But this is not the only way. To show this, consider the Lagrange density of the form

\[
I = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi \right) - \frac{1}{2} m^2 \phi^2 - \frac{1}{n!} \lambda \phi^n,
\]

in the space-time of dimensionality \(d\). Degree of divergence is computed as in QED example, but some formulas need to be modified.
Here is the derivation:

\[ D = \alpha L - 2 P \rho \quad D = (d-2) P \rho - dV + d \]

\[ L = P \rho - V + 1 \quad \Rightarrow \quad \frac{(d-2)}{2} (V \cdot n - N \rho) - dV + d \]

\[ V n = 2 P \rho + N \rho \quad = d + \left( \frac{(d-2)}{2} n - d \right) V - \frac{(d-2)}{2} N. \]

\[ \Rightarrow \quad D = d + \left[ \frac{(d-2)}{2} n - d \right] V - \frac{(d-2)}{2} N \rho \]

It is clear from that formula that

D becomes worse as "n" increases. If \( d=4 \),

\[ D = 4 + (n-4)V - N \rho, \] which means

that interactions \( \lambda \rho \) are

* non-renormalizable, \( n > 4 \)
* renormalizable, \( n = 4 \)
* super-renormalizable, \( n < 4 \).

We can notice that this feature is related to the mass dimension of the coupling constant \( \lambda \). Since \( \text{dim}[\rho]\sim \text{mass} \),

\( \text{dim}(\lambda) \sim \text{mass}^{4-n} \), since \( \text{dim}[\lambda \rho^n]\sim \text{mass}^n \).

Therefore, the renormalizable theory couplings are dimensionless, the super-renormalizable — couplings have positive mass dimensions and the non-renormalizable — negative mass dimensions.
As we will see, in renormalizable theories, it will be possible to make all Green's function finite (i.e. not divergent) by expressing them through a finite number of observables. For non-renormalizable theories, the number of observables required to make the theory prediction independent of the UV cut-off to all orders in perturbation theory, is infinite.

Let us also comment about the fact that we only considered overall divergences of Feynman graphs and Green's functions, in spite of the fact that subdivergences can exist as well. The reason is simple: subdivergences, if exist, will be related to Green's functions with lower number of loops. We imagine that, eventually, all divergences in Green's functions are treated recursively; therefore, when we discuss divergences of Green's functions at L-loops, we assume that all Green's functions at L-1 loops — and therefore all subdivergences of L-loop Green's functions — are already understood and removed by some procedure.