One-loop renormalization of the Yang-Mills theory, the running coupling constant and the $\beta$-function.

In this lecture we will discuss the one-loop renormalization of the YM theory. The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} + \frac{T}{2} (i \not{D}_\mu - m) \not{\psi} + \frac{g^2}{2} \delta_{ab}^{\alpha \beta} \not{D}_\mu \not{D}^{\alpha \beta} + \text{gauge fixing}$$

is interpreted as the Lagrangian written through bare fields, couplings & masses. With all the interactions terms expanded, $\mathcal{L}$ reads

$$\mathcal{L} = -\frac{1}{4} \left( \partial_\mu A^a_{\nu} - \partial_\nu A^a_{\mu} \right)^2 + \bar{\psi}_0 \left( i \not{D} - m_0 \right) \psi_0$$

$$- \frac{g_0}{4} \not{D}_0 \not{D}^0 + g_0 A^a_\mu \not{D}_\mu \not{D}^0 \psi_0$$

$$- g_0 f^{abc} \left( \partial_\mu A^a_\nu \right) A^b_\mu A^c_\nu$$

$$- \frac{1}{4} g_0^2 \left( f^{abc} A^a_\mu A^b_\nu \right) \left( f^{cde} A^e_\mu A^d_\nu \right)$$

$$- g_0 \frac{c_0}{4} f^{abc} \partial_\mu A^b_\nu \partial_\nu A^c_\mu$$

Now, we dropped the gauge-fixing term (justified by taking $\xi \to \infty$) & expressed everything in the Lagrangian through bare input parameters such as fields, the coupling constants & masses.
We now rescale fields as $A^a_{0 \nu} = Z_3^{1/2} A^a \nu$, $\psi = Z_2 \psi$, $-\frac{1}{2} \epsilon_0 = \frac{1}{Z_2} c$, and write the Lagrangian in terms of normalized fields and the counter-terms:

$$\mathcal{L} = -\frac{1}{4} \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right)^2 + \bar{\psi} \left(i \gamma^\mu - m \right) \psi - \frac{1}{2} c \partial^2 \psi^a$$

$$+ g A^a_\mu \bar{\psi} \gamma^\mu t^a \psi - g f^{abc} (\partial_\mu A^a_\nu) A^b_\rho A^c_\nu$$

$$- \frac{1}{4} g^2 f^{eab} A^a_\mu A^b_\nu A^c_\rho A^\rho_\nu - f^{eab} A^a_\mu A^b_\nu - g \sum f^{abc} f^{d^a} A^a_\mu c^c$$

$$- \frac{1}{4} \delta_3 \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right)^2 + \bar{\psi} \left(i \delta_2 \gamma_5 - \delta_2 \gamma_5 \right) \psi - \sum c \partial^2 c^a$$

$$+ g \delta_1 A^a_\rho \bar{\psi} \gamma^\rho t^a \psi - g \delta_1 f^{abc} (\partial_\mu A^a_\nu) A^b_\mu A^c_\nu$$

$$- \frac{1}{4} g^2 \delta_1 f^{abc} \left( f^{eab} A^a_\mu A^b_\nu \right) \left( f^{eac} A^c_\rho A^\rho_\nu \right)$$

$$- g \delta_1 c \sum f^{abc} \sum A^a_\mu c^c$$

Here, the counterterms are given by:

\[
\begin{align*}
\delta_2 &= \frac{Z_2 - 1}{Z_2} \\
\delta_3 &= \frac{Z_3 - 1}{Z_3} \\
\delta_2^c &= \frac{Z_2^c - 1}{Z_2^c} \\
\delta_m &= \frac{Z_2 m_0 - m}{Z_2} \\
\delta_1 &= \frac{g_0}{g} \frac{Z_2}{Z_3} \left( Z_3 \right)^{1/2} - 1 \\
\delta_1^c &= \frac{g_0}{g} \frac{Z_2^c}{Z_3^c} \left( Z_3^c \right)^{1/2} - 1 \\
\delta_1^{3/2} &= \frac{g_0}{g} \frac{Z_2^c (Z_3)^{3/2}}{Z_3} - 1 \\
\delta_1^{4/3} &= \frac{g_0^2}{g^2} \frac{Z_2^c (Z_3)^{4/3}}{Z_3} - 1 \\
\delta_1^c &= \frac{g_0}{g} \frac{Z_2^c (Z_3)^{1/2}}{Z_3^c} - 1.
\end{align*}
\]
We can read off the Feynman rules that are generated by the counter-term Lagrangian:

\[ \bar{c}^a \gamma^\mu \frac{\partial}{\partial \phi^\mu} c^b \delta^a_b = -i (k^2 g^{\mu\nu} - k^\mu k^\nu) \delta^{ab} \delta_3 \]

\[ \gamma^\mu \gamma^\nu c^a \frac{\partial}{\partial \phi^a} \gamma_\mu \gamma_\nu c^b \delta_3 = i g \epsilon^{ab} \gamma^\mu \gamma^\nu \delta_b \]

Note that after there are also counter-terms for other vertices - for example, the 3-gluon vertex receives a counter-term, which is

\[ \frac{1}{2} c^a \frac{\partial}{\partial \phi^a} c^b \gamma_\mu \gamma_\nu c^c \delta^b_{ct} = \delta^a_{ct} \frac{3g}{3!} \left( \begin{array}{c} 1 \end{array} \right) \]

and the four-gluon vertex, which is

\[ \frac{1}{4} \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^b} \frac{\partial}{\partial \phi^c} \frac{\partial}{\partial \phi^d} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma c^a c^b c^c c^d \delta_{ct} = \delta^a_{ct} \frac{4g}{4!} \left( \begin{array}{c} 2 \end{array} \right) \]

Also, as we see there are counter-terms for the vertices that involve ghosts and gauge fields, and ghost self-energy.

The important point is that there are more divergent counter-terms. Indeed,
$S^{3g}_1$ and $S^{4g}_1$ are fully fixed by $S_1$, $S_2$ and $S_3$, so that the renormalization of the 3-gluon vertex and the renormalization of the 4-gluon vertex are not independent and follow from the gauge symmetry of the theory.

Since we computed the quark self-energy, the gluon self-energy, and the divergence of the quark-gluon vertex in the previous lecture, we should be able to obtain the counter-terms and the relation between bare and renormalized coupling constant.

We will also describe a particular renormalization scheme, known as "minimal subtractions", which avoids introducing explicit subtraction point by defining renormalized Green's functions as the ones from which only (also this is almost true) are removed. Usually "Minimal subtractions"
imply using dimensional regularization, which in turn leads to the following nettlety: if we want the renormalized coupling constant $g$ to be dimensionless, we can not do what we just did because the bare coupling constant must have mass dimension. Indeed, the action is dimensionless ($\hbar = 1$), so dimensionality of $A$ and $\varphi$ is fixed from kinetic terms; then the dimensionality of the interaction term fixes $\dim [g_0] \sim M$. Hence, we must re-write the Lagrangian $g \rightarrow g \mu^\varepsilon$, including in the counter-terms $\delta_1 = \frac{g_0}{Z_2(Z_3-1)} \mu^\varepsilon$.

Now, let us go back to our calculation of the divergences in various Green's functions discussed in the previous lecture. Take the self-energy of the quark. We computed it to be

$$\left[ \frac{i}{p} \gamma_i \right] \text{div} = \delta_{ij} C_F \frac{i g^2}{(4\pi)^{d/2}} \frac{\Gamma(1+\varepsilon)}{\varepsilon} \rho + \text{finite}.$$

Now, the expansion parameter is not $g$, but rather $g \mu^\varepsilon$, so that

$$\left[ \frac{i}{p} \gamma_i \right] \text{div} \rightarrow \delta_{ij} C_F \frac{i g^2 \mu^{2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(1+\varepsilon)}{\varepsilon} \rho + \text{finite}.$$
To make this Green's function finite, we need to add the counter-term \( \vec{p} \otimes \vec{p} = i \gamma \delta \cdot \delta_{ij} \).

The "minimal subtraction" (MS) counterterm is required to remove poles in \( \epsilon \) from the Green's function:

\[
\left[ \frac{i}{\vec{p}} \right]_{\text{Ren}} = \left[ \frac{i}{\vec{p}} \right]_{\text{div}} + \text{finite} + i \gamma \delta \cdot \delta_{ij}
\]

and

\[
\delta_{2}^{\text{MS}} = -\frac{C_{F} \alpha^{2}}{(4\pi)^{2} \epsilon} \frac{1}{\epsilon}
\]

One unpleasant consequence of this subtraction scheme is the appearance of various unnecessary terms in the renormalized Green's function: they come from the expansion of

\[
\frac{\Gamma(1+\epsilon)}{(4\pi)^{d/2}} \ln \epsilon = \frac{\Gamma(1+\epsilon)}{(4\pi)^{d/2}} \epsilon \times \frac{1}{\epsilon^{2}} \times \left\{ \frac{1}{\epsilon} - \frac{1}{\epsilon} \right\}
\]

Both, the Euler \( \gamma_{E} \) and \( \ln (4\pi) \) are universal; they appear in all divergent Green's functions in exactly that combination. Therefore, it is useful to redefine the subtraction scheme, to remove them.
We modify this by writing $g^2 \to g \mu^2 e^{-\delta}$ and add the counter-term. We find

$$\delta_3 = \frac{g^2}{(4\pi)^2} \left( \frac{5}{3} C_A - \frac{4}{3} N_F T_R \right) \left( \frac{1}{\hat{\epsilon}} - \hat{\gamma}_E + \ln(4\pi) \right)$$

Finally, we find the counter-term for quark-gluon vertex by using the result of the calculation in the last lecture:

$$\langle \begin{array}{c} \xi \cr \xi \end{array} \rangle_{\text{div}} \left[ \begin{array}{c} \xi \cr \xi \end{array} \right] = \left[ \begin{array}{c} \xi \cr \xi \end{array} \right]_{\text{div}} = ig f_{\mu \nu} t^a \frac{g^2}{(4\pi)^2} \gamma(1+i\gamma_5) x \left[ C_F + C_A \right]$$

$$\delta_1^{\overline{MS}} = -\frac{g^2}{(4\pi)^2} \left( C_F + C_A \right) \left( \frac{1}{\hat{\epsilon}} - \hat{\gamma}_E + \ln 4\pi \right)$$

With these 3 counterterms, we can find a relationship between renormalization and bare coupling constants: $[\hat{\epsilon} = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi]$

$$g_0 = g \mu^e \left[ 1 + \delta_1 \right] = g \mu^e \left[ 1 + \delta_1 - \frac{1}{2} \delta_3 \right] = \frac{z_2 z_3^{-1/2}}{z_2}$$

$$= g \mu^e \left[ 1 + \frac{g^2}{(4\pi)^2} \hat{\epsilon} \right] \left( -C_F - C_A + C_F - \frac{5}{6} C_A + \frac{2}{3} N_F T_R \right)$$
The modified subtraction scheme is called $\overline{\text{MS}}$ (modified minimal subtraction scheme). It amounts to subtracting away $\left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right)$ from divergent Green's functions.

\[
\left[ \frac{1}{\varepsilon} \rightarrow \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} + \ln(4\pi) \right]
\]

Here, \[ \delta_2^{\overline{\text{MS}}} = -\frac{C_F g^2}{(4\pi)^2} \left[ \frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) \right]. \]

Next, we will trace the dependence of the renormalized Green's function on $\mu$. Since

\[
\left[ \begin{array}{c} i \varepsilon \\ \mu \end{array} \right] = \left[ \begin{array}{c} i \varepsilon \\ \mu \end{array} \right]_{\text{div}} + \text{finite} + i\hat{\mu} \delta_2 \delta_{ij}
\]

\[
= \delta_{ij} C_F \frac{i g^2}{(4\pi)^2} \log(\mu^2) \hat{\mu} + \text{finite, $\mu$-independent contribution}
\]

The meaning of the scale $\mu$ is similar to the subtraction point ($p^2=-\mu^2$), but the relationship is not exact.

Next, we will consider the gluon self-energy. We have seen that

\[
\left[ \Phi^{a^b}_{\mu} \right]_{\text{div}} = \frac{i g^2 \Gamma(1+\varepsilon)}{(4\pi)^{3/2} \varepsilon} \left( q^2 g^{\mu \nu} - q^\mu q^\nu \right) \delta^{ab} \times \left[ \frac{5}{3} C_A - \frac{4}{3} N_F Tr \right]
\]

$$g_0 = g \mu^E \left[ 1 + \frac{g^2}{(4\pi)^2 \hat{E}} \left( -\frac{11}{6} C_A + \frac{2}{3} N_f T_K \right) \right]$$

Conveniently, this relationship is written for the non-Abelian analog of the fine-structure constant:

$$\alpha_s^{(0)} = \frac{g_0^2}{(4\pi)}$$

We have

$$\alpha_s^{(0)} \equiv \alpha_s \mu^{2E} \left[ 1 - \frac{(\alpha_s)}{2\pi} \frac{1}{\hat{E}} \left( \frac{11}{6} C_A - \frac{2}{3} N_f T_K \right) + \ldots \right]$$

The term in $[\ldots]$ brackets is called the strong-coupling constant renormalization constant.

We can use the above equation to find an interesting result. The left-hand side of that equation is the bare coupling—it is independent of $\mu$. The right-hand side depends on $\mu$, both explicitly and implicitly since $\alpha_s \equiv \alpha_s(\mu)$. We write

$$\alpha_s^{(0)} = \alpha_s \mu^{2E} \left[ 1 - \frac{(\alpha_s)}{2\pi} \frac{1}{\hat{E}} \hat{E}_0 + O(\alpha_s^2) \right],$$

$$\frac{d\alpha_s^{(0)}}{d\mu} = 0 \quad \Rightarrow \quad 0 = \mu \frac{d\alpha_s}{d\mu} \mu^{2E} \left[ \ldots \right] + 2E \mu \frac{d\alpha_s}{d\mu} \left[ \ldots \right]$$

We will work to first non-trivial order in $\alpha_s$. 
\[ \mu \frac{d\alpha_{s}}{d\mu} = -2 \varepsilon \alpha_{s} \left[ 1 - \frac{\alpha_{s}}{2\pi \varepsilon} \beta_{0} + \cdots \right] = -10^{-\varepsilon \alpha_{s} \left[ 1 - \frac{\alpha_{s}}{2\pi \varepsilon} \beta_{0} + \frac{\alpha_{s}^{2}}{2\pi \varepsilon} \beta_{0}^{2} + \cdots \right]} = -2 \varepsilon \alpha_{s} \left[ 1 - \frac{\alpha_{s}}{2\pi \varepsilon} \beta_{0} \right] \left( 1 + \frac{\alpha_{s}}{\pi \varepsilon} \beta_{0} + O(\alpha_{s}^{2}) \right) = -2 \varepsilon \alpha_{s} \left[ 1 + \frac{\alpha_{s}}{2\pi \varepsilon} \beta_{0} + O(\alpha_{s}^{2}) \right] = -2 \varepsilon \alpha_{s} - \frac{\alpha_{s}^{2}}{2\pi} \beta_{0} + \cdots \]

Taking the limit \( \varepsilon \to 0 \), we find

\[ \mu \frac{d\alpha_{s}}{d\mu} = -\beta_{0} \cdot \frac{\alpha_{s}^{2}}{\pi} \]

where \( \beta_{0} = \frac{11}{6} C_{A} - \frac{2}{3} N_{f} T_{c} \).

Hence, we find an integro-differential equation for the dependence of the coupling constant on the renormalization scale.

What are the solutions? First,

\[ \beta_{0} = \frac{11}{6} C_{A} - \frac{2}{3} N_{f} T_{c} = \frac{\beta_{0}}{3} N_{f} - \frac{2}{3} N_{f} > 0 \] as long as \( N_{f} < 11 \), \( \beta_{0} > 0 \). For "real world,”

\( N_{f} \approx 2 \text{ or } 3 \) (up, down and maybe strange quarks), \( \infty \text{ } \beta_{0} \approx 3 \).

Now,

\[ \mu \frac{d\alpha_{s}}{d\mu} = -\beta_{0} \cdot \frac{\alpha_{s}^{2}}{\pi} \Rightarrow \frac{d\alpha_{s}}{d\mu} = -\frac{\beta_{0}}{\pi} \frac{d\mu}{\mu} \]

\[ \frac{1}{\alpha_{s}(\mu_{i})} - \frac{1}{\alpha_{s}(\mu_{f})} = -\frac{\beta_{0}}{\pi} \ln \frac{\mu_{f}}{\mu_{i}} \Rightarrow \]

\[ \frac{1}{\alpha_{s}(\mu_{i})} = \frac{1}{\alpha_{s}(\mu_{f})} + \frac{\beta_{0}}{\pi} \ln \frac{\mu_{f}}{\mu_{i}} \]

\[ \frac{1}{\alpha_{s}(\mu_{i})} + \frac{\beta_{0}}{\pi} \ln \frac{\mu_{f}}{\mu_{i}} = \frac{1}{\alpha_{s}(\mu_{f})} \]

\[ \alpha_{s}(\mu_{f}) = \frac{\alpha_{s}(\mu_{i})}{1 + \frac{\alpha_{s}(\mu_{i})}{2\pi} \beta_{0} \ln \left( \frac{\mu_{f}}{\mu_{i}} \right)^{2}} \]
Hence, this equation implies that if we fix the coupling constant at the scale $\mu = \mu_i$, the coupling constant at a larger scale $\mu > \mu_i$ will be smaller. The scale-dependent coupling is called the “running” coupling constant; $\beta_0$ is known as the beta-function and the phenomenon of the coupling decrease with scale is known as asymptotic freedom. Right now, this scale looks somewhat artificial, but it is not related to quantities like energies of the collisions, etc. We will make this connection in the next lectures; then it will become clear why $\Delta s(\mu)$ is an important quantity.