Problem 1 - One-loop UV divergences of two- and three-point functions in QCD

In this exercise we will compute the UV divergent part of one-loop corrections to two- and three-point functions in QCD.

1. We start by considering the gluon self-energy \(-i \Sigma_g(q)\), where \(q^\mu\) is the gluon momentum.
   a) Draw all one-loop Feynman diagrams which contribute to the gluon self-energy. You should find 4 different diagrams. Do not forget to consider, together with fermions and gluons, also ghost contributions!
   b) Note that the four diagrams do not have and IR divergence in \(d \to 4\) (or \(\epsilon \to 0\)), such that all poles must be only of UV nature.
   c) Compute the four contributions separately in dimensional regularisation, keeping only the \(1/\epsilon\) divergent pieces. Note that only 2 diagrams must be computed. One can be, in fact, recycled from QED (with an appropriate overall normalisation!), while another one is identically equal to zero in dimensional regularisation. Why?
   d) Verify that upon summing the four contributions one recovers that the gluon self-energy is transverse, i.e.

   \[
   -i \Sigma_g(q) = -i q^2 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) \frac{\Pi(q^2)}{\epsilon} \delta_{ab},
   \]

   where \(a, b\) are the color indices of the two external gluons. Note that this ensures that the mass of the gluons remains zero also under radiative corrections.

2. Consider now the one-loop corrections to the fermions self-energy \(-i \Sigma_f(q)\). In this case there is only one Feynman diagram contributing at one-loop. Compute its value in dimensional regularization neglecting the finite piece, i.e. retain only the terms proportional to \(1/\epsilon\)!

3. Consider finally the one-loop corrections to the quark-gluon vertex in QCD.
   a) Draw the two Feynman diagrams contributing at one-loop order.
   b) Compute the two contribution, again retaining only the divergent pieces proportional to \(1/\epsilon\)!
      After having carried out the color algebra, notice that the divergent contributions come from the region where the loop momentum becomes very large. In particular, since the diagrams diverge logarithmically as \(k \to \infty\), you can neglect all external momenta w.r.t the loop momentum \(k\).