1) The simplest system where the problem of a decay of a meta-stable state arises is that of Quantum Mechanics. Consider a potential

\[ V(q) = V(q_0) \]

There is a quasi-stationary bound state localized at \( q = q_0 \). However, since the height of the barrier to the right of it is too small, this state isn't stable; it tunnels and escapes to \( q = \infty \) (the true vacuum of the system).

The lifetime is given by \( \tau = \frac{1}{\Gamma} \)

\[ \Gamma = A e^{-\frac{S_B}{q_1}} \]

\[ S_B = 2 \int_{q_0}^{q_1} \sqrt{2M V(q)} \, dq \] , \( M \) is the mass of a particle.

This formula isn't generic - to simplify it, we have chosen \( V(q_0) = 0 \), so that the total energy of the bound state vanishes (we neglect small quantum effects).

An interesting point about \( S_B \) is that it can be thought of as the value that Euclidean action takes for a particle moving along the...
classical trajectory from the point \( q_0 \) to the point \( q_1 \) and back with zero energy.

Let's see exactly what that means. We take the "normal" action for this type of a problem is

\[
S = \int \left[ \frac{m}{2} \left( \frac{dq}{dt} \right)^2 - V(q) \right] dt.
\]

Let's make a change of variables \( t = -iz \), so that

\[
\frac{dq}{dt} = -i \frac{dz}{dt}, \quad dt = -i \, dz.
\]

Then,

\[
S = +i \, S_E,
\]

where \( S_E = \int dz \left[ \frac{m}{2} \left( \frac{dz}{dt} \right)^2 + V(q) \right] \)

This action \( S_E \) got its name because the Minkowski interval \( dt^2 - dq^2 \) becomes \( -(dt^2 + dq^2) \) under \( t = -iz \) change and \( dt^2 + dq^2 \) is the Euclidean interval.

Take \( S_E \) and find equations of motion:

\[
+ m \frac{d^2 q}{dt^2} + \frac{\partial V}{\partial q} = 0,
\]

which corresponds to the regular motion in the potential \( -V(q) \).

The equation of motion has an integral of motion

\[
\frac{m}{2} (\dot{q})^2 - V(q) = E,
\]

that we can call "Euclidean" energy.
Consider the \( \phi = 0 \) solution, that starts at \( \phi = q_0 \) at \( t = -\infty \), reaches \( q_1 \) and then returns back to \( q_0 \) at \( t = +\infty \). We can always assume that \( q_1 \) is reached at \( t = 0 \). Let's call such a solution an "a bounce" and denote it \( \phi_B(t) \).

Then, \( \frac{1}{2} \left( \frac{d\phi_B}{dt} \right)^2 = V(\phi_B(t)) \Rightarrow \)

\[
S^+_E [\phi_B(t)] = \int_0^t dt \left[ \frac{m}{2} \left( \frac{d\phi_B}{dt} \right)^2 + V(\phi_B(t)) \right] =
\]

\[
= 2 \int_{-\infty}^0 dt \left[ \frac{m}{2} \left( \frac{d\phi_B}{dt} \right)^2 + V(\phi_B(t)) \right] =
\]

\[
= 2 \int_{-\infty}^0 dt \left[ 2 V(\phi_B(t)) \right].
\]

From the Euclidean energy conservation law, we find

\[
\frac{m}{2} \sqrt{\frac{m}{2 V(q_B)}} dq_B \sqrt{\frac{m}{2 V(q_B)}} = dt \Rightarrow
\]

\[
S^+_E [\phi_B(t)] = 2 \int_{q_0}^{q_1} dq_B \sqrt{2m V(q_B)} = 2 \int_{q_0}^{q_1} dq \sqrt{2m V(q)}
\]

which coincides with the exponential decay, exponent \( q_0 \) of the standard expression for the tunneling rate in Quantum Mechanics.

Our goal, eventually, will be to apply similar \( \phi_B(t) \) formulas to Quantum field theory problems. Let's discuss why this is possible.
To understand this, we imagine that we have to study Quantum field theory with the action 

\[ S = \int dt d\vec{x} \left[ \frac{1}{2} \partial \phi \partial \phi^* - V(\phi) \right] \]

We now assume that the space-time is discretized and the spatial variable \( x \) is defined on a lattice, with a lattice spacing \( a \). \( \vec{x} = a \vec{n} \) \( \vec{n} \in \text{integers} \)

\[ \partial_{\mu} \phi \partial_{\mu} \phi^* = \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} \left( \frac{\partial \phi^*}{\partial t} \right)^2 - \frac{1}{a^2} (\phi_i \phi_i) (\phi_i \phi_i) \]

and \( d\vec{x} = a^3 \sum_i \). Hence,

\[ S = a^3 \sum_i \int dt \left[ \left( \frac{\partial \phi_i}{\partial t} \right)^2 - \frac{1}{a^2} (\phi_i \phi_i) (\phi_i \phi_i) - V(\phi_i) \right] \]

For any finite volume, this is an action of a classical problem with very large but finite number of freedom. Essentially one can view \( \phi(t, \vec{x}) \to \phi_i(t) \) as "coordinates" and "\( \vec{x} \)" as labels. This classical action can be studied using ordinary methods of Quantum Mechanics — and this includes relativistic varying regular formulas of vacuum stability. 

We have not discussed how to deal with vacuum stability issues.
In Quantum Mechanical problems with a large number of degrees of freedom.

However, the QFT problem we will be dealing with will have a high degree of symmetry that will allow us to focus on the non-trivial dependence of the action on just one "label". The problem then will be very similar to what we have in Quantum mechanics.

We will now try to apply this understanding to compute the rate for producing electron-positron pairs in a constant electric field in a vacuum. (Schwinger)

Consider an empty space with a constant electric field (along the z-axis) \( E \). The potential energy of a particle with charge \( e \) is

\[ U = -eEz. \]

When a particle moves in such a field, the total energy \( E = \sqrt{p^2c^2 + m^2c^4} - eEZ \) and the transverse momentum \( \vec{p}_\perp = (p_x, p_y, 0) \) are conserved.

We will assume that the system is in a vacuum state and that the vacuum wave function is given.
By a Dirac sea: all levels with \(-6\) energies \(E < -m\) filled with electrons and all levels with energies \(E > m\) empty. If a constant electric field \(E = 0\) is applied the energy levels tilt:

\[ m \rightarrow z \rightarrow z_1 \]  

As the result, any of the filled energy levels (at a point \(z_1\)) can be made free by electron tunneling from \(z_1\) to \(z_2\). If this happens, electron and a hole will be created with an interpretation that production of an \(e^+e^-\) pair occurred.

Let \(E = \sqrt{mc^2 + p^2c^2} - eEz\) be the energy of a particle in the Dirac sea. The longitudinal momentum

\[ p_x(z) = \frac{1}{c} \sqrt{(eEz + E)^2 - mc^4 - p^2c^2}. \]

\[ p_x(z) = 0 \quad \text{for} \quad z_{1,2} = -\frac{E}{eE} \pm \frac{\sqrt{m^2c^4 + p^2c^2}}{eE}. \]

A particle from the Dirac sea enters a barrier at \(z = z_1\) and leaves at \(z = z_2\).
The region \( z_1 < z < z_2 \) is kinematically forbidden, so to get from \( z_1 \) to \( z_2 \) requires tunneling.

So, the tunneling action is

\[
S_B = 2 \int_{z_1}^{z_2} |p(z)| \, dz = \frac{2}{c} \int_{z_1}^{z_2} \sqrt{\varepsilon(z - z_1)(z - z_2)} \, d\varepsilon
\]

\[
= \frac{2}{c} \varepsilon (z_2 - z_1)^2 \int_0^1 d\varepsilon \sqrt{\varepsilon(1 - \varepsilon)} = \\
= \frac{2}{c} \varepsilon (z_2 - z_1)^2 \cdot \frac{\pi}{8} = \frac{2}{c} \frac{(m^2 c^4 + p_1^2 c^2)}{\varepsilon} \cdot \frac{\pi}{2}
\]

\[
\Rightarrow S_B = \pi \left( \frac{m^2 c^2 + p_1^2}{\varepsilon} \right) c,
\]

The transition rate is given by

\[
W = \frac{\alpha}{\hbar} e^{-S_B/\hbar} = A \exp \left[ -\frac{\pi (m^2 c^2 + p_1^2)c}{\varepsilon \hbar} \right]
\]

We can now estimate the coefficient \( A \) by pre-exponential factor.

To this end, consider a particle in the element of momentum space \( d^3 p = dp_1 dp_2 dp_3 \). The space density is

\[
d\rho = \frac{2}{(2\pi\hbar)^3} \frac{dp_1}{\hbar} \frac{dp_2}{\hbar} \frac{dp_3}{\hbar},
\]

where a factor 2 corresponds to two spin orientations of the electron. The number of particles passing through an elementary area \( dx \) dy to the left of the barrier is
\[ dN = d\gamma_2 (z) \, dx \, dy \quad \text{with} \quad d\gamma_2 (z) = \gamma_2 (z) \frac{\partial \gamma_2}{\partial \mu_2} \frac{1}{\mu_2} \] \[ \text{This includes} \quad \gamma_2 (z) \, d\mu_2 = \frac{2E_{\text{kin}}}{\mu_2} \, d\mu_2 = dE_{\text{kin}} \] \[ \text{Hence} \quad dN = \frac{2d^2p_1^2}{(2\pi\hbar)^3} \, dE_{\text{kin}} \, dx \, dy = \frac{2d^2p_1^2}{(2\pi\hbar)^3} e^E \, dz \, d\theta \, d\phi \] \[ \text{Hence, the total number of pairs created in a given volume} \] \[ dV = dx \, dy \, dz, \quad \text{we need to multiply} \] \[ e^{-S_B / \hbar} \quad \text{with} \quad dN. \quad \text{Hence} \quad A \circ dN. \] \[ \text{We find for the total number of pairs created per unit time in a} \] \[ \text{unit volume} \] \[ P_{1/2} = \frac{\mu N}{dW} = 2 e^{E} \int \frac{d^2p_1^2}{(2\pi\hbar)^3} e^{-} \] \[ \frac{\pi (c^2 \mu^2 + p_1^2)}{e \hbar} = \] \[ P_{1/2} = e^{\frac{E \, c^2}{4 \pi^3 \hbar^2}} \exp \left[ - \frac{\pi c^2 \mu^2}{e E \hbar} \right] \] \[ \text{Our calculation is valid in the} \] \[ \text{quasi-classical approximation} \] \[ \text{while the exponent is small large} \] \[ \frac{m^2 c^2}{e \hbar} \gg 1. \quad \text{What does this mean physically? To see this, note that} \] \[ \text{of the tunneling path} |z_1 - z_2| \] \[ l \sim \frac{1}{\sqrt{z_2 - z_1}} \sim \frac{m c^2}{e E}. \quad \text{The Compton wave} \]
length of the electron is \( \lambda = \frac{\hbar}{mc} \). Therefore, the applicability of the quasi-classical approximation implies \( \ell/\lambda \gg 1 \).

The tunneling probability is very small.

Take the hydrogen atom: The Bohr radius is \( a_B \sim \frac{1}{\alpha_m} \); the electric field is
\[
e \frac{E_{at}}{a_B} \sim m^2 \alpha^3,
\]
Then (in c-1, \( \hbar \)-1 units)
\[
\frac{m^2}{e^2} \frac{\alpha}{a_B^2} \sim \frac{m^2 \alpha}{e^2 \alpha^2 m^2 \alpha^3} = \frac{1}{\alpha^3} \sim 10^4 \Rightarrow \text{pair creation}
\]
Probability of spontaneous emission is \( \sim \exp (-10^4 \times 10^{-4}) \ll 1 \).