More on the anomaly

We will now discuss a connection between a more traditional calculation of the anomaly and its infrared nature using the Schwinger model as an example. We want to prove an equation

\[ \partial_\mu j^{\mu,5} = -\frac{1}{2\pi} \varepsilon^{\mu\nu} F_{\mu\nu} \]

To see how this equation appears, let us first define the \textit{regularized} axial current

\[ j^{\mu,5}_{\text{reg}} \Rightarrow j^{\mu,5} = \overline{\Psi} \gamma^{\mu} \gamma_5 \Psi - \overline{R} j^{\mu,5}_{\text{reg}} \cdot R. \]  

Here \((\Psi, \overline{\Psi})\) are the massless fermion fields and \((R, \overline{R})\) are the massive fermion fields. The mass \(M_R\) is supposed to be taken to infinity at the end of the calculation. We would like to calculate the divergence of the current directly.

The fermion \(R\) is supposed to \textit{regularize} the current, so that we can apply equations of motion:

\[ \partial_\mu j^{\mu,5}_{\text{reg}} = \overline{\Psi} \gamma^{\mu} \gamma_5 \Psi - \overline{R} \partial_\mu \gamma_5 R + \overline{\Psi} \gamma^{\mu} \gamma_5 \Psi - \overline{R} \partial_\mu \gamma_5 R = -2iM_R \overline{R} \gamma_5 R \]

(to same result holds if \(\Psi \& \overline{R}\) couple to gauge fields).
Have we used the Dirac equations, e.g.

\[ i \gamma^\mu \partial_\mu \psi = 0, \quad i \gamma^R \psi = M_R \psi. \]

We obtain the same result:

\[ \nabla_{\mu} J_{\mu \nu} = -2i M_R R \overline{\psi} \gamma_5 \gamma_\nu \psi. \]

This is an operator equation and we want to calculate the matrix elements of both sides of this previous equation. There are several matrix elements that we can imagine:

\[ \langle 0 | -2i M_R R \overline{\psi} \gamma_5 | 0 \rangle; \quad \langle \gamma_5 \psi | -2i M_R R \overline{\psi} \gamma_5 | 0 \rangle. \]

\[ -2i M \gamma_5 \]

\[ a) \]

\[ \langle \gamma_1 \gamma_2 | -2i M_R R \overline{\psi} \gamma_5 | 0 \rangle; \quad \langle \gamma_1 \gamma_2 \gamma_3 | -2i M_R R \overline{\psi} \gamma_5 | 0 \rangle. \]

\[ -2i M \gamma_5 \]

\[ c) \]

We need to understand which contributions survive the \( M_R \to \infty \) limit. To this end, we will need a couple of formulas for traces with \( \gamma_5 \). They are:

\[ \text{Tr}(\gamma_5) = 0, \quad \text{Tr}(\gamma_5 \gamma_\mu) = 0, \quad \text{Tr}(\gamma_5 \gamma_\mu \gamma_\nu) = 2 \varepsilon_{\mu
u} a_{\mu} b_{\nu}, \]

with \( \varepsilon^{01} = 1 \) and \( \varepsilon_{\mu\nu} = -\varepsilon^{\mu\nu} \).

The first diagram \( a) \) is proportional to \( \text{Tr}(\gamma_5 (\gamma^\mu M_\mu)) \equiv 0 \). The fourth and all
higher multiplicity diagrams vanish by power count in \( M_R \to \infty \) limit. (Example (d):

4 propagators \( \sim \frac{1}{M_R^4} \), the integration measure \( \sim M_R^2 \),

and the vertex for \( -2iM_R \bar{\psi} \gamma^5 \psi \), all together,

\[ \lim_{M_R \to \infty} \frac{M_R^3}{M_R^4} = 0 \).

The third diagram (c) requires some consideration. Naïve power counting gives

\[ M_1^2 M_2^2 \frac{1}{M_R^2} \to 1 \],

and so doesn't show any decoupling. However, the coupling of the external vector field to heavy loop must be gauge invariant,

the matrix element must be proportional to

\[ \epsilon_a^i \bar{\epsilon}^a_i \]

for each of the photons.

Therefore, the diagram is proportional to

\[ \frac{M_1 M_2}{M_R^3} \to 0 \], as \( M_R \to \infty \).

The last diagram to check is (b):

The power counting works like this:

the mass dimension of this diagram is one. Thanks to gauge invariance,

it must be proportional to \[ \epsilon_a^i \bar{\epsilon}^a_i \epsilon a_i \epsilon a_i \]

the dimensions match, which means that the diagram (b) is mass \( M_R \) independent.

Let us check this by explicit calculation:
\[
\begin{align*}
-k+q & = (\mathbb{M}) \frac{2 i M_R}{\kappa} \int \frac{d^2 k}{(2\pi)^2} \frac{\text{Tr} [ \hat{f}_5(k^+ + M_R) \hat{E}(k^+ + M_R + q^+)]}{(k^2 - M_R^2)(k^0 + q^0 - M_R^2)} \ \\
& = + 2 M_R \int \frac{d^2 k}{(2\pi)^2} \frac{\text{Tr} [ \hat{f}_5(k^+ + M_R) \hat{E}(k^+ + M_R + q^+)]}{(k^2 - M_R^2)(k^0 + q^0 + M_R^2)} \\
& \quad + M_R \left[ \text{Tr} [ \hat{f}_5 \hat{E}(k^+ + q^0)] + \text{Tr} (\hat{f}_5 \hat{E}_g^i) \right] \\
& \quad + \text{Tr} (\hat{f}_5 \hat{E}_g^i(k^0 + q^0)) = M_R \text{Tr} (\hat{f}_5 \hat{E}_g^i) + \text{Tr} (\hat{f}_5 \hat{E}_g^i) \\
& \rightarrow 2 M_R \hat{E}^{\mu \nu} \epsilon_{\mu \nu \rho \sigma} q^\rho q^\sigma, \text{ and the last term can be neglected because it vanishes in } M_R \to \infty \text{ limit. Hence, we obtain } (f^{\mu \nu} = \epsilon^{\mu \nu \rho \sigma} q^\rho q^\sigma)
\end{align*}
\]

\[
\begin{align*}
-k+q & \Rightarrow + \int \frac{d^2 k}{(2\pi)^2} \left[ 2 M_R - \frac{2 M_R^2 \epsilon^{\mu \nu} \hat{E}_g^i(q^0)}{(k^0 + q^0 + M_R^2)^2} \right] \\
& = + 4 M_R \left[ \hat{f}_5 \hat{E}_g^i(q^0) \right] \int \frac{d^2 k}{(2\pi)^2} \frac{1}{(k^0 + M_R^2)^2} \\
& \text{Perform the Wick rotation and integrate; the result is } \int \frac{d^2 k}{(2\pi)^2} \frac{1}{(k^0 + M_R^2)^2} = \frac{i}{4\pi M_R^2} \\
& = + \frac{i}{2\pi} \epsilon^{\mu \nu} f_{\mu \nu}(q). \text{ In position space, this equation implies the anomaly equation:}
\end{align*}
\]

\[
\lim_{M_R \to \infty} \hat{f}_5 \hat{E}_g^i = \frac{1}{2\pi} \epsilon^{\mu \nu} f_{\mu \nu}
\]
This anomaly equation is now obtained from UV properties of the divergence of the current (non-decoupling of the UV regulator). The UV regulator is introduced in a way that keeps gauge invariance intact.

Next, we consider the calculation of the matrix element of the current itself:

$$\langle 0 | J_{5}^{a}(q, \xi) | q, \xi' \rangle$$

This matrix element should be transversal & parity odd.

As the result, there is just one term that we can write down:

$$\langle 0 | J_{5}^{a}(q, \xi) | q, \xi' \rangle = F(q^{2}) \frac{q^{a}}{q^{2}} e^\mu \phi^{\mu}_{a},$$

where $f = \epsilon_{\mu} \phi^\mu - \epsilon \phi^\mu$.

[Show that e.g. $\epsilon_{\mu} \phi_{a}^{\mu}$ can be written as $\frac{1}{2} q^{a} e^\mu \phi^{\mu}$ using Schouten identities.]

The importance feature of this equation is the appearance of the pole at $q^{2} = 0$. The $q^{2} \to 0$ singularities appear only because of the infrared behavior of the diagram. The infrared behavior of the diagram is, however, not ambiguous.

We find:
\[ \langle j^M_{\frac{3}{5}} \rangle = i^2(-1) \int \frac{d^2k}{(2\pi)^2} \text{Tr} \left[ j^M_{\frac{3}{5}} \frac{i k^\alpha}{k^2} j^S_{\frac{3}{5}} \frac{i (k+q)^\beta}{(k+q)^2} \right] \varepsilon_p \]

\[ = \int \frac{d^2k}{(2\pi)^2} \text{Tr} \left[ j^M_{\frac{3}{5}} k^\alpha j^S_{\frac{3}{5}} (k+q)^\beta \right] \frac{1}{k^2(k+q)^2} \varepsilon_p \]

Consider the tensor integral:

\[ \int \frac{d^2k}{(2\pi)^2} \frac{k^\alpha(k+q)^\beta}{k^2(k+q)^2} \]  

To extract the 1/q^2 pole from this integral, write

\[ \frac{1}{k^2(k+q)^2} = \int_0^1 dx \frac{dx}{[(k+q+x)^2 + q^2(1-x)]^2} \]

\[ \int \frac{d^2k}{(2\pi)^2} \frac{k^\alpha(k+q)^\beta}{k^2(k+q)^2} \]  

\[ = (k \rightarrow k-qx) = \int dx \int \frac{d^2q}{(2\pi)^2} \frac{k^\alpha k^\beta - q^\alpha q^\beta x(1-x)}{[k^2 + qx(1-x)]^2} \]

The first \( k^\alpha k^\beta \) term doesn't produce 1/q^2 and, in fact, is UV divergent. The second term is what we want to find. Performing the Wick rotation and integrating, we find

\[ \int \frac{d^2k}{(2\pi)^2} \frac{k^\alpha(k+q)^\beta}{k^2(k+q)^2} \rightarrow \frac{iq^\alpha q^\beta}{4\pi q^2} + \text{non-pole} \]

Using this in the matrix element, we find

\[ \langle j^M_{\frac{3}{5}} \rangle = -\text{Tr} \left[ j^M_{\frac{3}{5}} \frac{i}{4\pi q^2} j^S_{\frac{3}{5}} \right] \varepsilon_p \]

\[ = -\left( \frac{i}{4\pi q^2} \right) + \text{non-pole} = -\left( \frac{i}{4\pi q^2} \right) + \text{non-pole} \]
\[ = \lim_{\Gamma \to 0} \frac{i}{4\pi q^2} \left( \frac{q_\mu q \nu \mu}{2q_\rho \rho} + \frac{q_\rho q \nu \mu}{2q_\rho \rho} \right) n \text{/pole} \]

\[ = +2 \varepsilon^{\mu \nu} q_\rho q_\nu \frac{i}{4\pi q^2} \rho = \frac{\varepsilon^{\mu \nu} q_\rho}{\pi q^2} \left( q_\rho q_\nu \right) \text{ + n/pole}. \]

We use Schouten identities:

\[ q^\mu \varepsilon_{\mu \nu} + q^\nu \varepsilon_{\mu \nu} = 0 \]

To write (contr. 1):

\[ \varepsilon^{\mu \nu} q_\rho = q^\mu \varepsilon_{\mu \nu} q_\rho + n \text{-pole}. \]

\[ <j^5_{\mu}> = \frac{-i}{2\pi q^2} q^\mu \varepsilon_{\mu \nu} F_{\nu \rho} \]

Taking the divergence and going back to the position space, we again obtain:

\[ \Theta_{\mu} j^5_{\mu} = -\frac{1}{2\pi q^2} \varepsilon \cdot F_{\mu \nu} \]

Anomaly is as much UV phenomenon (regulate non-decoupling) as it is IR (level crossing, poles in \( q^2 \)). This connection is particularly useful in confining theories (QCD) through 't Hooft matching condition.