Problem 1 - Solution of the Bogomol’nyi equation for monopoles

We have seen in the lectures that magnetic monopole solutions can arise in the Georgi-Glashow (GG) model. The Lagrangian of the model is

\[ \mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^{\mu\nu,a} + \frac{1}{2} (\mathcal{D}_\mu \phi^a)(\mathcal{D}^\mu \phi^a) - \lambda (\phi^a \phi^a - v^2)^2. \]  

(1)

The energy functional following from this Lagrangian in the limit \( \lambda \to 0 \) reads

\[ E = \int d^3x \left[ \frac{1}{2g^2} B_i^a B_i^a + \frac{1}{2} (\mathcal{D}_i \phi^a)(\mathcal{D}^i \phi^a) \right], \]

(2)

where the magnetic field is defined as

\[ B_i^a = -\frac{1}{2} \epsilon_{ijk} G_{jk}^a, \]

and of course

\[ G_{jk}^a = \partial_j A_k^a - \partial_k A_j^a + \epsilon^{abc} A_j^b A_k^c. \]

1. Show that the energy functional can be rewritten as

\[ E = v Q_M + \int d^3x \left[ \frac{1}{2g} B_i^a (\mathcal{D}_i \phi^a) \left( \frac{1}{g} B_i^a - \mathcal{D}_i \phi^a \right) \right], \]

(3)

where the magnetic charge \( Q_M \) in a sphere of radius \( R \) is defined as the flux of the magnetic field through the surface of the sphere \( S_R \)

\[ Q_M = \frac{1}{g v} \int_{S_R} d^2 S_i B_i^a \phi^a. \]

2. Minimizing the energy functional is equivalent to imposing the Bogomol’nyi equation

\[ \frac{1}{g} B_i^a - \mathcal{D}_i \phi^a = 0. \]

(4)

By solving this set of \textit{nine} first-order differential equations we can find the field configurations for the magnetic monopole solution. You have seen in class that the proper \textit{Ansatz} for the fields is

\[ \phi^a = v n^a H(r), \quad A_i^a = \epsilon^{aij} \frac{1}{r} n^j F(r), \]

(5)

where \( n^i = x^i / r \) and \( H(r) \) and \( F(r) \) are two scalar functions of \( r \) only. Note that we have written the \textit{nine} different functions \( \phi^a \) and the \( A_i^a \) in terms of two scalar functions only! It is very non trivial that this Ansatz is enough to solve the nine equations (4). Show by explicit calculation that with the Ansatz above we have

\[ B_i^a = (\delta^{ai} - n^a n^i) \frac{1}{r} F' + n^a n^i \frac{1}{r^2} \left( 2F - F^2 \right) \]

\[ \mathcal{D}_i \phi^a = v \left[ (\delta^{ai} - n^a n^i) \frac{1}{r} H(1 - F) + n^a n^i H' \right], \]

(6)

with \( H' = dH(r)/dr, F' = dF(r)/dr \).
3. Using the result above show that the Ansatz goes through and Eqs. (4) are equivalent to the system of two differential equations

\[
\begin{align*}
\frac{dF}{d\rho} &= H(1 - F), \\
\frac{dH}{d\rho} &= \frac{1}{\rho^2} (2F - F^2),
\end{align*}
\]

(7)

in terms of the new rescaled variable \( \rho = gvr \).

4. You have seen in the lecture what the solution to equations (7) look like. Try and solve the equations explicitly imposing the correct boundary condition.

**Problem 2 - Invariance of monopole solution**

Consider the monopole solution found for the fields \( \phi^a \) and \( A_5^a \). Prove explicitly that the monopole solution stays intact under the combined action \( \vec{L} + \vec{T} \), where \( \vec{L} \) and \( \vec{T} \) denote respectively the generators of the spatial and \( SU(2) \) color rotations.