Problem 1 - Finite-spacing corrections to the scalar propagator

We have seen in the lecture that it is straightforward to put a scalar field on a lattice. If the lattice spacing is $a$ and the number of sites is $N$, the action for the field reads

$$S = \frac{a^4}{N^4} \sum_q \left\{ \frac{1}{2a^2} \sum_\mu (2 - 2 \cos \mu \cdot q) |\phi_q|^2 + \frac{m^2}{2} |\phi_q|^2 \right\}.$$  \hfill (1)

From this action we can read off the propagator in position space

$$\langle \phi_{x,n} \phi_{x,m} \rangle = \int_{\pi/a}^{\pi/a} d^4 q (2\pi)^4 \frac{e^{i q (x_m - x_n)}}{m^2 + \frac{1}{a^2} \sum_\mu (1 - \cos (a q \cdot \mu))},$$  \hfill (2)

In class we have seen that by expanding in $a$ for $a \approx 0$ we recover at zeroth order the usual dispersion relation for massive particles in Euclidean space

$$\sum_\mu q_\mu^2 + m^2 = 0 \implies \sqrt{-q_0^2} = E_0 = \sqrt{q^2 + m^2},$$  \hfill (3)

where $q_0$ is the euclidean energy $q_0 = i E_0$.

1. Starting from (2), compute the first non-vanishing corrections in the lattice spacing $a$, to the dispersion relation (3).

2. Are there any continuous symmetries violated by these corrections?

Problem 2 - Finite-spacing corrections to gluon action in QCD

In class we studied in detail how to put gauge fields on the lattice by taking a product of Wilson lines, or links, that form a plaquette and tracing. In this way we get a gauge invariant contribution to the action. The Wilson lines are defined as

$$U_{x,\mu} = P \exp \left( i g \int_0^a dt \, \hat{A}_\mu (x + \mu t) \right)$$  \hfill (4)

and the free lattice gauge action can be build starting from the plaquette operator

$$U_{\mu\nu}^P = \text{Tr} \left( U_{x,\mu} U_{x+\mu,nu} U_{x+\mu+v,\mu} U_{x+v,\mu} \right).$$  \hfill (5)

In the Abelian case the action is

$$S_{AB} = \frac{1}{g^2} \sum_{\text{sites}} \sum_{\mu < \nu} \text{Re} \left( 1 - U_{\mu\nu}^P \right)$$  \hfill (6)

while in the non-abelian case, for example for $SU(N)$, it reads

$$S_{SU(N)} = \frac{2N}{g^2} \sum_{\text{sites}} \sum_{\mu < \nu} \text{Re} \left( 1 - \frac{1}{2N} U_{\mu\nu}^P \right)$$  \hfill (7)
1. In class we showed that at order zero in \( a \) the abelian action (6) reduces to

\[
S^{AB} \approx \frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(a^2).
\]  

(8)

Repeat computation in the non-abelian case, i.e. starting from (7).

2. Compute the first corrections to Eq. (8), i.e. the term proportional to \( a^2 \).

3. The \( \mathcal{O}(a^2) \) correction can be seen as due to higher dimensional operators in a effective field theory approach. Are the new interactions renormalizable? Do they violate any symmetries of the continuum theory?

4. Suppose that our Universe were really defined on a lattice and that the space-time were therefore intrinsically \( \textit{not continuous} \). Can you imagine any physical process which would receive corrections induced by the new higher dimensional operators above, and would allow us to discover its discretized nature?