Power Counting

Suppose the coupling of terms in the Lagrangian go like $F^2/\Lambda_{CSB}^{2n}$, where $\Lambda_{CSB}$ is the chiral symmetry breaking scale, $2n$ is the number of derivatives. In this problem, we construct a power-counting argument to show that if a divergent loop diagram is cut off at momentum of order $\Lambda_{CSB} \approx 4\pi F$, it contributes terms to the effective action of the same order of magnitude as the bare interaction terms. We restrict ourself to the derivative interactions because terms with derivatives replaced by $B,M$ act the same way.

1) Show that a vertex has the form

$$ (2\pi)^4 \delta^4(\Sigma p_i) F^2 \left( \frac{p^2}{\Lambda_{CSB}^2} \right)^n \left( \frac{1}{F} \right)^m, $$

where $p$ is a typical momentum and $m$ is the number of $\pi$ lines emanating from the vertex.

2) We start by considering a concrete example.

Call the loop momenta $k$ and external momenta $p$. For each internal meson line, include a factor

$$ \frac{1}{k^2} d^4k. $$

3) Now, we would like to make the argument for a general loop diagram. Take $N$ such vertices for various $n$ and $m$. Suppose there are $l$ internal meson lines. Consider a contribution in which there is a $k^{2j}$ from the momentum factors and the rest involves only external momenta, $p^{2M}$, for $M = N + \Sigma n - j$. Show that there are terms that contribute at the same order of magnitude as the bare interaction terms.

The Axial $U(1)$ Problem

In class we looked at $SU(3)_L \times SU(3)_R$ symmetries to build up our chiral theory. The transformations we have imposed, e.g. $\psi_L \rightarrow U\psi_L$, are actually $U(3)$ transformations. Since we can write $U(3) = SU(3) \times U(1)$, we have an extra symmetry that needs to be considered.
Suppose the axial $U(1)$ were a symmetry of QCD. There would be an $SU(3)$ singlet Goldstone boson field, $\pi_0$, which in the $SU(3)$ symmetry limit transforms under a $U(1)$ transformation by a translation $\delta \pi_0 = c_0$. In the presence of $SU(3) \times SU(3)$ symmetry breaking, the analog of

$$L = \frac{F^2}{4} \text{tr} \left[ D^\mu \Sigma^\dagger D_\mu \Sigma \right] + B_0 \text{tr} \left[ \Sigma^\dagger M_Q \right] + B_0 \text{tr} \left[ \Sigma M_Q \right]$$

is

$$L(\pi) = \frac{1}{2} \partial^\mu \pi_0 \partial_\mu \pi_0 + \frac{F^2}{4} \text{tr} \left[ \partial^\mu \Sigma^\dagger \partial_\mu \Sigma \right] + B_0 \text{tr} \left[ M_Q \Sigma^\dagger e^{-i\lambda \pi_0/F} \right] + B_0 \text{tr} \left[ M_Q \Sigma e^{i\lambda \pi_0/F} \right],$$

where $\lambda$ is an unknown constant that determines the strength of the spontaneous breaking of the $U(1)$ symmetry. In Equation 4 $M_Q$ is transformed with the extra $U(1)$ transformations. The new Goldstone field $\pi_0$ appears in order to compensate for this transformation meaning that there are now 9 Goldstone bosons.

Equation 4 cannot describe the pseudo-scalar mesons in our world for any value of $\lambda$. Show this by computing the mass spectrum as done in class and demonstrate that it does not describe our physical world. Show that the lightest meson has a mass less than $k m_\pi$ for some $k$. What is $k$?