Problem 1 - Semileptonic Pion Decay

The hadronic decay of a $\pi^+$ to $\pi^0$ and leptons can be computed “exactly” because the momentum transfer to the leptons is very small compared to $\Lambda_{QCD}$. In this exercise, we will compute the decay rate and determine $|V_{ud}|$ using the known values:

$$\tau_{\pi^+} = 2.6033 \times 10^{-8} \text{s}, \quad m_{\pi^+} = 139.57 \text{MeV}, \quad m_{\pi^0} = 134.98 \text{MeV},$$

$$R(\pi^+ \to \pi^0 + e^+ + \nu) = 1.036 \times 10^{-8},$$

$$G_F = 1.1664 \times 10^{-5} \text{GeV}^{-2}.$$

1) Write down the amplitude for this decay in terms of the pion form factor $F_{\pi^+\pi^0}(q^2)$ in the limit $q^2 \to 0$. Note that the scale provided by the $W$ mass is large so we can disregard terms of order $\frac{q^2}{M_W^2}$ and the leptons can be considered massless.

2) Show that the squared matrix element can be written as

$$|M|^2 = 32 G_F^2 |V_{ud}|^2 m_{\pi^+}^2 E_\nu E_e (1 + \cos \theta), \quad (1)$$

where $\theta$ is the angle between the electron and neutrino.

3) Integrate over the phase space to find the decay rate $\Gamma(\pi^+ \to \pi^0 + e^+ + \nu)$.

4) Use the branching ratio $R(\pi^+ \to \pi^0 + e^+ + \nu)$ to extract a value of $|V_{ud}|$ from your result.

5) How does your result compare to the world average of $|V_{ud}| = 0.97425 \pm 0.00022$? We made a number of approximations, $q^2 \to 0$, $M_w \to \infty$, $m_e = 0$, which do you expect will provide the most significant correction to $|V_{ud}|$? Why?
Problem 2 - Energy Momentum Tensor

In class we used conservation and transformation properties of the hadronic matrix element to show that it can be written in terms of the form factor $F_+ (q^2)$,

$$\langle H_2 | \bar{u} \gamma^\mu (1 + \gamma^5) d | H_1 \rangle = F_+ (q^2) \left[ p_2^\mu + p_1^\mu - \frac{m_2^2 - m_1^2}{q^2} q^\mu \right].$$  (2)

The energy momentum tensor, $T^{\mu \nu}(x)$ is symmetric and conserved. Use arguments analogous to those from the notes for the computation of the hadronic current to show that the matrix elements of $T^{\mu \nu}$ between spin-less particle states can be written as

$$\langle H(p_2) | T^{\mu \nu} | H(p_1) \rangle = F_1 \left( q^2 g^{\mu \nu} - q^\mu q^\nu \right) + F_2 k^\mu k^\nu,$$  (3)

where $q = p_1 - p_2$ and $k = p_1 + p_2$.

The energy momentum tensor is also related to the 4-momentum operators by

$$P^\mu = \int T^{\mu 0}(x) d^4 x.$$  (4)

Use this identity with $T^{00}$ to show that at $q^2 = 0$, $F_2 = \frac{1}{2}$. 