

# Precise Charm and Bottom Quark Masses

Johann H. Kühn

in collaboration with

K. Chetyrkin, Y. Kiyo, A. Maier, P. Maierhöfer, P. Marquard, A. Smirnov,  
M. Steinhauser, C. Sturm and the HPQCD Collaboration

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# I. WHY precise masses?

B-decays:

$$\Gamma(B \rightarrow X_u l \bar{\nu}) \sim G_F^2 m_b^5 |V_{ub}|^2$$

$$\Gamma(B \rightarrow X_c l \bar{\nu}) \sim G_F^2 m_b^5 f(m_c^2/m_b^2) |V_{cb}|^2$$

$$B \rightarrow X_s \gamma$$

$\Upsilon$ -spectroscopy:

$$m(\Upsilon(1s)) = 2M_b - \left(\frac{4}{3}\alpha_s\right)^2 \frac{M_b}{4} + \dots$$

## Higgs decay (ILC)

$$\Gamma(H \rightarrow b\bar{b}) = \frac{G_F M_H}{4\sqrt{2}\pi} m_b^2(M_H) \tilde{R}$$

$$\tilde{R} = 1 + 5.6667 a_S + 29.147 a_S^2 + 41.758 a_S^3 - 825.7 a_S^4 \quad \left( a_S \equiv \frac{\alpha_S}{\pi} \right)$$

$a_S^4$ -term = 5-loop calculation [Baikov, Chetyrkin, JK]

## Yukawa Unification

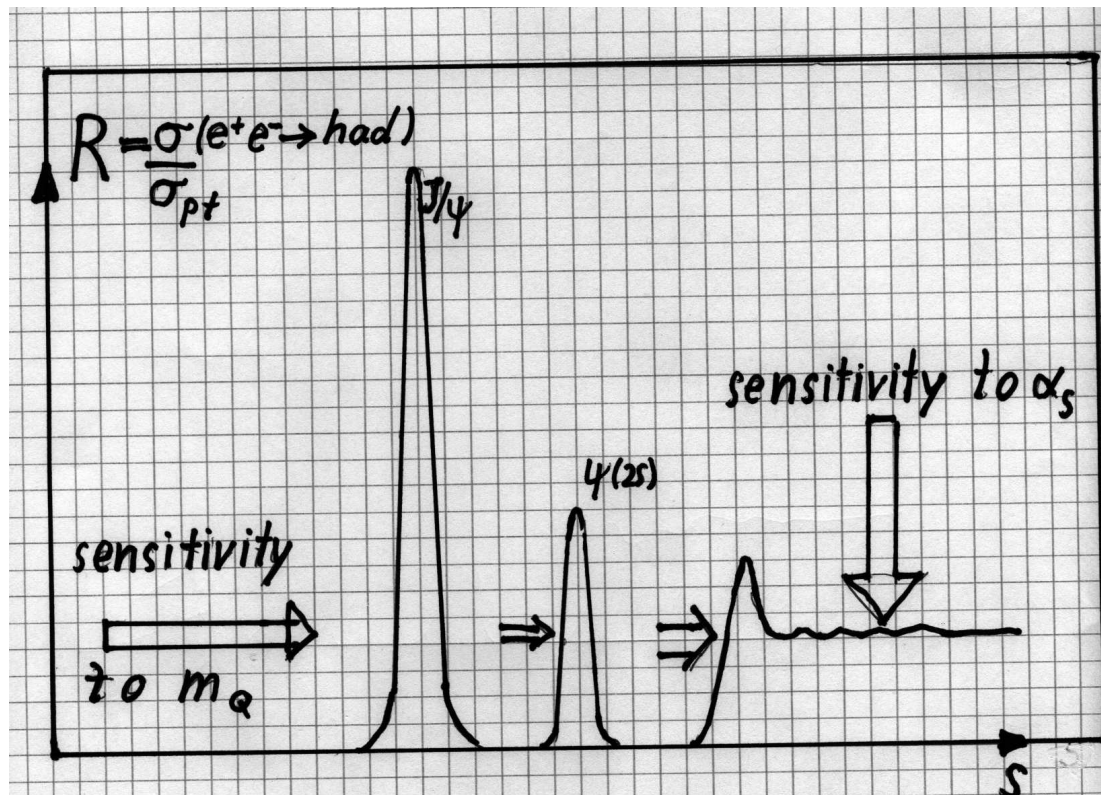
$$\lambda_\tau \sim \lambda_b \quad \text{or} \quad \lambda_\tau \sim \lambda_b \sim \lambda_t \quad \text{at GUT scale}$$

top-bottom  $\rightarrow m_t/m_b \sim$  ratio of vacuum expectation values

$$\text{request } \frac{\delta m_b}{m_b} \sim \frac{\delta m_t}{m_t} \Rightarrow \delta m_t \approx 1 \text{ GeV} \Rightarrow \delta m_b \approx 25 \text{ MeV}$$

## II. $m_Q$ from SVZ Sum Rules, Moments and Tadpoles

### Main Idea (SVZ)



Some definitions:

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right) \Pi(q^2) \equiv i \int d^4x e^{iqx} \langle T j_\mu(x) j_\nu(0) \rangle$$

with the electromagnetic current  $j_\mu$ .

$$R(s) = 12\pi \text{Im} \left[ \Pi(q^2 = s + i\epsilon) \right]$$

Taylor expansion:  $\Pi_Q(q^2) = Q_Q^2 \frac{3}{16\pi^2} \sum_{n \geq 0} \bar{C}_n z^n$

with  $z = q^2/(4m_Q^2)$  and  $m_Q = m_Q(\mu)$  the  $\overline{\text{MS}}$  mass.

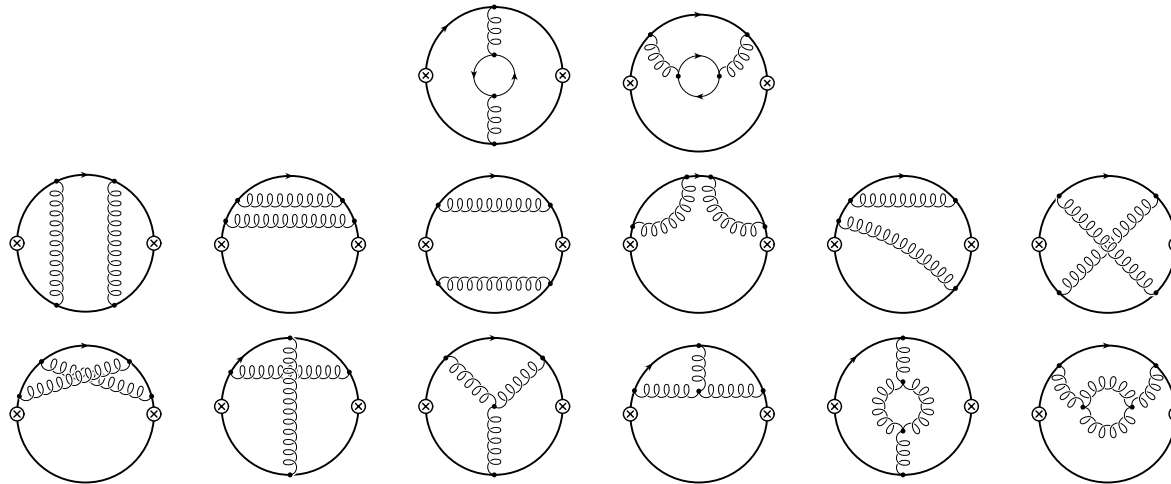
$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi} \bar{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}_n^{(3)} + \dots$$

generic form

$$\begin{aligned}\bar{C}_n = & \bar{C}_n^{(0)} \\ & + \frac{\alpha_S}{\pi} \left( \bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_c} \right) \\ & + \left( \frac{\alpha_S}{\pi} \right)^2 \left( \bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_c} + \bar{C}_n^{(22)} l_{m_c}^2 \right) \\ & + \left( \frac{\alpha_S}{\pi} \right)^3 \left( \bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_c} + \bar{C}_n^{(32)} l_{m_c}^2 + \bar{C}_n^{(33)} l_{m_c}^3 \right) \\ & + \dots\end{aligned}$$

## Analysis in NNLO

Coefficients  $\bar{C}_n$  from three-loop one-scale tadpole amplitudes with “arbitrary” power of propagators;





- FORM program MATAD

Coefficients  $\bar{C}_n$  up to  $n = 8$

(also for axial, scalar and pseudoscalar correlators)

(Chetyrkin, JK, Steinhauser, 1996)

- up to  $n = 30$  for vector correlator

(Boughezal, Czakon, Schutzmeier 2007)

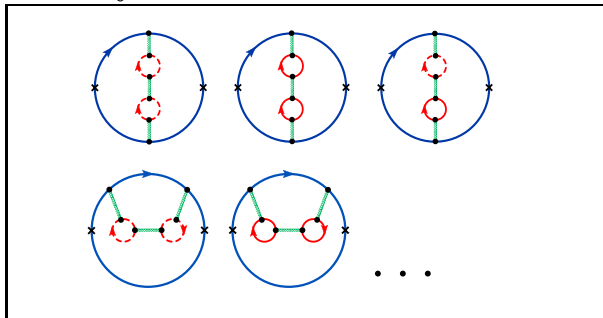
- up to  $n = 30$  for vector, axial, scalar and pseudoscalar correlators

(A. Maier, P. Maierhöfer, P. Marquard, 2007)

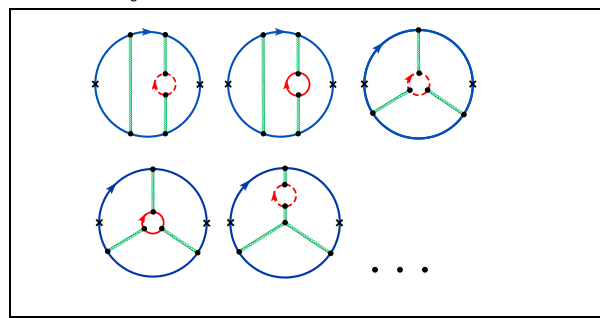
# Analysis in N<sup>3</sup>LO

Algebraic reduction to 13 master integrals (Laporta algorithm);  
numerical and analytical evaluation of master integrals

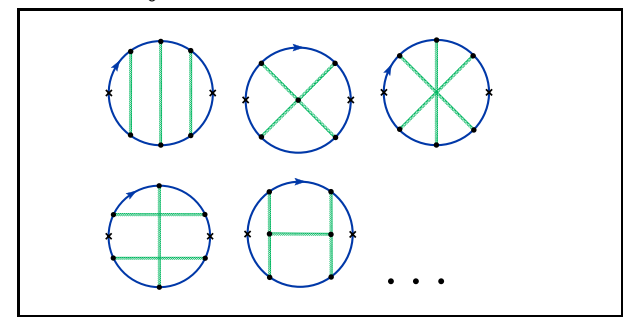
$n_f^2$ -contributions





$n_f^1$ -contributions



$n_f^0$ -contributions



 : heavy quarks,  : light quarks,

$n_f$ : number of active quarks

⇒ About **700 Feynman-diagrams**

$\bar{C}_0$  and  $\bar{C}_1$  in order  $\alpha_s^3$  (four loops!) (2006)

⇒ Reduction to master integrals

(Chetyrkin, JK, Sturm; Boughezal, Czakon, Schutzmeier)

$\bar{C}_2$  and  $\bar{C}_3$  (2008)

(Maier, Maierhöfer, Marquard, A. Smirnov)

All master integrals known analytically and double checked.

(Schröder + Vuorinen, Chetyrkin et al., Schröder + Steinhauser,

Laporta, Broadhurst, Kniehl et al.)

$\bar{C}_4 - \bar{C}_{10}$ : extension to higher moments by Padé method, using analytic information from low energy ( $q^2 = 0$ ), threshold ( $q^2 = 4m^2$ ), high energy ( $q^2 = -\infty$ ) (Kiyoyama, Maier, Maierhöfer, Marquard, 2009)

## Relation to measurements

$$\mathcal{M}_n^{\text{th}} \equiv \frac{12\pi^2}{n!} \left( \frac{d}{dq^2} \right)^n \Pi_c(q^2) \Big|_{q^2=0} = \frac{9}{4} Q_c^2 \left( \frac{1}{4m_c^2} \right)^n \bar{C}_n$$

Perturbation theory:  $\bar{C}_n$  is function of  $\alpha_s$  and  $\ln \frac{m_c^2}{\mu^2}$

dispersion relation:

$$\Pi_c(q^2) = \frac{q^2}{12\pi^2} \int ds \frac{R_c(s)}{s(s-q^2)} + \text{subtraction}$$

$$\Leftrightarrow \mathcal{M}_n^{\text{exp}} = \int \frac{ds}{s^{n+1}} R_c(s)$$

constraint:  $\mathcal{M}_n^{\text{exp}} = \mathcal{M}_n^{\text{th}}$

$$\Leftrightarrow m_c$$

## Ingredients (charm)

### experiment:

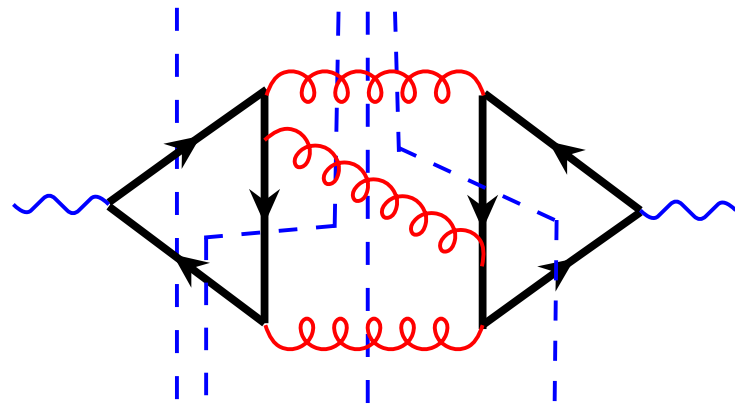
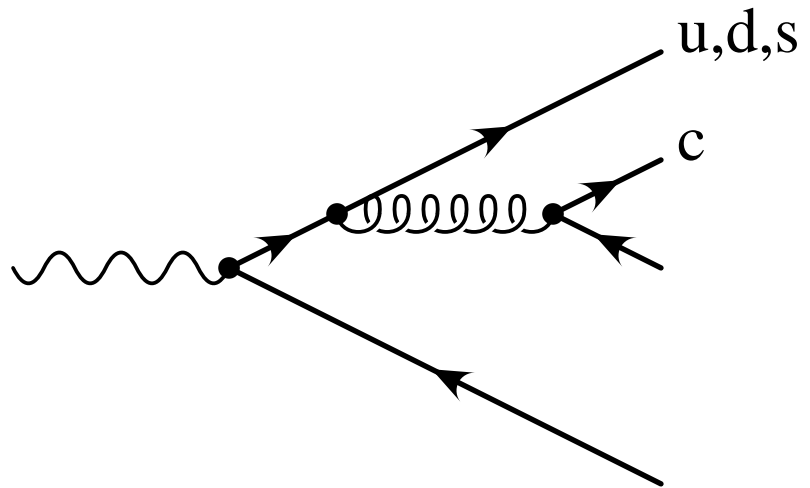
- $\Gamma_e(J/\psi, \psi')$  from BES & CLEO & BABAR (PDG)
- $\psi(3770)$  and  $R(s)$  from BES
- $\alpha_s = 0.1187 \pm 0.0020$

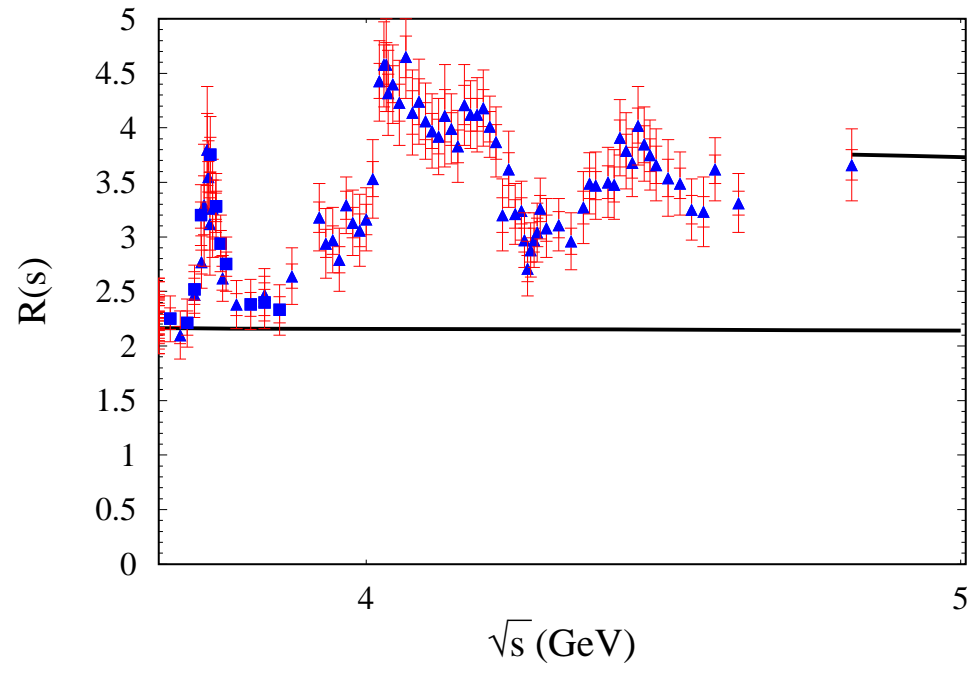
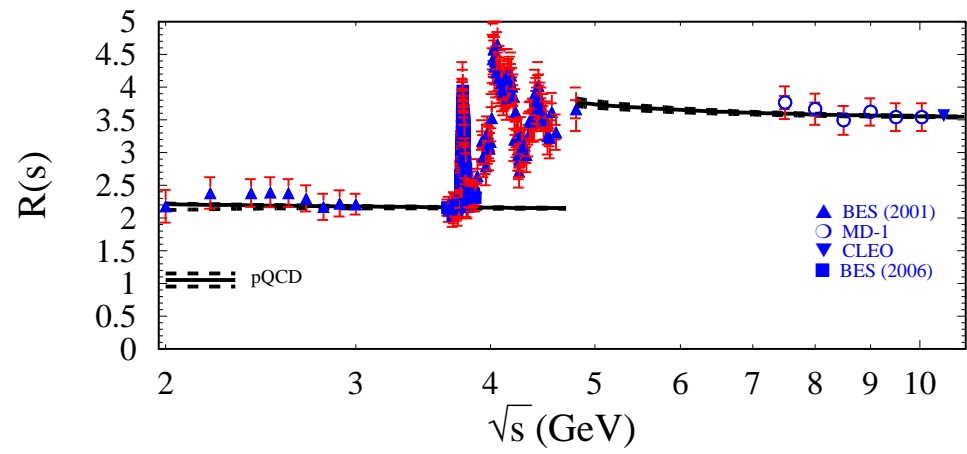
### theory:

- N<sup>3</sup>LO for  $n = 1, 2, 3, 4$
- include condensates

$$\delta\mathcal{M}_n^{\text{np}} = \frac{12\pi^2 Q_c^2}{(4m_c^2)^{(n+2)}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n \left( 1 + \frac{\alpha_s \bar{b}_n}{\pi} \right)$$

- estimate of non-perturbative terms (oscillations, based on Shifman)
- careful extrapolation of  $R_{uds}$
- careful definition of  $R_c$





Contributions from

- narrow resonances:  $R = \frac{9 \Pi M_R \Gamma_e}{\alpha^2(s)} \delta(s - M_R^2)$  (PDG)
- threshold region ( $2 m_D - 4.8$  GeV) (BESS)
- perturbative continuum ( $E \geq 4.8$  GeV) (Theory)

$n$	$\mathcal{M}_n^{\text{res}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{np}}$ $\times 10^{(n-1)}$
1	0.1201(25)	0.0318(15)	0.0646(11)	0.2166(31)	-0.0001(2)
2	0.1176(25)	0.0178(8)	0.0144(3)	0.1497(27)	0.0000(0)
3	0.1169(26)	0.0101(5)	0.0042(1)	0.1312(27)	0.0007(14)
4	0.1177(27)	0.0058(3)	0.0014(0)	0.1249(27)	0.0027(54)

Different relative importance of resonances vs. continuum for  $n = 1, 2, 3, 4$ .



## Results ( $m_c$ )

PRD 80: (2009) 074010

$n$	$m_c(3 \text{ GeV})$	exp	$\alpha_s$	$\mu$	np	total
1	986	9	9	2	1	13
2	976	6	14	5	0	16
3	978	5	15	7	2	17
4	1004	3	9	31	7	33

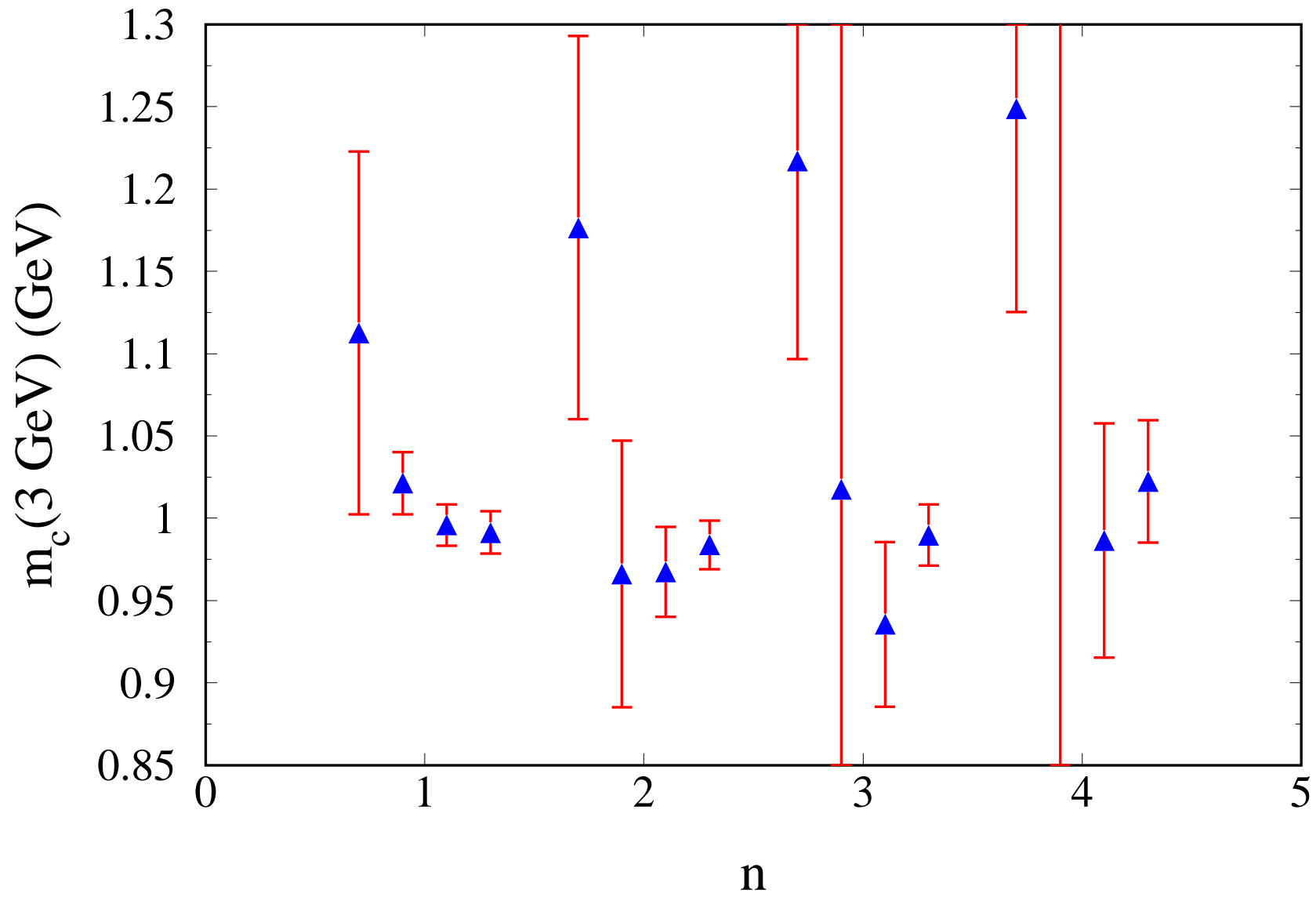
Remarkable consistency between  $n = 1, 2, 3, 4$

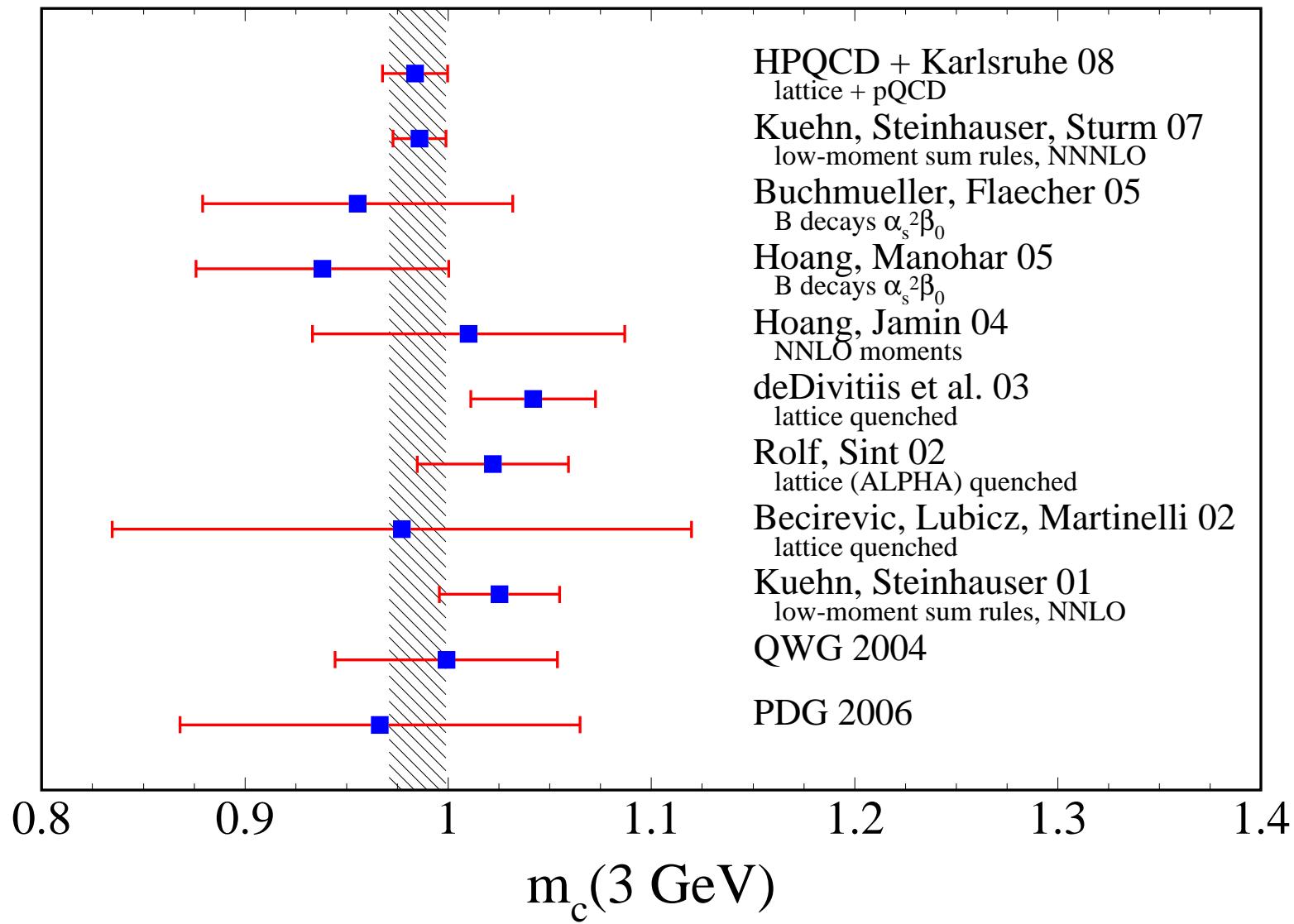
and stability ( $\mathcal{O}(\alpha_s^2)$  vs.  $\mathcal{O}(\alpha_s^3)$ );

preferred scale:  $\mu = 3 \text{ GeV}$ ,

conversion to  $m_c(m_c)$ :

- $m_c(3 \text{ GeV}) = 986 \pm 13 \text{ MeV}$
- $m_c(m_c) = 1279 \pm 13 \text{ MeV}$



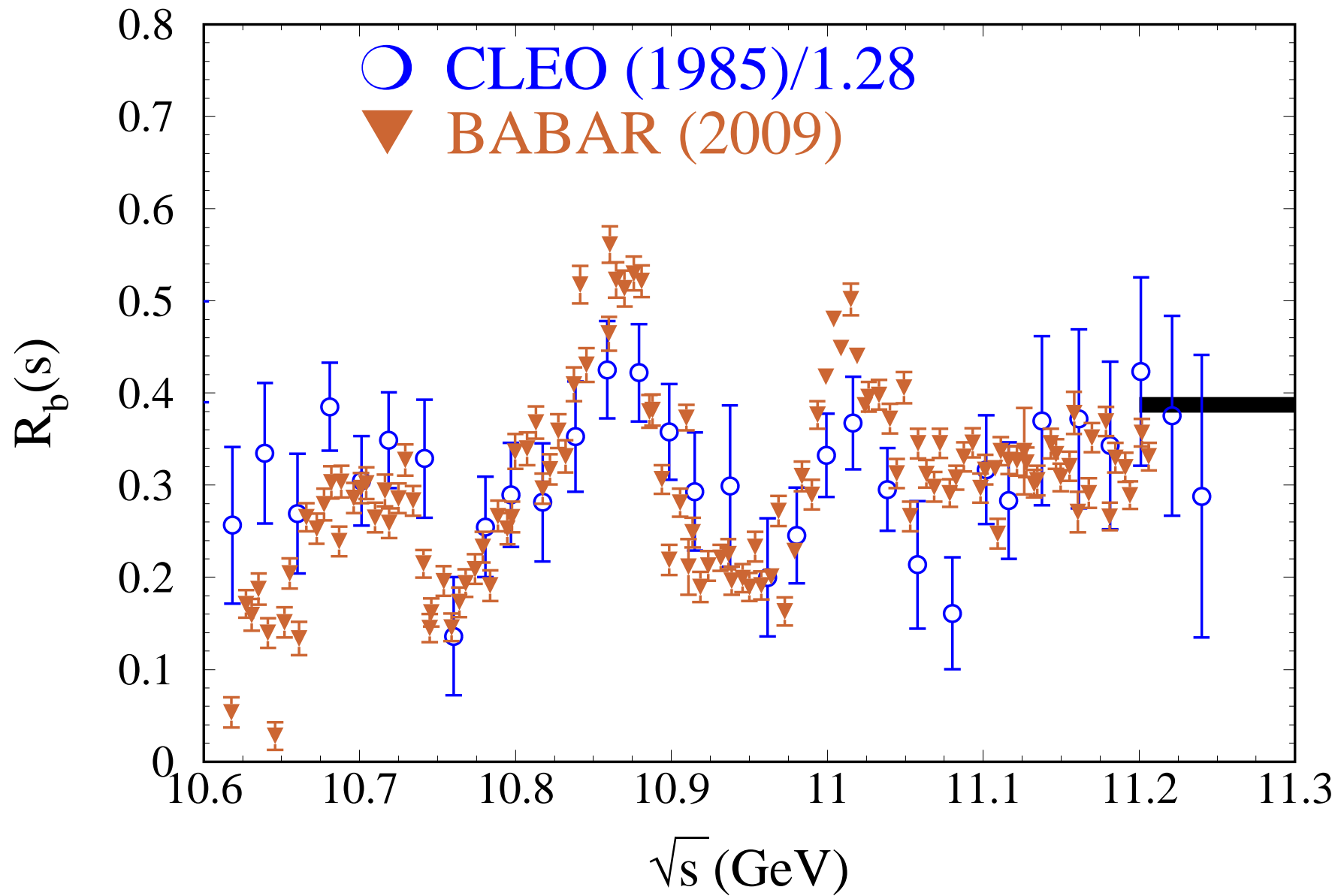


## Experimental Ingredients for $m_b$

Contributions from

- narrow resonances ( $\Upsilon(1S) - \Upsilon(4S)$ ) (PDG)
- threshold region (10.618 GeV – 11.2 GeV) (BABAR 2009)
- perturbative continuum ( $E \geq 11.2$  GeV) (Theory)
- different relative importance of resonances vs. continuum for  $n = 1, 2, 3, 4$

$n$	$\mathcal{M}_n^{\text{res},(1S-4S)}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(2n+1)}$
1	1.394(23)	0.287(12)	2.911(18)	4.592(31)
2	1.459(23)	0.240(10)	1.173(11)	2.872(28)
3	1.538(24)	0.200(8)	0.624(7)	2.362(26)
4	1.630(25)	0.168(7)	0.372(5)	2.170(26)

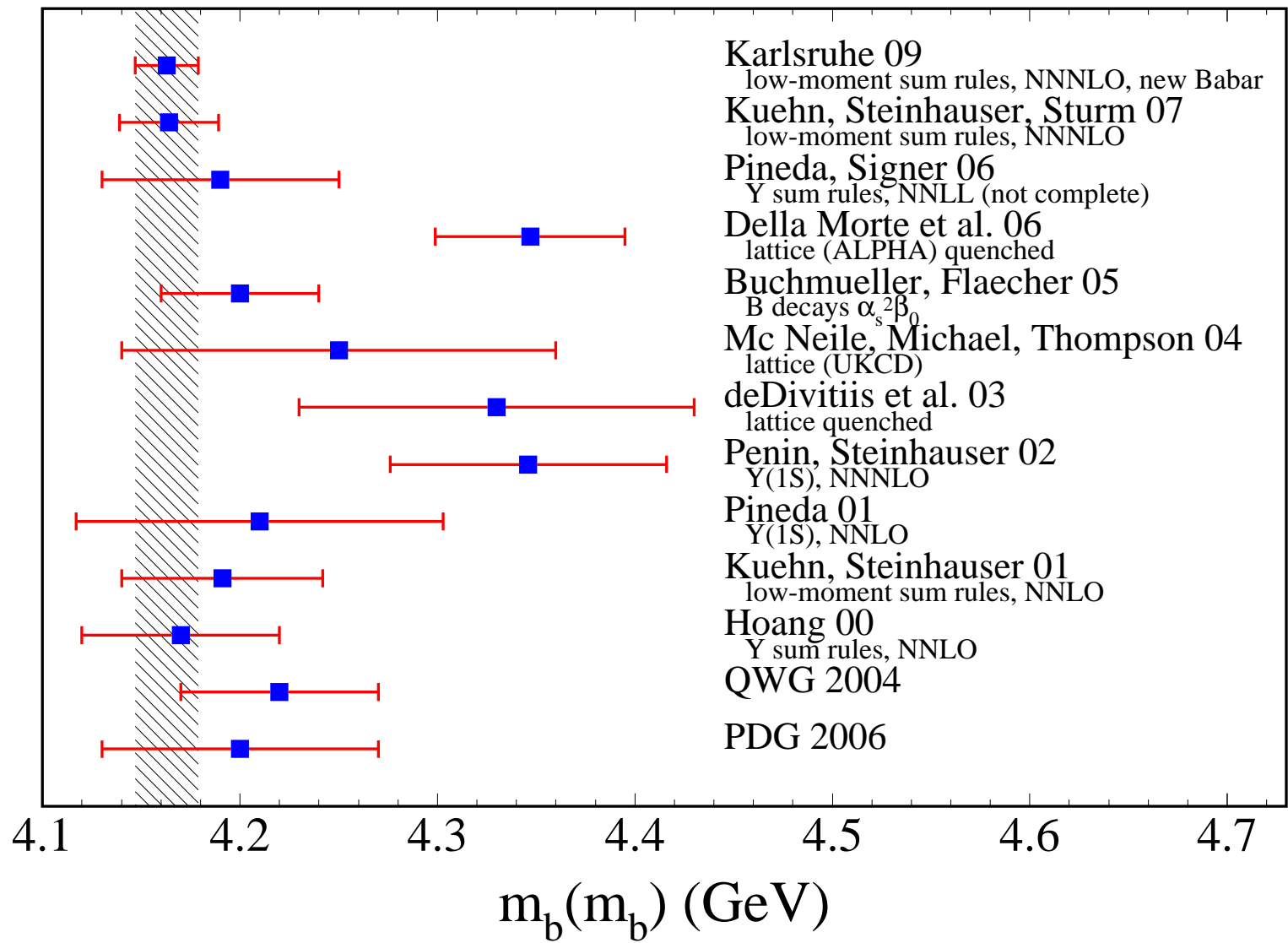


$n$	$m_b(10 \text{ GeV})$	exp	$\alpha_s$	$\mu$	total	$m_b(m_b)$
1	3597	14	7	2	16	4151
2	3610	10	12	3	16	4163
3	3619	8	14	6	18	4172
4	3631	6	15	20	26	4183

Consistency ( $n = 1, 2, 3, 4$ ) and stability ( $\mathcal{O}(\alpha_s^2)$  vs.  $\mathcal{O}(\alpha_s^3)$ );  
(slight dependence on  $n$  could result from input into  $\mathcal{M}_{\text{exp}}^n$ )

- $m_b(10 \text{ GeV}) = 3610 \pm 16 \text{ MeV}$
- $m_b(m_b) = 4163 \pm 16 \text{ MeV}$

well consistent with **KSS 2007**



## $\alpha_s$ -dependence

$$m_c(3 \text{ GeV}) = \left( 986 - \frac{\alpha_s - 0.1189}{0.002} \cdot 9 \pm 10 \right) \text{ MeV}$$

$$m_b(10 \text{ GeV}) = \left( 3610 - \frac{\alpha_s - 0.1189}{0.002} \cdot 12 \pm 11 \right) \text{ MeV}$$

$$m_b(m_b) = \left( 4163 - \frac{\alpha_s - 0.1189}{0.002} \cdot 12 \pm 11 \right) \text{ MeV}$$

$$m_b(M_Z) = \left( 2835 - \frac{\alpha_s - 0.1189}{0.002} \cdot 27 \pm 8 \right) \text{ MeV}$$

$$m_b(161.8 \text{ GeV}) = \left( 2703 - \frac{\alpha_s - 0.1189}{0.002} \cdot 28 \pm 8 \right) \text{ MeV}$$



## lattice & pQCD

(HPQCD + Karlsruhe, Phys. Rev. D78, 054513)

lattice evaluation of pseudoscalar correlator

⇒ replace experimental moments by lattice simulation

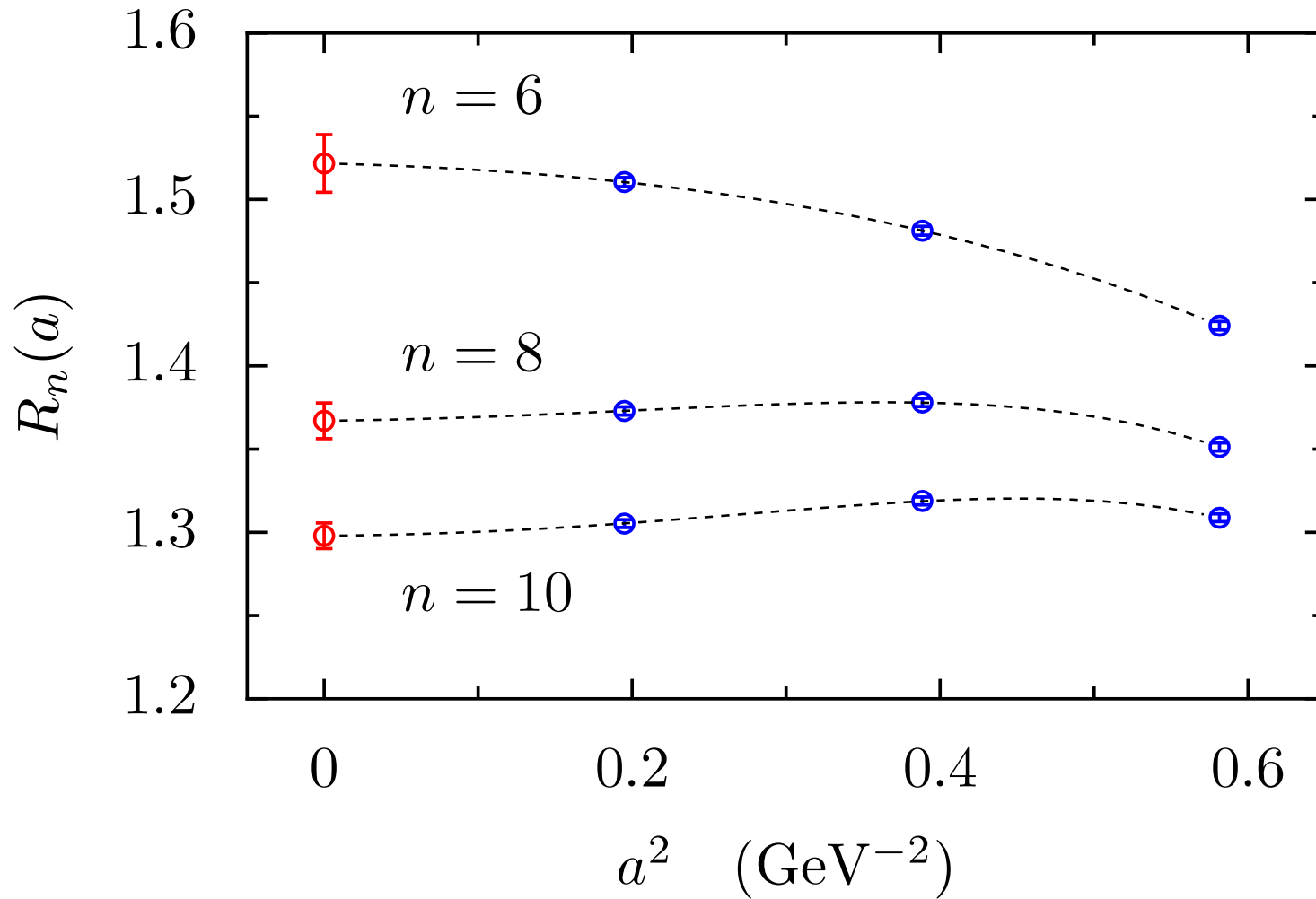
input:  $M(\eta_c) \hat{=} m_c$ ,  $M(\Upsilon(1S)) - M(\Upsilon(2S)) \hat{=} \alpha_s$

pQCD for pseudoscalar correlator available:

“all” moments in  $\mathcal{O}(\alpha_s^2)$

three lowest moments in  $\mathcal{O}(\alpha_s^3)$ .

lowest moment: dimensionless:  $\sim \left( \bar{C}(0) + \frac{\alpha_s}{\pi} \bar{C}(1) + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}(2) + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}(3) + \dots \right)$



Reduced moments  $R_n$  from lattice simulations with different lattice spacings  $a$ . The dashed lines show the functions used to fit the lattice results. These extrapolation functions were used to obtain the  $a = 0$  results shown in the plot.

	$R_6$	$R_8$	$R_{10}$	$R_{12}$
$a^2$ extrapolation	1.3%	0.9%	0.7%	0.5%
pert'n theory	0.4	0.9	1.3	1.6
$\alpha_{\overline{MS}}$ uncertainty	0.3	0.6	1.0	1.3
gluon condensate	0.3	0.0	0.3	0.7
statistical errors	0.0	0.0	0.0	0.0
relative scale errors	0.4	0.4	0.3	0.3
overall scale errors	0.6	0.6	0.7	0.7
sea quarks	0.2	0.2	0.1	0.1
finite volume	0.1	0.1	0.3	0.4
Total	1.6%	1.6%	2.0%	2.4%

Sources of uncertainty in the determinations of  $m_c(\mu = 3 \text{ GeV})$  from different reduced moments  $R_n$  of the pseudoscalar correlator. The uncertainties listed are percentages of the final result  $0.984(16) \text{ GeV}$ .

$$\Rightarrow \alpha_s(3\text{GeV}) \Rightarrow \alpha_s(M_Z) = 0.1174(12)$$

higher moments:  $\sim m_c^2 \times \left(1 + \dots \frac{\alpha_s}{\pi} \dots\right)$

$$\Rightarrow m_c(3\text{GeV}) = 986(10) \text{ MeV}$$

to be compared with 986(13) MeV from  $e^+e^-$  !

update: HPQCD 2010

$$\alpha_s(3\text{GeV}) \Rightarrow \alpha_s(M_Z) = 0.1183(7)$$

$$m_c(3\text{GeV}) = 986(6) \text{ MeV}$$

$$m_b(10\text{GeV}) = 3617(25) \text{ MeV}$$

# SUMMARY

$$m_c(3 \text{ GeV}) = 986(13) \text{ MeV}$$

$e^+e^- + \text{pQCD}$

$$m_c(3 \text{ GeV}) = 986(10) \text{ MeV}$$

lattice + pQCD

$$m_b(10 \text{ GeV}) = 3610(16) \text{ MeV}$$

$$m_b(m_b) = 4163(16) \text{ MeV}$$

$e^+e^- + \text{pQCD}$