Precise Charm and Bottom Quark Masses

Johann H. Kühn

Universität Karlsruhe (TH)
Forschungsuniversität • gegründet 1825

in collaboration with

K. Chetyrkin, Y. Kiyo, A. Maier, P. Maierhöfer, P. Marquard, A. Smirnov,

M. Steinhauser, C. Sturm and the HPQCD Collaboration

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PLB 669 (2008) 88
NPB 823 (2009) 269
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arXiv: 0907.2117

I. WHY precise masses?

B-decays:

$\Gamma(B \to X_{\mathrm{u}} l \bar{\nu}) \sim G_{\mathrm{F}}^2 \, m_{\mathrm{b}}^5 \, |V_{\mathrm{ub}}|^2$

$\Gamma(B \to X_{\mathrm{c}} l \bar{\nu}) \sim G_{\mathrm{F}}^2 \, m_{\mathrm{b}}^5 \, f(m_{\mathrm{c}}^2/m_{\mathrm{b}}^2) \, |V_{\mathrm{cb}}|^2$

$B \to X_{\mathrm{s}} \gamma$

Υ-spectroscopy:

$$m\left(\Upsilon(1\mathrm{s})\right) = 2M_{\mathrm{b}} - \left(\frac{4}{3}\alpha_{\mathrm{s}}\right)^2 \frac{M_{\mathrm{b}}}{4} + \ldots$$

Higgs decay (ILC)

$$\Gamma(H \to b\bar{b}) = \frac{G_F\, M_H}{4\sqrt{2}\pi}\, m_b^2(M_H)\, \tilde{R}$$

$$\tilde{R} = 1 + 5.6667 a_s + 29.147 {a_s}^2 + 41.758 {a_s}^3 - 825.7 {a_s}^4 \qquad \left(a_s \equiv \frac{\alpha_s}{\pi} \right)$$

${a_s}^4$-term = 5-loop calculation [Baikov, Chetyrkin, JK]

Yukawa Unification

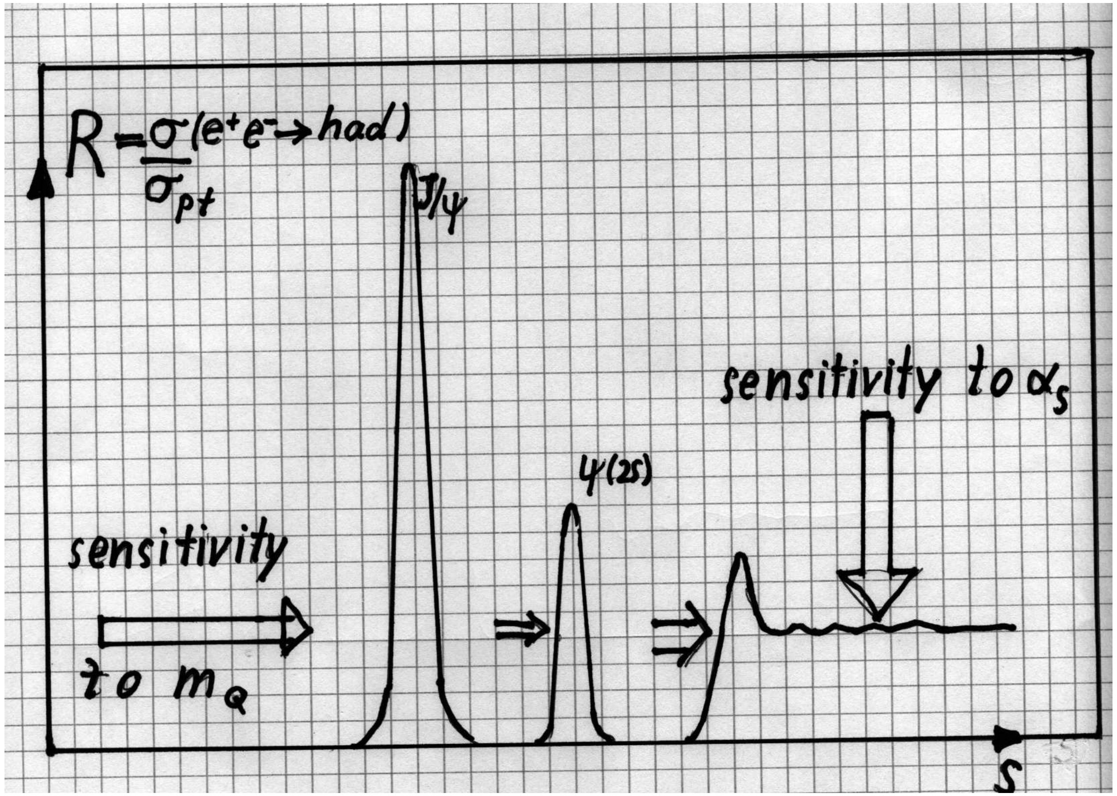
$\lambda_\tau \sim \lambda_b$ or $\lambda_\tau \sim \lambda_b \sim \lambda_t$ at GUT scale

top-bottom $\to$ m_t / m_b $\sim$ ratio of vacuum expectation values

request $\frac{\delta m_b}{m_b} \sim \frac{\delta m_t}{m_t} \Rightarrow \delta m_t \approx 1\,\text{GeV} \Rightarrow \delta m_b \approx 25\,\text{MeV}$

II. m_Q from SVZ Sum Rules, Moments and Tadpoles

Main Idea (SVZ)



Some definitions:

$$\left(-q^2 g_{\mu\nu} + q_\mu\, q_\nu\right)\,\Pi(q^2) \;\equiv\; i\int \mathrm{d}x\, e^{iqx}\langle T j_\mu(x) j_\nu(0)\rangle$$

with the electromagnetic current j_μ.

$$R(s) \;=\; 12\pi\,\mathrm{Im}\left[\Pi(q^2 = s + i\epsilon)\right]$$

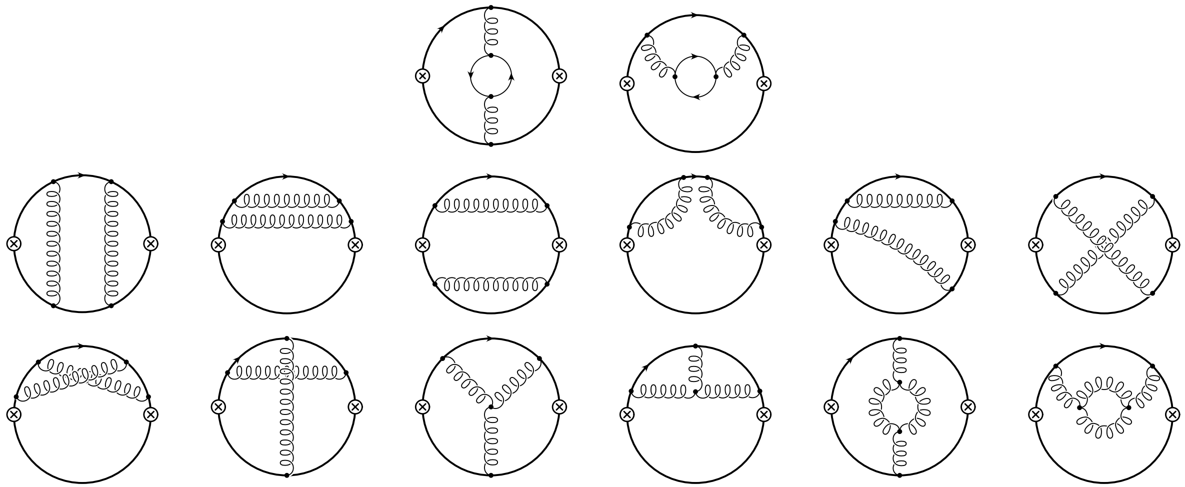
Taylor expansion: $\Pi_Q(q^2) \;=\; Q_Q^2\,\frac{3}{16\pi^2}\sum_{n\geq 0}\bar{C}_n\, z^n$

with $z = q^2/(4m_Q^2)$ and $m_Q = m_Q(\mu)$ the $\overline{\mathrm{MS}}$ mass.

$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi}\bar{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}_n^{(3)} + \ldots$$

Analysis in NNLO

Coefficients $\bar{C}_n$ from three-loop one-scale tadpole amplitudes with "arbitrary" power of propagators;



- FORM program MATAD

  Coefficients $\bar{C}_n$ up to $n = 8$

  (also for axial, scalar and pseudoscalar correlators)

  (Chetyrkin, JK, Steinhauser, 1996)

- up to $n = 30$ for vector correlator

  (Boughezal, Czakon, Schutzmeier 2007)

- up to $n = 30$ for vector, axial, scalar and pseudoscalar correlators

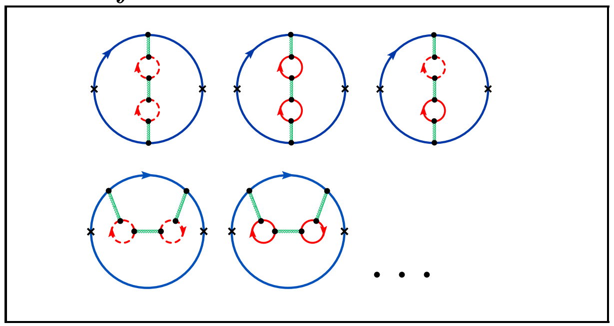
  (A. Maier, P. Maierhöfer, P. Marquard, 2007)

Analysis in N^{3}LO

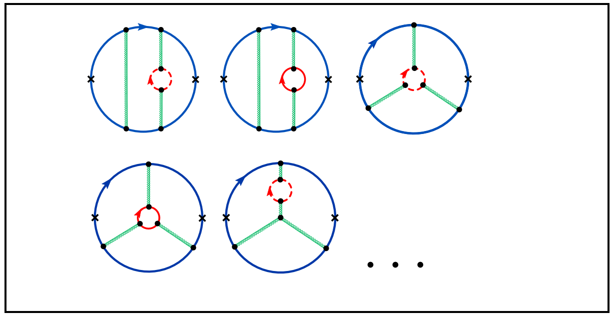
Algebraic reduction to 13 master integrals (Laporta algorithm);

numerical and analytical evaluation of master integrals

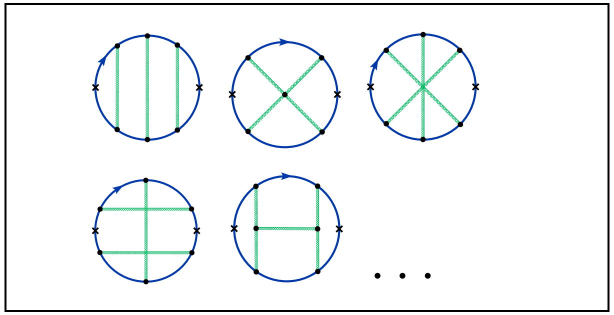
n_f^2-contributions



n_f^1-contributions



n_f^0-contributions



 : heavy quarks,  : light quarks,

n_f: number of active quarks

$\Rightarrow$ About 700 Feynman-diagrams

$\bar{C}_0$ and $\bar{C}_1$ in order α_s^3 (four loops!) (2006)

➪ Reduction to master integrals

(Chetyrkin, JK, Sturm; Boughezal, Czakon, Schutzmeier)

➪ Evaluation of master integrals numerically

or analytically in terms of transcendentals.

All master integrals known analytically and double checked.

(Schröder + Vuorinen, Chetyrkin et al., Schröder + Steinhauser,

Laporta, Broadhurst, Kniehl et al.)

Similar approach: four-loop ρ parameter

(K. G. Chetyrkin, M. Faisst, JK, P. Maierhöfer, C. Sturm)

New developments

- $\bar{C}_2$, $\bar{C}_3$
  (Maier, Maierhöfer, Marquard, A. Smirnov, 2008)

- $\bar{C}_4 - \bar{C}_{10}$: extension to higher moments by Padé method, using analytic information from low energy ($q^2 = 0$), threshold ($q^2 = 4m^2$), high energy ($q^2 = -\infty$) (Kiyo, Maier, Maierhöfer, Marquard, 2009)

Relation to measurements

$$\mathcal{M}_n^{\text{th}} \equiv \frac{12\pi^2}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_c(q^2)\bigg|_{q^2=0} = \frac{9}{4}Q_c^2\left(\frac{1}{4m_c^2}\right)^n \bar{C}_n$$

Perturbation theory: $\bar{C}_n$ is function of α_s and $\ln\frac{m_c^2}{\mu^2}$

dispersion relation:

$$\Pi_c(q^2) = \frac{q^2}{12\pi^2}\int \mathrm{d}s\,\frac{R_c(s)}{s(s-q^2)} + \text{subtraction}$$

$$\Rightarrow \mathcal{M}_n^{\text{exp}} = \int \frac{\mathrm{d}s}{s^{n+1}} R_c(s)$$

constraint: $\mathcal{M}_n^{\text{exp}} = \mathcal{M}_n^{\text{th}}$

$\Rightarrow$ m_c

Ingredients (charm)

experiment:

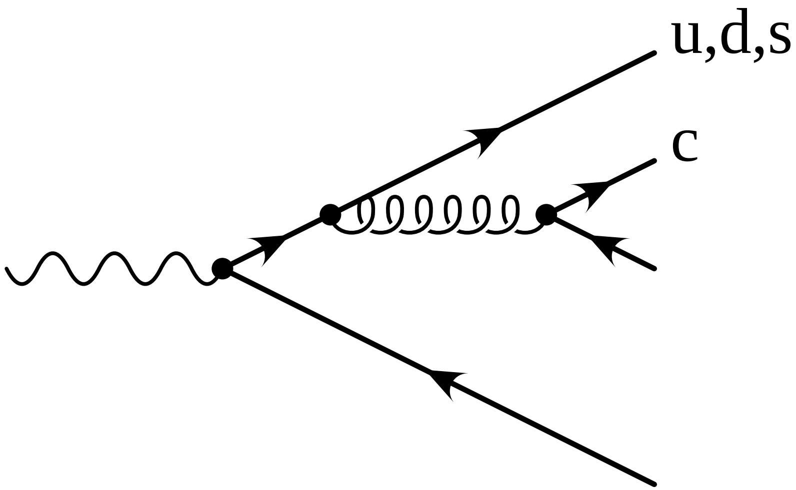
- $\Gamma_e(J/\psi, \psi')$ from BES & CLEO & BABAR
- $\psi(3770)$ and $R(s)$ from BES
- $\alpha_s = 0.1187 \pm 0.0020$

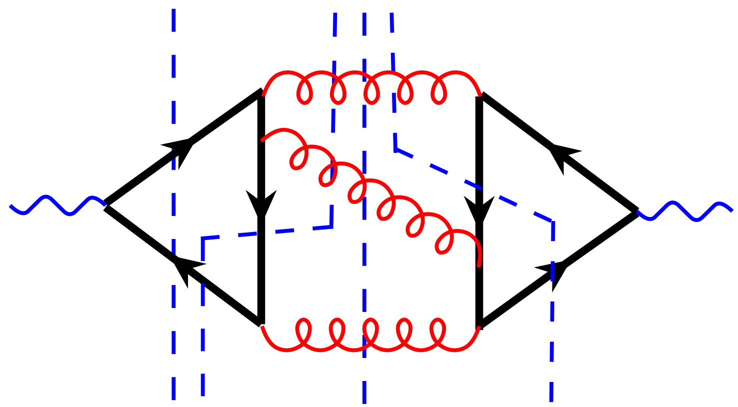
theory:

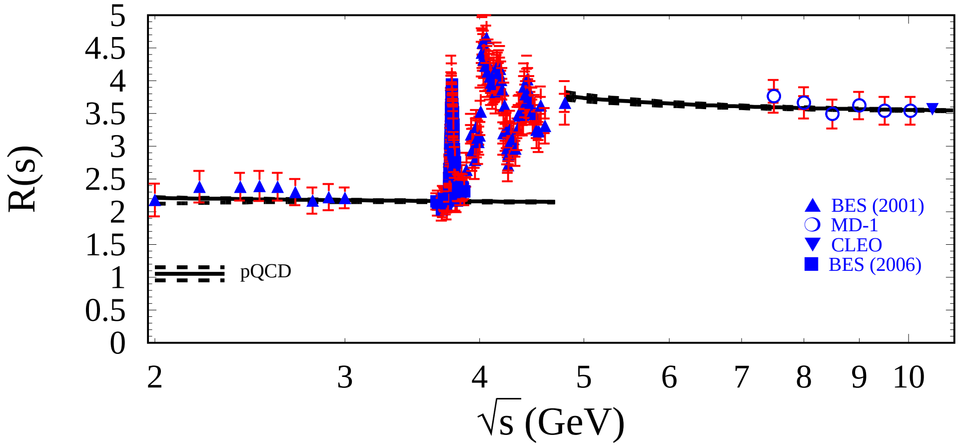
- N^{3}LO for $n = 1, 2, 3, 4$
- include condensates

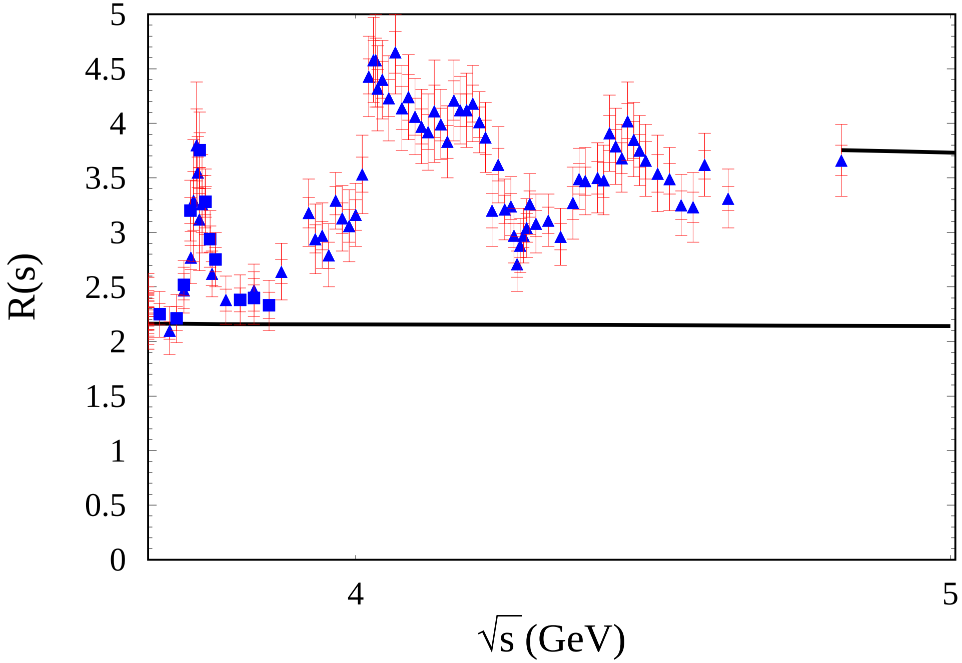
$$\delta\mathcal{M}_n^{\text{np}} = \frac{12\pi^2 Q_c^2}{(4m_c^2)^{(n+2)}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n \left(1 + \frac{\alpha_s}{\pi} \bar{b}_n\right)$$

- estimate of non-perturbative terms (oscillations, based on Shifman)
- careful extrapolation of R_{uds}
- careful definition of R_c


u,d,s
c




R(s)
√s (GeV)
pQCD
BES (2001)
MD-1
CLEO
BES (2006)


R(s)
√s (GeV)

Contributions from

- narrow resonances: $R = \frac{9\,\Pi\,M_R\,\Gamma_e}{\alpha^2(s)}\,\delta(s - M_R^2)$ (PDG)
- threshold region ($2\,m_D$ – 4.8 GeV) (BESS)
- perturbative continuum ($E \geq$ 4.8 GeV) (Theory)

n	$\mathcal{M}_n^{\text{res}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{np}}$ $\times 10^{(n-1)}$
1	0.1201(25)	0.0318(15)	0.0646(11)	0.2166(31)	−0.0001(2)
2	0.1176(25)	0.0178(8)	0.0144(3)	0.1497(27)	0.0000(0)
3	0.1169(26)	0.0101(5)	0.0042(1)	0.1312(27)	0.0007(14)
4	0.1177(27)	0.0058(3)	0.0014(0)	0.1249(27)	0.0027(54)

Different relative importance of resonances vs. continuum for $n = 1, 2, 3, 4$.

Results (m_c)

arXiv: 0907:2110

n	$m_c(3\,\mathrm{GeV})$	exp	α_s	μ	np	total
1	986	9	9	2	1	13
2	976	6	14	5	0	16
3	978	5	15	7	2	17
4	1004	3	9	31	7	33

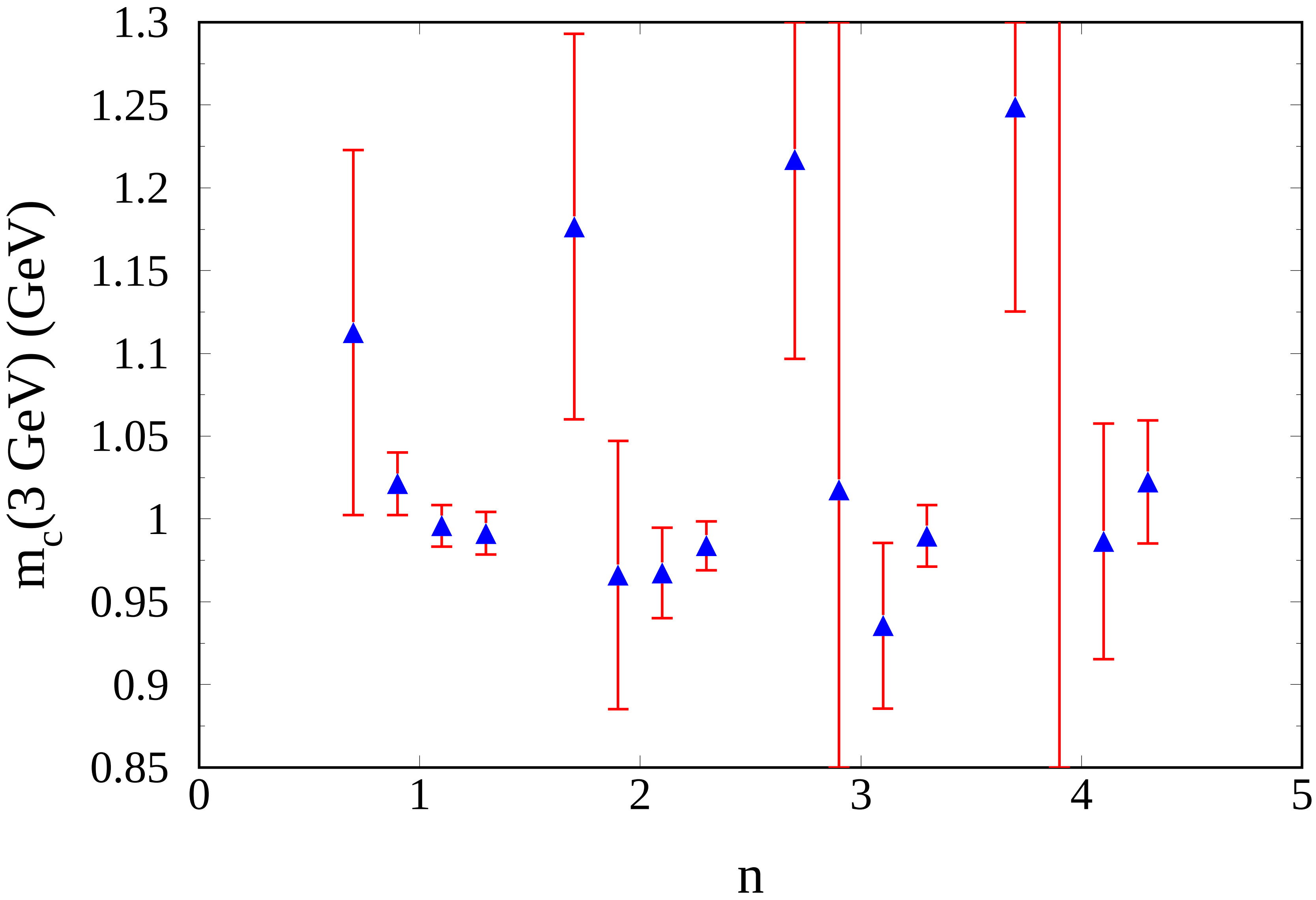
Remarkable consistency between $n = 1, 2, 3, 4$

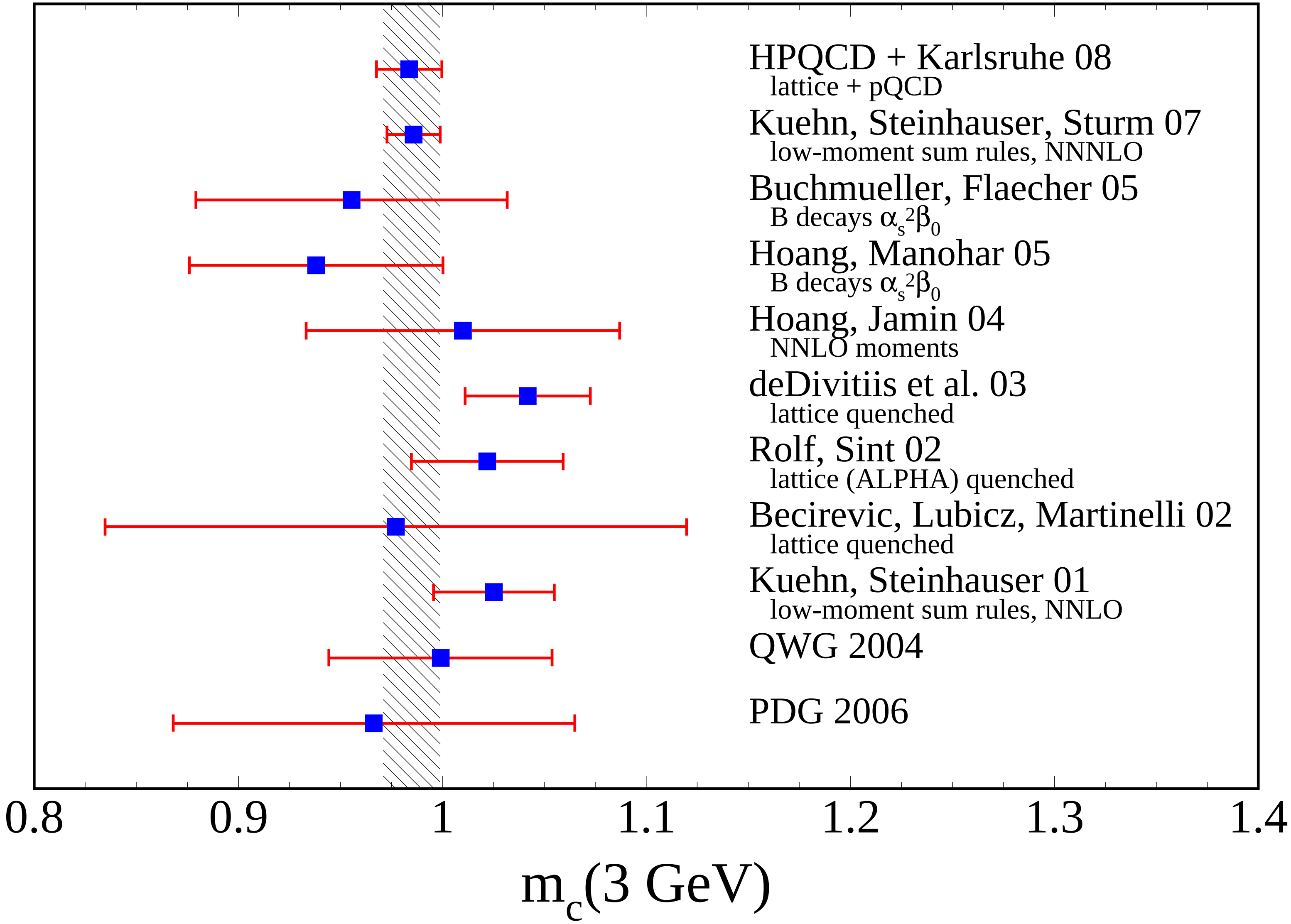
and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);

prefered scale: $\mu = 3\,\mathrm{GeV}$,

conversion to $m_c(m_c)$:

- $m_c(3\,\mathrm{GeV}) = 986 \pm 13\,\mathrm{MeV}$
- $m_c(m_c) = 1279 \pm 13\,\mathrm{MeV}$


1.3
1.25
1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
m_c(3 GeV) (GeV)
0
1
2
3
4
5
n

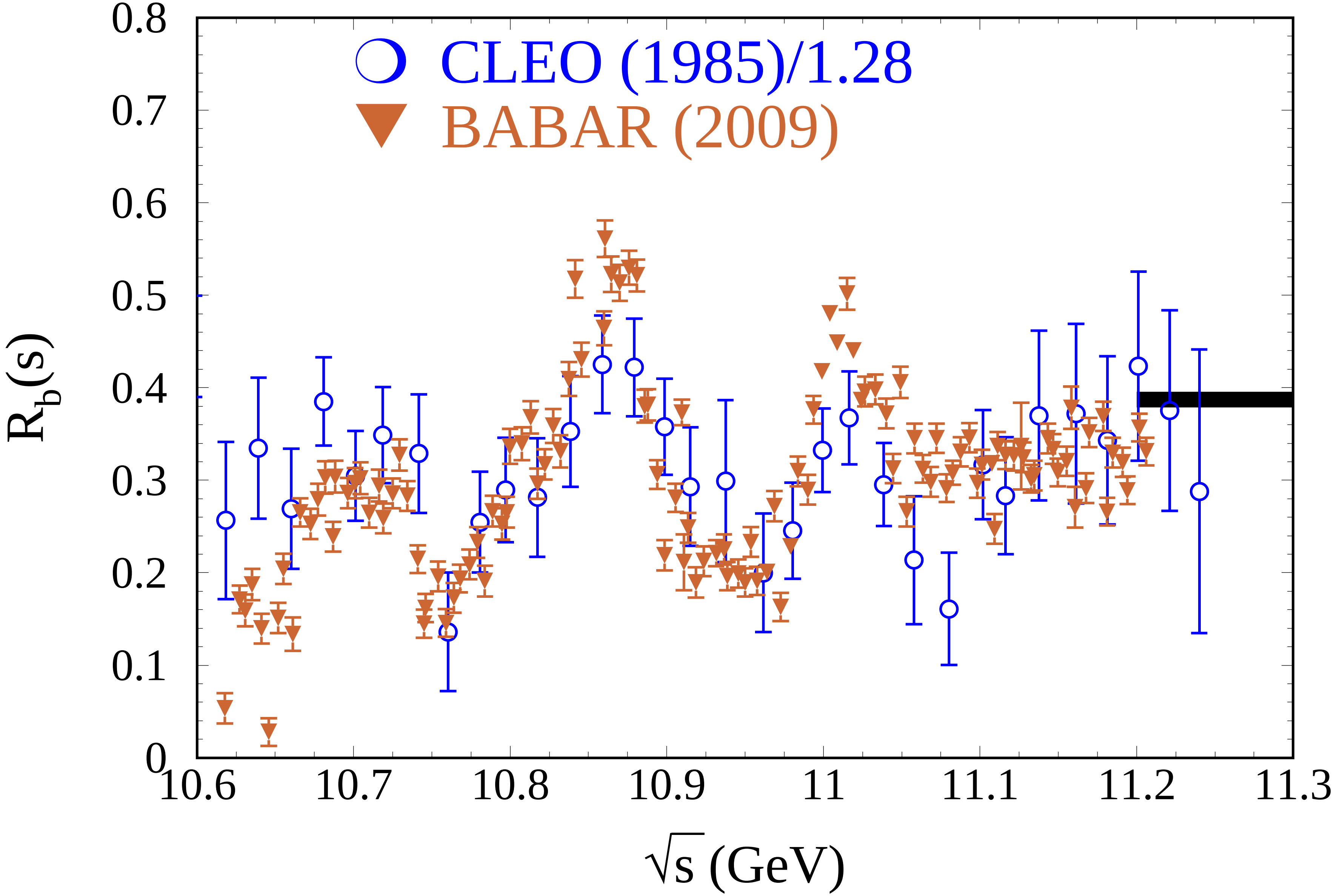

HPQCD + Karlsruhe 08
lattice + pQCD
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Hoang, Manohar 05
B decays $\alpha_s^2\beta_0$
Hoang, Jamin 04
NNLO moments
deDivitiis et al. 03
lattice quenched
Rolf, Sint 02
lattice (ALPHA) quenched
Becirevic, Lubicz, Martinelli 02
lattice quenched
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
QWG 2004
PDG 2006
0.8
0.9
1
1.1
1.2
1.3
1.4
m_c(3 GeV)

Experimental Ingredients for m_b

Contributions from

- narrow resonances ($\Upsilon(1S) - \Upsilon(4S)$) (PDG)
- threshold region (10.618 GeV − 11.2 GeV) (BABAR 2009)
- perturbative continuum ($E \geq 11.2$ GeV) (Theory)
- different relative importance of resonances vs. continuum for $n = 1, 2, 3, 4$

n	$\mathcal{M}_n^{\text{res},(1S-4S)}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(2n+1)}$
1	1.394(23)	0.287(12)	2.911(18)	4.592(31)
2	1.459(23)	0.240(10)	1.173(11)	2.872(28)
3	1.538(24)	0.200(8)	0.624(7)	2.362(26)
4	1.630(25)	0.168(7)	0.372(5)	2.170(26)


CLEO (1985)/1.28
BABAR (2009)
$R_b(s)$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
10.6
10.7
10.8
10.9
11
11.1
11.2
11.3
$\sqrt{s}$ (GeV)

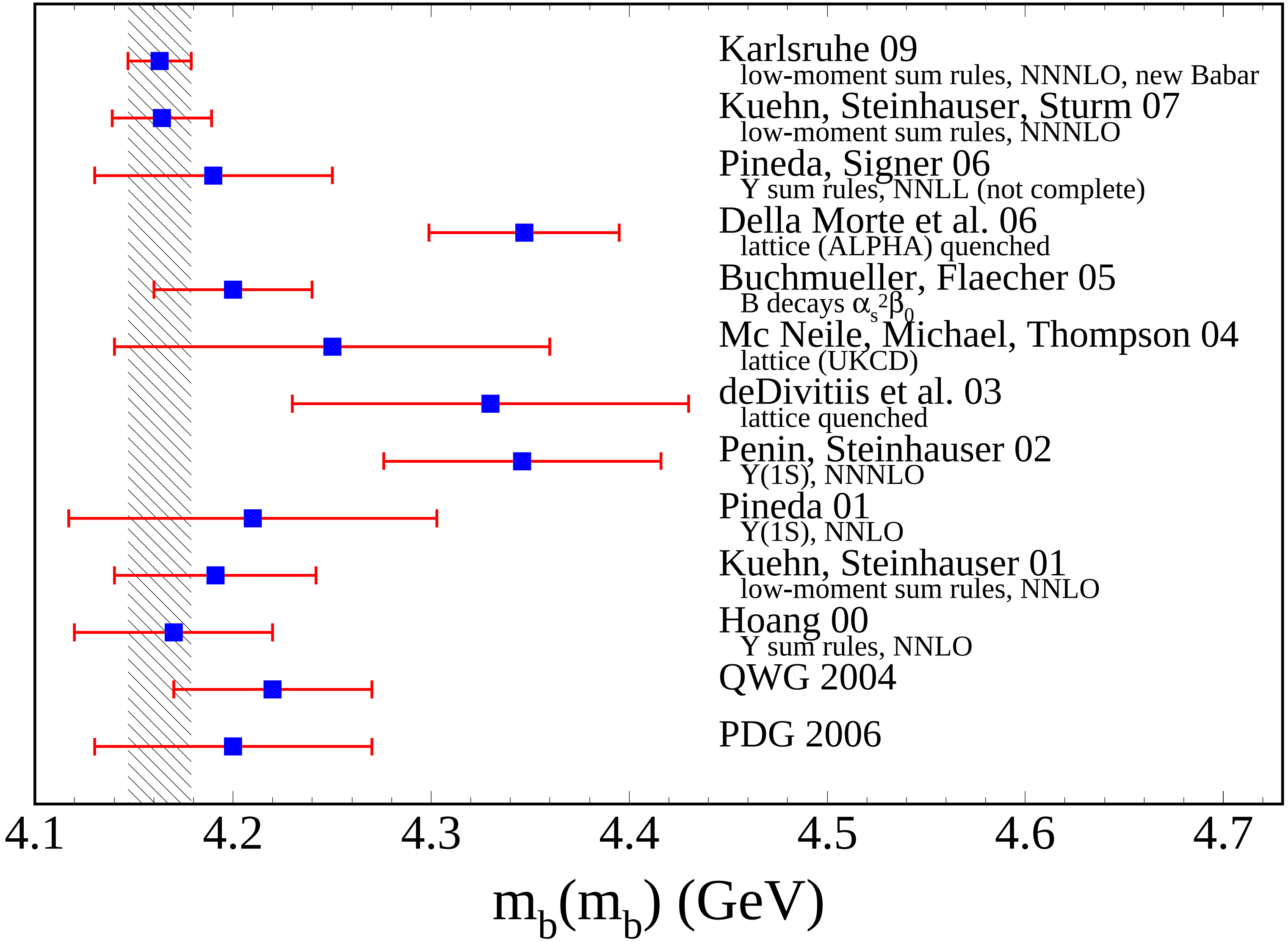
n	$m_b(10\,\text{GeV})$	exp	α_s	μ	total	$m_b(m_b)$
1	3597	14	7	2	16	4151
2	3610	10	12	3	16	4163
3	3619	8	14	6	18	4172
4	3631	6	15	20	26	4183

Consistency ($n = 1, 2, 3, 4$) and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);

(slight dependence on n could result from input into $\mathcal{M}^n_{\text{exp}}$)

- $m_\text{b}(10\,\text{GeV}) = 3610 \pm 16\,\text{MeV}$
- $m_\text{b}(m_\text{b}) = 4163 \pm 16\,\text{MeV}$

well consistent with KSS 2007


Karlsruhe 09
low-moment sum rules, NNNLO, new Babar
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Pineda, Signer 06
Υ sum rules, NNLL (not complete)
Della Morte et al. 06
lattice (ALPHA) quenched
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Mc Neile, Michael, Thompson 04
lattice (UKCD)
deDivitiis et al. 03
lattice quenched
Penin, Steinhauser 02
Υ(1S), NNNLO
Pineda 01
Υ(1S), NNLO
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
Hoang 00
Υ sum rules, NNLO
QWG 2004
PDG 2006
4.1
4.2
4.3
4.4
4.5
4.6
4.7
$m_b(m_b)$ (GeV)

α_s-dependence

$$m_c(3\ \text{GeV}) = \left(986 - \frac{\alpha_s - 0.1189}{0.002} \cdot 9 \pm 10\right)\ \text{MeV}$$

$$m_b(10\ \text{GeV}) = \left(3610 - \frac{\alpha_s - 0.1189}{0.002} \cdot 12 \pm 11\right)\ \text{MeV}$$

$$m_b(m_b) = \left(4163 - \frac{\alpha_s - 0.1189}{0.002} \cdot 12 \pm 11\right)\ \text{MeV}$$

$$m_b(M_Z) = \left(2835 - \frac{\alpha_s - 0.1189}{0.002} \cdot 27 \pm 8\right)\ \text{MeV}$$

$$m_b(161.8\ \text{GeV}) = \left(2703 - \frac{\alpha_s - 0.1189}{0.002} \cdot 28 \pm 8\right)\ \text{MeV}$$

lattice & pQCD

(HPQCD + Karlsruhe, Phys. Rev. D78, 054513)

lattice evaluation of pseudoscalar correlator

$\Rightarrow$ replace experimental moments by lattice simulation

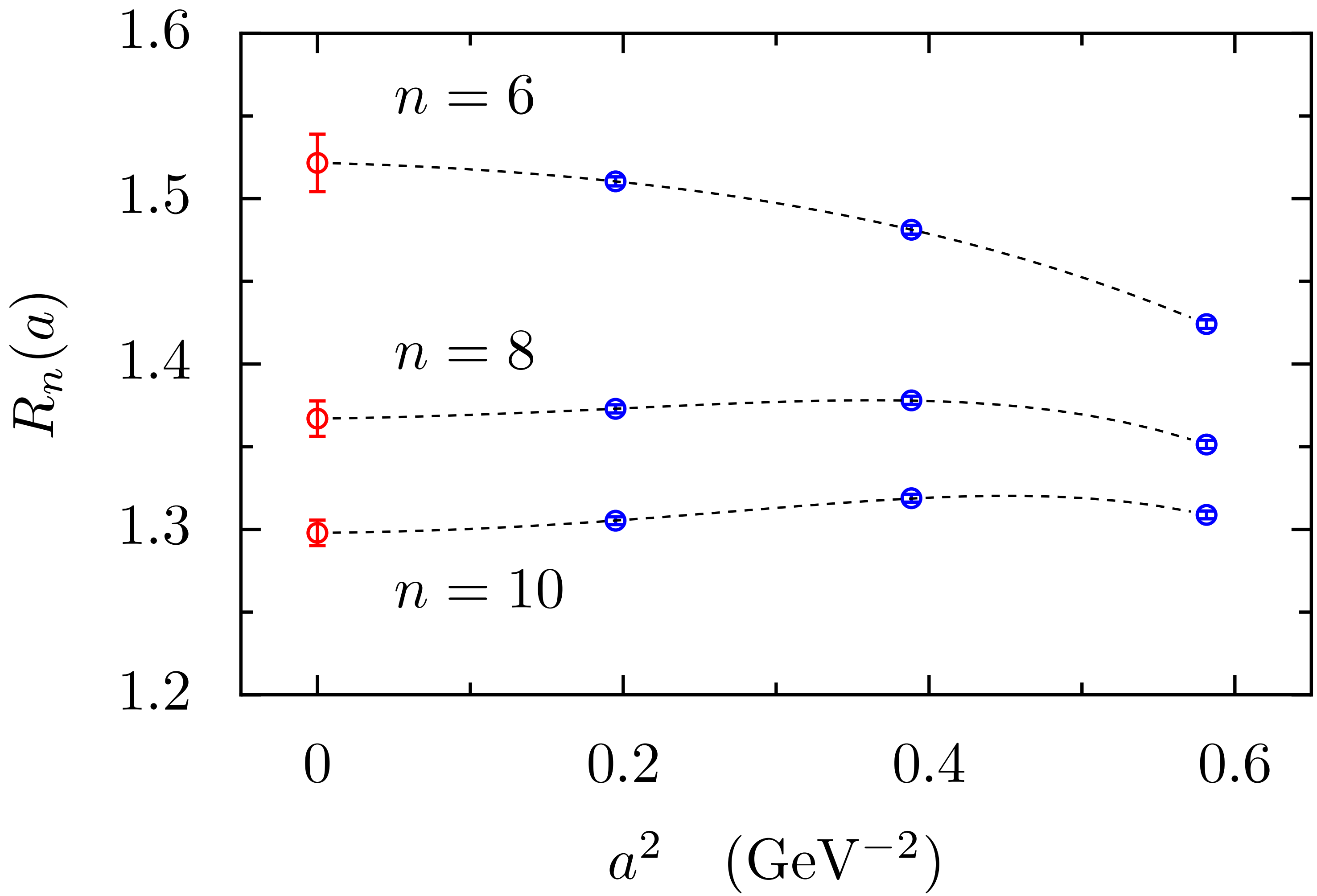
input: $M(\eta_c) \hat{=} m_c \,, \quad M(\Upsilon(1S)) - M(\Upsilon(2s)) \hat{=} \alpha_s$

pQCD for pseudoscalar correlator available:

"all" moments in $\mathcal{O}(\alpha_s^2)$

three lowest moments in $\mathcal{O}(\alpha_s^3)$.

lowest moment: dimensionless: $\sim \left(\bar{C}^{(0)} + \frac{\alpha_s}{\pi} \bar{C}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}^{(3)} + \ldots \right)$



Reduced moments R_n from lattice simulations with different lattice spacings a. The dashed lines show the functions used to fit the lattice results. These extrapolation functions were used to obtain the $a = 0$ results shown in the plot.

	R_6	R_8	R_{10}	R_{12}
a^2 extrapolation	1.3%	0.9%	0.7%	0.5%
pert'n theory	0.4	0.9	1.3	1.6
$\alpha_{\overline{\text{MS}}}$ uncertainty	0.3	0.6	1.0	1.3
gluon condensate	0.3	0.0	0.3	0.7
statistical errors	0.0	0.0	0.0	0.0
relative scale errors	0.4	0.4	0.3	0.3
overall scale errors	0.6	0.6	0.7	0.7
sea quarks	0.2	0.2	0.1	0.1
finite volume	0.1	0.1	0.3	0.4
Total	1.6%	1.6%	2.0%	2.4%

Sources of uncertainty in the determinations of $m_c(\mu = 3\,\text{GeV})$ from different reduced moments R_n of the pseudoscalar correlator. The uncertainties listed are percentages of the final result 0.984 (16) GeV.

$\Rightarrow$ $\alpha_s(3\text{GeV}) \Rightarrow \alpha_s(M_Z) = 0.1174(12)$

higher moments: $\sim m_c^2 \times \left(1 + ...\frac{\alpha_s}{\pi}...\right)$

$\Rightarrow$ $m_c(3\text{GeV}) = 986(10)$ MeV

to be compared with 986(13) MeV from e^+e^- !

SUMMARY

$m_c(3\,\text{GeV}) = 986(13)$ MeV	$e^+e^- +$ pQCD
$m_c(3\,\text{GeV}) = 986(10)$ MeV	lattice $+$ pQCD
$m_b(10\,\text{GeV}) = 3610(16)$ MeV $m_b(m_b) = 4163(16)$ MeV	$e^+e^- +$ pQCD