Charm and Bottom Quark Masses: An Update

Johann H. Kühn

Universität Karlsruhe (TH)
Forschungsuniversität • gegründet 1825

in collaboration with

K. Chetyrkin, Y. Kiyo, A. Maier, P. Maierhöfer, P. Marquard, A. Smirnov,

M. Steinhauser, C. Sturm and the HPQCD Collaboration

NPB 619 (2001) 588
EPJ C48 (2006) 107
NPB 778 (2008) 05413
PLB 669 (2008) 88
arXiv: 0907.2110
arXiv: 0907.2117
arXiv: 0907.2120

I. WHY precise masses?

B-decays:

$\Gamma(B \to X_{\text{u}} l \bar{\nu}) \sim G_{\text{F}}^2 \, m_{\text{b}}^5 \, |V_{\text{ub}}|^2$

$\Gamma(B \to X_{\text{c}} l \bar{\nu}) \sim G_{\text{F}}^2 \, m_{\text{b}}^5 \, f(m_{\text{c}}^2/m_{\text{b}}^2) \, |V_{\text{cb}}|^2$

$B \to X_{\text{s}} \gamma$

Υ-spectroscopy:

$$m\,(\Upsilon(1\text{s})) = 2M_{\text{b}} - \left(\tfrac{4}{3}\alpha_{\text{s}}\right)^2 \frac{M_{\text{b}}}{4} + \ldots$$

Higgs decay (ILC)

$$\Gamma(H \to b\bar{b}) = \frac{G_{\mathrm{F}}\, M_{\mathrm{H}}}{4\sqrt{2}\pi}\, m_{\mathrm{b}}^2(M_{\mathrm{H}})\, \tilde{R}$$

$$\tilde{R} = 1 + 5.6667 a_{\mathrm{s}} + 29.147 {a_{\mathrm{s}}}^2 + 41.758 {a_{\mathrm{s}}}^3 - 825.7 {a_{\mathrm{s}}}^4 \qquad \left(a_{\mathrm{s}} \equiv \frac{\alpha_{\mathrm{s}}}{\pi}\right)$$

${a_{\mathrm{s}}}^4$-term = 5-loop calculation [Baikov, Chetyrkin, JK]

Yukawa Unification

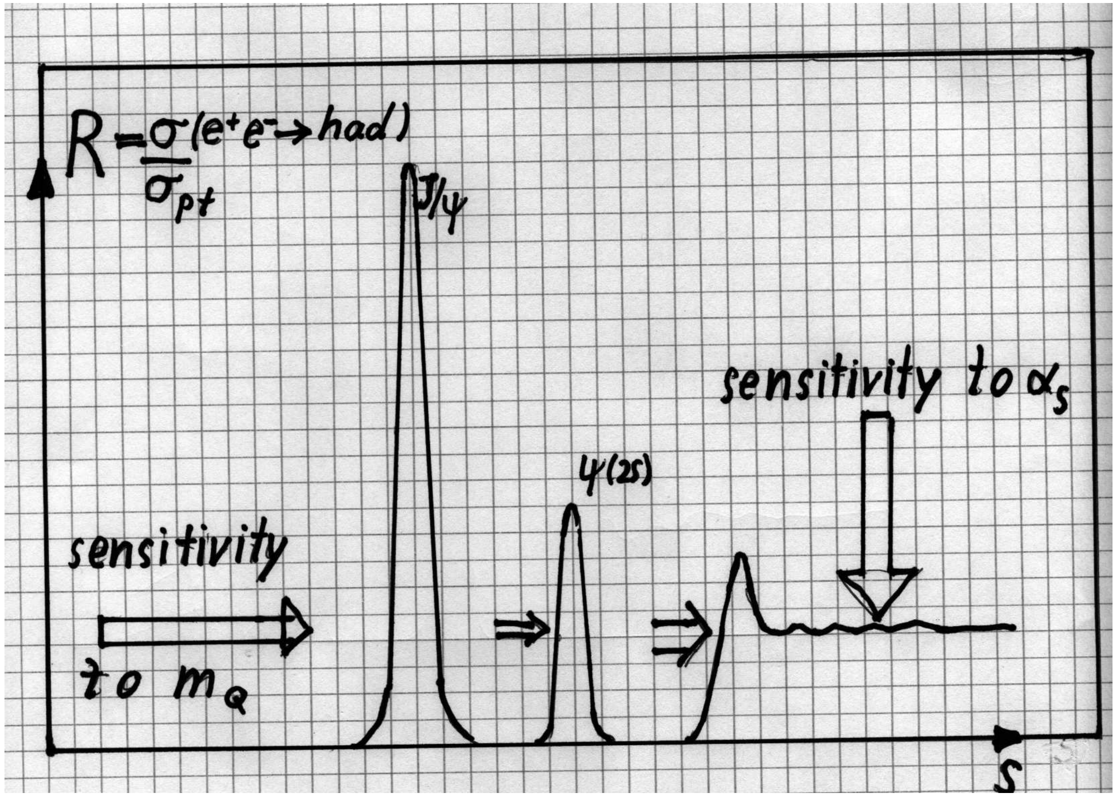
$\lambda_\tau \sim \lambda_{\mathrm{b}}$ or $\lambda_\tau \sim \lambda_{\mathrm{b}} \sim \lambda_{\mathrm{t}}$ at GUT scale

top-bottom $\to$ $m_{\mathrm{t}} / m_{\mathrm{b}} \sim$ ratio of vacuum expectation values

request $\frac{\delta m_{\mathrm{b}}}{m_{\mathrm{b}}} \sim \frac{\delta m_{\mathrm{t}}}{m_{\mathrm{t}}} \Rightarrow \delta m_{\mathrm{t}} \approx 1\,\mathrm{GeV} \Rightarrow \delta m_{\mathrm{b}} \approx 25\,\mathrm{MeV}$

II. m_Q from SVZ Sum Rules, Moments and Tadpoles

Main Idea (SVZ)



Some definitions:

$$\left(-q^2 g_{\mu\nu} + q_\mu\, q_\nu\right) \Pi(q^2) \;\; \equiv \;\; i \int \mathrm{d}x\, e^{iqx} \langle T j_\mu(x) j_\nu(0) \rangle$$

with the electromagnetic current j_μ.

$$R(s) \;\; = \;\; 12\pi\, \mathrm{Im}\left[\Pi(q^2 = s + i\epsilon)\right]$$

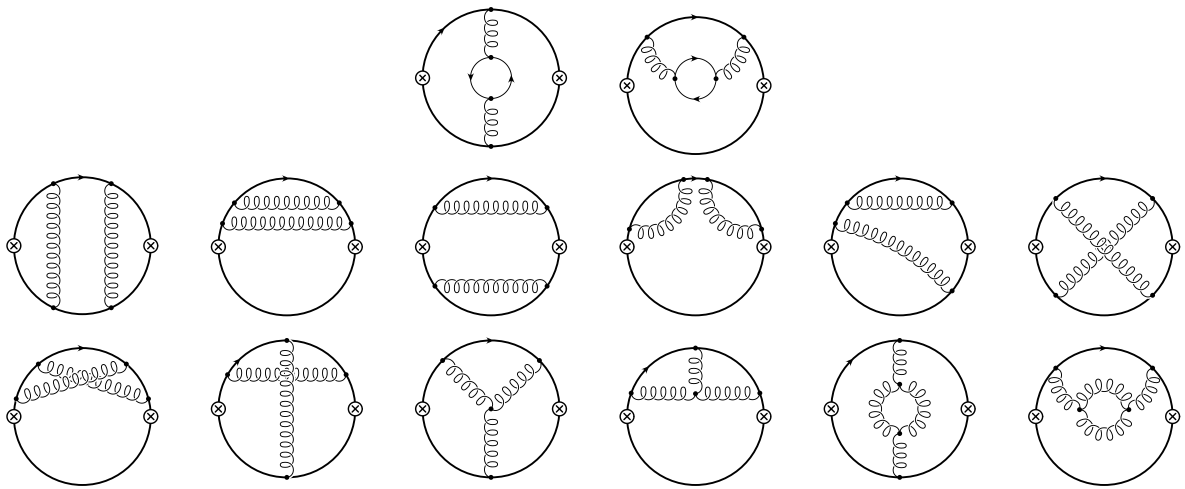
Taylor expansion: $\Pi_Q(q^2) \;\; = \;\; Q_Q^2 \, \frac{3}{16\pi^2} \sum_{n \geq 0} \bar{C}_n \, z^n$

with $z = q^2/(4m_Q^2)$ and $m_Q = m_Q(\mu)$ the $\overline{\mathrm{MS}}$ mass.

$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi} \bar{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}_n^{(3)} + \ldots$$

Analysis in NNLO

Coefficients $\bar{C}_n$ from three-loop one-scale tadpole amplitudes with "arbitrary" power of propagators;



- **FORM program MATAD**

  Coefficients $\bar{C}_n$ up to $n = 8$ known analytically in order α_s^2

  (also for axial, scalar and pseudoscalar correlators)

  (Chetyrkin, JK, Steinhauser, 1996)

- up to $n = 30$ for vector correlator

  (Boughezal, Czakon, Schutzmeier 2007)

- up to $n = 30$ for vector, axial, scalar and pseudoscalar correlators

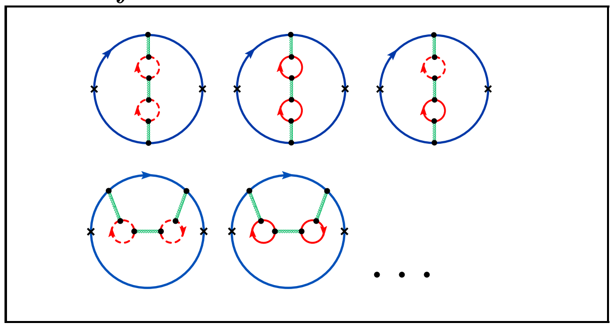
  (A. Maier, P. Maierhöfer, P. Marquard, 2007)

Analysis in N^{3}LO

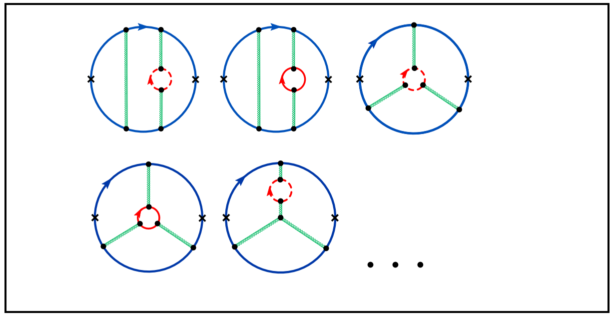
Algebraic reduction to 13 master integrals (Laporta algorithm);

numerical and analytical evaluation of master integrals

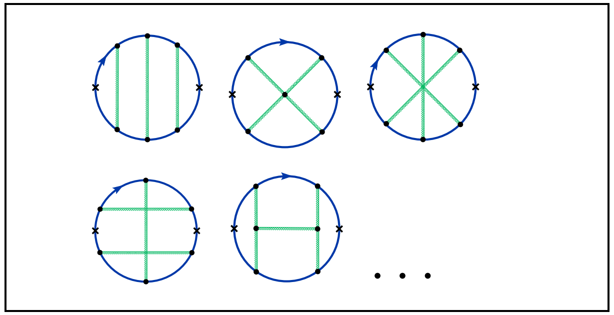
n_f^2-contributions



n_f^1-contributions



n_f^0-contributions



 : heavy quarks,  : light quarks,

n_f: number of active quarks

$\Rightarrow$ About 700 Feynman-diagrams

$\bar{C}_0$ and $\bar{C}_1$ in order α_s^3 (four loops!) (2006)

➪ Reduction to master integrals through Laporta algorithm

(Chetyrkin, JK, Sturm; confirmed by Boughezal, Czakon, Schutzmeier)

➪ Evaluation of master integrals numerically through difference equations

(30 digits) or Padé method or analytically in terms of transcendentals.

All master integrals known analytically and double checked.

(Schröder + Vuorinen, Chetyrkin et al., Schröder + Steinhauser, Laporta, Broadhurst, Kniehl et al.)

Similar approach: four-loop ρ parameter

(K. G. Chetyrkin, M. Faisst, JK, P. Maierhöfer, C. Sturm)

New developments

- $\bar{C}_2$: improved Laporta algorithm
  (Maier, Maierhöfer, Marquard, 2008)

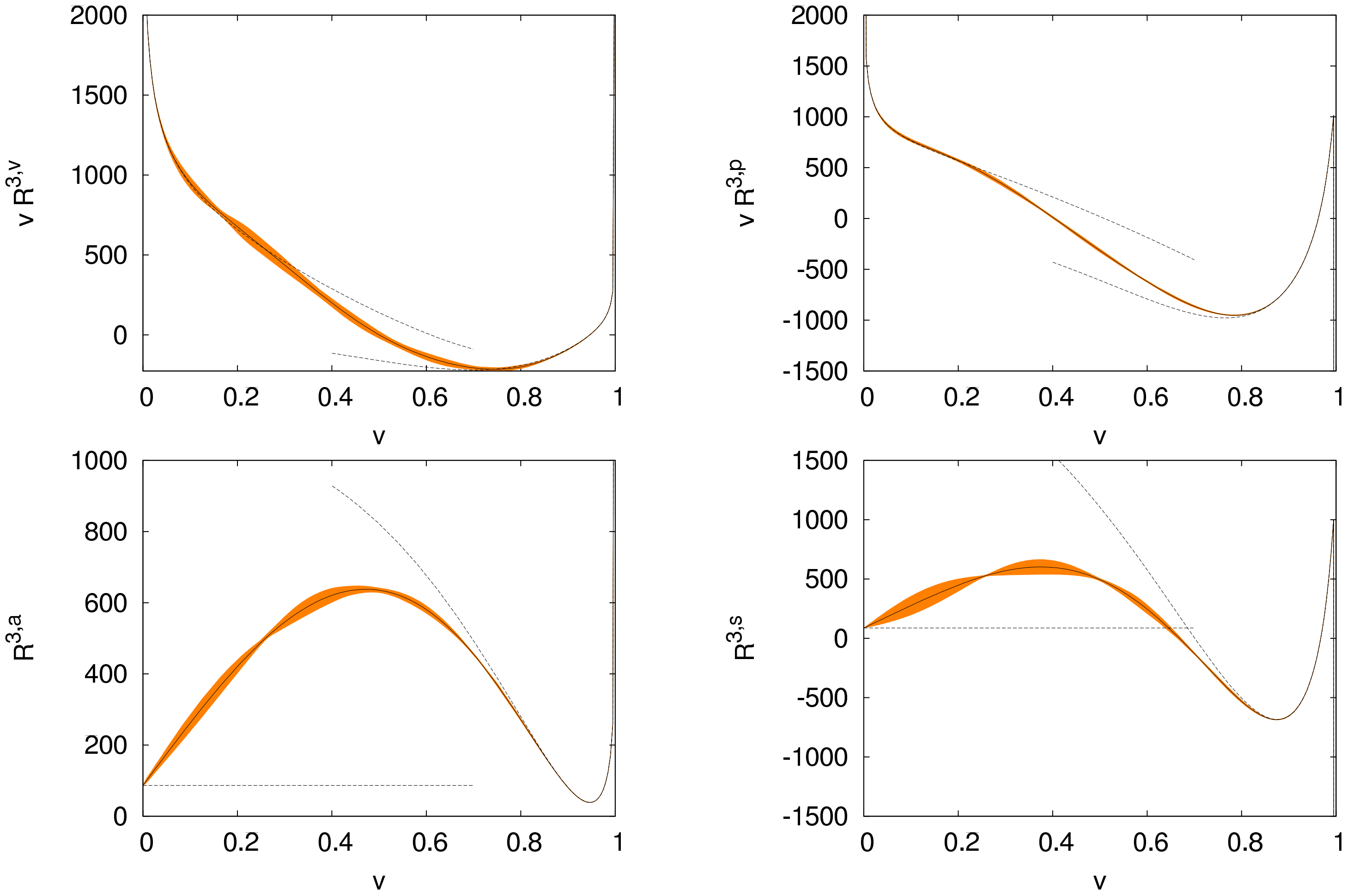
- $\bar{C}_3$ (all correlators): special treatment of self energy insertions
  (Maier, Maierhöfer, Marquard, A. Smirnov, 2008)

- $\bar{C}_4 - \bar{C}_{10}$: extension to higher moments by Padé method, using analytic information from low energy ($q^2 = 0$), threshold ($q^2 = 4m^2$), high energy ($q^2 = -\infty$) (Kiyo, Maier, Maierhöfer, Marquard, 2009)

current	low energy	threshold	high energy	total
vector	3	2	2+2	9
axial-vector	3	0	2+2	7
scalar	3	0	2+2	7
pseudo-scalar	4	2	2+2	10

- $10^3 - 10^4$ "good approximations"; analysis in "on shell" scheme

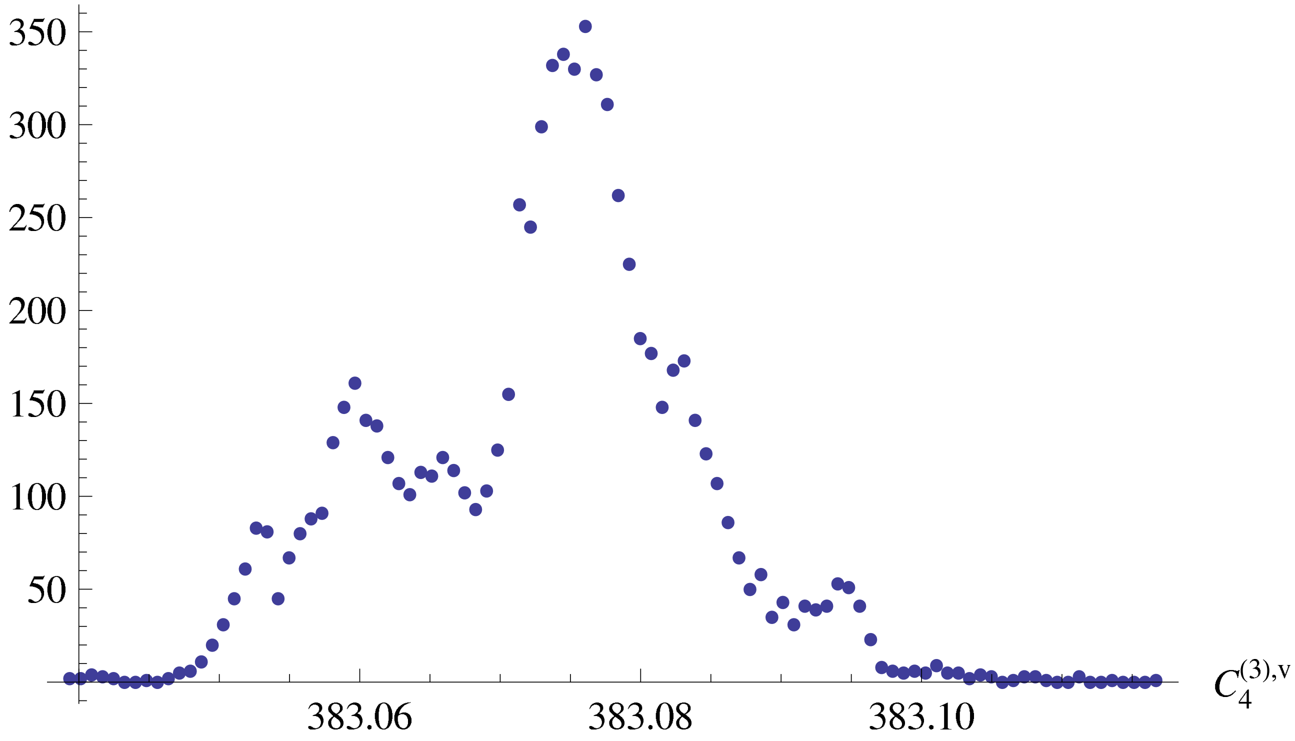
Results



Imaginary parts of the four loop contributions to the vector, pseudoscalar, axialvector and scalar correlators above the charm threshold ($v = \sqrt{1 - 1/z}$). Mean values, 3σ bands, threshold and high energy expansions (dashed).

Extraction of approximate moments

e. g. $C_4^{(3),v}$



	$n_l = 3$	$n_l = 4$	$n_l = 5$
$C_1^{(3),v}$	366.1748	308.0188	252.8399
$C_2^{(3),v}$	381.5091	330.5835	282.0129
$C_3^{(3),v}$	385.2331	338.7065	294.2224
$C_4^{(3),v}$	383.073(11)	339.913(10)	298.576(9)
$C_5^{(3),v}$	378.688(32)	338.233(32)	299.433(27)
$C_6^{(3),v}$	373.536(61)	335.320(63)	298.622(54)
$C_7^{(3),v}$	368.23(9)	331.90(10)	296.99(9)
$C_8^{(3),v}$	363.03(13)	328.33(14)	294.94(12)
$C_9^{(3),v}$	358.06(17)	324.78(18)	292.72(16)
$C_{10}^{(3),v}$	353.35(20)	321.31(22)	290.44(19)
$K_0^{(3),v}$	17(11)	17(29)	16(10)
$D_2^{(3),v}$	2.0(42)	1.2(83)	1.4(21)

➪ translation to $\overline{\text{MS}}$ scheme

	$\bar{C}_3^{(3),v}$	$\bar{C}_4^{(3),v}$	$\bar{C}_5^{(3),v}$	$\bar{C}_6^{(3),v}$	$\bar{C}_7^{(3),v}$	$K_0^{(3),v}$
Hoang *et al.*	-3.28 ± 0.57	-4.2 ± 1.2	-5.0 ± 1.7	-5.3 ± 2.0	-5.2 ± 2.3	-10 ± 11
KMMM	-2.840 (exact)	$-3.349(11)$	$-3.737(32)$	$-3.735(61)$	$-3.39(10)$	17(11)

similar results for axial, scalar and pseudoscalar

n	1	2	3	4
charm	−5.6404	−3.4937	−2.8395	−3.349(11)
lower ‖ upper limits	—	−6.0 ‖ 7.0	−6.0 ‖ 5.2	−6.0 ‖ 3.1
bottom	−7.7624	−2.6438	−1.1745	−1.386(10)
lower ‖ upper limits	—	−8.0 ‖ 9.5	−8.0 ‖ 8.3	−8.0 ‖ 7.4

⇒ reliable predictions for the lowest four moments

⇒ consistency check:

different role of threshold vs. continuum recall: $\Pi_Q(q^2) \;=\; Q_Q^2\,\frac{3}{16\pi^2}\sum_{n\geq 0}\bar{C}_n\,z^n$

$\bar{C}_n$ depend on the charm quark mass through $l_{mc} \equiv \ln(m_c^2(\mu)/\mu^2)$

$$\begin{aligned}\bar{C}_n \;=\; & \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}\left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{mc}\right) \\ & + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{mc} + \bar{C}_n^{(22)} l_{mc}^2\right) \\ & + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3\left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{mc} + \bar{C}_n^{(32)} l_{mc}^2 + \bar{C}_n^{(33)} l_{ms}^3\right)\end{aligned}$$

n	$\bar{C}_n^{(0)}$	$\bar{C}_n^{(10)}$	$\bar{C}_n^{(11)}$	$\bar{C}_n^{(20)}$	$\bar{C}_n^{(21)}$	$\bar{C}_n^{(22)}$	$\bar{C}_n^{(30)}$	$\bar{C}_n^{(31)}$	$\bar{C}_n^{(32)}$	$\bar{C}_n^{(33)}$
1	1.0667	2.5547	2.1333	2.4967	3.3130	− 0.0889	−5.6404	4.0669	0.9590	0.0642
2	0.4571	1.1096	1.8286	2.7770	5.1489	1.7524	−3.4937	6.7216	6.4916	−0.0974
3	0.2709	0.5194	1.6254	1.6388	4.7207	3.1831	−2.8395	7.5736	13.1654	1.9452
4	0.1847	0.2031	1.4776	0.7956	3.6440	4.3713	−3.349(11)	4.9487	17.4612	5.5856

previous estimate $-6 < C_n^{(30)} < 6$ for $n = 2, 3, 4$

confirmed by exact calculation ($n = 2, 3$) and Padé estimate ($n = 4$)

Relation to measurements

$$\mathcal{M}_n^{\text{th}} \equiv \frac{12\pi^2}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_c(q^2)\bigg|_{q^2=0} = \frac{9}{4}Q_c^2\left(\frac{1}{4m_c^2}\right)^n \bar{C}_n$$

Perturbation theory: $\bar{C}_n$ is function of α_s and $\ln\frac{m_c^2}{\mu^2}$

dispersion relation:

$$\Pi_c(q^2) = \frac{q^2}{12\pi^2}\int \mathrm{d}s\,\frac{R_c(s)}{s(s-q^2)} + \text{subtraction}$$

➪ $\mathcal{M}_n^{\text{exp}} = \int \frac{\mathrm{d}s}{s^{n+1}} R_c(s)$

constraint: $\mathcal{M}_n^{\text{exp}} = \mathcal{M}_n^{\text{th}}$

➪ m_c

Ingredients (charm)

experiment:

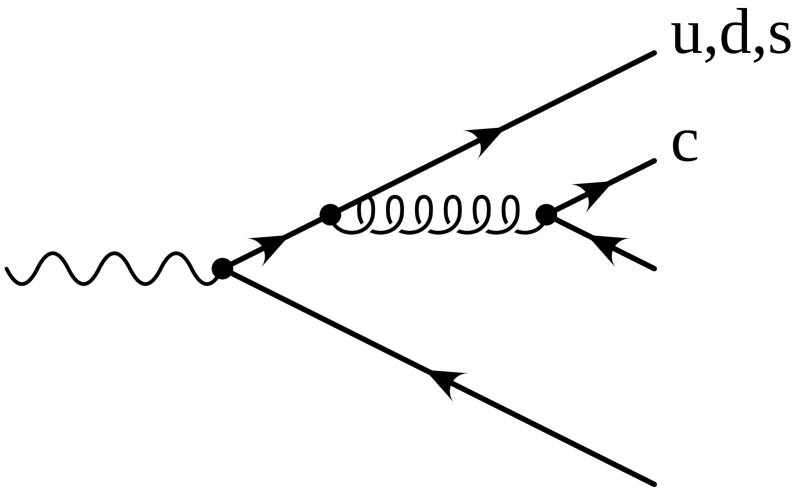
- $\Gamma_e(J/\psi, \psi')$ from BES & CLEO & Babar
- $\psi(3770)$ and $R(s)$ from BES
- $\alpha_s = 0.1187 \pm 0.0020$

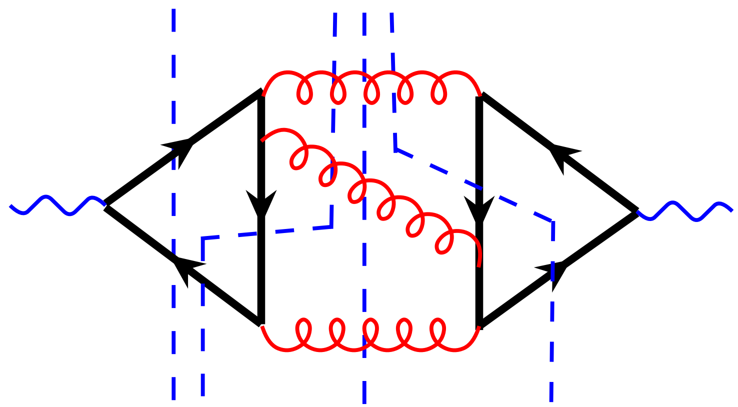
theory:

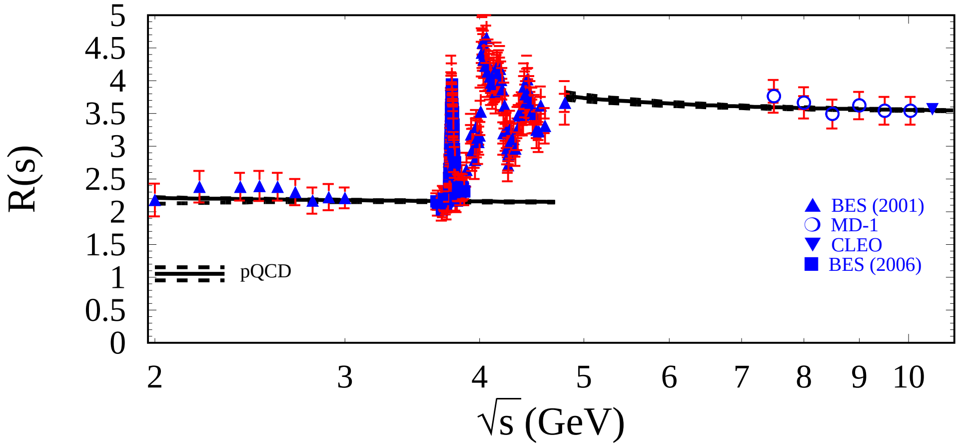
- N^3LO for $n = 1, 2, 3$
- N^3LO - estimate for $n = 4$
- include condensates

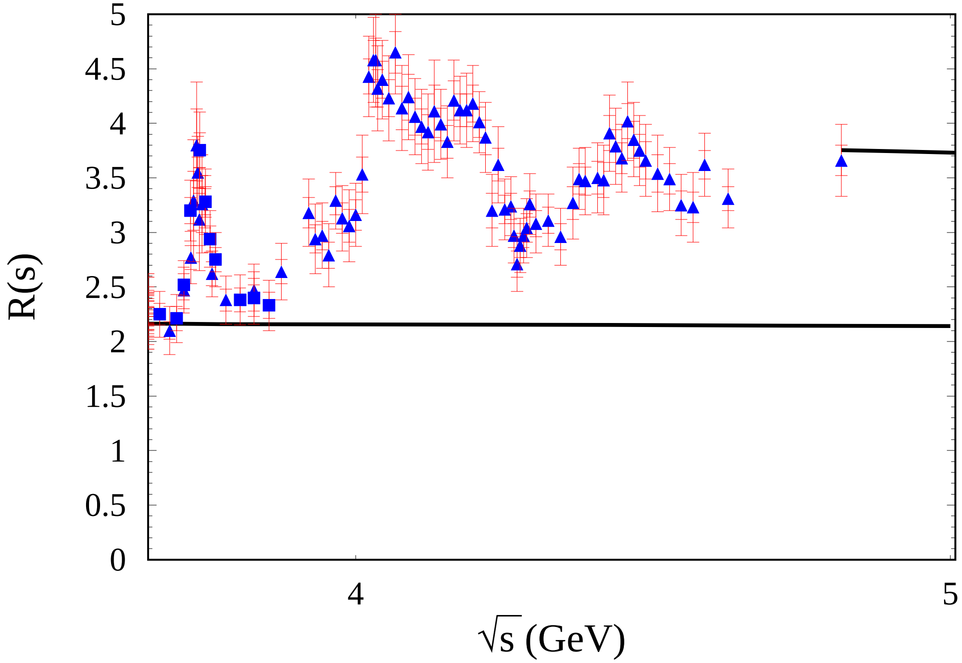
$$\delta\mathcal{M}_n^{\rm np} \;=\; \frac{12\pi^2 Q_c^2}{(4m_c^2)^{(n+2)}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n \left(1 + \frac{\alpha_s}{\pi}\bar{b}_n\right)$$

- estimate of non-perturbative terms (oscillations, based on Shifman)
- careful extrapolation of R_{uds}
- careful definition of R_c


u,d,s
c




R(s)
√s (GeV)
pQCD
BES (2001)
MD-1
CLEO
BES (2006)


R(s)
√s (GeV)

Contributions from

- narrow resonances: $R = \frac{9\,\Pi\,M_R\,\Gamma_e}{\alpha^2(s)}\,\delta(s - M_R^2)$
- threshold region $(2\,m_D - 4.8\,\text{GeV})$
- perturbative continuum $(E \geq 4.8\,\text{GeV})$

n	$\mathcal{M}_n^{\text{res}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{np}}$ $\times 10^{(n-1)}$
1	0.1201(25)	0.0318(15)	0.0646(11)	0.2166(31)	−0.0001(2)
2	0.1176(25)	0.0178(8)	0.0144(3)	0.1497(27)	0.0000(0)
3	0.1169(26)	0.0101(5)	0.0042(1)	0.1312(27)	0.0007(14)
4	0.1177(27)	0.0058(3)	0.0014(0)	0.1249(27)	0.0027(54)

Different relative importance of resonances vs. continuum for $n = 1, 2, 3, 4$.

Results (m_c)

arXiv: 0907:2110

n	$m_c(3\,\text{GeV})$	exp	α_s	μ	np	total
1	986	9	9	2	1	13
2	976	6	14	5	0	16
3	978	5	15	7	2	17
4	1004	3	9	31	7	33

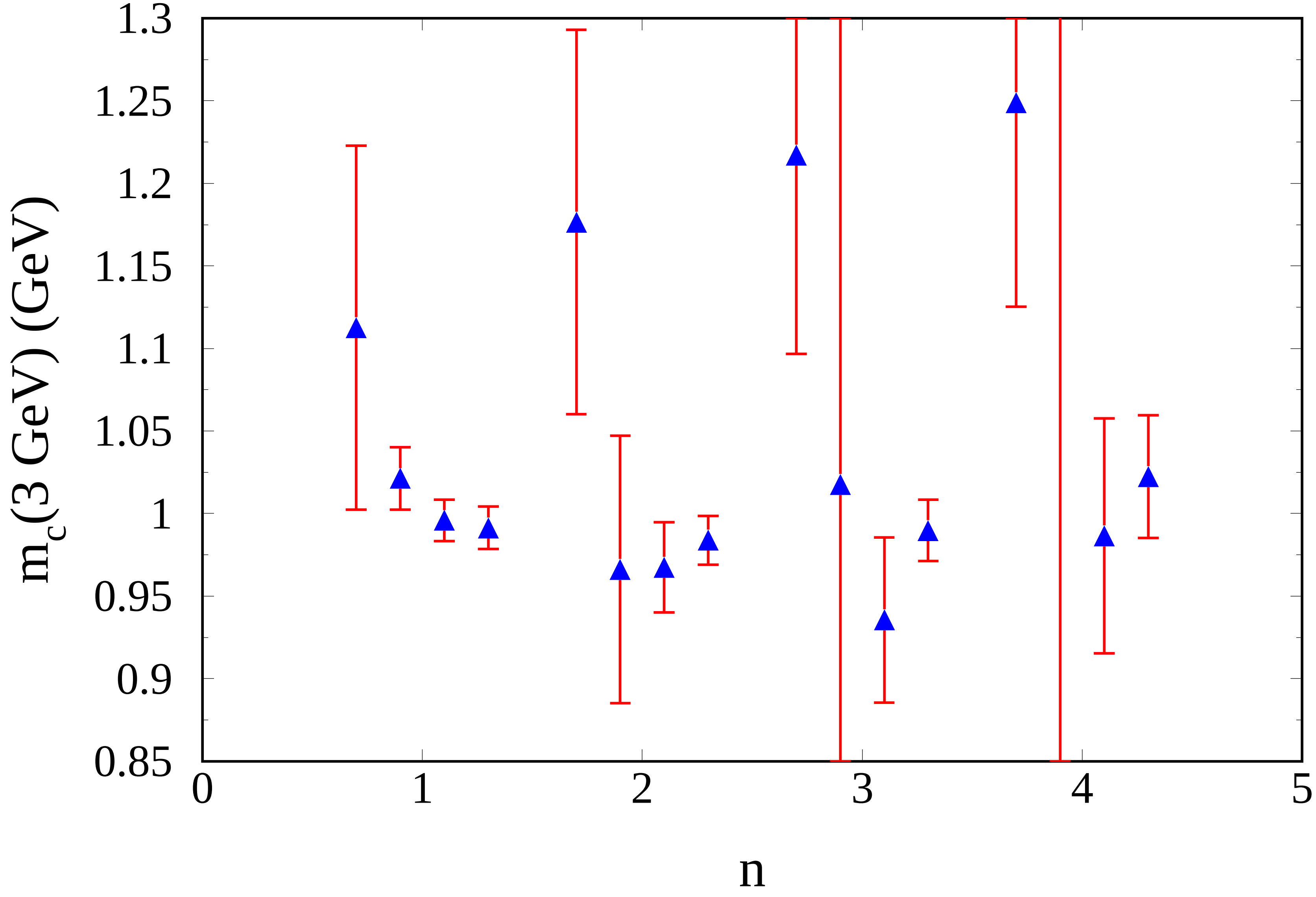
Remarkable consistency between $n = 1, 2, 3, 4$

and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);

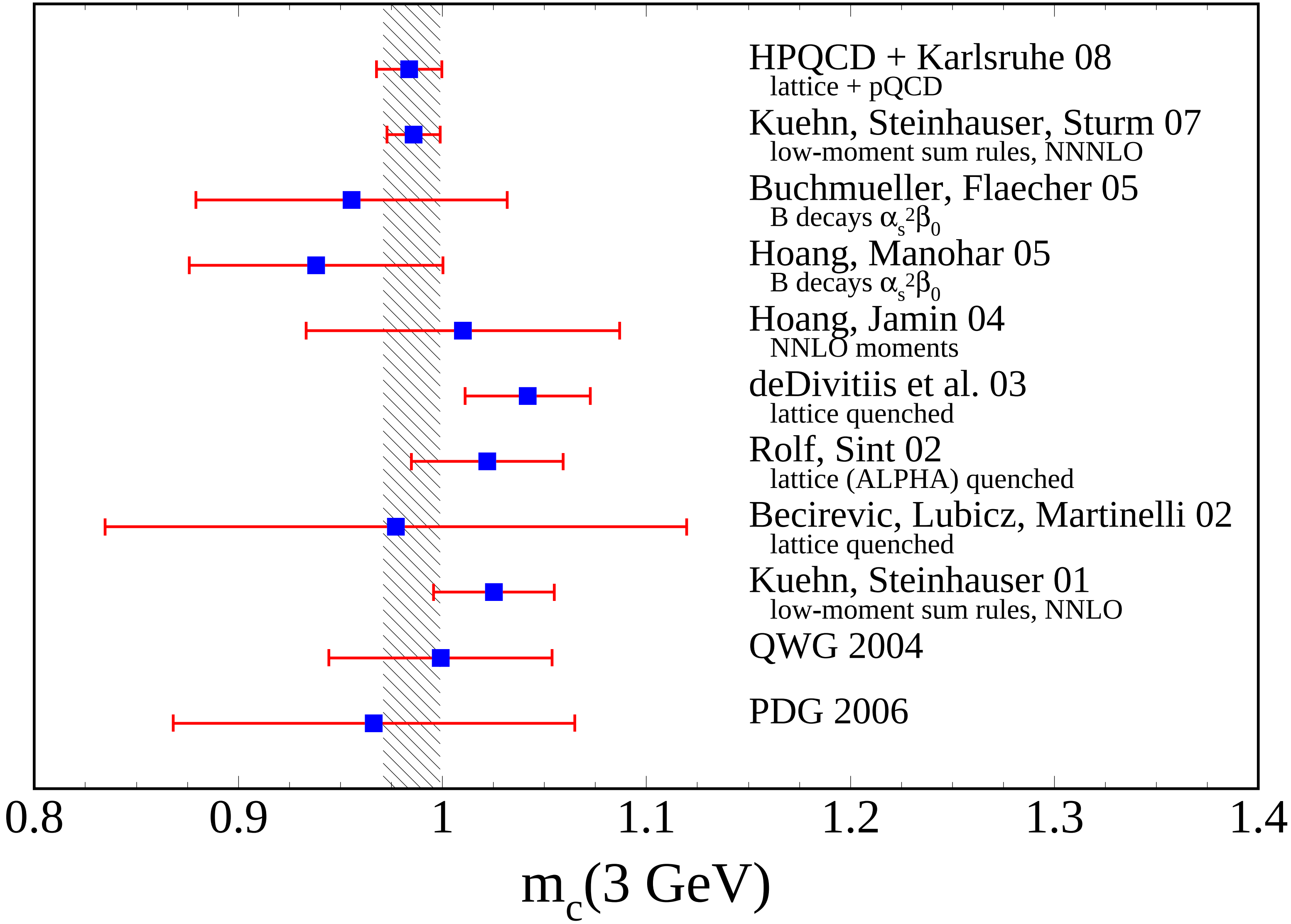
prefered scale: $\mu = 3\,\text{GeV}$,

conversion to $m_c(m_c)$:

- $m_c(3\,\text{GeV}) = 986 \pm 13\,\text{MeV}$
- $m_c(m_c) = 1279 \pm 13\,\text{MeV}$



1.3
1.25
1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
m_c(3 GeV) (GeV)
0
1
2
3
4
5
n


HPQCD + Karlsruhe 08
lattice + pQCD
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Hoang, Manohar 05
B decays $\alpha_s^2\beta_0$
Hoang, Jamin 04
NNLO moments
deDivitiis et al. 03
lattice quenched
Rolf, Sint 02
lattice (ALPHA) quenched
Becirevic, Lubicz, Martinelli 02
lattice quenched
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
QWG 2004
PDG 2006
0.8
0.9
1
1.1
1.2
1.3
1.4
m_c(3 GeV)

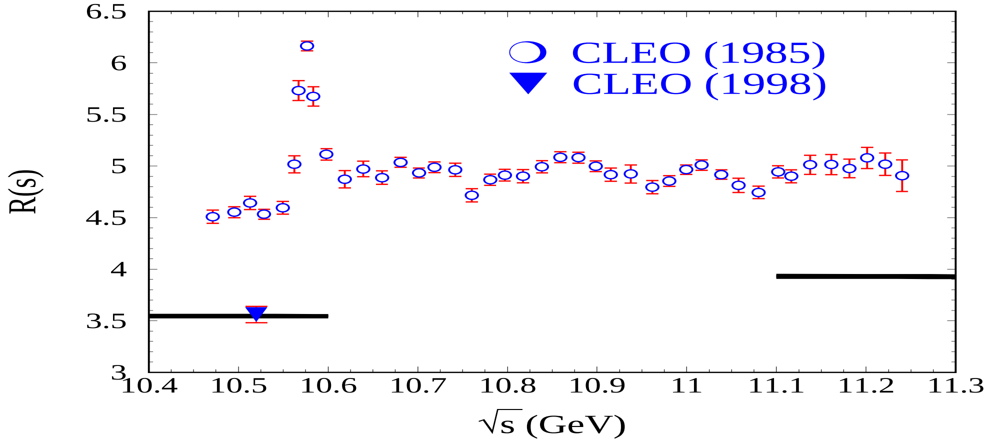
Experimental Ingredients for m_b

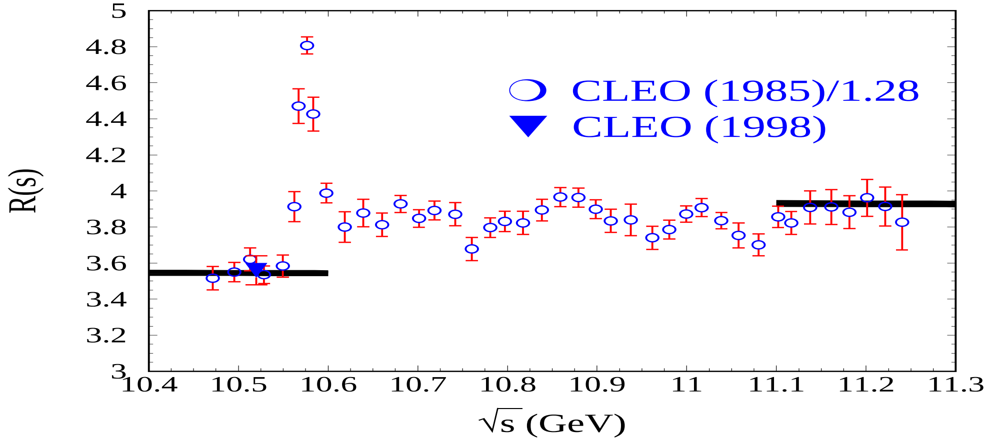
Contributions from

- narrow resonances ($\Upsilon(1S) - \Upsilon(4S)$)
- threshold region (10.618 GeV − 11.2 GeV)
- perturbative continuum ($E \geq 11.2$ GeV)
- different relative importance of resonances vs. contimuum for $n = 1, 2, 3, 4$

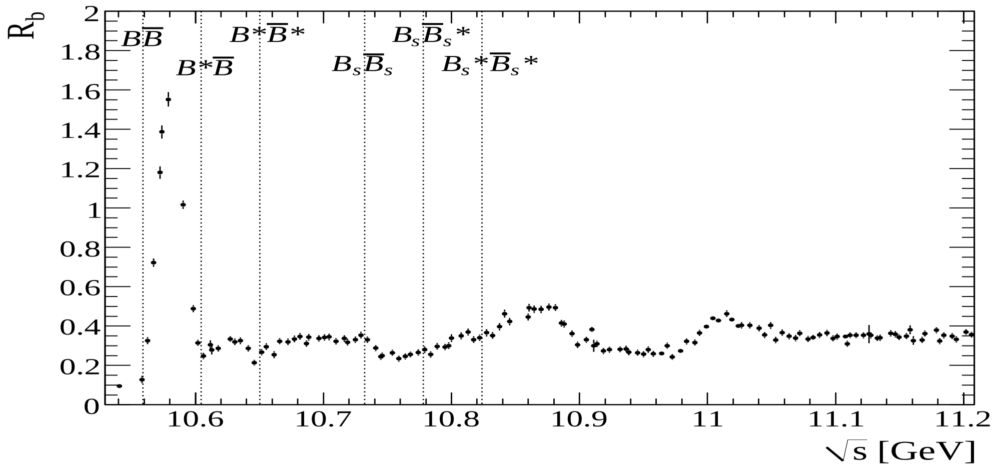
n	$\mathcal{M}_n^{\text{res},(1S-4S)}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(2n+1)}$
1	1.394(23)	0.287(12)	2.911(18)	4.592(31)
2	1.459(23)	0.240(10)	1.173(11)	2.872(28)
3	1.538(24)	0.200(8)	0.624(7)	2.362(26)
4	1.630(25)	0.168(7)	0.372(5)	2.170(26)

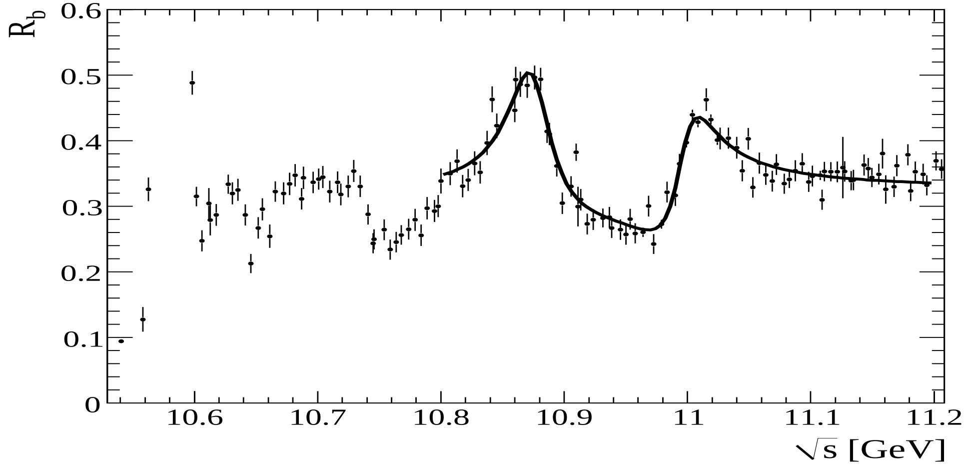
Improvements Based on Recent Babar Results



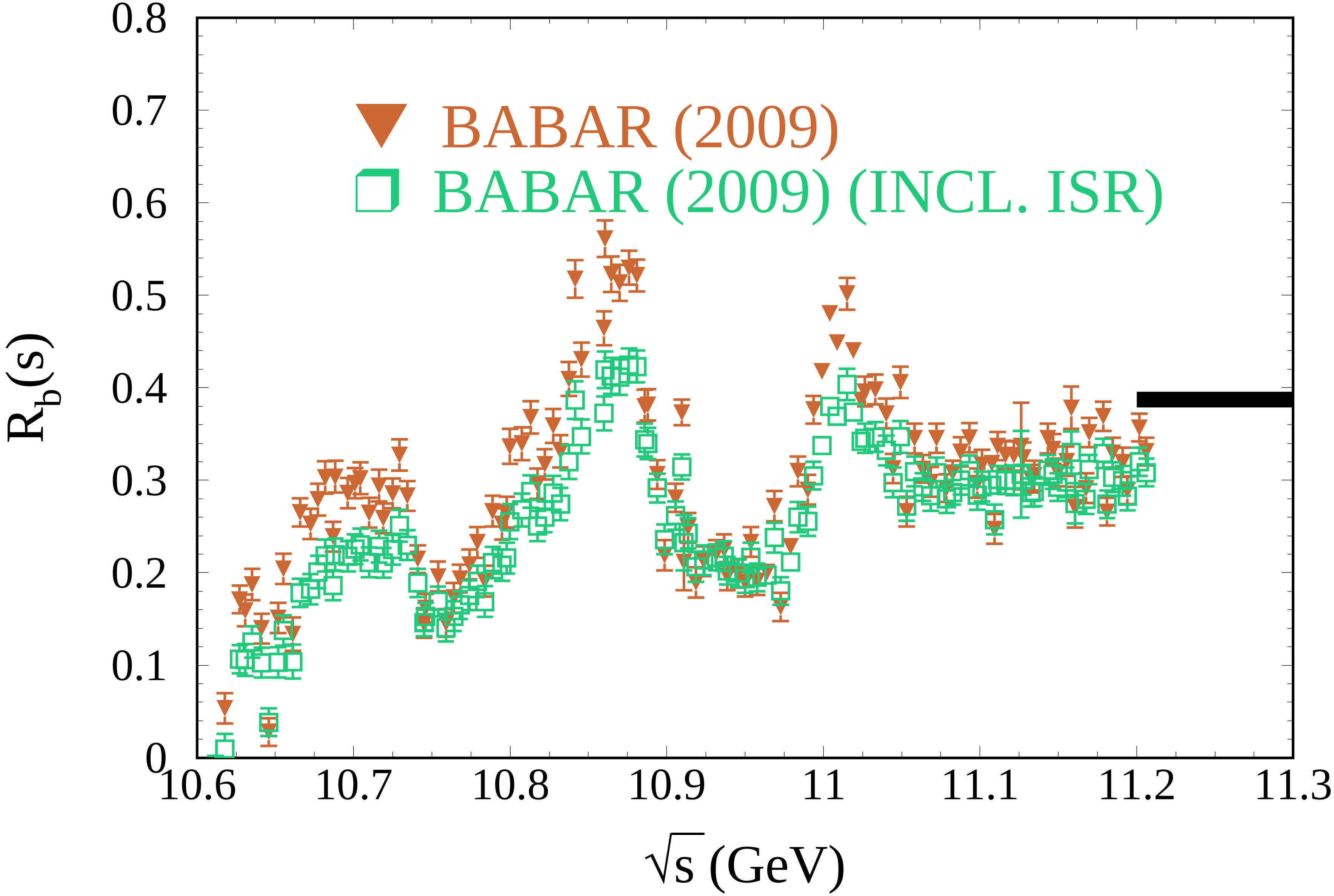


uncertainties after "renormalization" estimated to be 10%
⇒ dominant contribution to error

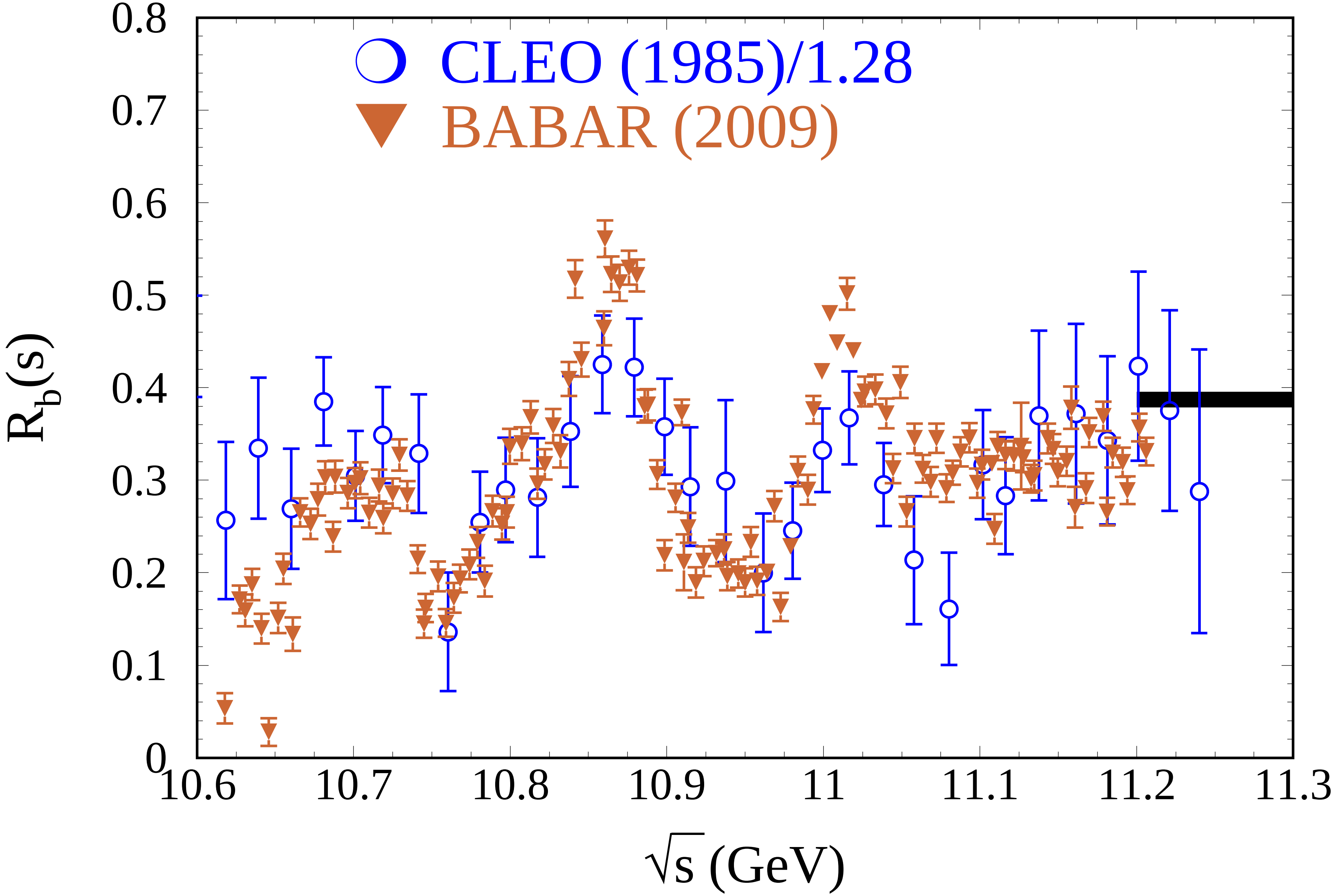




2% systematic experimental error;
Deconvolute ISR and apply radiative corrections



BABAR (2009)
BABAR (2009) (INCL. ISR)
$R_b(s)$
$\sqrt{s}$ (GeV)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
10.6
10.7
10.8
10.9
11
11.1
11.2
11.3



CLEO (1985)/1.28
BABAR (2009)
$R_b(s)$
$\sqrt{s}$ (GeV)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
10.6
10.7
10.8
10.9
11
11.1
11.2
11.3

n	$\mathcal{M}_n^{\text{threshold}}$ $\times 10^{(2n+1)}$ CLEO	$\mathcal{M}_n^{\text{threshold}}$ $\times 10^{(2n+1)}$ BABAR
1	0.296(32)	0.287(12)
2	0.249(27)	0.240(10)
3	0.209(22)	0.200(8)
4	0.175(19)	0.168(7)

- consistency between BABAR and CLEO
- reduction of experimental error in this region by factor 3, total error by factor 2/3

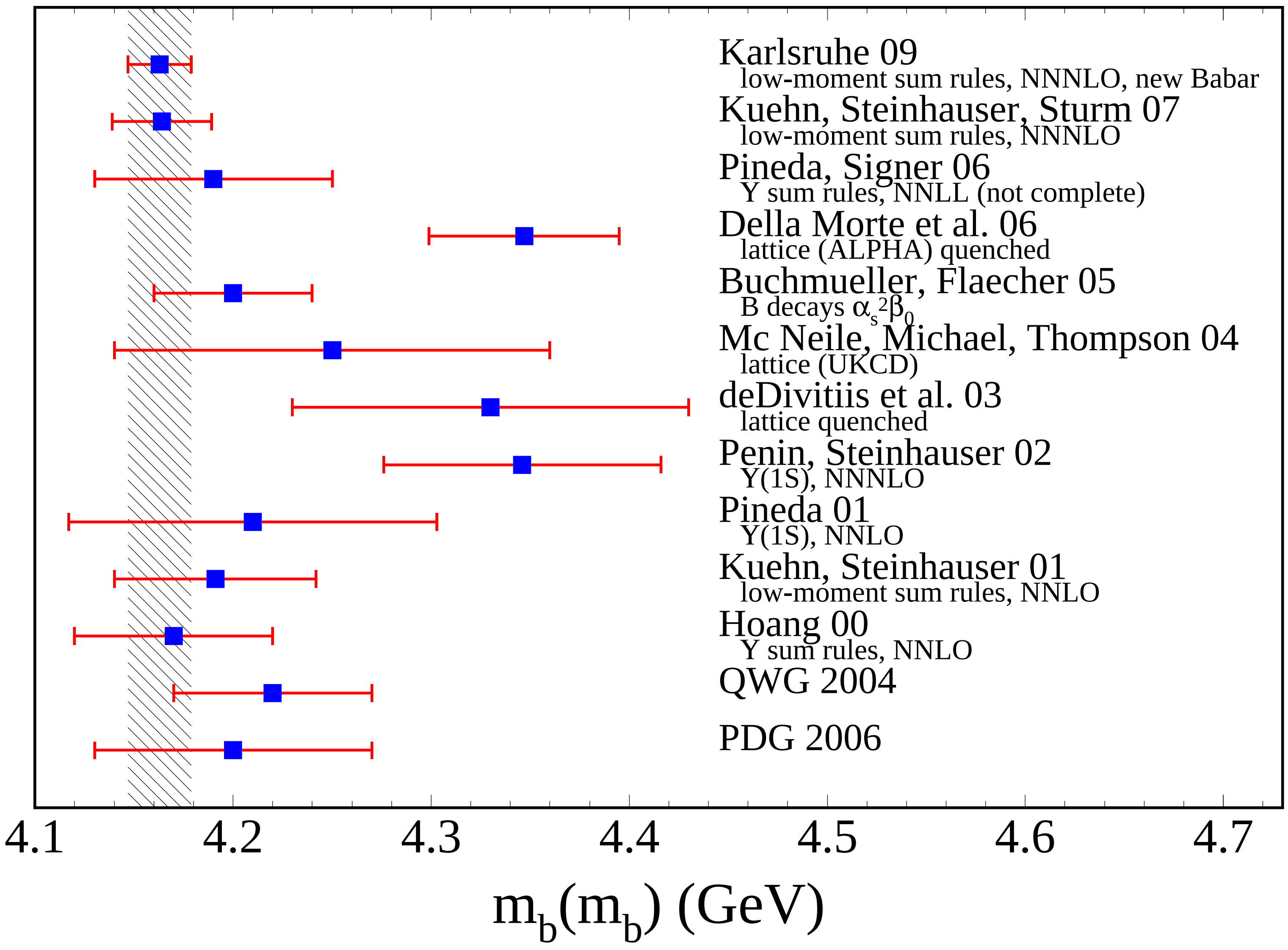
n	$m_b(10\,\text{GeV})$	exp	α_s	μ	total	$m_b(m_b)$
1	3597	14	7	2	16	4151
2	3610	10	12	3	16	4163
3	3619	8	14	6	18	4172
4	3631	6	15	20	26	4183

Consistency ($n = 1, 2, 3, 4$) and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);

(slight dependence on n could result from input into $\mathcal{M}^n_{\text{exp}}$)

- $m_{\text{b}}(10\,\text{GeV}) = 3610 \pm 16\,\text{MeV}$
- $m_{\text{b}}(m_{\text{b}}) = 4163 \pm 16\,\text{MeV}$

well consistent with KSS 2007



Karlsruhe 09
low-moment sum rules, NNNLO, new Babar
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Pineda, Signer 06
Y sum rules, NNLL (not complete)
Della Morte et al. 06
lattice (ALPHA) quenched
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Mc Neile, Michael, Thompson 04
lattice (UKCD)
deDivitiis et al. 03
lattice quenched
Penin, Steinhauser 02
Y(1S), NNNLO
Pineda 01
Y(1S), NNLO
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
Hoang 00
Y sum rules, NNLO
QWG 2004
PDG 2006
4.1
4.2
4.3
4.4
4.5
4.6
4.7
$m_b(m_b)$ (GeV)

lattice & pQCD

(HPQCD + Karlsruhe, Phys. Rev. D78, 054513)

lattice evaluation of pseudoscalar correlator

$\Rightarrow$ replace experimental moments by lattice simulation

input: $M(\eta_c) \widehat{=} m_c\,, \quad M(\Upsilon(1S)) - M(\Upsilon(2s)) \widehat{=} \alpha_s$

pQCD for pseudoscalar correlator available:

"all" moments in $\mathcal{O}(\alpha_s^2)$

three lowest moments in $\mathcal{O}(\alpha_s^3)$.

lowest moment: dimensionless: $\sim \left(\bar{C}^{(0)} + \frac{\alpha_s}{\pi} \bar{C}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}^{(3)} + ... \right)$

$\Rightarrow$ $\alpha_s(3\text{GeV}) \Rightarrow \alpha_s(M_Z) = 0.1174(12)$

higher moments: $\sim m_c^2 \times \left(1 + ... \frac{\alpha_s}{\pi} ...\right)$

$\Rightarrow$ $m_c(3\text{GeV}) = 986(10)$ MeV

to be compared with 986(13) MeV from e^+e^- !

$$m_c(3\ \text{GeV}) = \left(986 - \frac{\alpha_s - 0.1189}{0.002} \cdot 9 \pm 10\right)\ \text{MeV}$$

$$m_b(10\ \text{GeV}) = \left(3610 - \frac{\alpha_s - 0.1189}{0.002} \cdot 12 \pm 11\right)\ \text{MeV}$$

$$m_b(m_b) = \left(4163 - \frac{\alpha_s - 0.1189}{0.002} \cdot 12 \pm 11\right)\ \text{MeV}$$

$$m_b(M_Z) = \left(2835 - \frac{\alpha_s - 0.1189}{0.002} \cdot 27 \pm 8\right)\ \text{MeV}$$

$$m_b(161.8\ \text{GeV}) = \left(2703 - \frac{\alpha_s - 0.1189}{0.002} \cdot 28 \pm 8\right)\ \text{MeV}$$

SUMMARY

$$m_c(3\,\text{GeV}) = 986(13)\ \text{MeV}$$

$e^+e^- +$ pQCD

$$m_c(3\,\text{GeV}) = 986(10)\ \text{MeV}$$

lattice + pQCD

$$m_b(10\,\text{GeV}) = 3610(16)\ \text{MeV}$$
$$m_b(m_b) = 4163(16)\ \text{MeV}$$

$e^+e^- +$ pQCD