Charm and Bottom Quark Masses, and Multi-Loop Results

Johann H. Kühn

Karlsruhe Institute of Technology

Universität Karlsruhe (TH)
Forschungsuniversität • gegründet 1825

SFB TR9 Computational Theoretical Particle Physics

I. WHY precise masses?

B-decays:

$$\Gamma(B \to X_{\mathrm{u}} l \bar{\nu}) \sim G_{\mathrm{F}}^2 \, m_{\mathrm{b}}^5 \, |V_{\mathrm{ub}}|^2$$

$$\Gamma(B \to X_{\mathrm{c}} l \bar{\nu}) \sim G_{\mathrm{F}}^2 \, m_{\mathrm{b}}^5 \, f(m_{\mathrm{c}}^2/m_{\mathrm{b}}^2) \, |V_{\mathrm{cb}}|^2$$

$$B \to X_{\mathrm{s}} \gamma$$

Υ-spectroscopy:

$$m\,(\Upsilon(1\mathrm{s})) = 2M_{\mathrm{b}} - \left(\tfrac{4}{3}\alpha_{\mathrm{s}}\right)^2 \frac{M_{\mathrm{b}}}{4} + \ldots$$

Higgs decay (ILC)

$$\Gamma(H \to b\bar{b}) = \frac{G_F\, M_H}{4\sqrt{2}\pi}\, m_b^2(M_H)\, \tilde{R}$$

$$\tilde{R} \;=\; 1 + 5.6667 a_s + 29.147 {a_s}^2 + 41.758 {a_s}^3 - 825.7 {a_s}^4 \qquad \left(a_s \equiv \frac{\alpha_s}{\pi}\right)$$

${a_s}^4$-term = 5-loop calculation [Baikov,...]

Yukawa Unification

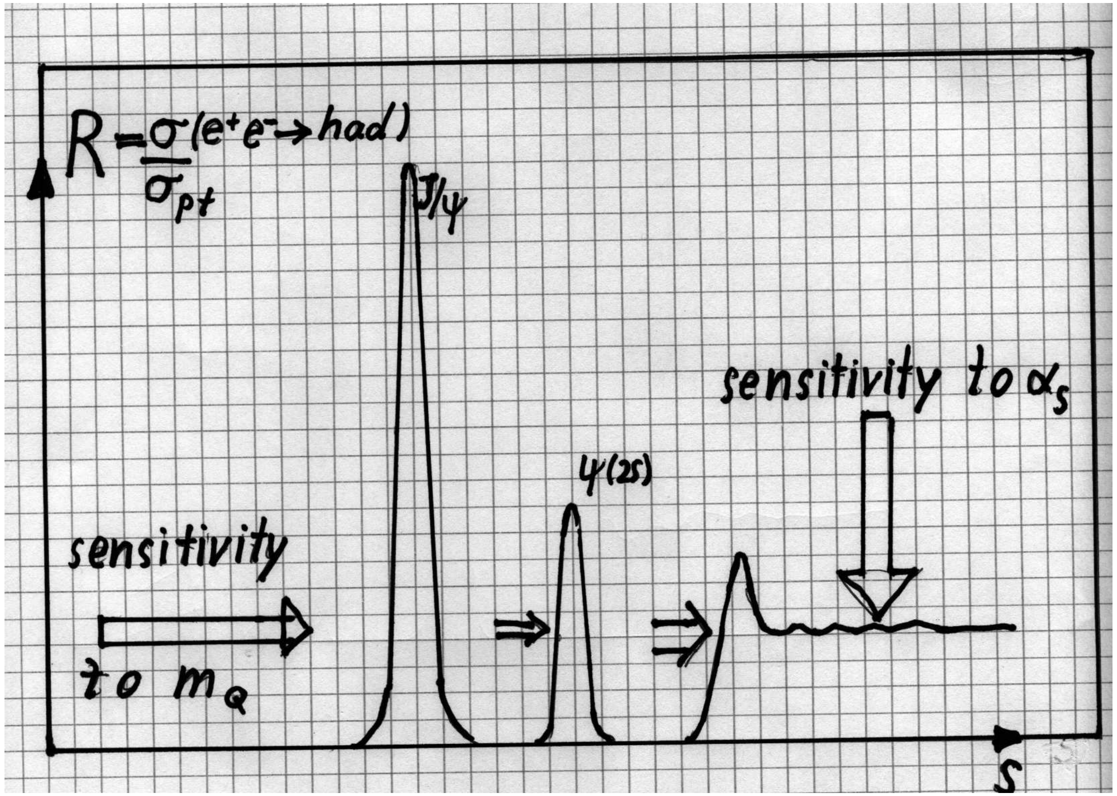
$\lambda_\tau \sim \lambda_b$ or $\lambda_\tau \sim \lambda_b \sim \lambda_t$ at GUT scale

top-bottom $\to m_t / m_b \sim$ ratio of vacuum expectation values

request $\frac{\delta m_b}{m_b} \sim \frac{\delta m_t}{m_t} \Rightarrow \delta m_t \approx 1$ GeV $\Rightarrow \delta m_b \approx 25$ MeV

II. m_q from SVZ Sum Rules, Moments and Tadpoles

Main Idea (SVZ)



Some definitions:

$$R(s) \;=\; 12\pi\,\mathrm{Im}\left[\Pi(q^2 = s + i\epsilon)\right]$$

$$\left(-q^2 g_{\mu\nu} + q_\mu\, q_\nu\right)\,\Pi(q^2) \;\equiv\; i\int \mathrm{d}x\, e^{iqx}\langle T j_\mu(x) j_\nu(0)\rangle$$

with the electromagnetic current j_μ

Taylor expansion: $\Pi_Q(q^2) \;=\; Q_Q^2\,\frac{3}{16\pi^2}\sum_{n\geq 0}\bar{C}_n\, z^n$

with $z = q^2/(4m_Q^2)$ and $m_Q = m_Q(\mu)$ the $\overline{\mathrm{MS}}$ mass.

$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi}\bar{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}_n^{(3)} + \ldots$$

Coefficients $\bar{C}_n$ up to $n = 8$ known analytically in order α_s^2

(Chetyrkin, JK, Steinhauser, 1996)

up to $n = 30$ (Boughezal, Czakon, Schutzmeier 2007)

| also $\bar{C}_0$ and $\bar{C}_1$ in order α_s^3 (four loops!) | (2006)

➪ reduction to master integrals through Laporta algorithm

(Chetyrkin, JK, Sturm; confirmed by Boughezal, Czakon, Schutzmeier)

evaluation of master integrals numerically through difference equations

(30 digits) or Padé method or analytially in terms of transcendentals

(Schröder + Vuorinen, Chetyrkin et al., Schröder + Steinhauser,

Laporta, Broadhurst, Kniehl et al.)

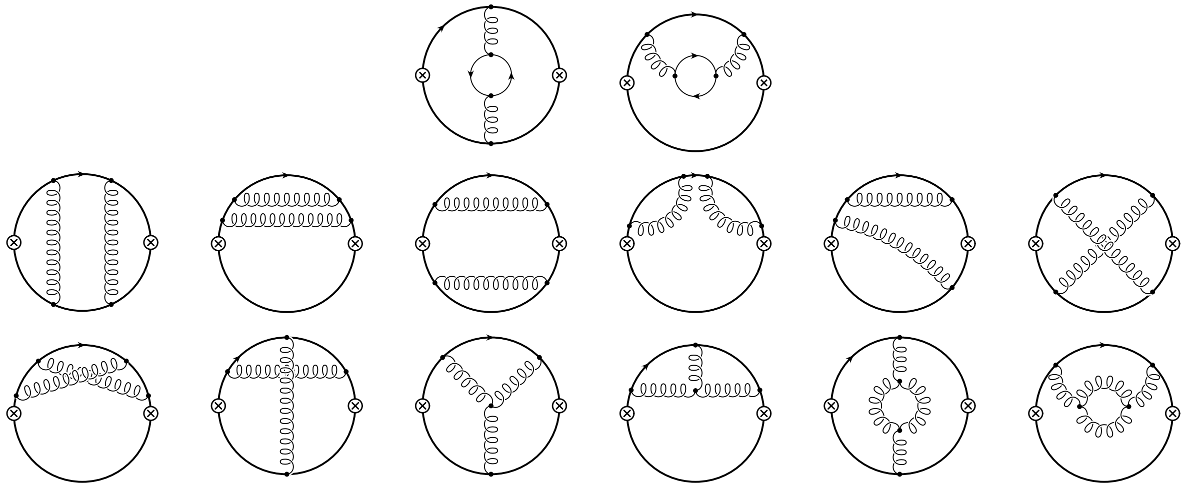
recently also $\bar{C}_2$ (Maier, Maierhöfer, Marquard, arXiv:0806.3405 [hep-ph])

and $\bar{C}_3$ (in preparation)

Analysis in NNLO

Coefficients $\bar{C}_n$ from three-loop one-scale tadpole amplitudes with "arbitrary" power of propagators;

FORM-program MATAD

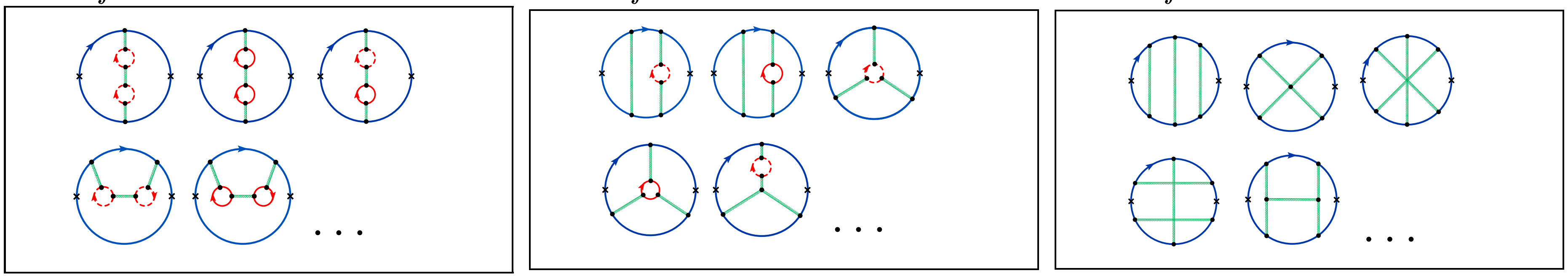


Analysis in N^{3}LO

Algebraic reduction to 13 master integrals (Laporta algorithm);

numerical and analytical evaluation of master integrals

n_f^2-contributions $\qquad$ n_f^1-contributions $\qquad$ n_f^0-contributions



 : heavy quarks,  : light quarks,

n_f: number of active quarks

$\Longrightarrow$ About 700 Feynman-diagrams recall: $\Pi_Q(q^2) \;=\; Q_Q^2\,\frac{3}{16\pi^2}\sum_{n\geq 0}\bar{C}_n\,z^n$

$\bar{C}_n$ depend on the charm quark mass through $l_{mc} \equiv \ln(m_c^2(\mu)/\mu^2)$

$$\begin{aligned}\bar{C}_n \;=\; & \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}\left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{mc}\right) \\ & + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{mc} + \bar{C}_n^{(22)} l_{mc}^2\right) \\ & + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3\left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{mc} + \bar{C}_n^{(32)} l_{mc}^2 + \bar{C}_n^{(33)} l_{ms}^3\right)\end{aligned}$$

n	$\bar{C}_n^{(0)}$	$\bar{C}_n^{(10)}$	$\bar{C}_n^{(11)}$	$\bar{C}_n^{(20)}$	$\bar{C}_n^{(21)}$	$\bar{C}_n^{(22)}$	$\bar{C}_n^{(30)}$	$\bar{C}_n^{(31)}$	$\bar{C}_n^{(32)}$	$\bar{C}_n^{(33)}$
1	1.0667	2.5547	2.1333	2.4967	3.3130	− 0.0889	−5.6404	4.0669	0.9590	0.0642
2	0.4571	1.1096	1.8286	2.7770	5.1489	1.7524	−3.4937	6.7216	6.4916	−0.0974
3	0.2709	0.5194	1.6254	1.6388	4.7207	3.1831	—	7.5736	13.1654	1.9452
4	0.1847	0.2031	1.4776	0.7956	3.6440	4.3713	—	4.9487	17.4612	5.5856

estimate $-6 < C_n^{(30)} < 6$, $\quad n = 3, 4$

confirmed by exact calculation (n=3) and Padé estimate (n=4)

Define the moments

$$\mathcal{M}_n^{\text{th}} \equiv \frac{12\pi^2}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_c(q^2)\Big|_{q^2=0} = \frac{9}{4}Q_c^2\left(\frac{1}{4m_c^2}\right)^n \bar{C}_n$$

Perturbation theory: $\bar{C}_n$ is function of α_s and $\ln\frac{m_c^2}{\mu^2}$

dispersion relation:

$$\Pi_c(q^2) = \frac{q^2}{12\pi^2}\int \mathrm{d}s\, \frac{R_c(s)}{s(s-q^2)} + \text{subtraction}$$

$$\Rightarrow \mathcal{M}_n^{\text{exp}} = \int \frac{\mathrm{d}s}{s^{n+1}} R_c(s)$$

constraint: $\mathcal{M}_n^{\text{exp}} = \mathcal{M}_n^{\text{th}}$

$\Rightarrow$ m_c

SVZ:

$\mathcal{M}_n^{\text{th}}$ can be reliably calculated in pQCD:

low n: dominated by scales of $\mathcal{O}(2m_Q)$

- fixed order in α_s is sufficient, in particular no resummation of $1/v$ - terms from higher orders required
- condensates are unimportant
- pQCD in terms of short distance mass : m_c(3 GeV) ⇨ $m_c(m_c)$ stable expansion : no pole mass or closely related definition (1S-mass, potential-subtracted mass) involved
- moments available in NNLO
- and $\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3$ in N^{3}LO

Ingredients (charm)

experiment:

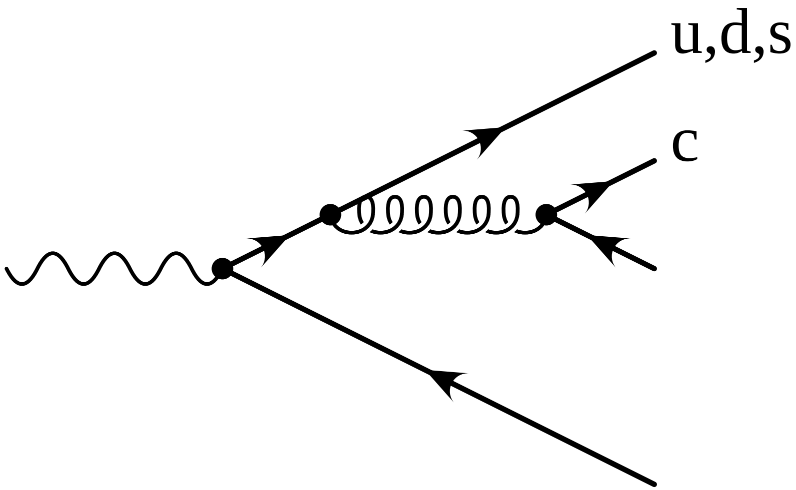
- $\Gamma_e(J/\psi, \psi')$ from BES & CLEO & Babar
- $\psi(3770)$ and $R(s)$ from BES
- $\alpha_s = 0.1187 \pm 0.0020$

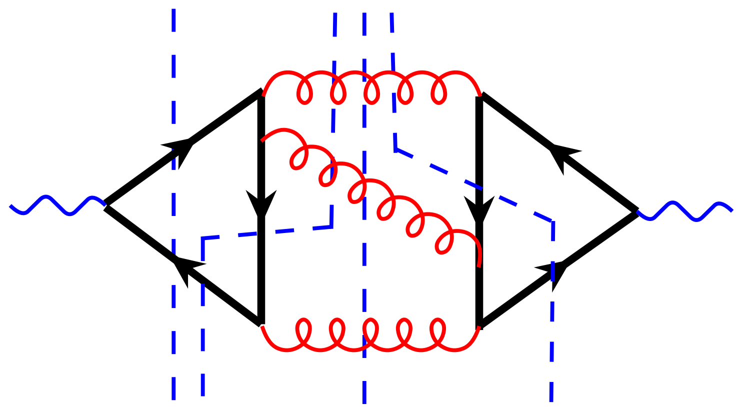
theory:

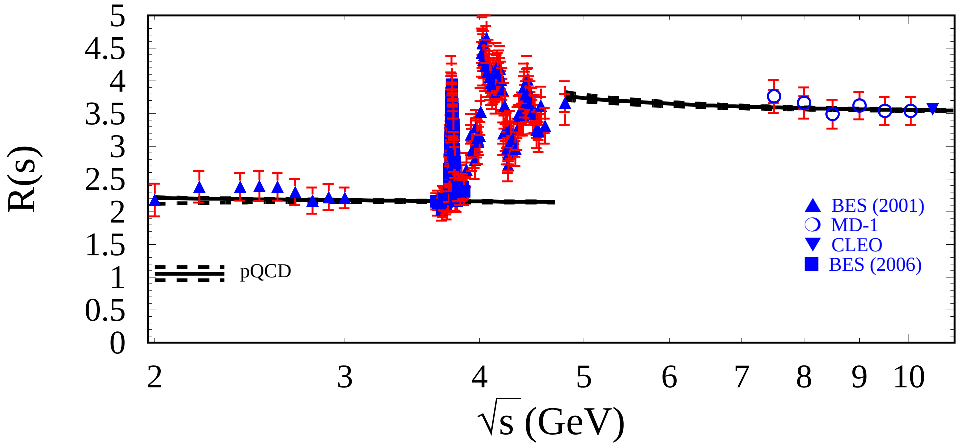
- N^{3}LO for $n = 1, 2, 3$
- N^{3}LO - estimate for $n = 4$
- include condensates

$$\delta \mathcal{M}_n^{\text{np}} = \frac{12\pi^2 Q_c^2}{(4m_c^2)^{(n+2)}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n \left(1 + \frac{\alpha_s}{\pi} \bar{b}_n\right)$$

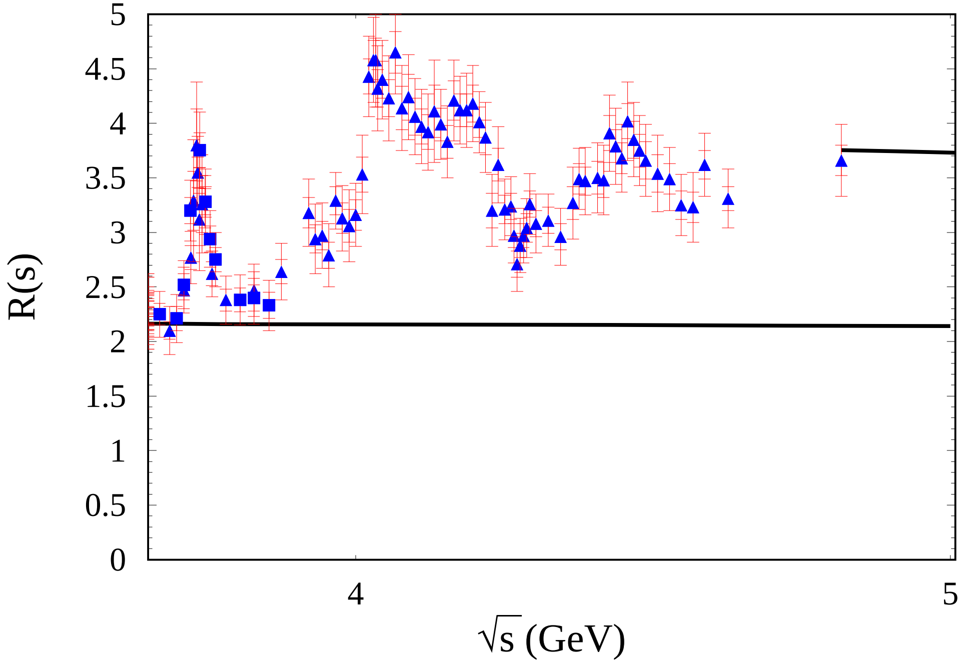
- estimate of non-perturbative terms (oscillations, based on Shifman)
- careful extrapolation of R_{uds}
- careful definition of R_c


u,d,s
c





R(s)
√s (GeV)
BES (2001)
MD-1
CLEO
BES (2006)
pQCD



R(s)
√s (GeV)

Contributions from

- narrow resonances: $R = \frac{9\,\Pi\,M_R\,\Gamma_e}{\alpha^2(s)}\,\delta(s - M_R^2)$
- threshold region ($2\,m_D$ – 4.8 GeV)
- perturbative continuum ($E \geq 4.8$ GeV)

n	$\mathcal{M}_n^{\text{res}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{np}}$ $\times 10^{(n-1)}$
1	0.1201(25)	0.0318(15)	0.0646(11)	0.2166(31)	−0.0001(2)
2	0.1176(25)	0.0178(8)	0.0144(3)	0.1497(27)	0.0000(0)
3	0.1169(26)	0.0101(5)	0.0042(1)	0.1312(27)	0.0007(14)
4	0.1177(27)	0.0058(3)	0.0014(0)	0.1249(27)	0.0027(54)

Results (m_c)

n	$m_c(3\ \text{GeV})$	exp	α_s	μ	np	total	$\delta\bar{C}_n^{30}$	$m_c(m_c)$
1	0.986	0.009	0.009	0.002	0.001	0.013	—	1.286
2	0.976	0.006	0.014	0.005	0.000	0.016	—	1.277
3	0.978	0.005	0.014	0.007	0.002	0.016	—	1.278
4	1.012	0.003	0.008	0.030	0.007	0.032	0.016	1.309

$n = 1$:

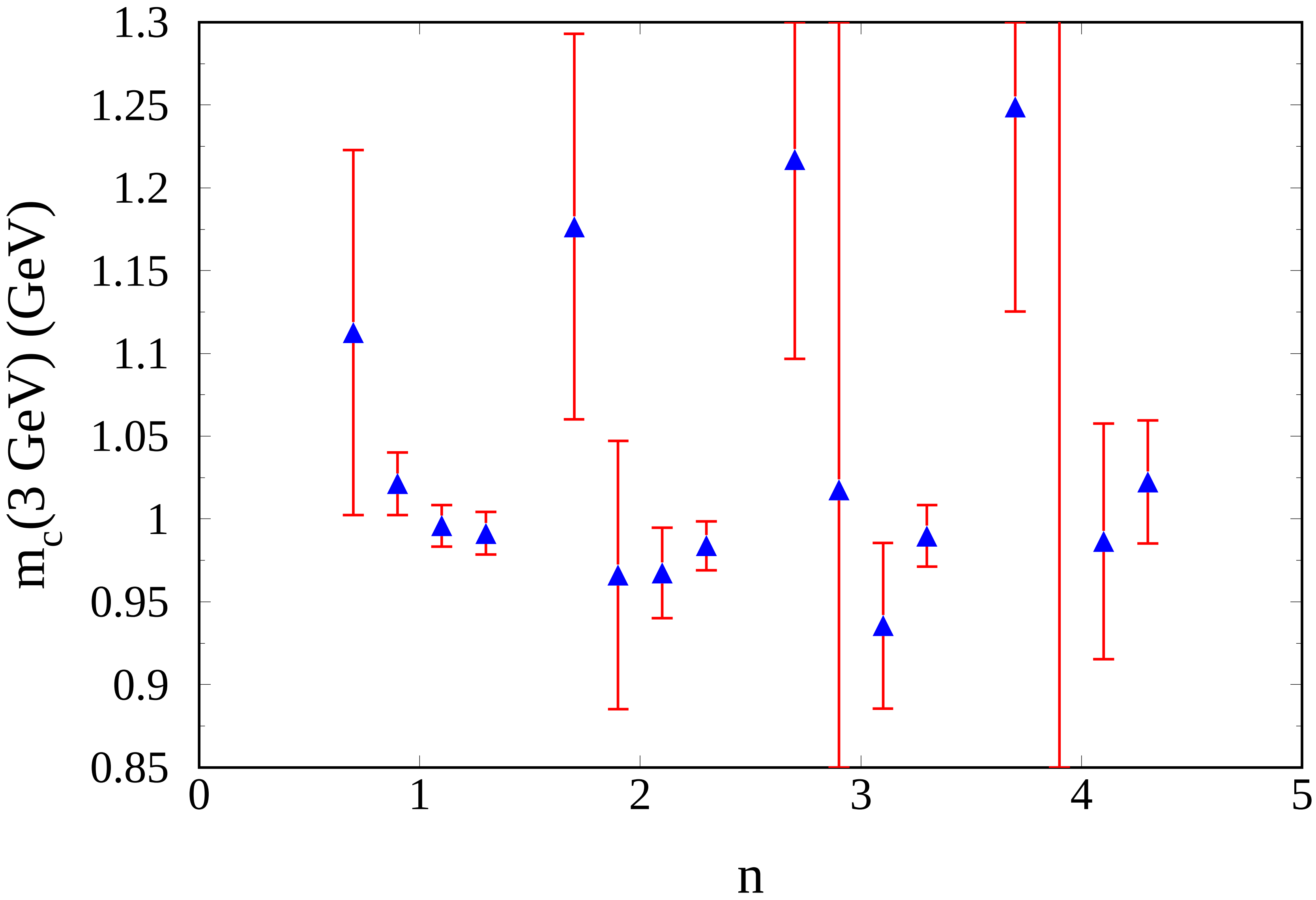
- $m_c(3\,\text{GeV}) = 986 \pm 13\,\text{MeV}$
- $m_c(m_c) = 1286 \pm 13\,\text{MeV}$

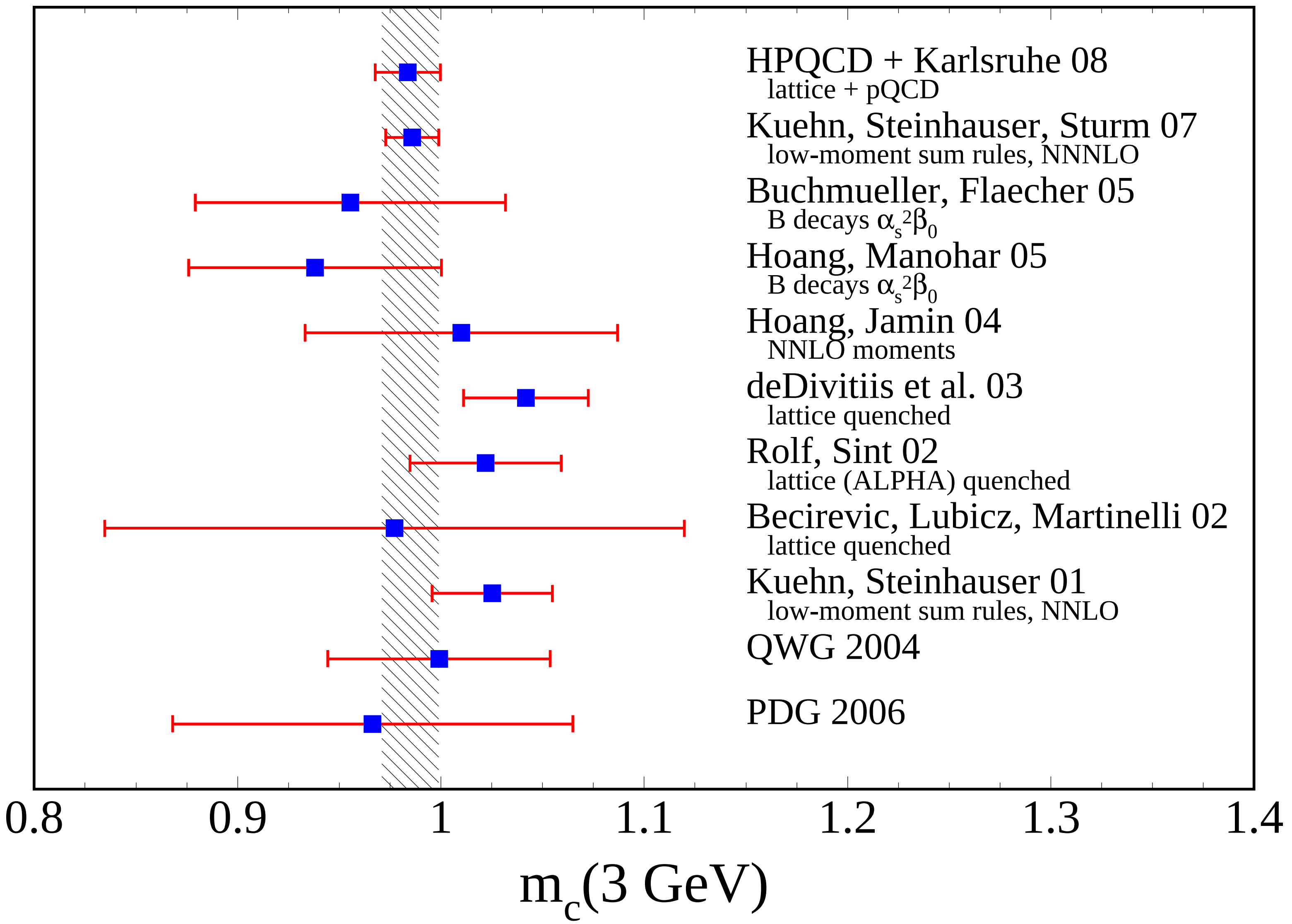
$n = 2$:

- $m_c(3\,\text{GeV}) = 976 \pm 16\,\text{MeV}$
- $m_c(m_c) = 1277 \pm 16\,\text{MeV}$

$n = 3$:

- $m_c(3\,\text{GeV}) = \begin{cases} 982 \pm 26\,\text{MeV} \;\text{old},\; C_3^{(3)} \;\text{estimated} \\ 978 \pm 16\,\text{MeV} \;\text{new},\; C_3^{(3)} \;\text{calculated} \end{cases}$
- $m_c(m_c) = 1278 \pm 16\,\text{MeV}$


1.3
1.25
1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
m_c(3 GeV) (GeV)
0
1
2
3
4
5
n


HPQCD + Karlsruhe 08
lattice + pQCD
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Hoang, Manohar 05
B decays $\alpha_s^2\beta_0$
Hoang, Jamin 04
NNLO moments
deDivitiis et al. 03
lattice quenched
Rolf, Sint 02
lattice (ALPHA) quenched
Becirevic, Lubicz, Martinelli 02
lattice quenched
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
QWG 2004
PDG 2006
0.8
0.9
1
1.1
1.2
1.3
1.4
m_c(3 GeV)

Experimental Ingredients for m_b

Contributions from

- narrow resonances ($\Upsilon(1S) - \Upsilon(4S)$)
- threshold region (10.618 GeV − 11.2 GeV)
- perturbative continuum ($E \geq 11.2$ GeV)

n	$\mathcal{M}_n^{\text{res},(1S-4S)} \times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{thresh}} \times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{cont}} \times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{exp}} \times 10^{(2n+1)}$
1	1.394(23)	0.296(32)	2.911(18)	4.601(43)
2	1.459(23)	0.249(27)	1.173(11)	2.881(37)
3	1.538(24)	0.209(22)	0.624(7)	2.370(34)
4	1.630(25)	0.175(19)	0.372(5)	2.178(32)

Results (m_b)

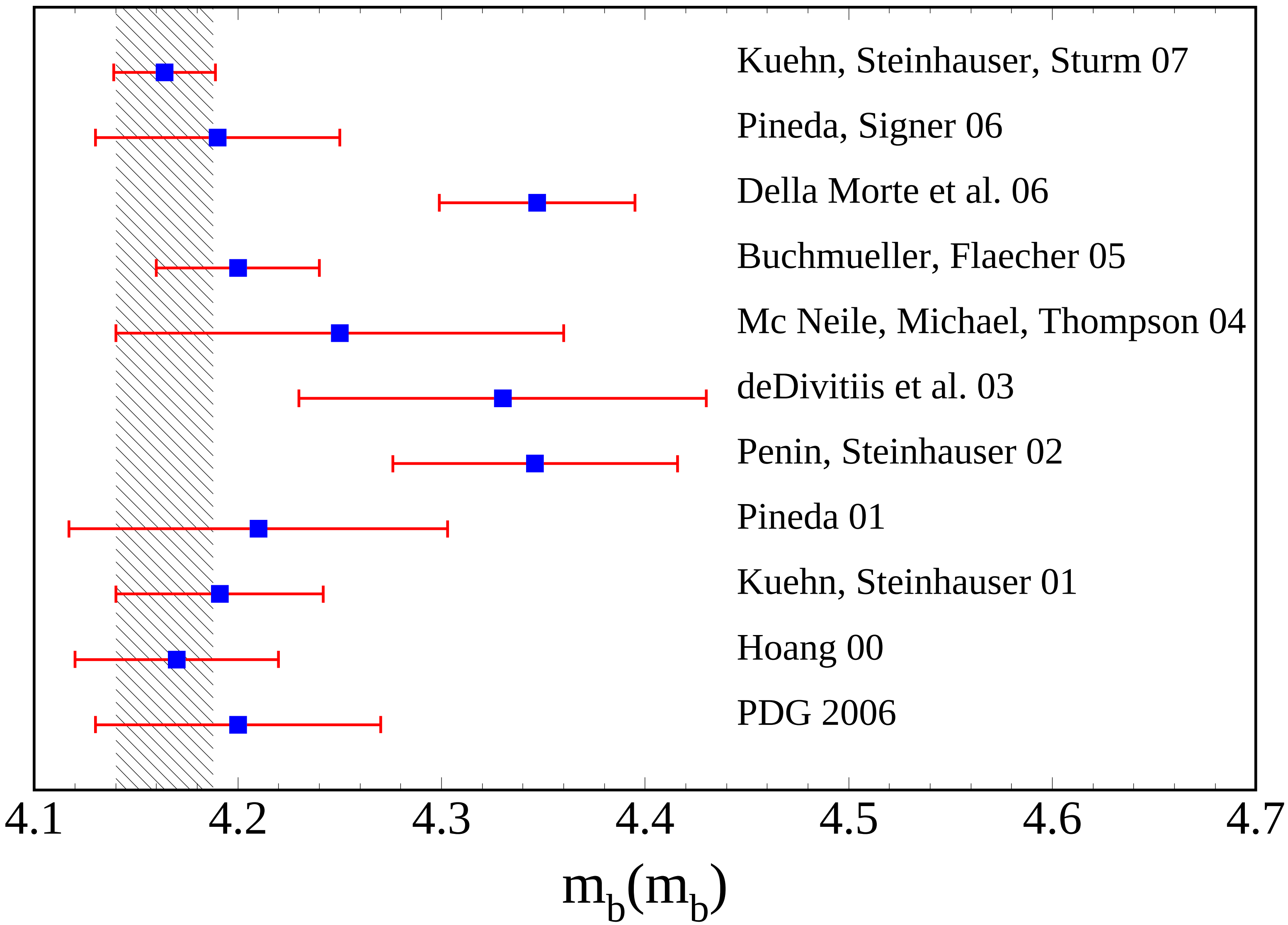
n	m_b(10 GeV)	exp	α_s	μ	total	$\delta\bar{C}_n^{30}$	$m_b(m_b)$
1	3.593	0.020	0.007	0.002	0.021	—	4.149
2	3.607	0.014	0.012	0.003	0.019	—	4.162
3	3.617	0.010	0.014	0.006	0.019	—	4.172
4	3.631	0.008	0.015	0.021	0.027	0.012	4.185

$n = 2$:

- $m_c(3\,\text{GeV}) = 3607 \pm 19\,\text{MeV}$
- $m_c(m_c) = 4162 \pm 19\,\text{MeV}$

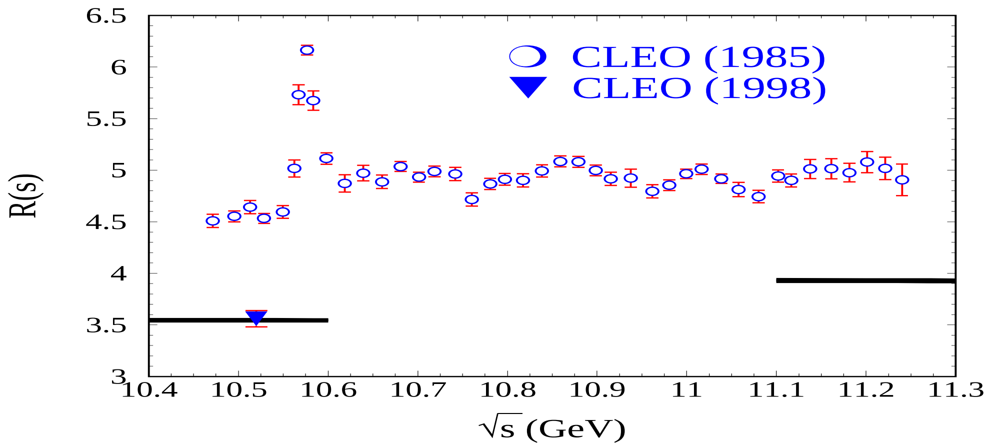
$n = 3$:

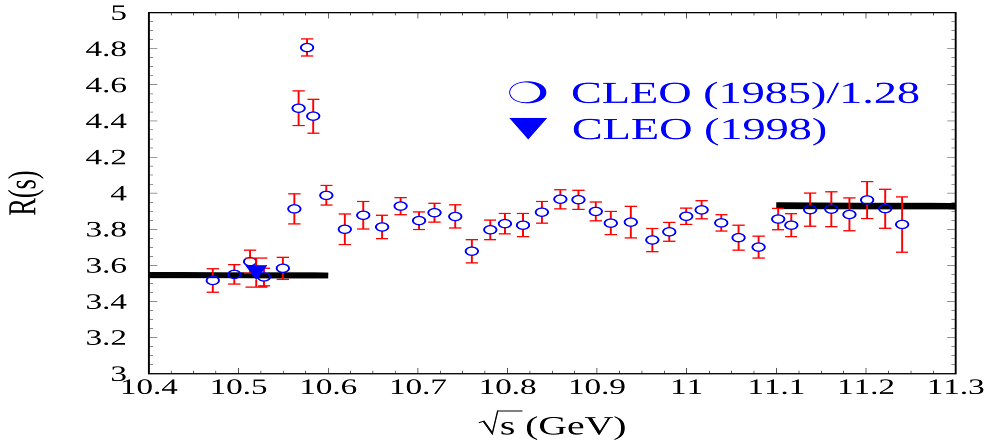
- $m_b(10\,\text{GeV}) = \begin{cases} 3618 \pm 27\,\text{MeV} \text{ old, } C_3^{(3)} \text{ estimated} \\ 3617 \pm 19\,\text{MeV} \text{ new, } C_3^{(3)} \text{ calculated} \end{cases}$
- $m_b(m_b) = 4172 \pm 19\,\text{MeV}$



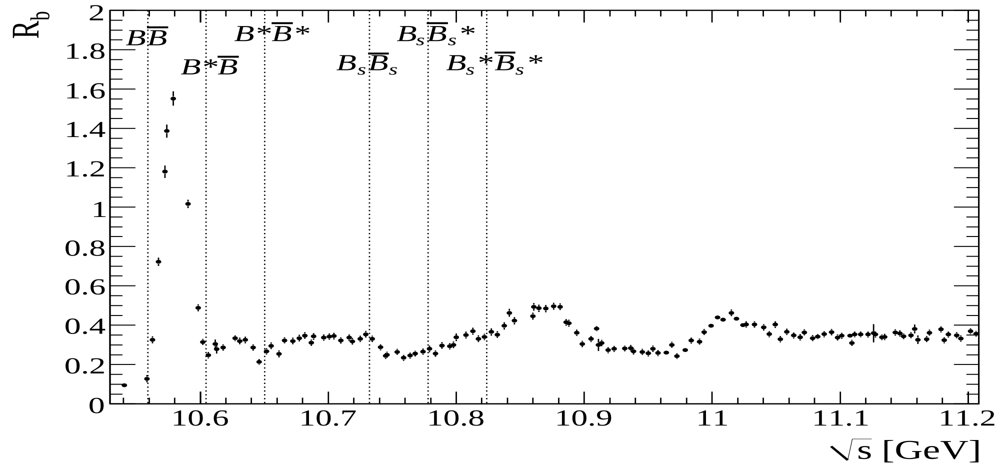
Kuehn, Steinhauser, Sturm 07
Pineda, Signer 06
Della Morte et al. 06
Buchmueller, Flaecher 05
Mc Neile, Michael, Thompson 04
deDivitiis et al. 03
Penin, Steinhauser 02
Pineda 01
Kuehn, Steinhauser 01
Hoang 00
PDG 2006
4.1
4.2
4.3
4.4
4.5
4.6
4.7
$m_b(m_b)$

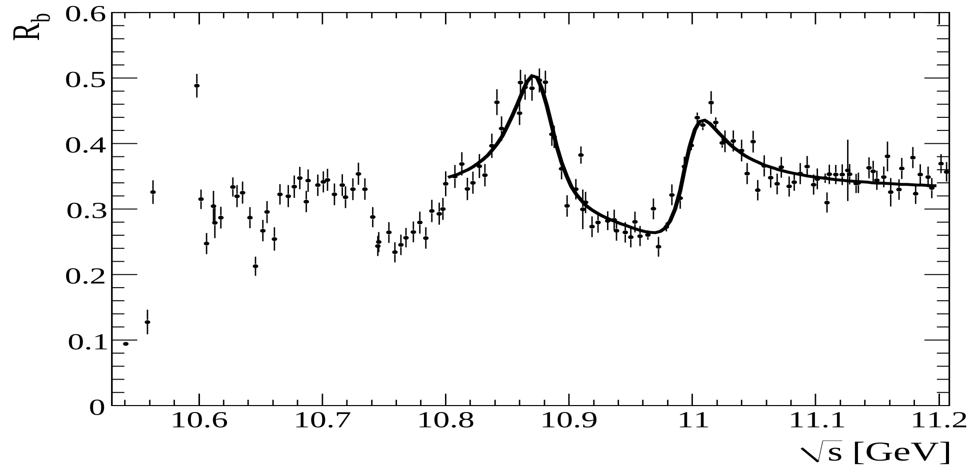
Improvements Based on Recent Babar Results (Preliminary)





uncertainties after "renormalization" estimated to be 10%
⇒ dominant contribution to error





2% systematic experimental error
Deconvolute ISR and apply radiative corrections ⇒

Preliminary Analysis

n	$\mathcal{M}_n^{\text{threshold}}$ $\times 10^{(2n+1)}$ CLEO	$\mathcal{M}_n^{\text{threshold}}$ $\times 10^{(2n+1)}$ BABAR
1	0.296(32)	0.282(9)
2	0.249(27)	0.235(7)
3	0.209(22)	0.197(6)
4	0.175(19)	0.165(5)

- consistency between BABAR and CLEO

- reduction of experimental error in this region by factor 3,
  total error by factor 2/3

- slight upwards shift of m_b by $5 - 10$ MeV

recent analysis: lattice & pQCD

(HPQCD + Karlsruhe, Phys. Rev. D78, 054513)

lattice evaluation of pseudoscalar correlator

$\Rightarrow$ replace experimental moments by lattice simulation

input: $M(\eta_c) \widehat{=} m_c\,, \quad M(\Upsilon(1S)) - M(\Upsilon(2s)) \widehat{=} \alpha_s$

pQCD for pseudoscalar correlator available:

"all" moments in $\mathcal{O}(\alpha_s^2)$

three lowest moments in $\mathcal{O}(\alpha_s^3)$.

lowest moment: dimensionless: $\sim \left(\bar{C}^{(0)} + \frac{\alpha_s}{\pi}\bar{C}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}^{(3)} + ...\right)$

$\Rightarrow$ $\alpha_s(3\text{GeV}) \Rightarrow \alpha_s(M_Z) = 0.1174(12)$

higher moments: $\sim m_c^2 \times \left(1 + ...\frac{\alpha_s}{\pi}...\right)$

$\Rightarrow$ $m_c(3\text{GeV}) = 986(10)$ MeV

to be compared with 986(13) MeV from e^+e^- !

SUMMARY

$m_c(3\ \text{GeV}) = 0.986(13)\ \text{GeV}$

$e^+e^- +$ pQCD

$m_c(3\ \text{GeV}) = 0.986(10)\ \text{GeV}$

lattice + pQCD

$m_b(10\ \text{GeV}) = 3.607(19)\ \text{GeV}$

$e^+e^- +$ pQCD $+ \bar{C}_2^{(3)}$

confirmed by recent BABAR analysis