Four Loop Calculations as Tool for Precise

Quark Mass Determination

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SFB TR-9

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I. Introduction and Motivation

Definitions :

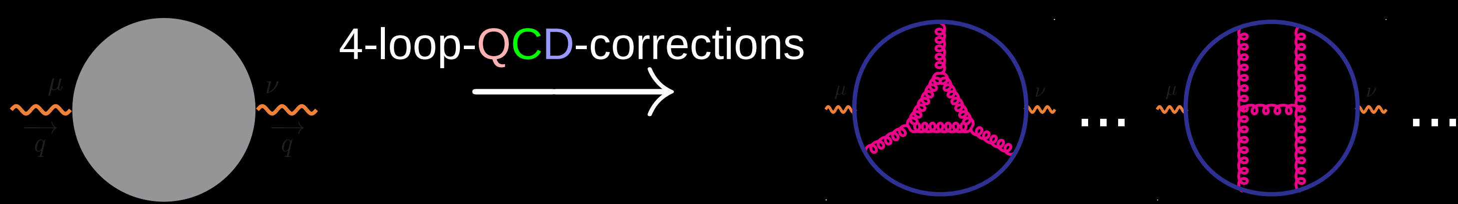
- Correlator of two currents:

$$\Pi^{\mu\nu}(q,j) = i \int dx\, e^{iqx} \langle 0|T j^\mu(x) j^\nu(0)|0\rangle$$

here: $j^\mu(x)$ electromagnetic heavy quark current

- Diagrammatically:

$$i\Pi^{\mu\nu}(q) =$$



- Polarization function $\Pi(q^2)$:

$$\Pi^{\mu\nu}(q) = \left(-g^{\mu\nu}\, q^2 + q^\mu\, q^\nu\right)\ \Pi(q^2)$$

Theory $\leftrightarrow$ Experiment

- Dispersion-relations

$$\Pi(q^2) = \Pi(q^2 = 0) + \frac{q^2}{12\,\pi^2} \int ds\, \frac{R(s)}{s\,(s - q^2)}$$

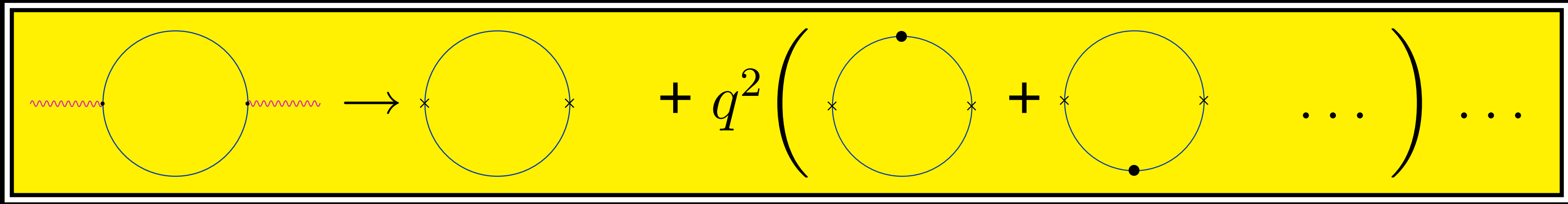
Moments $\mathcal{M}_n$ related to derivatives of $\Pi(q^2)$ at $q^2 = 0$:

$$\mathcal{M}_n^{\text{exp}} = \int \frac{ds}{s^{n+1}} R^{\text{exp}}(s) = \frac{12\,\pi}{n!} \left(\frac{d}{dq^2}\right)^n \Pi(q^2)\bigg|_{q^2=0}$$

- Taylor expansion:

$$\Pi(q^2) = \frac{3\,Q_f^2}{16\,\pi^2} \sum_n \overline{C}_n \left(\frac{q^2}{4\,m^2}\right)^n$$

- Diagrammatic Expansion:



- First and higher derivatives of $\Pi(q^2)$

  ➪ extraction of charm- and bottom-quark mass:

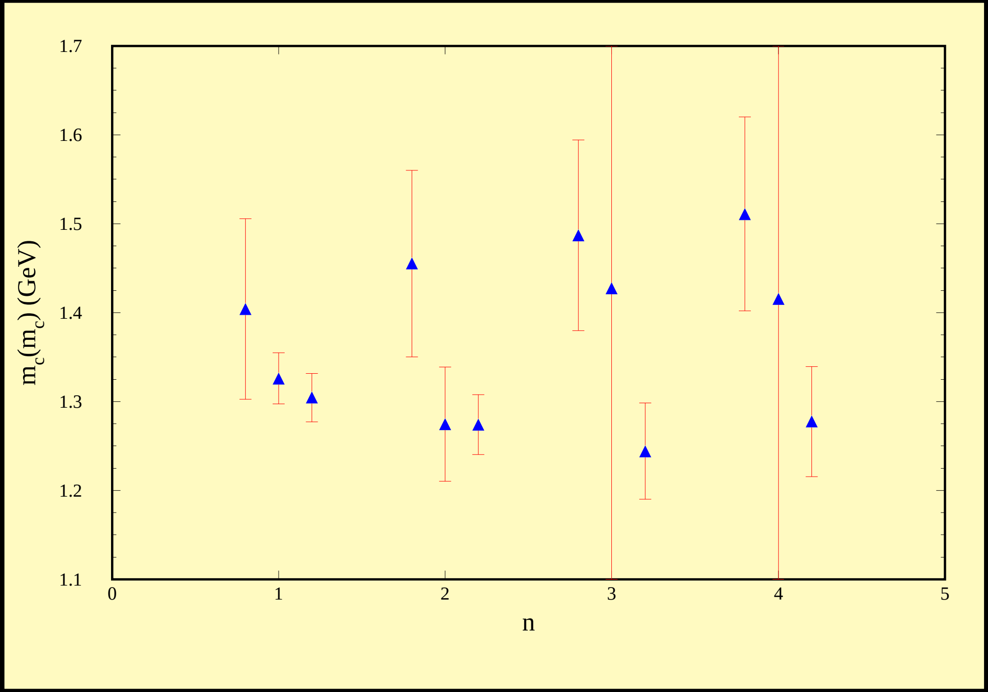
$$m(\mu) = \frac{1}{2}\left(Q_f^2\,\frac{9}{4}\frac{\overline{C}_n}{\mathcal{M}_n^{\text{exp}}}\right)^{1/(2\,n)}$$

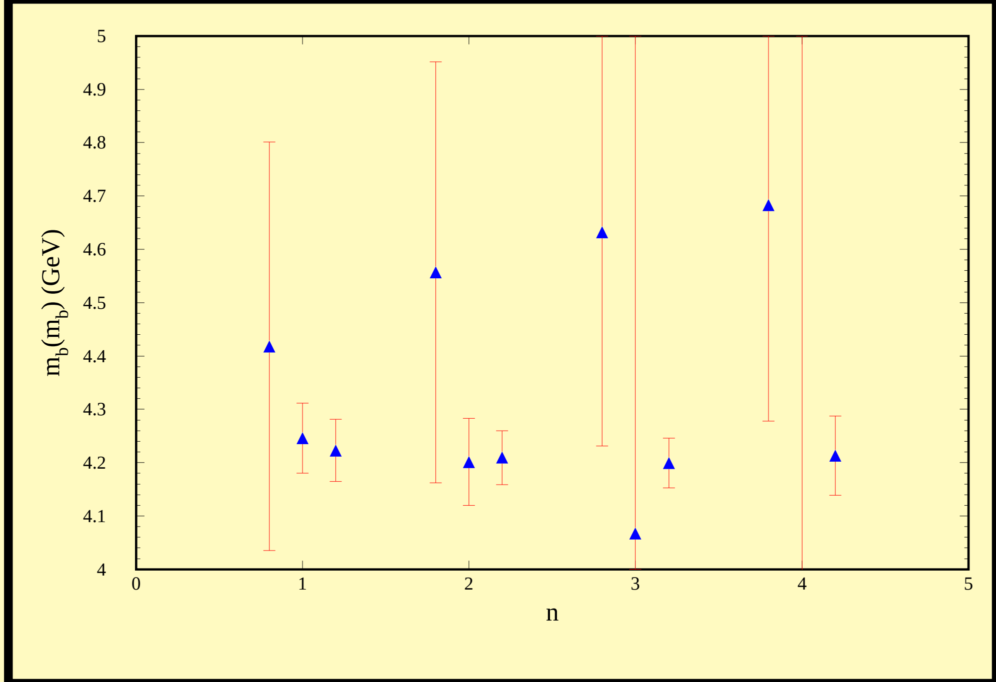
← Theory

← Experiment

c-quarks: Novikov, et al. ’78; b-quarks: Reinders, et al. ’85

...at 3-loop-order:





J.H. Kühn, M. Steinhauser '01

$\overline{\text{MS}}$ – mass: $m_b(m_b) = 4.191(51)$ GeV

. . . analog for charm-quarks:

$m_c(m_c) = 1.304(27)$ GeV

$\Pi(q^2 = 0)$:

Relates electromagnetic coupling constant in on-shell and $\overline{\text{MS}}$ renormalization schemes

$$\alpha_{em} = \frac{\overline{\alpha}_{em}(\mu)}{1 + \frac{\overline{\alpha}_{em}(\mu)}{4\,\pi}\, 3\, \overline{C}_0}$$

Status:

$$\Pi(q^2) = \Pi^{(0)}(q^2) + \frac{\alpha_s}{\pi}\Pi^{(1)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)^2 \Pi^{(2)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)^3 \Pi^{(3)}(q^2) + \ldots$$

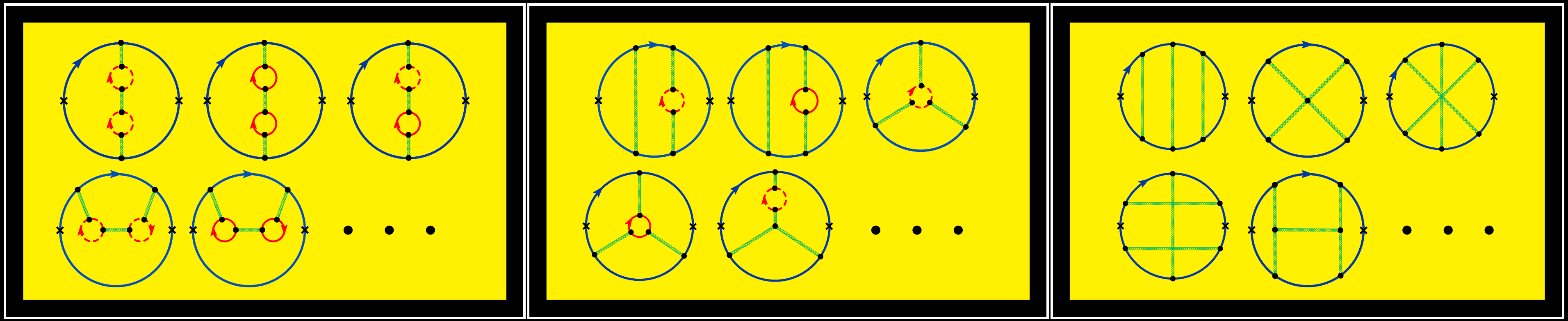
G. Källén, A. Sabry '55; J. Schwinger '73
K.G. Chetyrkin, J.K., M. Steinhauser '96

– Low energy limit $\longleftarrow$ in this talk
First two Taylor coefficients in Order α_s^3 $\longleftarrow$ new

II. Strategy & Methods:

Classification according to the number of inserted Fermion loops

n_f^2-contributions n_f^1-contributions n_f^0-contributions



○ : heavy quarks, ◌ : light quarks,

n_f: number of active quarks

$\Longrightarrow$ About 700 Feynman-diagrams

Integration-by-parts:

Chetyrkin, Tkachov

$$0 = \int d^d k_1 \ldots d^d k_4 \quad \partial_{(k_j)_\mu} \left(k_l^\mu \, I_{\alpha\beta} \right) \;, \qquad j,l = 1,\ldots,\text{loops=4}$$

$I_{\alpha\beta}$: Generic integrand with propagator powers $\alpha = \{\alpha_1, \ldots\}$
and scalar-product powers $\beta = \{\beta_1, \ldots\}$

Laporta-Algorithm:

Laporta, Remiddi

Idea: – IBP-identities for explicit numerical values of α, β
– Introduction of an order among the integrals
– Solving a linear system of equations

Problem: Dramatic growth of number of equations

Here: >31 million IBP-identities generated and solved
$\rightsquigarrow$ Integral-tables with solutions for around
5 million integrals, expressed through 13 masters

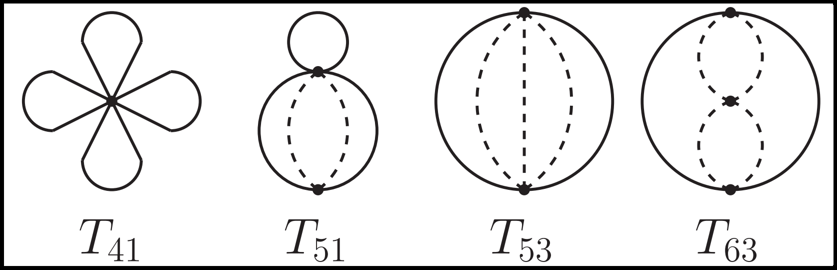
Consider symmetries of diagrams
$\rightsquigarrow$ Smaller number of IBP-equations,
$\rightsquigarrow$ Keep size of integral-tables under control

Automation:

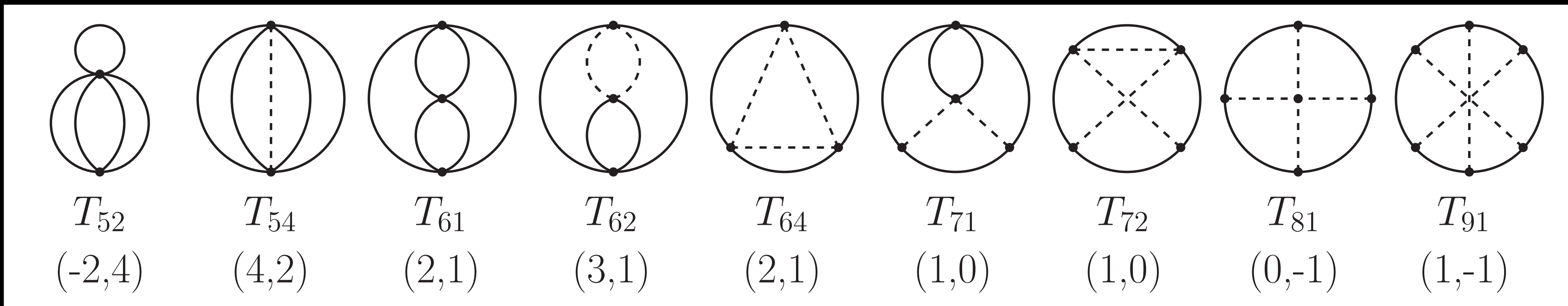
Generation & solution of the system of linear equations with:
– Implementation based on `FORM3` J.A.M. Vermaseren
– Simplification of rational functions in d by `FERMAT` Lewis
– with the use of `GateToFermat` Tentyoukov, Vermaseren

Evaluation of 13 master integrals(MI):

four are simple ...



Master integrals



Solution with high precision numerics Schröder, Vuorinen
with difference equation method Laporta
other contributions: Broadhurst; Laporta; Kniehl, Kotikov; Schröder, Steinhauser

Difficulty:

- Solving IBP-identities $\Rightarrow$ Division by $(d-4)$ can appear
  $\rightsquigarrow$ "spurious" poles
  $\rightsquigarrow$ Master integrals with spurious poles as coefficient
  need to be known in higher order in ε

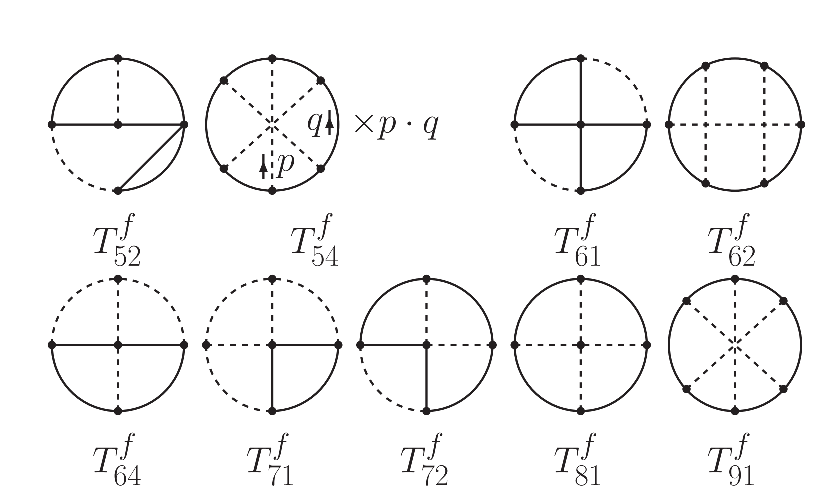
ε-finite basis:

Chetyrkin, Faisst, Sturm, Tentyukov

Choice of master integrals is not unique

$\rightsquigarrow$ Select a new basis with ε-finite coefficients

This ε-finite basis can be found in the set of initial integrals F_i



Prescription:

1.) Select F_i having the $1/\varepsilon^{n_{max}}$-pole with highest power n_{max}

2.) Solve for master integral $T_{j,n_{max}}$ and replace $T_{j,n_{max}} \longrightarrow F_i \equiv T^f_{j,n_{max}}$

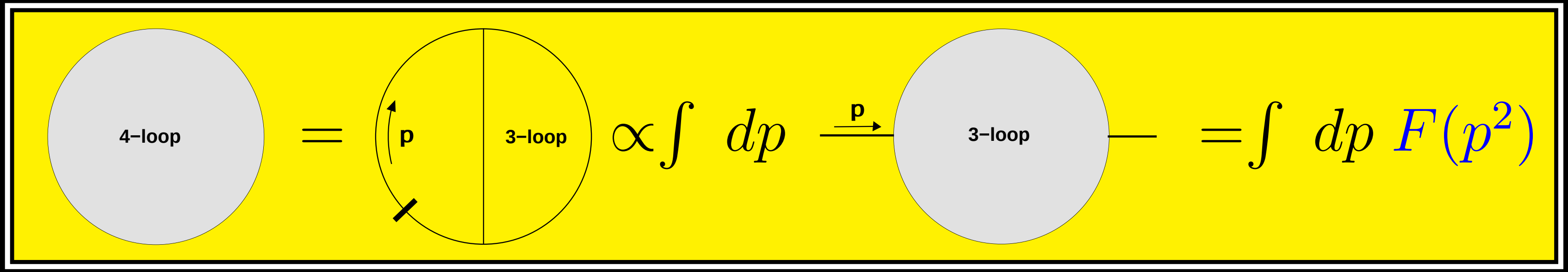
3.) Repeat 1.) and 2.) until no spurious poles survive

Padé-method

complimentary to difference equations

Prescription:

Integration with respect to 3 loop momenta "semi-analytical" and with respect to the 4-th numerical



Integrand $F(p^2)$: Determined through Padé-approximation

Pole-part analytically, finite part numerically

$$T^f_{52} = \frac{3\,\zeta_3}{2\,\epsilon^2} + \frac{15\,\zeta_3}{2\,\epsilon} - \frac{91\,\pi^4}{360\,\epsilon} - \frac{4\,\pi^2\,\log^2 2}{3\,\epsilon} + \frac{4\log^4 2}{3\,\epsilon} + \frac{32\,\mathrm{Li}_4\left(\frac{1}{2}\right)}{\epsilon} + 27.30068067(5),$$

$$T^f_{54} = \frac{5\,\zeta_5}{4\,\epsilon} - 6.7622402238547(2),$$

$$T^f_{61} = \frac{5\,\zeta_5}{\epsilon} - 29.703462427815(3),$$

$$T^f_{62} = 2.4403458233350757(6),$$

$$T^f_{64} = \frac{5\,\zeta_5}{\epsilon} - 18.026245978729184(4),$$

$$T^f_{71} = \frac{5\,\zeta_5}{\epsilon} - 5.19831391(6),$$

$$T^f_{72} = \frac{5\,\zeta_5}{\epsilon} - 26.4964794044474738(5),$$

$$T^f_{81} = \frac{5\,\zeta_5}{\epsilon} - 22.53760348(3),$$

$$T^f_{91} = 1.808879546207(2).$$

III. Results:

$$\Pi(q^2) = \frac{3}{16\,\pi^2}\Bigg\{ \left(\frac{\alpha_s}{\pi}\right) 1.4444 + \left(\frac{\alpha_s}{\pi}\right)^2 \left(1.5863 + 0.1387\, n_h + 0.3714\, n_l\right)$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^3 \left(0.0257\, n_l^2 - 0.0309\, n_h^2 + 0.0252\, n_h\, n_l - 1.2112\, n_l - 3.3426\, n_h + 1.4186\right)$$

$$+ \left(\frac{q^2}{4\,\overline{m}^2}\right) \Bigg[1.0667 + \left(\frac{\alpha_s}{\pi}\right) 2.5547 + \left(\frac{\alpha_s}{\pi}\right)^2 \left(0.2461 + 0.2637\, n_h + 0.6623\, n_l\right)$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^3 \left(0.0961\, n_l^2 + 0.0130\, n_h^2 + 0.1658\, n_h\, n_l - 2.9605\, n_l - 6.4188\, n_h + 8.2846\right) \Bigg] + \ldots \Bigg\}$$

numerically evaluated for scale $\mu = \overline{m}$

Charm

$n = 1$ N^{2}LO:

$m_c(3\text{ GeV}) = 1017 \pm 14(\text{exp}) \pm 7(\alpha_s) \pm 3(\text{scale})$

$m_c(m_c) = 1301 \pm 15$

$n = 1$ N^{3}LO:

$m_c(3\text{ GeV}) = 1012 \pm 14(\text{exp}) \pm 8(\alpha_s) \pm 0.76(\text{scale})$

$m_c(m_c) = 1297 \pm 15$

combine with $n = 2$ (N^{2}LO):

$m_c(3\text{ GeV}) = 1007 \pm 16$

$m_c(m_c) = 1290 \pm 15$, old result: 1304 ± 27

Bottom

$n = 1$ N^{2}LO:

$m_b(10\text{ GeV}) = 3644 \pm 40(\text{exp}) \pm 7(\alpha_s) \pm 4(\text{scale})$

$m_b(m_b) = 4191 \pm 40$

$n = 1$ N^{3}LO:

$m_b(10\text{ GeV}) = 3644 \pm 40(\text{exp}) \pm 7(\alpha_s) \pm 1(\text{scale})$

$m_b(m_b) = 4191 \pm 40$

combine with $n = 2, 3$ (N^{2}LO):

$m_b(10\text{ GeV}) = 3630 \pm 35$

$m_b(m_b) = 4180 \pm 35$, old result: 4210 ± 50

ρ-parameter

- $t\bar{t}$, $b\bar{b}$ and $t\bar{b}$ contributions at four loops

$$\delta\rho = \frac{\Pi_T^Z(0)}{M_Z^2} - \frac{\Pi_T^W(0)}{M_W^2}$$

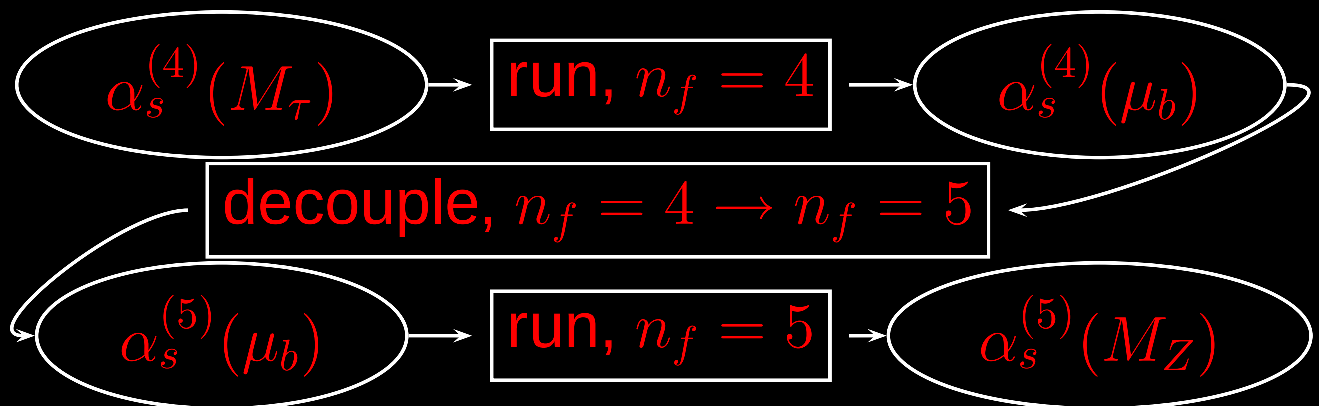
- algebraic reduction "relatively" simple
- large number of new masters evaluated with difference equations and Padé method

$$\delta\rho = 3\frac{G_F M_t^2}{8\pi^2\sqrt{2}}\left(1 - 2.86\left(\frac{\alpha_s}{\pi}\right) - 14.59\left(\frac{\alpha_s}{\pi}\right)^2 - 93.15\left(\frac{\alpha_s}{\pi}\right)^3\right.$$

$\Rightarrow$ $\Delta M_W \approx 2$ MeV (confirmed by Czakon, ...)

Decoupling

Schröder, Steinhauser Chetyrkin, JK, Sturm

- Evaluation of $\alpha_s^{(5)}(M_Z)$ from $\alpha_s^{(4)}(M_\tau)$
- cross b quark threshold:
  $n_l = 4, \mu = M_\tau \approx 1.77$ GeV: $m_b \gg \mu$
  $n_l = 5, \mu = M_Z \approx 91.2$ GeV: $m_b \ll \mu$
- 

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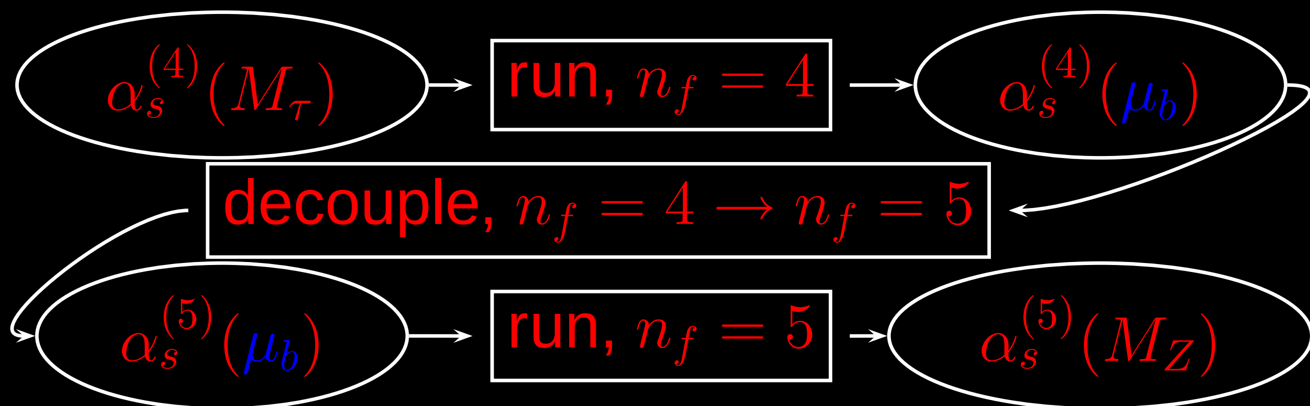
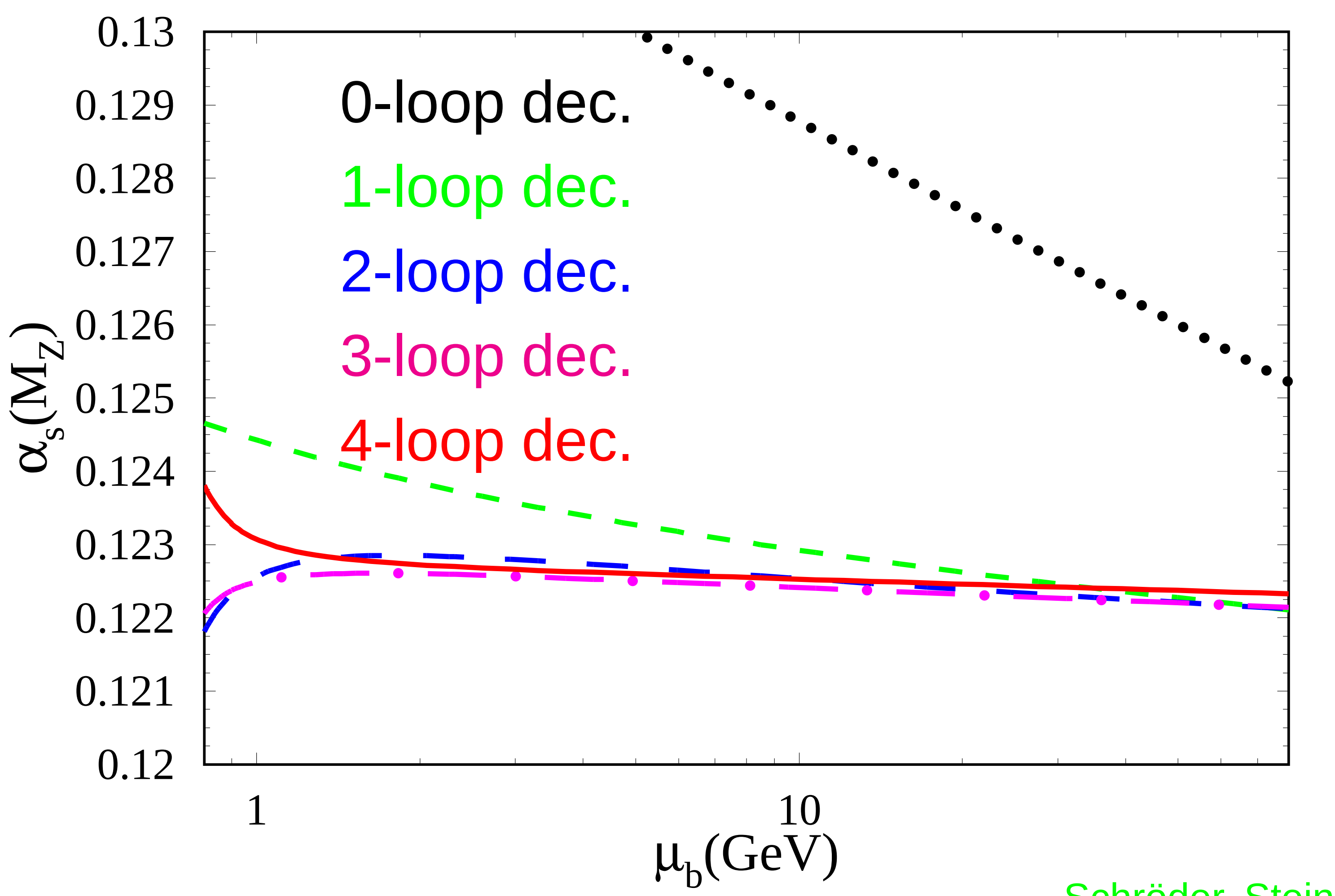
- 

- Dependence of $\alpha_s^{(5)}(M_Z)$ on μ_b
  for 0-, 1-, 2-, 3- and 4-loop decoupling
- Note: n-loop decoupling $\Leftrightarrow$ $(n+1)$-loop running
- set 5-loop coefficient of β function to zero

$\alpha_s^{(4)}(M_\tau) \Rightarrow \alpha_s^{(5)}(M_Z)$



Schröder, Steinhauser

IV. Summary & Conclusion

- Calculation of higher Taylor-coefficients of the polarization function allows a precise determination of the charm- and bottom-quark mass
- Significant progress in the evaluation of 4 loop vakuum integrals during the last year:
  - algebraic reduction and masters
  - phenomenological applications:
    sum rules, decoupling, ρ-parameter
  - all results confirmed by two independent calculations
- New data:

$$m_c(3\text{ GeV}) = 1.007(16)\text{ GeV}, \quad m_b(10\text{ GeV}) = 3.630(30)\text{ GeV}$$
$$m_c(m_c) = 1.300(15)\text{ GeV}, \quad m_b(m_b) = 4.179(35)\text{ GeV}$$