Charm and bottom masses at NNLO from electron-positron annihilation at low energies

J.H. Kühn, M. Steinhauser

I. Experimental Results for $R$ below $B\bar{B}$-Threshold $\leftrightarrow \alpha_s$

II. Sum Rules to NNLO with Massive Quarks $\leftrightarrow m_Q(m_Q)$
   updates based on recent data

III. Summary
I. Experimental Results for $R$ below $B\bar{B}$-Threshold $\Rightarrow \alpha_s$

- data
- $\alpha_s$
Data vs. Theory

\begin{figure}
\centering
\includegraphics[width=\textwidth]{data_vs_theory.png}
\caption{Comparison of experimental data (red points with error bars) from BES, MD-1, and CLEO with theoretical predictions.}
\end{figure}
<table>
<thead>
<tr>
<th>experiment</th>
<th>energy [GeV]</th>
<th>date</th>
<th>systematic error</th>
</tr>
</thead>
<tbody>
<tr>
<td>BES</td>
<td>2 — 5</td>
<td>2001</td>
<td>4%</td>
</tr>
<tr>
<td>MD-1</td>
<td>7.2 — 10.34</td>
<td>1996</td>
<td>4%</td>
</tr>
<tr>
<td>CLEO</td>
<td>10.52</td>
<td>1998</td>
<td>2%</td>
</tr>
<tr>
<td>PDG</td>
<td>$J/\psi$</td>
<td></td>
<td>$(7%)$ 3%</td>
</tr>
<tr>
<td>PDG</td>
<td>$\psi'$</td>
<td></td>
<td>$(9%)$ 5.7%</td>
</tr>
<tr>
<td>PDG</td>
<td>$\psi''$</td>
<td></td>
<td>15%</td>
</tr>
</tbody>
</table>

pQCD and data agree well in the regions

$2 — 3.73$ GeV and $5 — 10.52$ GeV
pQCD includes full $m_Q$-dependence up to $\mathcal{O}(\alpha_s^2)$
and terms of $\mathcal{O}(\alpha_s^3(m^2/s)^n)$ with $n = 0, 1, 2$

can we deduce $\alpha_s$ from the low energy data?

**Result:**

BES below 3.73 GeV: $\alpha_s^{(3)}(3 \text{ GeV}) = 0.369^{+0.047+0.123}_{-0.046-0.130}$

BES at 4.8 GeV: $\alpha_s^{(4)}(4.8 \text{ GeV}) = 0.183^{+0.059+0.053}_{-0.064-0.057}$

MD-1: $\alpha_s^{(4)}(8.9 \text{ GeV}) = 0.193^{+0.017+0.127}_{-0.017-0.107}$

CLEO: $\alpha_s^{(4)}(10.52 \text{ GeV}) = 0.186^{+0.008+0.061}_{-0.008-0.057}$
combined, assuming uncorrelated errors:

\[ \alpha_s^{(4)}(5 \text{ GeV}) = 0.235 \pm 0.047 \]

Evolve up to \( M_Z \) : \( \alpha_s^{(5)}(M_Z) = 0.124^{+0.011}_{-0.014} \)

confirmation of running!

Result consistent with LEP, but not competitive
(precision of 0.4% at 3.7 GeV (0.7% at 2 GeV) would be required)

The evaluation of \( R(s) \) in order \( \alpha_s^4 \) is within reach
(Baikov, Chetyrkin, JK)
II. Sum Rules to NNLO with Massive Quarks

- $m_Q$ from SVZ Sum Rules, Moments and Tadpoles
- Tadpoles at Three Loop
- Results for Charm and Bottom Masses
Some definitions

\[ R(s) = 12\pi \text{Im} \left[ \Pi(q^2 = s + i\epsilon) \right] \]

\[ \left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right) \Pi(q^2) \equiv i \int dx \, e^{iqx} \langle T j_\mu(x) j_\nu(0) \rangle \]

with the electromagnetic current \( j_\mu \)

Taylor expansion: \( \Pi_c(q^2) = Q_c^2 \frac{3}{16\pi^2} \sum_{n \geq 0} \tilde{C}_n \, z^n \)

with \( z = q^2/(4m_c^2) \) and \( m_c = m_c(m) \) the \( \overline{\text{MS}} \) mass.
Coefficients $\tilde{C}_n$ up to $n = 8$ known analytically in order $\alpha_s^2$ (Chetyrkin, JK, Steinhauser)

recently also $\tilde{C}_0$ in order $\alpha_s^3$ (four loops!) (Chetyrkin, JK, Sturm)

$\tilde{C}_1$ to order $\alpha_s^3$ is within reach
all three-loop – one-scale tadpole amplitudes can be calculated with “arbitrary” power of propagators (Broadhurst; Chetyrkin, JK, Steinhauser); FORM-program MATAD (Steinhauser)

Three-loop diagrams contributing to $\Pi^{(2)}_l$ (inner quark massless) and $\Pi^{(2)}_F$ (both quarks with mass $m$).

Purely gluonic contribution to $\Theta(\alpha_s^2)$
\( \tilde{C}_n \) depend on the charm quark mass through

\[
l_{mc} \equiv \ln(m_c^2(\mu)/\mu^2)
\]

\[
\tilde{C}_n = \tilde{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi} \left( \tilde{C}_n^{(10)} + \tilde{C}_n^{(11)} l_{mc} \right)
\]

\[
+ \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left( \tilde{C}_n^{(20)} + \tilde{C}_n^{(21)} l_{mc} + \tilde{C}_n^{(22)} l_{mc}^2 \right)
\]

| \( n \) | \( \tilde{C}_n^{(0)} \) | \( \tilde{C}_n^{(10)} \) | \( \tilde{C}_n^{(11)} \) | \( \tilde{C}_n^{(20)} \) | \( \tilde{C}_n^{(21)} \) | \( \tilde{C}_n^{(22)} \) |
|---|---|---|---|---|---|
| 1 | 1.0667 | 2.5547 | 2.1333 | 2.4967 | 3.3130 | -0.0889 |
| 2 | 0.4571 | 1.1096 | 1.8286 | 2.7770 | 5.1489 | 1.7524 |
| 3 | 0.2709 | 0.5194 | 1.6254 | 1.6388 | 4.7207 | 3.1831 |
| 4 | 0.1847 | 0.2031 | 1.4776 | 0.7956 | 3.6440 | 4.3713 |
Define the moments

\[
\mathcal{M}_n^{\text{th}} \equiv \frac{12\pi^2}{n!} \left( \frac{d}{dq^2} \right)^n \Pi_c(q^2) \bigg|_{q^2=0} = \frac{9}{4} Q_c^2 \left( \frac{1}{4m_c^2} \right)^n \bar{C}_n
\]

dispersion relation:

\[
\Pi_c(q^2) = \frac{q^2}{12\pi^2} \int ds \frac{R_c(s)}{s(s - q^2)} + \text{subtraction}
\]

\[
\mathcal{M}_n^{\exp} = \int \frac{ds}{s^{n+1}} R_c(s)
\]

constraint:  \[
\mathcal{M}_n^{\exp} = \mathcal{M}_n^{\text{th}}
\]

\[
\Rightarrow m_c
\]
SVZ:
$M_n^{\text{th}}$ can be reliably calculated in pQCD: low $n$:

- fixed order in $\alpha_s$ is sufficient, in particular no resummation of $1/\nu$ - terms from higher orders required
- condensates are unimportant
- pQCD in terms of short distance mass: $m_c(3 \text{ GeV}) \Rightarrow m_c(m_c)$
  stable expansion: no pole mass or closely related definition (1S-mass, potential-subtracted mass) involved
- moments available in NNLO
- and soon $\tilde{C}_0$, $\tilde{C}_1$, $\tilde{C}_2(?)$ in $N^3\text{LO}$
input for $R(s)$

- resonances ($J/\psi$, $\psi'$)
- continuum below 4.8 GeV (BES)
- continuum above 4.8 GeV (theory)

Experimental error of the moments dominated by resonances

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_c(3 \text{ GeV})$</td>
<td>1.027(30)</td>
<td>0.994(37)</td>
<td>0.961(59)</td>
<td>0.997(67)</td>
</tr>
<tr>
<td>$m_c(m_c)$</td>
<td>1.304(27)</td>
<td>1.274(34)</td>
<td>1.244(54)</td>
<td>1.277(62)</td>
</tr>
</tbody>
</table>

Error in $m_c$ dominated by experiment for $n=1$, by theory (variation of $\mu$, $\alpha_s$) for $n = 3, 4, \ldots$
stability: compare LO, NLO, NNLO ⇔ clear improvement

\[ m_c(m_c) \text{ for } n = 1, 2, 3, 4 \text{ in LO, NLO, NNLO}. \]
anatomy of errors and update: $m_c$

old results and

new results (update on $\Gamma_e(J/\psi, \psi')$ and $\alpha_s = 0.1187 \pm 0.0020$)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$J/\psi, \psi'$</th>
<th>charm threshold region</th>
<th>continuum</th>
<th>sum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_{n}^{\text{exp, res}} \times 10^{(n-1)}$</td>
<td>$M_{n}^{\text{exp, cc}} \times 10^{(n-1)}$</td>
<td>$M_{n}^{\text{cont}} \times 10^{(n-1)}$</td>
<td>$M_{n}^{\text{exp}} \times 10^{(n-1)}$</td>
</tr>
<tr>
<td>1</td>
<td>0.1114(82)</td>
<td>0.0313(15)</td>
<td>0.0638(10)</td>
<td>0.2065(84)</td>
</tr>
<tr>
<td>1</td>
<td>0.1138(40)</td>
<td>0.0313(15)</td>
<td>0.0639(10)</td>
<td>0.2090(44)</td>
</tr>
<tr>
<td>2</td>
<td>0.1096(79)</td>
<td>0.0174(8)</td>
<td>0.0142(3)</td>
<td>0.1412(80)</td>
</tr>
<tr>
<td>2</td>
<td>0.1121(38)</td>
<td>0.0174(8)</td>
<td>0.0142(3)</td>
<td>0.1437(39)</td>
</tr>
</tbody>
</table>
old:

\[ m_c(m_c) = \begin{cases} 
1.304(27) \text{ GeV (from } n=1) \\
1.274(34) \text{ GeV (from } n=2) 
\end{cases} \]

new:

\[ m_c(m_c) = \begin{cases} 
1.300(15) \text{ GeV (from } n=1), \text{ error dominated by exp.} \\
1.269(25) \text{ GeV (from } n=2), \text{ error dominated by th.} 
\end{cases} \]

new results consistent with old results; smaller error
Similar analysis for the **bottom quark**: resonances include $\Upsilon(1)$ up to $\Upsilon(3)$, “continuum” starts at 11.2 GeV

$m_b(m_b)$ for $n = 1, 2, 3$ and 4 in LO, NLO and NNLO
<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_b(10 \text{ GeV})$</td>
<td>3.665(60)</td>
<td>3.651(52)</td>
<td>3.641(48)</td>
<td>3.655(77)</td>
</tr>
<tr>
<td>$m_b(m_b)$</td>
<td>4.205(58)</td>
<td>4.191(51)</td>
<td>4.181(47)</td>
<td>4.195(75)</td>
</tr>
</tbody>
</table>
anatomy of errors and update: $m_b$

old results and update ($\Gamma_e$ from CLEO; $\alpha_s$)

<table>
<thead>
<tr>
<th>n</th>
<th>$\mathcal{M}_n^\text{exp, res}$ $\times 10^{(2n+1)}$</th>
<th>$\mathcal{M}_n^\text{exp, thr}$ $\times 10^{(2n+1)}$</th>
<th>$\mathcal{M}_n^\text{cont}$ $\times 10^{(2n+1)}$</th>
<th>$\mathcal{M}_n^\text{exp}$ $\times 10^{(2n+1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.237(63)</td>
<td>0.306(62)</td>
<td>2.913(21)</td>
<td>4.456(121)</td>
</tr>
<tr>
<td>1</td>
<td>1.271(24)</td>
<td>0.306(62)</td>
<td>2.918(16)</td>
<td>4.494(84)</td>
</tr>
<tr>
<td>2</td>
<td>1.312(65)</td>
<td>0.261(54)</td>
<td>1.182(12)</td>
<td>2.756(113)</td>
</tr>
<tr>
<td>2</td>
<td>1.348(25)</td>
<td>0.261(52)</td>
<td>1.185(9)</td>
<td>2.795(75)</td>
</tr>
<tr>
<td>3</td>
<td>1.399(68)</td>
<td>0.223(44)</td>
<td>0.634(8)</td>
<td>2.256(108)</td>
</tr>
<tr>
<td>3</td>
<td>1.437(26)</td>
<td>0.223(44)</td>
<td>0.636(6)</td>
<td>2.296(68)</td>
</tr>
</tbody>
</table>
old:

\[ m_b(m_b) = \begin{cases} 
4.205(58) \text{ GeV (from } n=1) \text{ error dominated by exp.} \\
4.191(51) \text{ GeV (from } n=2) \\
4.181(47) \text{ GeV (from } n=3) \text{ error dominated by exp.}
\end{cases} \]

new:

\[ m_b(m_b) = \begin{cases} 
4.191(40) \text{ GeV (from } n=1) \text{ error dominated by exp.} \\
4.179(35) \text{ GeV (from } n=2) \\
4.170(33) \text{ GeV (from } n=3) \text{ equal distr. of exp., } \alpha_s, \text{th.}
\end{cases} \]
III. Summary

\[ \alpha_s^{(4)}(5 \text{ GeV}) = 0.235 \pm 0.047 \]

\[ \Rightarrow \alpha_s^{(5)}(M_Z) = 0.124^{+0.011}_{-0.014} \]

\(\Rightarrow\) drastic improvement in \(\delta m_c, \delta m_b\) from moments with low \(n\) in \(N^2\)LO

\(\Rightarrow\) direct determination of short-distance mass

old results:

\[
\begin{align*}
m_c(m_c) &= 1.304(27) \text{ GeV} \\
m_b(m_b) &= 4.19(5) \text{ GeV}
\end{align*}
\]
improved measurements of $\Gamma_e(J/\psi, \psi')$ and $\Gamma_e(\Upsilon, \Upsilon', \Upsilon'')$ lead to significant improvements

preliminary results:

\begin{align*}
\begin{array}{ll}
m_c(m_c) & = 1.300(15) \text{ GeV} \\
M_c & = 1.696(19) \text{ GeV} \\
m_b(m_b) & = 4.179(35) \text{ GeV} \\
M_b & = 4.815(40) \text{ GeV}
\end{array}
\end{align*}