FOUR- and FIVE-LOOP CALCULATIONS and QCD PRECISION STUDIES

α_s at GIGA-Z

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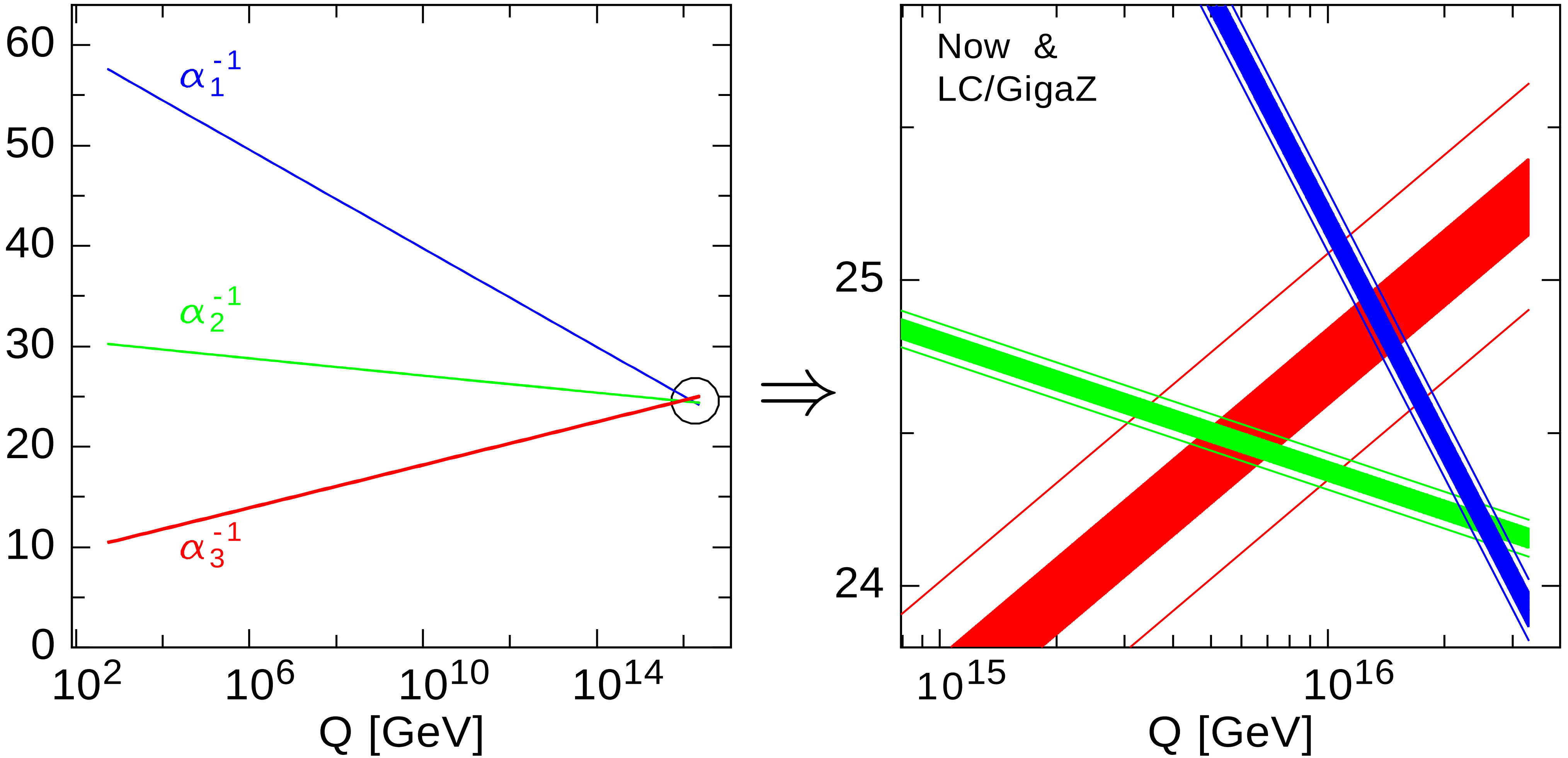
from Zerwas



$\alpha_s = 0.1183 \pm 0.0027$ vs ± 0.0009

α_s from LEP and GIGA-Z

α_s from LEP

$$R_\ell = \Gamma_h/\Gamma_\ell = 20.767 \pm 0.025 \qquad (1.2\ {}^0\!/_{00}\ !)$$

($\sim 10^6$ leptonic events)

$$\alpha_s = 0.1226 \pm 0.0038\ {}^{+0.0028}_{-0.0}\left(M_H = {}^{900}_{100}\text{GeV}\right)$$

$$\sigma_\ell \sim \frac{\Gamma_\ell^2}{\Gamma_{tot}^2} = 2.003 \pm 0.0027\text{pb}$$

(luminosity)

$$\alpha_s = 0.1183 \pm 0.0030\ {}^{+0.0022}_{-0.0}\left(M_H = {}^{900}_{100}GeV\right)$$

SM-fit: $\alpha_s = 0.1188 \pm 0.0027$

based on $\sim 10^7$ Z-events/experiment

GIGA-Z: 10^9 events

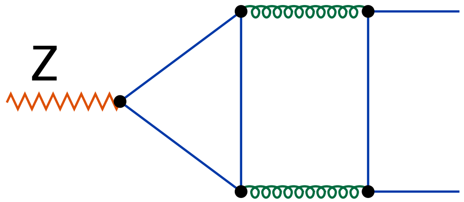
$\Longrightarrow \delta\alpha_s = 0.0009$ M. Winter

α_s based on

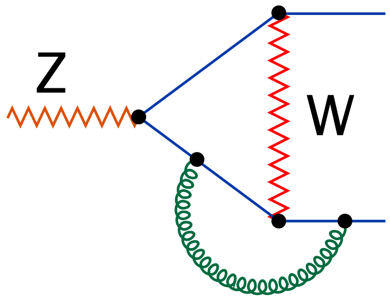
$$\Gamma_{\text{had}} = \Gamma_0 \left(1 + \frac{\alpha_s}{\pi} + 1.409 \frac{\alpha_s^2}{\pi^2} - 12.767 \frac{\alpha_s^3}{\pi^3} \right)$$

$$+ \ \text{corrections}$$

$\Big(\sim m_b^2/M_z^2$;

singlett terms 

$+ \ldots.$

mixed QCD $\otimes$ electroweak 

$+ \ldots \Big)$

dominant theory error:
uncalculated higher orders! α_s^4
substitute:
optimization schemes (PMS,FAC): $-97\left(\frac{\alpha_s}{\pi}\right)^4$ (Kataev)

estimates for uncertainty:

- conservative: last calculated term (α_s^3)
  $\Longrightarrow \delta\alpha_s = 0.002; \quad \delta\alpha_s/\alpha_s = 1.7\%$
- "standard" (optimistic): estimated α_s^4 term
  $\Longrightarrow \delta\alpha_s = 0.0006; \quad \delta\alpha_s/\alpha_s = 0.5\%$
- scale variation: $\mu = \frac{1}{3}\sqrt{s} - 3\sqrt{s}$

$\delta\alpha_s = {}^{+0.002}_{+0.00016}$ $\qquad \delta\alpha_s/\alpha_s = 1.7 - 0.1\%$ (asymmetric!)

theory error

smaller than **present** experimental error (but not much!)

highly relevant for GIGA-Z

$\Longrightarrow$ reliable information on α_s^4-term required for GIGA-Z

and even more so for $\Gamma(\tau \to \nu \text{ had})/\Gamma(\tau \to e\nu\nu)$

comment on R at 10.52 GeV:

$$\frac{\delta\alpha_s(M_Z)}{\alpha_s(M_Z)} = 3\%$$

would require $\delta R_{exp}/R_{exp} \approx 0.3\%$ at 10.52 GeV

$\approx 0.5\%$ at 3.7 GeV

Theory:
The long march towards α_s^4

Massless Correlators: Technicalities

Correlator of two currents $\boldsymbol{j = \bar{q}\,\Gamma\,q}$ and $\boldsymbol{j^\dagger}$

$$\boldsymbol{\Pi^{jj}(q^2=-Q^2) = i\int \mathrm{d}x e^{iqx}\langle 0|T[\;\; j(x) j^\dagger(0)\;]|0\rangle}$$

related to the corresponding absorptive part $R(s)$ through

$$R^{jj}(s) \approx \Im\,\Pi^{jj}(s-i\delta)$$

RG equation $(a_s \equiv \alpha_s/\pi)$

$$\Pi^{jj} = Z^{jj} + \Pi^B(-Q^2, \alpha_s^B)$$

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta(a_s)\frac{\partial}{\partial a_s}\right)\Pi = \gamma^{jj}(a_s)$$

extremely useful for determining the absorptive part of Π^{jj}

For Π at $(L+1)$ loop

$$\frac{\partial}{\partial \log(\mu^2)}\Pi = \gamma^{jj}(a_s) - \left(\beta(a_s)\frac{\partial}{\partial a_s}\right)\Pi$$

anom.dim. at a_s^L
(L+1) loop integrals

L-loop integrals only contribute due to the factor of $\beta(a_s)$

- to find Log-dependent part of Π at (L+1)-loops one needs (L+1)-loop anomalous dimension γ^{jj} and L-loop Π (BUT! including its constant part)

- (L+1) loop anom.dim. reducible to L-loop p-integrals

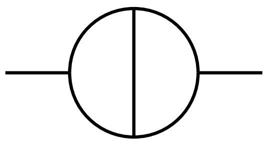
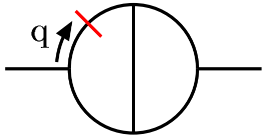
Strategy

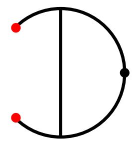
α_s^4 requires absorptive part of 5-loop correlator

$\widehat{=}$ divergent part $(1/\epsilon)$ of 5-loop correlator

A finite part of 4-loop $\Rightarrow$ div. part of 5-loop

systematic, automatized algorithm (Chetyrkin)

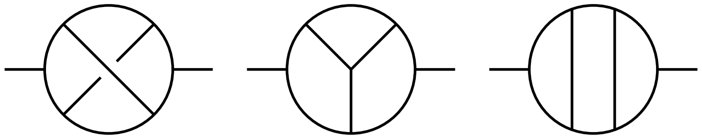
div  $\widehat{=} \int dq^2$ 

requires 

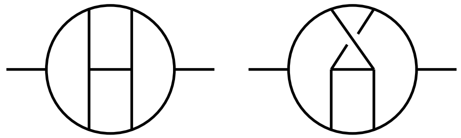
B finite part of 4-loop massless propagators difficult!

compare 3- and 4-loop calculation

3 topologies without insertions

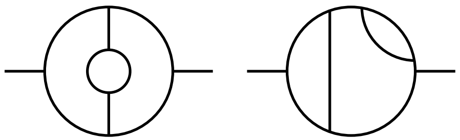


11 topologies without insertion



. . .

14 topologies with+without insertions



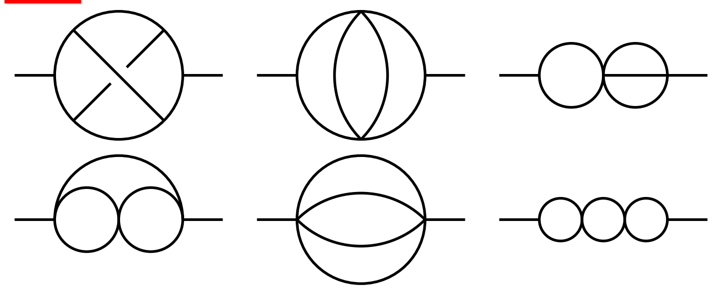
. . .

∼150 topologies with+without insertions

. . .

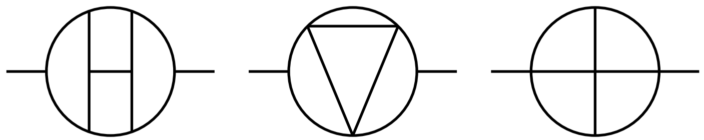
reduction to master integrals:
MINCER

6 master integrals



reduction to master integrals ???

28 master integrals



. . .

MINCER: 3-loop (Larin, Tkatchov, Vermaseren)

recursion relations based on integration by parts identities!

reduction algorithm and program constructed "manually" for 14 topologies.

4-loop:

more complicated identities

$\sim$ 150 topologies . . .

straightforward generalization of MINCER difficult

$\Rightarrow$ fully automatized construction of program; new concept?

C Baikov: recursion relations can be solved "mechanically" in the limit of large dimension d:

consider amplitude f:

$$f(\text{topology, power of prop, } d)$$
$$= \sum_{\alpha=\text{masters}} C^{(\alpha)}(\text{topology, power of prop, } d) \star f^{(\alpha)}(d)$$

$f^{(\alpha)}$: 28 masters, analytically or numerically solvable

$C^{(\alpha)}$: rational function $\frac{P^n(d)}{Q^m(d)}$, to be calculated

expand $C^{(\alpha)}$:

$$C^{(\alpha)} = \sum_k c_k^{(\alpha)}(\text{topology, power of prop})(1/d)^k + \ldots$$

sufficiently many terms $c_k^{(\alpha)} \Rightarrow C^{(\alpha)}$ m, n depend on power of propagators!

evaluation of $c_k^{(\alpha)}$:

handling of polynomials of 9 variables of degree k

$\frac{(9+k)!}{9!\,k!}$ terms $\quad k = 40 \;\Rightarrow\; 2 \cdot 10^9$ terms

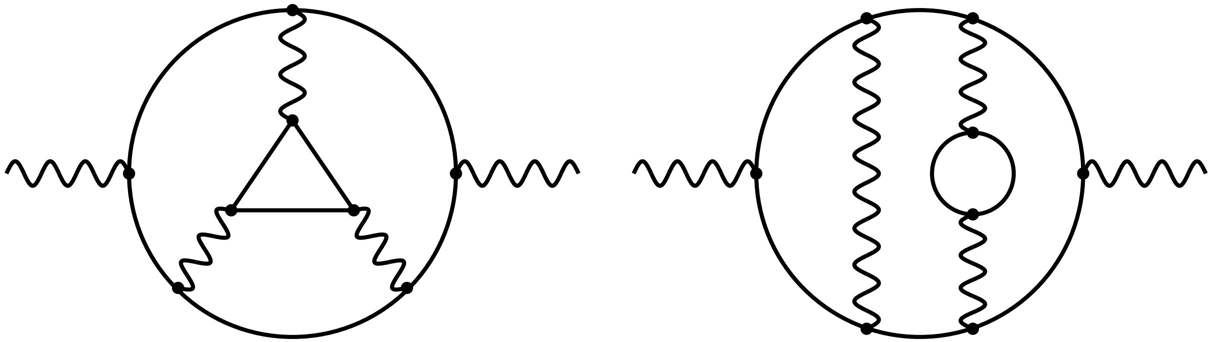
$k = 24 \;\Rightarrow\; 4 \cdot 10^7$ terms (4 GB disk $\longrightarrow$ 40 GB)

weeks of runtime

addition information on structure of $P^n(d)$, $Q^m(d)$ may lead to drastic reduction of hardware requirements

Results

four-loop n_f-terms done (2002)



$\Rightarrow$ leading and subleading n_f terms for $R_{e^+e^-}$, R_τ, m^2/s-terms:

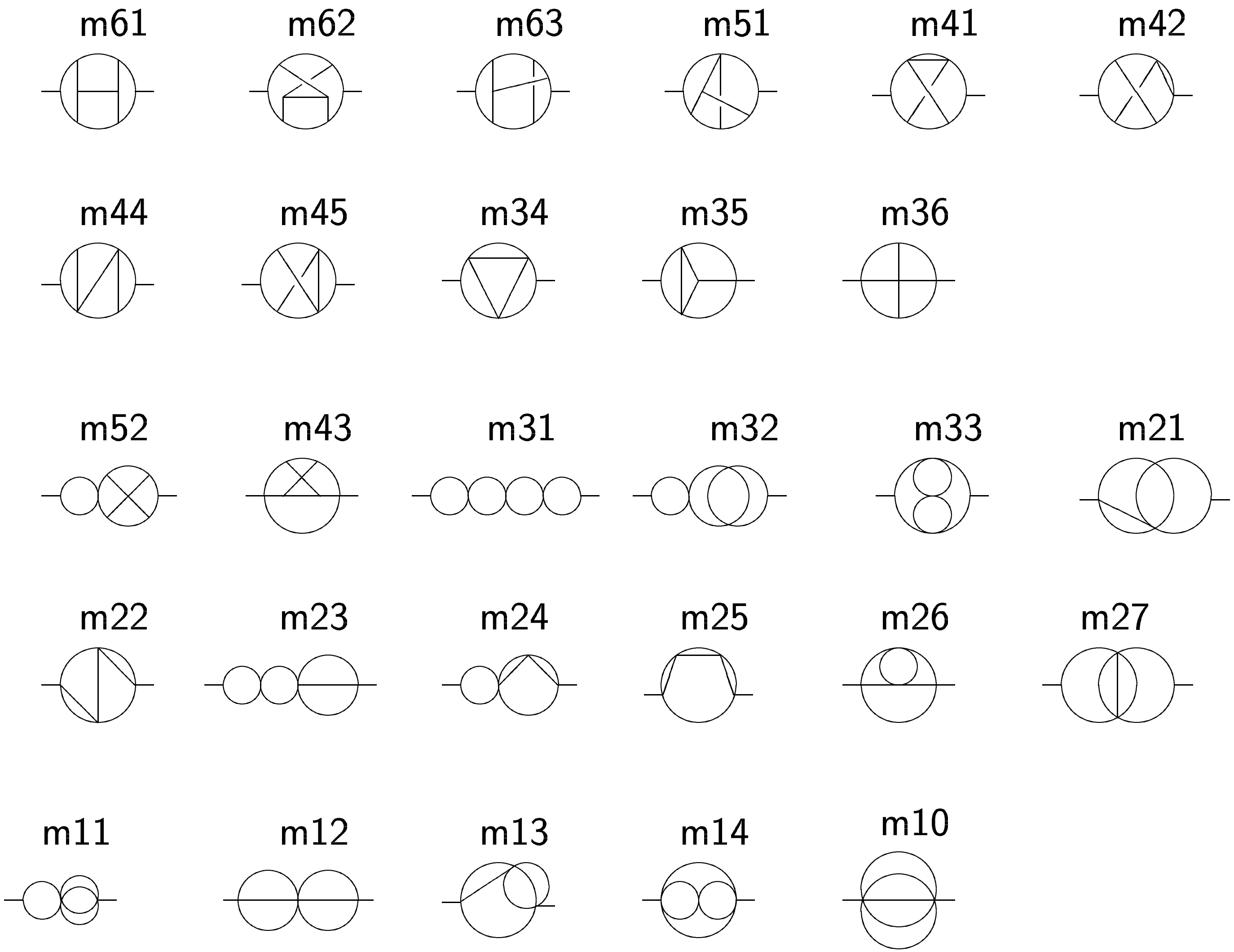
$\alpha_s^4 n_f^3$ (renormalon chain)

$\alpha_s^4 n_f^2$ new

$$R = 1 + ...(?+?n_f - 0.7974\, n_f^2 + 0.02152\, n_f^3)\ \left(\frac{\alpha_s}{\pi}\right)^4$$

NEW All relevant Master Integrals solved (2004)

(method: “glue and cut” (Chetyrkin, Tkachov))



$\Longrightarrow$ Complete 4-loop mass correction

Define

$$\begin{aligned}\Pi_{\mu\nu} &= (-g_{\mu\nu}\, q^2 + q_\mu q_\nu)\Pi(q^2) \\ \Pi(q^2) &= \Pi_0(q^2) + \frac{m^2}{q^2}\Pi_2(q^2)\end{aligned}$$

finite part of Π_2 in $\mathcal{O}(\alpha_s^n)$ + RG

$\Rightarrow \log$ part in $\mathcal{O}(\alpha_s^{n+1})$

$\Rightarrow Im$ part in $\mathcal{O}(\alpha_s^{n+1})$

Result for constant part in Π_2

$$\Pi_2 = -8 \quad - \quad \frac{64}{3} a_s + a_s^2 \left\{ \frac{95}{9} n_f + -\frac{18923}{54} - \frac{784}{27}\zeta_3 + \frac{4180}{27}\zeta_5 \right\}$$
$$+ \quad a_s^3 \left\{ \left[-\frac{5161}{1458} - \frac{8}{27}\zeta_3 \right] n_f^2 + \left[\frac{62893}{162} + \frac{424}{27}\zeta_3^2 - \frac{4150}{243}\zeta_3 \right. \right.$$
$$\left. \left. + \frac{20}{3}\zeta_4 - \frac{28880}{243}\zeta_5 \right] n_f + k_{2,0}^{[V]3} \right\}$$

$$k^{[V]3} = -\frac{10499303}{1944} + \frac{66820}{81}\zeta_3 - \frac{7225}{27}\zeta_3^2 + \frac{281390}{81}\zeta_5 - \frac{1027019}{648}\zeta_7$$

Numerically,

$$\Pi_2 \quad = \quad -8 - 21.333\, a_s + a_s^2\,(10.56\, n_f - 224.80)$$
$$+ a_s^3\,(-3.896\, n_f^2 + 274.37\, n_f - 2791.81)$$

Result for r_2^V

Define

$$R(s) = 3\left\{r_0^V + \frac{m^2}{s} r_2^V\right\} + \cdots = 3\left\{\sum_{i\geq 0} a_s^i \left(r_0^{V,i} + \frac{m^2}{s} r_2^{V,i}\right)\right\} + \ldots$$

$$r_2^V = 12\, a_s + a_s^2\,(-4.3333\; n_f + 126.5) + a_s^3\left(1.2182\, n_f^2 - 104.167\, n_f + 1032.14\right)$$

$$+a_s^4\left(-0.20345\, n_f^3 + 49.0839\, n_f^2 + \;\; r_{2,1}^{V,4}\, n_f \;\; + {r^{V,4}}_{2,0}\right)$$

For, say, $n_f = 4$ we get

$$r_2^V/12 \text{ (exact)} = a_s + 9.09722 a_s^2 + 52.913 a_s^3 + 128.499 a_s^4$$

$$r_2^V/12 \text{ (PMS)} = a_s + 9.09722 a_s^2 + 52.913 a_s^3 + 177 a_s^4$$

$$r_2^V/12 \text{ (FAC)} = a_s + 9.09722 a_s^2 + 52.913 a_s^3 + 197 a_s^4$$

New RESULT ($N_F = 3$)

Five-loop anomalous dimension of the SS-correlator:

$$\gamma_q^{SS} = -6 + a_s - 10a_s^2\left[-\frac{383}{12} + 3\,\zeta_3\right] + a_s^3\left[-\frac{122935}{864} + \frac{2365}{36}\zeta_3 + \frac{21}{4}\zeta_4 - \frac{65}{2}\zeta_5\right]$$

$$+ \quad a_s^4\left[-\frac{39620177}{41472} + \frac{1288967}{3456}\zeta_3 - \frac{29425}{288}\zeta_3^2 - \frac{44971}{384}\zeta_4 - \frac{124045}{576}\zeta_5 + \frac{16375}{64}\zeta_6 + \frac{97895}{384}\zeta_7\right]$$

$\zeta(6)$ and $\zeta(7)$ appear first at five loop level

$$R^{SS} = 1 + 5.667\,a_s + 31.86\,a_s^2 + 89.16\,a_s^3 - 536.84\,a_s^4$$

Some Details about Complexity of the Calculation of the γ_q^{SS} in five loops

- CPU-time consumption (very approximately) $3 \cdot 10^8$ sec (about 10 years) for a 1.5GH PC

- Due to the heavy use of the SGI cluster
  (of 32 parallel SMP CPU of 1.5 GH frequency each)
  the calculation took about 15 calendar months

- Vector case:
  optimistically vector is 3 times more complicated,
  pessimistically 10 times

- Experience $\otimes$ improved programs $\otimes$ better hardware
  $\Longrightarrow$ 1-2 years

SUMMARY

α_s^4 term needed for GIGA-Z (and τ decays)

systematic reduction to master integrals (Baikov)

- Master integrals solved
- ImΠ^{SS} available (H$\longrightarrow b\bar{b}$)
- Mass terms of $\mathcal{O}\left(\alpha_s^4\right)$ available
- Im$\Pi^{VV} = R^V$ within reach
- Huge demands on computing

Other applications:

- Coefficient functions in OPE (DIS!)
- Sum rules
- m_s

⋮