Progress on Two-Loop Non-Propagator Integrals

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Introduction

Experimental precision for many $2 \to 2$ and $1 \to 3$ reactions has reached a level that demands theoretical predictions at next-to-next-to-leading order.

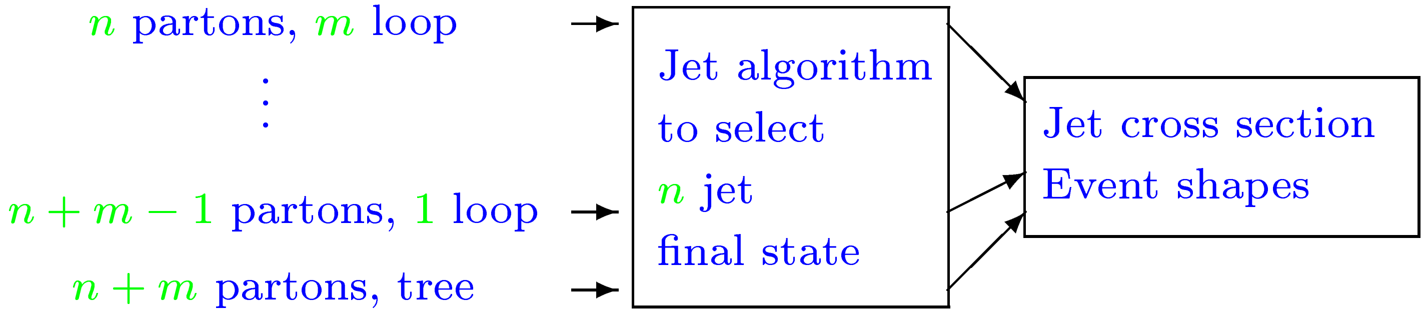
Examples:

- Bhabha-Scattering: $e^+e^- \to e^+e^-$
- Three Jet observables in e^+e^-
- DIS $(2{+}1)$-Jet production
- Hadron-Hadron 2-Jet and $V + 1$-Jet production

Calculation of Jet Observables

General Structure:

n jets, m–th order in perturbation theory



- Jet algorithm acts differently on different partonic final states
  $\longrightarrow$ Optical theorem can not be applied
- Divergencies from real and virtual contributions must be extracted before application of jet algorithm

Calculation of Jet Observables

Techniques for combining real and virtual contributions:

- Phase space slicing
  G. Kramer et al.; W. Giele, N. Glover
- Subtraction
  K. Ellis, D. Ross, A. Terrano; S. Catani, M. Seymour
- Hybrid Subtraction
  N. Glover, M. Sutton

All techniques require the analytic calculation of the amplitudes for all subprocesses (or at least of their divergent parts).

Major missing ingredient for three-jet type observables in e^+e^- at NNLO:

Two-loop four-point integrals with massless propagators and one off-shell leg

Reduction of Scalar Two-Loop Four-Point Functions

Generic structure of scalar two-loop integrals:

$$I_{t,r,s}(p_1,\dots,p_n) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{D_1^{m_1}\dots D_t^{m_t}} S_1^{n_1}\dots S_q^{n_q}$$

D_i	:	massless scalar propagators
S_i	:	scalar products involving loop momenta
t	:	number of different propagators
$r=\sum_i m_i$	:	dimension of denominator
$s=\sum_i n_i$	:	dimension of numerator

Topology of Feynman graph defined by specifying the set of different propagators

$$\{D_1,\dots,D_t\}$$

Reduction of Scalar Two-Loop Four-Point Functions

Identities:

- Integration-by-parts (IBP)
  K. Chetyrkin, F. Tkachov

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{\partial}{\partial a^\mu} \left[b^\mu f(k,l,p_i) \right] = 0$$

with: $a^\mu = k^\mu, l^\mu$ and $b^\mu = k^\mu, l^\mu, p_i^\mu$

- Lorentz Invariance (LI)
  E. Remiddi, TG

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \delta\varepsilon^\mu_\nu \left(\sum_i p_i^\nu \frac{\partial}{\partial p_i^\mu} \right) f(k,l,p_i) = 0$$

For each two-loop four-point integral, one has 10 IBP and 3 LI identities.

Reduction of Scalar Two-Loop Four-Point Functions

The IBP and LI identities for $I_{t,r,s}$ relate:

$I_{t,r,s}$	:	the integral itself
$I_{t-1,r,s}$	:	simpler topology
$I_{t,r+1,s}, I_{t,r+1,s+1}$	:	same topology, more complicated than $I_{t,r,s}$
$I_{t,r-1,s}, I_{t,r-1,s-1}$	:	same topology, simpler than $I_{t,r,s}$

In numbers:

$\boxed{t = 7}$

different $I_{t,r,s}$

$r \backslash s$	0	1	2	3	4
7	1	2	3	4	5
8	7	14	21	28	35
9	28	56	84	112	140
10	84	168	252	336	420

accumulated equations / unknowns

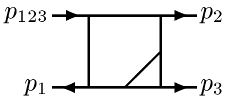
$r \backslash s$	0	1	2	3
7	13 22	39 45	78 76	130 115
8	104 106	312 213	624 354	1040 535
9	468 358	1404 717	2808 1196	4680 1795

\# equations grows faster than # unknowns

$\longrightarrow$ Reduction possible (using MAPLE and FORM)

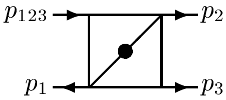
Reduction of Scalar Two-Loop Four-Point Functions

Example of a reducible integral:

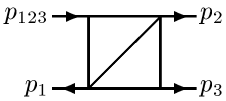


$$=$$

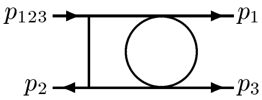
$$\frac{2(2d-9)}{(d-4)(d-6)}\,\frac{2s_{12}+s_{13}+s_{23}}{s_{13}s_{23}}$$



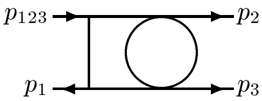
$$-\frac{3(d-4)}{d-6}\,\frac{(s_{13}+s_{23})^2}{s_{13}^2 s_{23}^2}$$



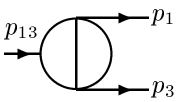
$$-\frac{6(d-3)(3d-14)}{(d-4)(d-6)}\,\frac{(s_{12}+s_{23})^2}{s_{13}^2 s_{23}^2}$$



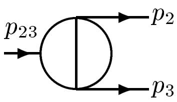
$$-\frac{6(d-3)(3d-14)}{(d-4)(d-6)}\,\frac{(s_{12}+s_{13})^2}{s_{13}^2 s_{23}^2}$$



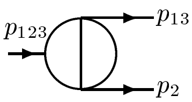
$$+\frac{3(d-3)(3d-10)(3d-14)}{2(d-4)^2(d-5)(d-6)}\,\frac{(2d-10)s_{12}+(3d-14)s_{23}}{s_{13}^2 s_{23}^2}$$



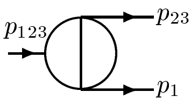
$$+\frac{3(d-3)(3d-10)(3d-14)}{2(d-4)^2(d-5)(d-6)}\,\frac{(2d-10)s_{12}+(3d-14)s_{13}}{s_{13}^2 s_{23}^2}$$



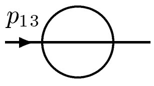
$$-\frac{3(d-3)(3d-10)}{(d-4)^2(d-6)}\,\frac{(3d-14)s_{12}+(4d-18)s_{13}-(d-4)s_{23}}{s_{13}^2 s_{23}^2}$$



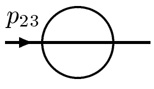
$$-\frac{3(d-3)(3d-10)}{(d-4)^2(d-6)}\,\frac{(3d-14)s_{12}-(d-4)s_{13}+(4d-18)s_{23}}{s_{13}^2 s_{23}^2}$$



$$-\frac{3(d-3)(3d-8)(3d-10)}{(d-4)^3(d-5)(d-6)}\,\frac{(d-5)(3d-14)(s_{12}+s_{13})+(d-4)^2 s_{23}}{s_{13}^3 s_{23}^2}$$



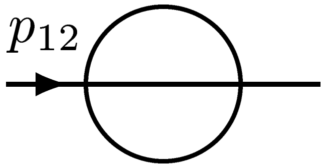
$$-\frac{3(d-3)(3d-8)(3d-10)}{(d-4)^3(d-5)(d-6)}\,\frac{(d-5)(3d-14)(s_{12}+s_{23})+(d-4)^2 s_{13}}{s_{13}^2 s_{23}^3}$$



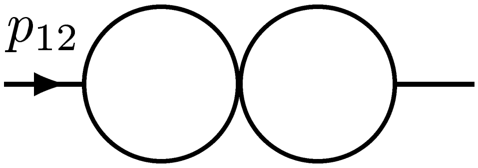
Calculation of Master Integrals

Genuine one-scale master integrals:

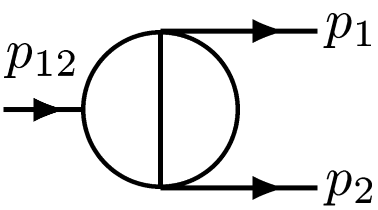
R. Gonsalves; G. Kramer, B. Lampe



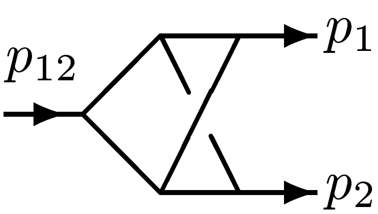
$$= \quad A_3 \left(-p_{12}^2\right)^{d-3}$$



$$= \quad A_{2,LO}^2 \left(-p_{12}^2\right)^{d-4}$$



$$= \quad A_4 \left(-p_{12}^2\right)^{d-4}$$



$$= \quad A_6 \left(-p_{12}^2\right)^{d-6}$$

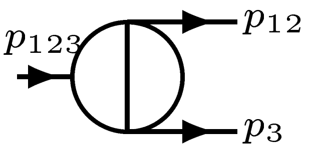
Calculation of Master Integrals

E. Remiddi

Multi-scale master integrals fulfil inhomogeneous differential equations in their external invariants.

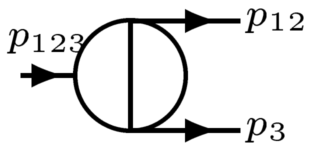
For example:

$$s_{123} \frac{\partial}{\partial s_{123}}$$

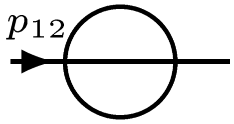


$$=$$

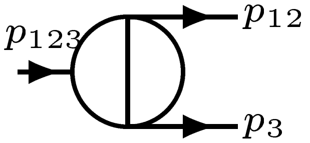
$$+ \frac{d-4}{2} \frac{2s_{123} - s_{12}}{s_{123} - s_{12}}$$



$$- \frac{3d-8}{2} \frac{1}{s_{123} - s_{12}}$$

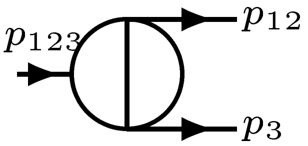


$$s_{12} \frac{\partial}{\partial s_{12}}$$

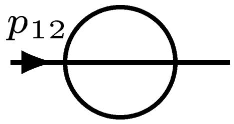


$$=$$

$$- \frac{d-4}{2} \frac{s_{12}}{s_{123} - s_{12}}$$



$$+ \frac{3d-8}{2} \frac{1}{s_{123} - s_{12}}$$



Calculation of Master Integrals

E. Remiddi, TG

Two-loop four-point functions with one off-shell leg depend on three invariants: s_{12}, s_{13}, s_{23}

Computation from differential equations:

- express differential equations in s_{123} (trivial homogeneous rescaling relation) and $y = s_{13}/s_{123}$, $z = s_{23}/s_{123}$ (inhomogeneous equations)

- solve differential equations with product ansatz
$$\mathcal{R}(y, z; s_{123}, \epsilon)\mathcal{H}(y, z; \epsilon)$$

- prefactor $\mathcal{R}$: rational function, can be determined from homogeneous part of equations in y and z

- Laurent-series $\mathcal{H}$: expansion in ϵ, with coefficients containing two-dimensional harmonic polylogarithms $H(\vec{m}(z); y)$

Harmonic Polylogarithms

E. Remiddi, J. Vermaseren

Harmonic Polylogarithms (HPL) $H(\vec{m};x)$ are an extension of the Nielsen polylogarithms $\mathrm{Li}_i(x)$ and $\mathrm{S}_{i,j}(x)$. They have the following properties:

- HPL are linear independent
- HPL fulfill a product algebra:

$$H(\vec{a};x)H(\vec{b};x) = \sum H(\vec{a} \oplus \vec{b};x)$$

- HPL form a closed set under class of integrations

$$\int_0^x \mathrm{d}x \left(\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}\right) H(\vec{b};x)$$

Harmonic Polylogarithms

Extension to Two-dimensional Harmonic Polylogarithms (2dHPL) $H(\vec{m}(z); y)$ is made by construction. They form a closed set under

$$\int_0^y \mathrm{d}y \left(\frac{1}{y}, \frac{1}{1-y}, \frac{1}{1-y-z}, \frac{1}{y+z} \right) H(\vec{b}(z); y)$$

2dHPL with up to three components in $\vec{m}(z)$ can be expressed in terms of Nielsen's polylogarithms. At four-component level: one-dimensional integral representation

2dHPL are the basis functions for two-loop four-point functions with one off-shell leg

$\longrightarrow$ computation of master integrals from differential equations reduces to algebraic determination of coefficients in the ansatz

Results

E. Remiddi, TG

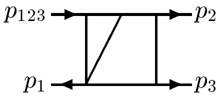
All eight two-loop four-point master integrals corresponding to planar topologies have been computed from their differential equations

- Analytic expressions for all divergent parts
- One-dimensional integral representation for finite parts
- Confirm Smirnov's recent result on two-loop double box with one off-shell leg

Work on non-planar master integrals is in progress

Results

For example:



$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \frac{(-s_{123})^{-2\epsilon}}{(s_{12}+s_{13})s_{23}} \sum_{i=0}^{4} \frac{f_{6.1,i}\left(\frac{s_{13}}{s_{123}}, \frac{s_{23}}{s_{123}}\right)}{\epsilon^i} + \mathcal{O}(\epsilon),$$

with:

$$\begin{aligned}
f_{6.1,4}(y,z) &= 0 \\
f_{6.1,3}(y,z) &= -\frac{1}{2}H(0;y) \\
f_{6.1,2}(y,z) &= +H(0,0;y) - H(1,0;y) - \frac{\pi^2}{6} \\
f_{6.1,1}(y,z) &= +H(0;y)H(1,0;z) + H(0;z)H(1-z,0;y) + H(0,0;z)H(0;y) - 2H(0,0,0;y) \\
&\quad -H(0,1,0;y) + H(0,1,0;z) + H(1,0;z)H(1-z;y) + 2H(1,0,0;y) - 2H(1,1,0;y) \\
&\quad +H(1,1,0;z) + H(1-z,1,0;y) - 3\zeta_3 \\
&\quad +\frac{\pi^2}{6}\left[+H(0;z) - 2H(1;y) + H(1;z) + H(1-z;y)\right] \\
f_{6.1,0}(y,z) &= -3H(0;y)H(1,0,0;z) - H(0;y)H(1,1,0;z) + 2H(0;z)H(1,1-z,0;y) \\
&\quad -2H(0;z)H(1-z,0,0;y) - H(0;z)H(1-z,1-z,0;y) - 2H(0,0;y)H(0,0;z) \\
&\quad -2H(0,0;y)H(1,0;z) + 2H(0,0;z)H(1,0;y) - 3H(0,0;z)H(1-z,0;y) \\
&\quad -3H(0,0,0;z)H(0;y) + 4H(0,0,0,0;y) + 2H(0,0,1,0;y) - 3H(0,0,1,0;z) \\
&\quad -H(0,1,0;z)H(0;y) + 2H(0,1,0;z)H(1;y) - 3H(0,1,0;z)H(1-z;y) + 2H(0,1,0,0;y) \\
&\quad -3H(0,1,0,0;z) + H(0,1,1,0;y) - 2H(0,1-z;y)H(1,0;z) - 2H(0,1-z,0;y)H(0;z) \\
&\quad -2H(0,1-z,1,0;y) + 2H(1,0;y)H(1,0;z) + 2H(1,0;z)H(1,1-z;y) \\
&\quad -H(1,0;z)H(1-z,0;y) - H(1,0;z)H(1-z,1-z;y) - 3H(1,0,0;z)H(1-z;y) \\
&\quad -4H(1,0,0,0;y) - 2H(1,0,1,0;y) - 3H(1,0,1,0;z) + 2H(1,1,0;z)H(1;y) \\
&\quad +4H(1,1,0,0;y) - 3H(1,1,0,0;z) - 4H(1,1,1,0;y) + 2H(1,1-z,1,0;y) \\
&\quad -2H(1-z,0,1,0;y) - 2H(1-z,1,0,0;y) + 2H(1-z,1,1,0;y) - H(1-z,1-z,1,0;y) \\
&\quad +\frac{\pi^4}{72} + \zeta_3\left[-2H(0;y) - 3H(0;z) - 6H(1;y) - 3H(1;z) - 3H(1-z;y)\right] \\
&\quad +\frac{\pi^2}{6}\Big[-H(0;y)H(0;z) - H(0;y)H(1;z) + 2H(0;z)H(1;y) - H(0;z)H(1-z;y) \\
&\quad -H(0,0;z) + H(0,1;y) - 2H(0,1-z;y) + 2H(1;y)H(1;z) - H(1,0;z) - 4H(1,1;y) \\
&\quad +2H(1,1-z;y) - H(1-z,0;y) + 2H(1-z,1;y) - H(1-z,1-z;y)\Big]
\end{aligned}$$

Summary and Outlook

Reduction of multi-leg integrals to a small set of master integrals using Integration-by-parts method and Lorentz-invariance

Calculation of master integrals using differential equations in external invariants

Solution of differential equations by construction of a linearly independent set of basis functions

Applications:

- Master integrals for massless two-loop four-point functions with one off-shell leg:
  $e^+e^- \to 3$ Jets, $ep \to (2+1)$ Jets,
  $pp \to (W^\pm, Z^0, \gamma^*) + 1$ Jet
  planar topologies: known
  non-planar topologies: work in progress