Introduction to renormalons

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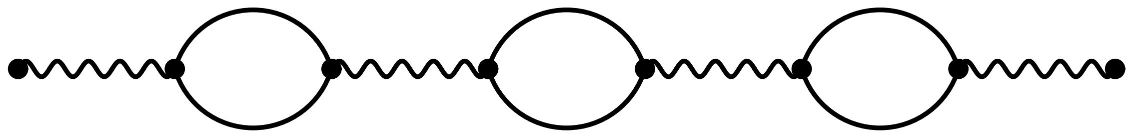
Dubna Calc-2000

$$
\begin{aligned}
A_0 =& 1 + \sum_{L=1}^{\infty} \sum_{n=0}^{L-1} a'_{Ln} n_f^n \left(\frac{g_0^2}{(4\pi)^{d/2}} \right)^L = 1 + \sum_{L=1}^{\infty} \sum_{n=0}^{L-1} a_{Ln} \beta_0^n \left(\frac{g_0^2}{(4\pi)^{d/2}} \right)^L \\
=& 1 + \frac{1}{\beta_0} f_0 \left(\frac{\beta_0 g_0^2}{(4\pi)^{d/2}} \right) + \mathcal{O}\left(\frac{1}{\beta_0^2} \right)
\end{aligned}
$$

$$
\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f
$$

Large β_0 limit: $n_f \to -\infty$

Naive nonabelianization: $a_{21}\beta_0 + a_{20}$ well approximated by $a_{21}\beta_0$



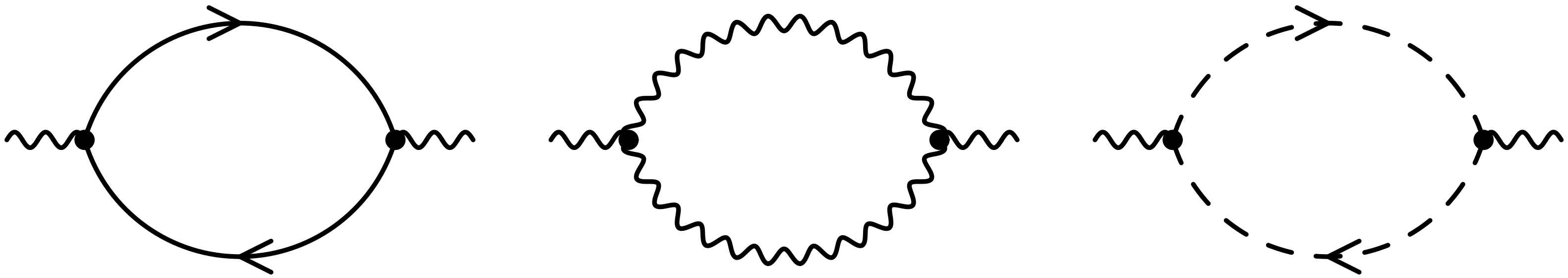
$$D_{0\mu\nu}^{(L-1)}(p) = -\frac{1}{(-p^2)^{1+(L-1)\epsilon}}\left(g_{\mu\nu} + \frac{p_\mu p_\nu}{-p^2}\right)$$

$$\times\left(-\frac{4}{3}T_F n_f \frac{g_0^2}{(4\pi)^{d/2}}\frac{D(\epsilon)}{\epsilon}e^{-\gamma\epsilon}\right)^{L-1}$$

$$D(\epsilon) = 6e^{\gamma\epsilon}\Gamma(1+\epsilon)B(2-\epsilon, 2-\epsilon) = 1 + \tfrac{5}{3}\epsilon + \cdots$$

$$a_{L,L-1} = \left(\frac{D(\epsilon)}{\epsilon}e^{-\gamma\epsilon}\right)^{L-1} a_1\big(1 + (L-1)\epsilon\big)$$

Alternative view



$$-\frac{4}{3}T_F n_f \to \beta_0 - \frac{C_A}{3}\left\{8\epsilon + \frac{3-2\epsilon}{2(1-\epsilon)}(a_0+3)\left[1-\frac{\epsilon}{2}(a_0+3)\right]\right\}$$

Gauge $a_0 = -3$

$\overline{\mathrm{MS}}$ coupling constant renormalization

$$\alpha_s = \overline{\mu}^{\,-2\epsilon} \frac{g^2}{4\pi} \quad \overline{\mu}^{\,2} = \mu^2 \frac{e^{\gamma}}{4\pi} \quad \text{or} \quad \frac{\alpha_s}{4\pi} = \mu^{-2\epsilon} Z_\alpha^{-1} \frac{g_0^2}{(4\pi)^{d/2}} e^{-\gamma\epsilon}$$

$$\frac{d\log\alpha_s}{d\log\mu} = -2\epsilon - 2\beta(\alpha_s) \quad \beta(\alpha_s) = \frac{1}{2}\frac{d\log Z_\alpha}{d\log\mu} = \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

In large β_0 limit $\beta_1 \sim \beta_0$, $\beta_2 \sim \beta_0^2 \ldots$

$$\beta = \frac{\beta_0 \alpha_s}{4\pi} \qquad \frac{d\log Z_\alpha}{d\log\beta} = -\frac{\beta}{\epsilon+\beta} \qquad Z_\alpha = \frac{1}{1+\beta/\epsilon}$$

$$
\begin{aligned}
A_0 =& 1 + \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{F(\epsilon, L\epsilon)}{L} \left(\frac{\beta}{\epsilon+\beta}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \\
F(\epsilon, u) =& u e^{\gamma\epsilon} a_1 \big(1 + (L-1)\epsilon\big) \mu^{2u} D(\epsilon)^{u/\epsilon - 1}
\end{aligned}
$$

We expand

$$
\left(\frac{\beta}{\epsilon+\beta}\right)^L = \left(\frac{\beta}{\epsilon}\right)^L \left[1 - L\frac{\beta}{\epsilon} + \frac{(L)_2}{2}\left(\frac{\beta}{\epsilon}\right)^2 - \frac{(L)_3}{3!}\left(\frac{\beta}{\epsilon}\right)^3 + \cdots\right]
$$

and $F(\epsilon, u) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{nm} \epsilon^n u^m$: quadruple sum

Renormalization $A_0 = ZA$

$$Z = 1 + \frac{Z_{11}}{\epsilon}\frac{\alpha_s}{4\pi} + \left(\frac{Z_{22}}{\epsilon^2} + \frac{Z_{21}}{\epsilon}\right)\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots = 1 + \frac{Z_1}{\epsilon} + \frac{Z_2}{\epsilon^2} + \cdots$$

$$\gamma = \frac{d\log Z}{d\log\mu} = -2\frac{dZ_1}{d\log\alpha_s}$$

$1/\epsilon$ terms

$$
\begin{aligned}
&\beta F_{00} - \beta^2(F_{10} + F_{01}) + \beta^3(F_{20} + F_{11} + F_{02}) \\
&\qquad\qquad - \beta^4(F_{30} + F_{21} + F_{12} + F_{03}) + \cdots \\
&+ \tfrac{1}{2}\beta^2(F_{10} + 2F_{01}) - \beta^3(F_{20} + 2F_{11} + 4F_{02}) \\
&\qquad\qquad + \tfrac{3}{2}\beta^4(F_{30} + 2F_{21} + 4F_{12} + 8F_{03}) + \cdots \\
&+ \tfrac{1}{3}\beta^3(F_{20} + 3F_{11} + 9F_{02}) - \beta^4(F_{30} + 3F_{21} + 9F_{12} + 27F_{03}) + \cdots \\
&+ \tfrac{1}{4}\beta^4(F_{30} + 4F_{21} + 16F_{12} + 64F_{03}) + \cdots \\
&+ \cdots \\
&= \beta F_{00} - \frac{\beta^2}{2}F_{10} + \frac{\beta^3}{3}F_{20} - \frac{\beta^4}{4}F_{30} + \cdots
\end{aligned}
$$

ϵ^0 terms

$$
\begin{aligned}
&\beta(F_{10} + F_{01}) - \beta^2(F_{20} + F_{11} + F_{02}) + \beta^3(F_{30} + F_{21} + F_{12} + F_{03}) \\
&\qquad\qquad\qquad - \beta^4(F_{40} + F_{31} + F_{22} + F_{13} + F_{04}) + \cdots \\
&+ \tfrac{1}{2}\beta^2(F_{20} + 2F_{11} + 4F_{02}) - \beta^3(F_{30} + 2F_{21} + 4F_{12} + 8F_{03}) \\
&\qquad\qquad\qquad + \tfrac{3}{2}\beta^4(F_{40} + 2F_{31} + 4F_{22} + 8F_{13} + 16F_{04}) + \cdots \\
&+ \tfrac{1}{3}\beta^3(F_{30} + 3F_{21} + 9F_{12} + 27F_{03}) \\
&\qquad\qquad\qquad - \beta^4(F_{40} + 3F_{31} + 9F_{22} + 27F_{13} + 81F_{04}) + \cdots \\
&+ \tfrac{1}{4}\beta^4(F_{40} + 4F_{31} + 16F_{22} + 64F_{13} + 256F_{04}) + \cdots \\
&+ \cdots \\
&= \beta F_{10} - \frac{\beta^2}{2}F_{20} + \frac{\beta^3}{3}F_{30} - \frac{\beta^4}{4}F_{40} + \cdots \\
&\qquad + \beta F_{01} + \beta^2 F_{02} + 2\beta^3 F_{03} + 6\beta^4 F_{04} + \cdots
\end{aligned}
$$

Anomalous dimension

$$\gamma = -\frac{2\beta}{\beta_0} F(-\beta, 0) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

Renormalized quantity

$$\begin{aligned} A(\mu) =& 1 + \frac{1}{\beta_0} \int_{-\beta}^{0} d\epsilon \frac{F(\epsilon, 0) - F(0,0)}{\epsilon} \\ & + \frac{1}{\beta_0} \int_{0}^{\infty} du\, e^{-u/\beta} \frac{F(0,u) - F(0,0)}{u} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \end{aligned}$$

where

$$\beta = \frac{\beta_0 \alpha_s(\mu)}{4\pi}$$

Renormalization group

$$
\begin{aligned}
\frac{dA(\mu)}{d\log\mu} + \gamma\big(\alpha_s(\mu)\big)A(\mu) = 0 \\
A(\mu) =& A(m)\exp\left[-\int\limits_{\alpha_s(m)}^{\alpha_s(\mu)} \frac{\gamma(\alpha)}{2\beta(\alpha)}\frac{d\alpha}{\alpha}\right] \\
=& \alpha_s(m)^{\frac{\gamma_0}{2\beta_0}}\hat{A}(m)\cdot\alpha_s(\mu)^{-\frac{\gamma_0}{2\beta_0}}K(\mu) \\
\hat{A}(m) =& A(m)K^{-1}(m) \\
K(\mu) =& \exp\left[-\int\limits_{0}^{\alpha_s(\mu)}\left(\frac{\gamma(\alpha)}{2\beta(\alpha)} - \frac{\gamma_0}{2\beta_0}\right)\frac{d\alpha}{\alpha}\right]
\end{aligned}
$$

$$\hat{A}(m) = 1 + \frac{1}{\beta_0} \int\limits_0^\infty du\, e^{-u/\beta} S(u) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$S(u) = \left. \frac{F(0,u) - F(0,0)}{u} \right|_{\mu=m}$$

$$\beta = \frac{\beta_0 \alpha_s(m)}{4\pi} \qquad \alpha_s(m) = \frac{2\pi}{\beta_0 \log \frac{m}{\Lambda_{\overline{\mathrm{MS}}}}} \qquad e^{-u/\beta} = \left(\frac{\Lambda_{\overline{\mathrm{MS}}}}{m}\right)^{2u}$$

Borel transform

$$\hat{A}(m) = 1 + \frac{1}{\beta_0} \sum_{L=1}^{\infty} c_L \beta^L \qquad S(u) = \sum_{L=1}^{\infty} \frac{c_L u^{L-1}}{(L-1)!}$$

$$c_L = \left(\frac{d}{du}\right)^{L-1} S(u)\Big|_{u=0}$$

Renormalon pole at $u = u_0$

$$S = \frac{1}{u_0 - u} \quad \Rightarrow \quad c_L = \frac{(L-1)!}{u_0^L}$$

Renormalon ambiguity: residue $\exp(-u_0/\beta) = \left(\Lambda_{\overline{\mathrm{MS}}}/m\right)^{2u_0}$
Minimum term of series $c_L \beta^L$ at $L \approx u_0/\beta$
is $\exp(-u_0/\beta) = \left(\Lambda_{\overline{\mathrm{MS}}}/m\right)^{2u_0}$
$u_0 = n/2$ — power correction $\left(\Lambda_{\overline{\mathrm{MS}}}/m\right)^n$
$u_0 < 0$ — alternating, Borel summable

$$\hat{A}(m) = 1 + \int\limits_0^\infty \frac{d\tau}{\tau} w(\tau) \frac{\alpha_s\left(\sqrt{\tau} m\right)}{4\pi} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$w(\tau)$ — virtuality distribution function

$$\alpha_s\left(\sqrt{\tau} m\right) = \alpha_s(m) \sum_{n=0}^{\infty} (-\beta \log \tau)^n$$

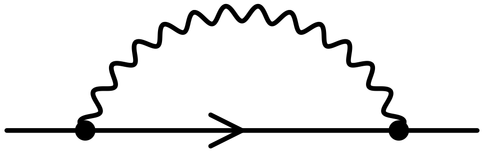
$$c_L = \int\limits_0^\infty \frac{d\tau}{\tau} w(\tau) (-\log \tau)^{L-1}$$

$$S(u) = \int\limits_0^\infty \frac{d\tau}{\tau} w(\tau) \tau^{-u} \quad \Rightarrow \quad w(\tau) = \frac{1}{2\pi i} \int\limits_{u_1 - i\infty}^{u_1 + i\infty} du\, S(u) \tau^u$$

$$S(u) = \frac{1}{u_0 - u} \qquad w(\tau) = \tau^{u_0}$$

Landau pole $\tau = \left(\Lambda_{\overline{\mathrm{MS}}}/m\right)^2$ residue $\left(\Lambda_{\overline{\mathrm{MS}}}/m\right)^{2u_0}$

On-shell mass



$$\Sigma(p) = m\Sigma_1(p^2) + (\not{p} - m)\,\Sigma_2(p^2)$$

$$S_0(p) = \frac{1}{(1-\Sigma_2(p^2))\,(\not{p}-m) + \delta m - m\Sigma_1(p^2)}$$

$\delta m = m - m_0$ — mass counterterm

m — on-shell mass: $\delta m = m\Sigma_1(m^2)$

$$\Sigma_1(m^2) = \frac{1}{4m}\,\mathrm{Tr}(\not{v}+1)\Sigma(mv)$$

$$= -iC_F g_0^2 \int \frac{d^dk}{(2\pi)^d} \frac{\frac{1}{4m}\,\mathrm{Tr}(\not{v}+1)\gamma^\mu(\not{k}+m\not{v}+m)\gamma^\nu \left(g_{\mu\nu} + \dfrac{k_\mu k_\nu}{-k^2}\right)}{(m^2-(k+mv)^2)(-k^2)^{1+(L-1)\epsilon}}$$

$$= -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(m^2 - (k+mv)^2)(-k^2)^{(L-1)\epsilon}} \left(\frac{2}{-k^2} - \frac{d-2}{2m^2} \right)$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(m^2 - (k+mv)^2)^{n_1}(-k^2)^{n_2}} = \frac{i m^{d-2(n_1+n_2)}}{(4\pi)^{d/2}} M(n_1, n_2)$$

$$M(n_1, n_2) = \frac{\Gamma_{\mathrm{IR}}(d - n_1 - 2n_2)\Gamma_{\mathrm{UV}}(-d/2 + n_1 + n_2)}{\Gamma(n_1)\Gamma(d - n_1 - n_2)}$$

$$\Sigma_1(m^2) = C_F \frac{g_0^2}{(4\pi)^{d/2}} m^{-2u} \left[2M(1, 1+u-\epsilon) - (1-\epsilon)M(1, u-\epsilon) \right]$$

$$F(u, \epsilon) = \left(\frac{\mu}{m} \right)^{2u} e^{\gamma\epsilon} \frac{\Gamma_{\mathrm{IR}}(1-2u)\Gamma_{\mathrm{UV}}(1+u)}{\Gamma(3-u-\epsilon)} N(u, \epsilon)$$

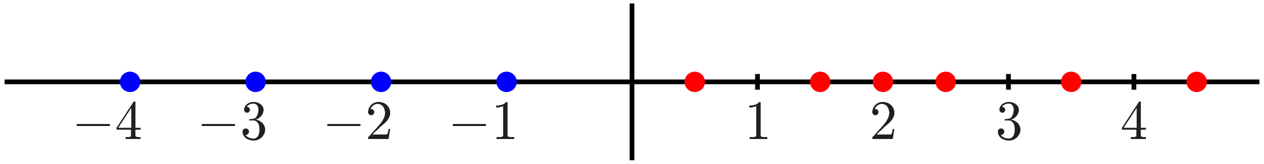
$$N(u, \epsilon) = 2C_F(1-u)(3-2\epsilon)$$

$$S(u) = C_F \left[2e^{\frac{5}{3}u} \frac{\Gamma_{\rm IR}(1-2u)\Gamma_{\rm UV}(u)}{\Gamma(3-u)}(1-u) - \frac{1}{u} \right]$$

Power counting

UV: $4 - 1 - 2(1+u) = 1 - 2u$ (leading divergence vanishes)

IR: $1 + 2(1+u) - 4 = 2u - 1$



Renormalon ambiguity

$$\Delta m = \frac{4}{3} C_F e^{5/6} \Lambda_{\overline{\rm MS}}$$