Loop calculations in HQET

Andrey Grozin
Institut für Theoretische Teilchenphysik
Universität Karlsruhe
grozin@particle.uni-karlsruhe.de

Dubna Calc-2000

HQET

Single heavy quark $|\vec{p}| \lesssim \omega$, $|p_0 - m| \lesssim \omega$

Light quarks and gluons $|\vec{k}_i| \lesssim \omega$, $|k_{0i}| \lesssim \omega$

HQET = QCD expanded to some order in ω/m

Residual energy $\widetilde{p}_0 = p_0 - m$ ("vacuum" has energy m)

Dispersion law $\widetilde{p}_0 = \sqrt{m^2 + \vec{p}^{\,2}} - m \approx 0$

Two-component spinor $\widetilde{Q}$ (or 4-component $\gamma_0 \widetilde{Q} = \widetilde{Q}$)

$$L = \widetilde{Q}^+ i D_0 \widetilde{Q} + \cdots$$

Propagator

$$\widetilde{S}(\widetilde{p}) = \frac{1}{\widetilde{p}_0 + i0}\,, \quad \widetilde{S}(x) = \widetilde{S}(x_0)\delta(\vec{x})\,, \quad \widetilde{S}(t) = -i\theta(t)\,.$$

Vertex $ig\delta_0^\mu t^a$

Loops vanish

Covariant notations

$$L_v = \overline{\widetilde{Q}}_v iv \cdot D \widetilde{Q}_v + \cdots$$

$$p = mv + \widetilde{p} \qquad |\widetilde{p}^\mu| \ll m \qquad \not{v} \widetilde{Q}_v = \widetilde{Q}_v$$

$$\widetilde{S}(\widetilde{p}) = \frac{1+\not{v}}{2} \frac{1}{\widetilde{p} \cdot v + i0}$$

Vertex $igv^\mu t^a$

QCD propagator

$$S(p) = \frac{\not{p} + m}{p^2 - m^2} = \frac{m(1+\not{v}) + \not{\widetilde{p}}}{2m\widetilde{p} \cdot v + \widetilde{p}^{\,2}} = \frac{1+\not{v}}{2} \frac{1}{\widetilde{p} \cdot v} + \mathcal{O}\left(\frac{\widetilde{p}}{m}\right) .$$

Vertex $ig\gamma^\mu t^a \to igv^\mu t^a$ between $(1+\not{v})/2$

Renormalization properties differ from QCD

Ultraviolet region in HQET: $\omega \ll k \ll m$

Heavy quark spin does not interact at $m \to \infty$:

chromomagnetic moment $\sim 1/m$.

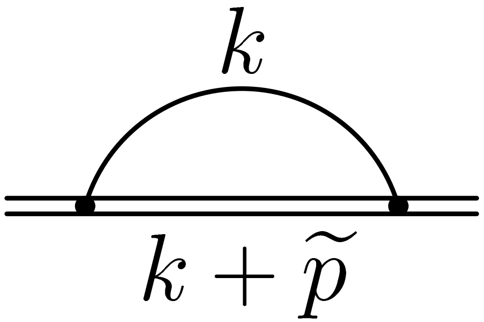
$SU(2)$ spin symmetry, superflavour symmetry

Lattice QCD: $a \ll 1/\Lambda \ll Na$

Heavy quark: $a \ll 1/m$

Lattice HQET

One-loop HQET diagram



$$I = \int \frac{d^d k}{(-(k+\widetilde{p})_0 - i0)^{n_1} (-k^2 - i0)^{n_2}}$$

α-parametrization

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int\limits_0^\infty e^{-a\alpha}\,\alpha^{n-1}\,d\alpha$$

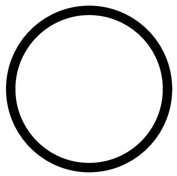
$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int e^{-a_1\alpha_1 - a_2\alpha_2}\,\alpha_1^{n_1-1}\alpha_2^{n_2-1}\,d\alpha_1\,d\alpha_2$$

HQET Feynman parametrization $\alpha_1 = y\alpha$, $\alpha_2 = \alpha$

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int\limits_0^\infty \frac{y^{n_1-1}\,dy}{[a_1 y + a_2]^{n_1+n_2}}$$

Combining denominators and shifting $k \to k - yv/2$

$$I = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int \frac{y^{n_1-1}\,dy\,d^d k}{[-k^2 + y(y/4-\omega) - i0]^{n_1+n_2}}$$



Massive vacuum bubble

$$\int \frac{d^d k}{(-k^2 + m^2 - i0)^n} = i\pi^{d/2} \frac{\Gamma(-d/2 + n)}{\Gamma(n)} (m^2)^{d/2-n}$$

Our integral $\omega < 0$

$$I = i\pi^{d/2} \frac{\Gamma(-d/2 + n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^\infty y^{n_1 - 1} \left[y \left(\frac{y}{4} - \omega \right) \right]^{d/2-n_1-n_2} dy$$

Substitutions $z = y/(-\omega)$, $z/4 + 1 = 1/x$

$$\int_0^\infty y^{d/2-n_2-1} \left(\frac{y}{4} - \omega\right)^{d/2-n_1-n_2} dy =$$

$$2^{d-2n_2}(-\omega)^{d-n_1-2n_2} \frac{\Gamma(-d+n_1+2n_2)\Gamma(d/2-n_2)}{\Gamma(-d/2+n_1+n_2)}$$

Final result

$$\int \frac{d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2}(-2\omega)^{d-2n_2} I(n_1, n_2)$$

$$D_1 = (kv+\omega)/\omega \qquad D_2 = -k^2$$

$$I(n_1, n_2) = \frac{\Gamma(-d+n_1+2n_2)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)}$$

UV: $\Gamma(-d+n_1+2n_2)$ IR: $\Gamma(d/2-n_2)$ Cut $\omega > 0$

Fourier transforms

$$\int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{(-\omega - i0)^n} \frac{d\omega}{2\pi} = \frac{i^n}{\Gamma(n)} t^{n-1} \theta(t)$$

$$\int_{0}^{+\infty} e^{i\omega t} t^n \, dt = \frac{(-i)^{n+1}\Gamma(n+1)}{(-\omega - i0)^{n+1}}$$

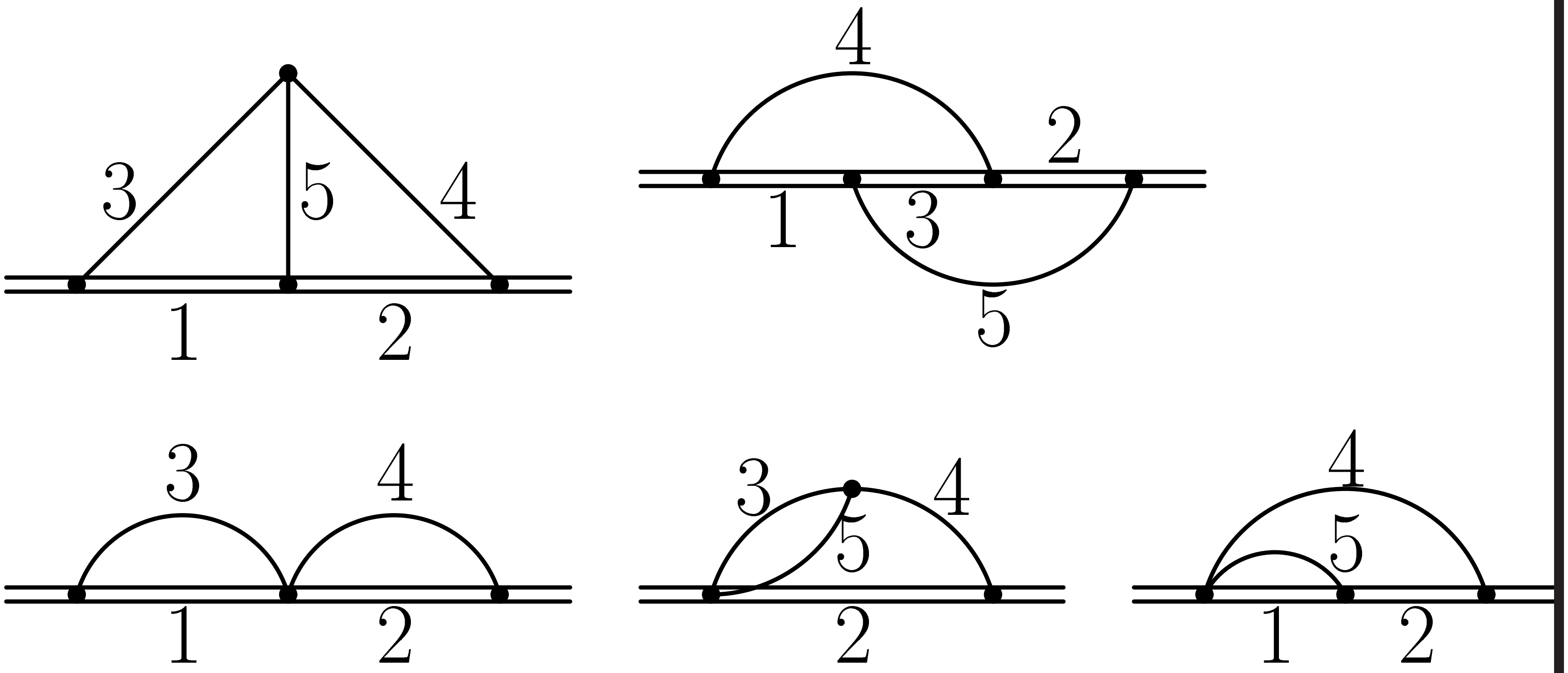
$$\int \frac{e^{-ipx}}{(-p^2 - i0)^n} \frac{d^d p}{(2\pi)^d} = i 2^{d-2n} \pi^{d/2} \frac{\Gamma(d/2 - n)}{\Gamma(n)} \frac{1}{(-x^2 + i0)^{d/2-n}}$$

$$\int \frac{e^{ipx}}{(-x^2 + i0)^n} d^d x = -i 2^{-2n} \pi^{-d/2} \frac{\Gamma(d/2 - n)}{\Gamma(n)} \frac{1}{(-p^2 - i0)^{d/2-n}}$$

Coordinate space

$$-2^{d-2n_2} \pi^{d/2} \frac{\Gamma(d/2 - n_2)}{\Gamma(n_1)\Gamma(n_2)} (it)^{n_1 + 2n_2 - d - 1} \theta(t)$$

Two-loop HQET diagrams



$$\int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = \\ - \pi^d (-2\omega)^{2(d-n_3-n_4-n_5)} I(n_1, n_2, n_3, n_4, n_5)$$

$$D_1 = (k_1 + \widetilde{p}) \cdot v/\omega \qquad D_2 = (k_2 + \widetilde{p}) \cdot v/\omega$$

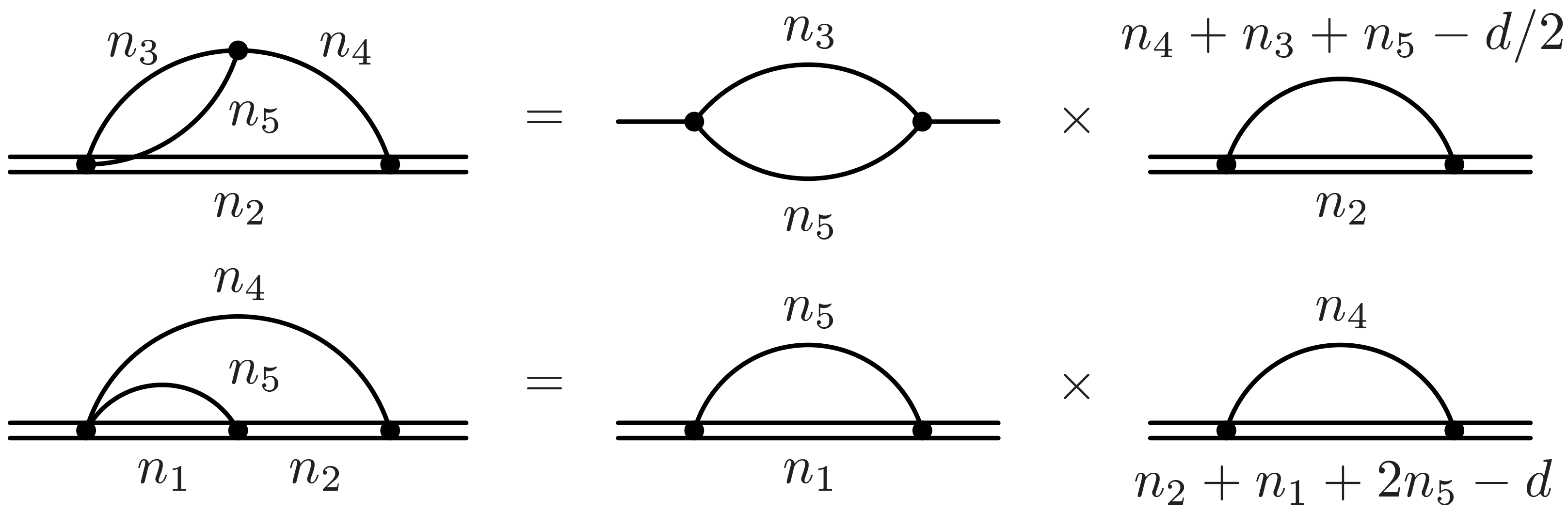
$$D_3 = -k_1^2 \qquad D_4 = -k_2^2 \qquad D_5 = -(k_1 - k_2)^2$$

Trivial cases

$$I(n_1,n_2,n_3,n_4,0) = I(n_1,n_3)I(n_2,n_4)$$

$$I(0,n_2,n_3,n_4,n_5) = G(n_3,n_5)I(n_2,n_4+n_3+n_5-d/2)$$

$$I(n_1,n_2,0,n_4,n_5) = I(n_1,n_5)I(n_2+n_1+2n_5-d,n_4)$$



$$G(n_1, n_2) = \frac{\Gamma(-d/2 + n_1 + n_2)\Gamma(d/2 - n_1)\Gamma(d/2 - n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d - n_1 - n_2)}$$

Integration by parts: apply $(\partial/\partial k_2) \cdot (k_2 - k_1)$

$$\frac{\partial}{\partial k_2} \to -\frac{n_2}{D_2}\frac{v}{\omega} + \frac{n_4}{D_4} 2k_2 + \frac{n_5}{D_5} 2(k_2 - k_1)$$

$$\frac{\partial}{\partial k_2} \cdot (k_2 - k_1) \to d - n_2 - n_4 - 2n_5 + \frac{n_2}{D_2} D_1 + \frac{n_4}{D_4} (D_3 - D_5)$$

because $k_2 v/\omega = D_2 - 1$, $2(k_2 - k_1) \cdot k_2 = D_3 - D_4 - D_5$

$$\left[d - n_2 - n_4 - 2n_5 + n_2 \mathbf{2}^+ \mathbf{1}^- + n_4 \mathbf{4}^+ (\mathbf{3}^- - \mathbf{5}^-) \right] I = 0$$

where, e. g.,

$$\mathbf{1}^{\pm} I(n_1, n_2, n_3, n_4, n_5) = I(n_1 \pm 1, n_2, n_3, n_4, n_5)$$

$$I = \frac{\left[n_4 \mathbf{4}^+ (\mathbf{5}^- - \mathbf{3}^-) - n_2 \mathbf{2}^+ \mathbf{1}^-\right] I}{d - n_2 - n_4 - 2n_5}$$

reduces $n_1 + n_3 + n_5$ by 1 $\Rightarrow$ trivial cases

Linear combination of I_1^2, I_2

$$I_n = \Gamma(1 + 2n\epsilon)\Gamma^n(1 - \epsilon)$$

$$\int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} =$$
$$- \pi^d (-2\omega)^{2(d-n_4-n_5)} J(n_1, n_2, n_3, n_4, n_5)$$

$$D_1 = (k_1 + \widetilde{p}) \cdot v/\omega \qquad D_2 = (k_2 + \widetilde{p}) \cdot v/\omega$$
$$D_3 = (k_1 + k_2 + \widetilde{p}) \cdot v/\omega$$
$$D_4 = -k_1^2 \qquad D_5 = -k_2^2$$

Trivial cases

$$J(n_1, n_2, 0, n_4, n_5) = I(n_1, n_4) I(n_2, n_5)$$
$$J(0, n_2, n_3, n_4, n_5) = I(n_3, n_4) I(n_2 + n_3 + 2n_4 - d, n_5)$$

Heavy denominators are linear dependent: $D_1 + D_2 - D_3 = 1$

$$J = (\mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-) J$$

Renormalization of HQET

HQET Lagrangian

$$L = \overline{\widetilde{Q}}_0 iv \cdot D_0 \widetilde{Q}_0 + \sum_i \overline{q}_{i0}(i\not{D}_0 - m_{i0})q_{i0} - \frac{1}{4} G^a_{0\mu\nu} G^{a\mu\nu}_0$$

$$- \frac{1}{2a_0} (\partial_\mu A^\mu_0)^2 + (D^\mu_0 h^a_0)^* (\partial_\mu h_0)$$

$$D_{0\mu} = \partial_\mu - i g_0 A^a_{0\mu} t^a$$

$$\widetilde{Q}_0 = \widetilde{Z}^{1/2}_Q \widetilde{Q} \qquad q_{i0} = Z^{1/2}_q q_i \qquad A_0 = Z^{1/2}_A A \qquad h_0 = Z^{1/2}_h h$$

$$g_0 = Z^{1/2}_\alpha g \qquad m_{i0} = Z_m m_i \qquad a_0 = Z_A a$$

Minimal renormalization constants

$$Z = 1 + \frac{Z_{11}}{\epsilon} \frac{\alpha_s}{4\pi} + \left(\frac{Z_{22}}{\epsilon^2} + \frac{Z_{21}}{\epsilon} \right) \left(\frac{\alpha_s}{4\pi} \right)^2 + \cdots$$

Lagrangian has dimensionality $[L] = d$
because action $\int L d^d x$ is dimensionless
Gluon kinetic term $(\partial A_0)^2 \Rightarrow [A_0] = 1 - \epsilon$; similarly $[q_{i0}] = \dfrac{3}{2} - \epsilon$
$[D_0] = 1 \Rightarrow [g_0] = \epsilon$

$$\alpha_s = \overline{\mu}^{\,-2\epsilon} \frac{g^2}{4\pi} \quad \overline{\mu}^{\,2} = \mu^2 \frac{e^\gamma}{4\pi} \quad \text{or} \quad \frac{\alpha_s}{4\pi} = \mu^{-2\epsilon} Z_\alpha^{-1} \frac{g_0^2}{(4\pi)^{d/2}} e^{-\gamma\epsilon}$$

$$\frac{d \log \alpha_s}{d \log \mu} = -2\epsilon - 2\beta(\alpha_s)$$

$$\beta(\alpha_s) = \frac{1}{2} \frac{d \log Z_\alpha}{d \log \mu} = \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f$$

$$Z_\alpha = 1 - \beta_0 \frac{\alpha_s}{4\pi\epsilon} + \left(\beta_0^2 - \tfrac{1}{2}\beta_1 \epsilon\right) \left(\frac{\alpha_s}{4\pi\epsilon}\right)^2 + \cdots$$

Anomalous dimensions

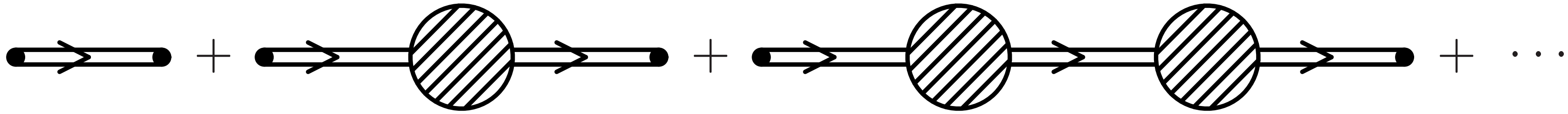
$$\gamma(\alpha_s) = \frac{d\log Z}{d\log\mu} = \gamma_0 \frac{\alpha_s}{4\pi} + \gamma_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

If Z is gauge invariant, $Z_{11} = -\frac{1}{2}\gamma_0$,
$Z_{22} = \frac{1}{8}\gamma_0(\gamma_0 + 2\beta_0)$, $Z_{21} = -\frac{1}{4}\gamma_1$
If γ_0 contains a linearly

$$Z = 1 - \frac{1}{2}\gamma_0 \frac{\alpha_s}{4\pi\epsilon} + \frac{1}{8}\left[\gamma_0(\gamma_0 + 2\beta_0) + \gamma_{A0}\frac{d\gamma_0}{da}a - 2\gamma_1\epsilon\right]\left(\frac{\alpha_s}{4\pi\epsilon}\right)^2 + \cdots$$

where

$$\gamma_A = \left[\left(a - \tfrac{13}{3}\right)C_A + \tfrac{8}{3}T_F n_f\right]\frac{\alpha_s}{4\pi} + \cdots$$

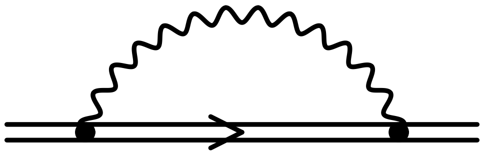


Bare heavy-quark propagator

$$i\widetilde{S}_0(\omega) = i\widetilde{S}^0(\omega) + i\widetilde{S}^0(\omega)(-i)\widetilde{\Sigma}(\omega)i\widetilde{S}^0(\omega)$$
$$+ i\widetilde{S}^0(\omega)(-i)\widetilde{\Sigma}(\omega)i\widetilde{S}^0(\omega)(-i)\widetilde{\Sigma}(\omega)i\widetilde{S}^0(\omega) + \cdots$$

$$\widetilde{S}^0(\omega) = \frac{1}{\omega} \qquad \widetilde{S}_0(\omega) = \frac{1}{\omega - \widetilde{\Sigma}(\omega)} = \widetilde{Z}_Q \widetilde{S}(\omega)$$

One loop



$$\widetilde{\Sigma}(\omega) = -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} v^\mu \frac{1}{kv+\omega} v^\nu \frac{1}{k^2} \left(g_{\mu\nu} - (1-a_0)\frac{k_\mu k_\nu}{k^2} \right)$$

$$= -C_F \frac{g_0^2 (-2\omega)^{1-2\epsilon}}{(4\pi)^{d/2}} \left[2I(1,1) + \tfrac{1}{2}(1-a_0) I(1,2) \right]$$

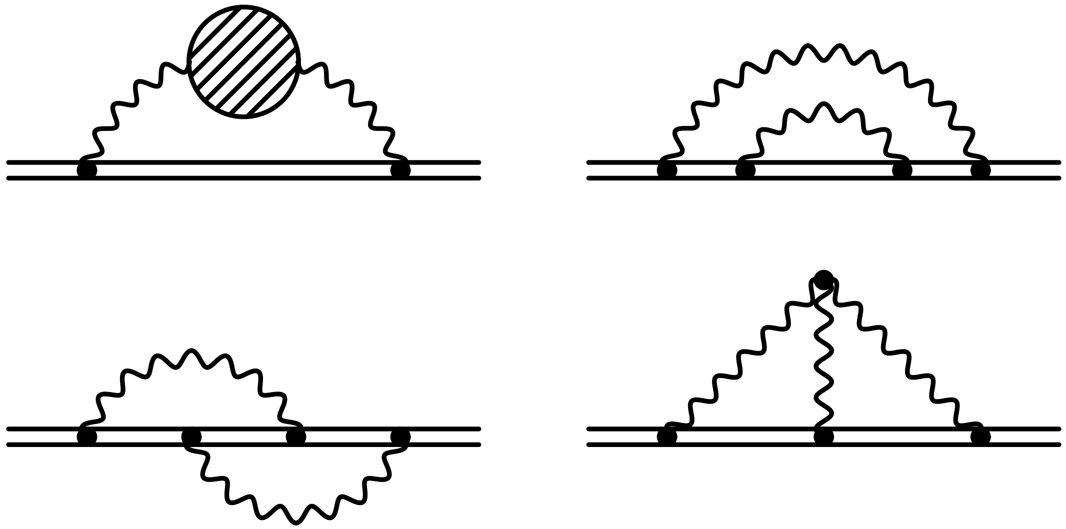
$$= C_F \frac{g_0^2 (-2\omega)^{1-2\epsilon}}{(4\pi)^{d/2}} \frac{I_1}{d-4} A \qquad A = a_0 - 1 - \frac{2}{d-3}$$

$$\omega \widetilde{S}_0(\omega) = 1 + C_F \frac{g_0^2(-2\omega)^{-2\epsilon}}{(4\pi)^{d/2}} \frac{I_1}{d-4} 2A$$

$$\widetilde{S}_0(t) = \widetilde{S}^0(t) \left[1 + C_F \frac{g_0^2(it/2)^{2\epsilon}}{(4\pi)^{d/2}} \Gamma(-\epsilon) A + \cdots \right] \qquad \widetilde{S}^0(t) = -i\theta(t)$$

$$\widetilde{\gamma}_Q = 2(a-3) C_F \frac{\alpha_s}{4\pi} + \cdots$$

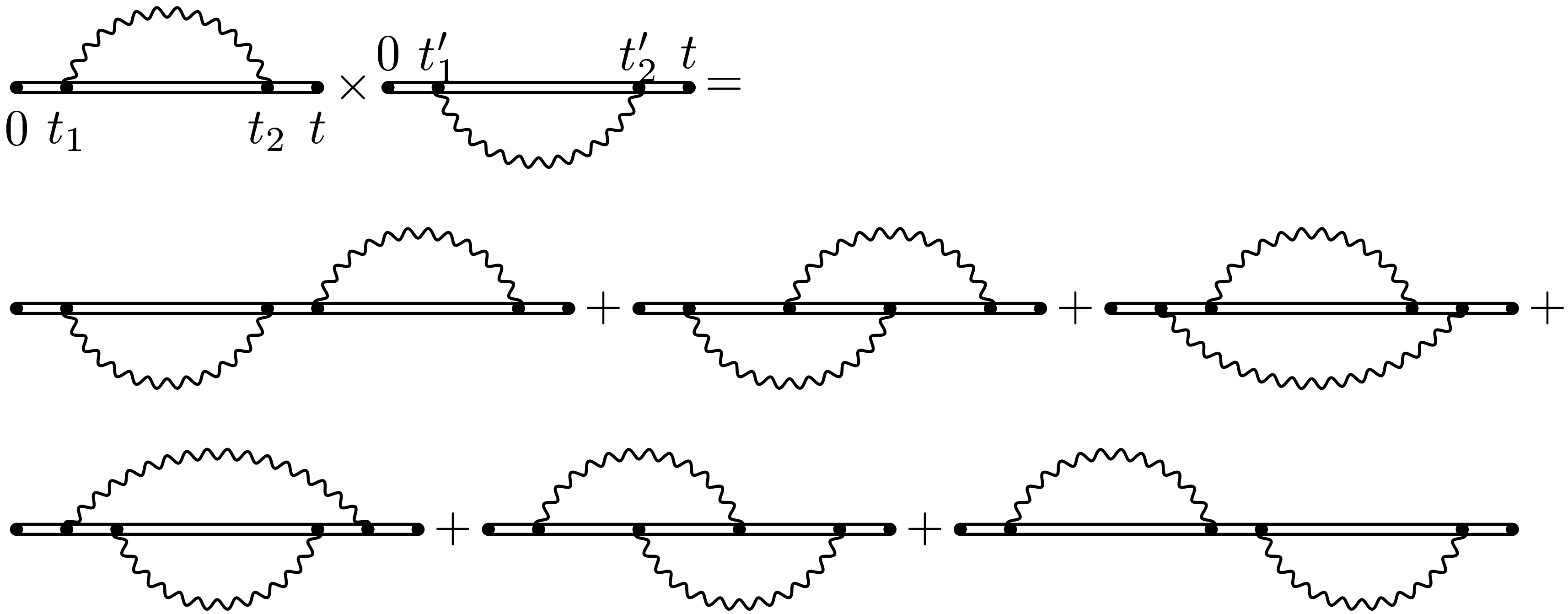
Two loops



$$\widetilde{\gamma}_Q = 2(a-3)C_F\frac{\alpha_s}{4\pi} + C_F\left(\frac{3a^2+24a-179}{6}C_A + \frac{32}{3}T_F n_f\right)\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

$$\widetilde{\gamma}_Q - \gamma_q = -6C_F\frac{\alpha_s}{4\pi} - C_F\left(\frac{127}{3}C_A - 3C_F - \frac{44}{3}T_F n_f\right)\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

Exponentiation theorem



$$\widetilde{S}_0(t) = \widetilde{S}^0(t)\exp\left[\frac{e_0^2(it/2)^{2\epsilon}}{(4\pi)^{d/2}}\Gamma(-\epsilon)A\right]$$

No electron loops: $Z_A = 1$

Vertex function $ie_0 v^\mu (1 + \widetilde{\Lambda})$

Ward identtity

$$\widetilde{\Lambda} = -\frac{\widetilde{\Sigma}(\omega') - \widetilde{\Sigma}(\omega)}{\omega' - \omega} \qquad \text{or} \qquad 1 + \widetilde{\Lambda} = \frac{\widetilde{S}_0^{-1}(\omega') - \widetilde{S}_0^{-1}(\omega)}{\omega' - \omega}$$

Renormalized vertex $g_0(1 + \widetilde{\Lambda}) Z_A^{1/2} \widetilde{Z}_Q$ is finite $\Rightarrow$ $Z_\alpha = 1$

$$\widetilde{Z}_Q = \exp\left[-(a-3)\frac{\alpha}{4\pi\epsilon}\right] \qquad \widetilde{\gamma}_Q = 2(a-3)\frac{\alpha}{4\pi}$$

$1/m$ corrections to HQET Lagrangian

HQET Lagrangian = series in $1/m$ containing all operators having the necessary symmetries, with arbitrary coefficients. Coefficients are tuned to reproduce QCD S-matrix elements expanded to some degree in ω/m.

Field redefinitions $\Rightarrow$ can eliminate $D_0\widetilde{Q}$

Reparametrization invariance: $\delta v \lesssim \omega/m$

Heavy quark spin symmetry and superflavour symmetry are violated by the chromomagnetic interaction

Hyperfine splittings: $B - B^*$

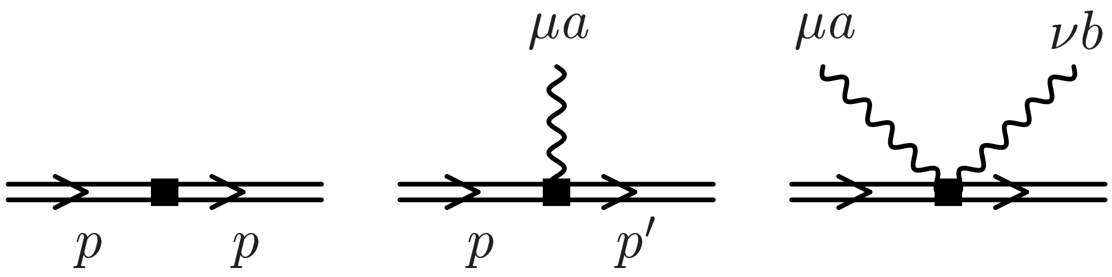
Scalar heavy quark

$$L = \widetilde{Q}^{+} i D_0 \widetilde{Q} + \frac{C_k}{2m} \widetilde{Q}^{+} \vec{D}^{\,2} \widetilde{Q} + \cdots$$

$$L_v = \widetilde{Q}_v^{+} i v \cdot D \widetilde{Q}_v - \frac{C_k}{2m} \widetilde{Q}_v^{+} D_\perp^2 \widetilde{Q}_v + \cdots$$

where $D_\perp = D - v(v \cdot D)$

Mass shell $\widetilde{p}_0 = p_0 - m = C_k \dfrac{\vec{p}^{\,2}}{2m} \Rightarrow$ at tree level $C_k = 1$



$$i \frac{C_k}{2m} \widetilde{p}_\perp^2 \qquad i \frac{C_k}{2m} g t^a (p + p')_\perp^\mu \qquad i \frac{C_k}{2m} g^2 (t^a t^b + t^b t^a) g_\perp^{\mu\nu}$$

where $g_\perp^{\mu\nu} = g^{\mu\nu} - v^\mu v^\nu$

Sum of 1PI self-energy diagrams at $1/m$: $-i\dfrac{C_k}{2m}\widetilde{\Sigma}_k(\omega, p_\perp^2)$

Variation of $\widetilde{\Sigma}$ at $v \to v + \delta v$:

- propagators $1/(\widetilde{p} \cdot v + i0) \Rightarrow i\widetilde{p}_i \cdot \delta v$
- vertices $igt^a v^\mu \Rightarrow igt^a \delta v^\mu$

Variation of $\widetilde{\Sigma}_k$ at $p_\perp \to p_\perp + \delta p_\perp$:

- no-gluon kinetic vertex $\Rightarrow i\dfrac{C_k}{m}\widetilde{p}_i \cdot \delta p_\perp$
- one-gluon kinetic vertex $\Rightarrow i\dfrac{C_k}{m}gt^a \delta p_\perp^\mu$
- two-gluon kinetic vertex $\Rightarrow$ don't change

They coincide if $m\delta v = C_k \delta p_\perp$

Ward identity of reparametrization invariance

$$\frac{\partial \widetilde{\Sigma}_k}{\partial p_\perp^\mu} = 2\frac{\partial \widetilde{\Sigma}}{\partial v^\mu} \qquad \text{or} \qquad \frac{\partial \widetilde{\Sigma}_k}{\partial p_\perp^2} = \frac{d\widetilde{\Sigma}}{d\omega}$$

$$\widetilde{\Sigma}_k(\omega, p_\perp^2) = \frac{d\widetilde{\Sigma}(\omega)}{d\omega} p_\perp^2 + \widetilde{\Sigma}_{k0}(\omega)$$

Diagrammatically: $p_\perp^2$ term comes from diagrams with no-gluon kinetic vertex, $i\dfrac{C_k}{2m}$ times unit insertion.

Mass shell — pole of propagator

$$\widetilde{S}_0(p) = \frac{1}{\omega - \widetilde{\Sigma}(\omega) - \dfrac{C_k}{2m}\left[\vec{p}^{\,2} - \dfrac{d\widetilde{\Sigma}(\omega)}{d\omega}\vec{p}^{\,2} + \widetilde{\Sigma}_{k0}(\omega)\right]}$$

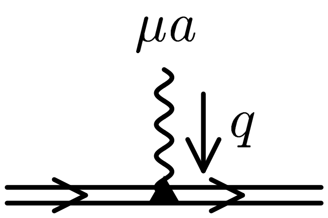
$$\left[1 - \frac{d\widetilde{\Sigma}(\omega)}{d\omega}\right]_{\omega=0} \omega = \frac{C_k}{2m}\left[1 - \frac{d\widetilde{\Sigma}(\omega)}{d\omega}\right]_{\omega=0} \vec{p}^{\,2}$$

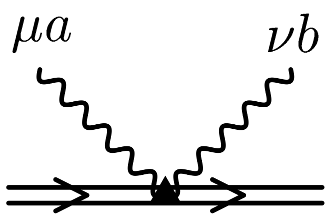
$$C_k(\mu) = 1$$

Spin $\frac{1}{2}$ heavy quark

$$L = \widetilde{Q}^+ iD_0 \widetilde{Q} + \frac{C_k}{2m} \widetilde{Q}^+ \vec{D}^{\,2} \widetilde{Q} - \frac{C_m}{2m} \widetilde{Q}^+ \vec{B} \cdot \vec{\sigma} \widetilde{Q} + \cdots$$

$$L_v = \overline{\widetilde{Q}}_v iv \cdot D \widetilde{Q}_v - \frac{C_k}{2m} \overline{\widetilde{Q}}_v D_\perp^2 \widetilde{Q}_v + \frac{C_m}{4m} \overline{\widetilde{Q}}_v G_{\mu\nu} \sigma^{\mu\nu} \widetilde{Q}_v + \cdots$$

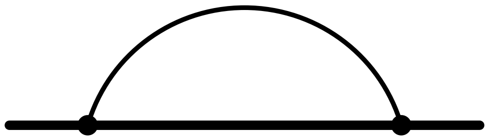




$$\frac{C_m}{2m} g t^a \sigma^{\nu\mu} q_\nu \qquad \frac{C_m}{2m} g^2 [t^a, t^b] \sigma^{\mu\nu}$$

On-shell calculations in QCD

One loop



$$\int \frac{d^d k}{\left[m^2 - (k+mv)^2 - i0\right]^{n_1} \left(-k^2 - i0\right)^{n_2}} = i\pi^{d/2} m^{d-2(n_1+n_2)} M(n_1, n_2)$$

$$\int \frac{d^d K}{(K^2 - 2iK_0)^{n_1} (K^2)^{n_2}} = \pi^{d/2} M(n_1, n_2) \qquad k_0 = ik_{E0} \quad K = k_E/m$$

$$\int \frac{d^d K}{(1 - 2iK_0)^{n_1} (K^2)^{n_2}} = \pi^{d/2} I(n_1, n_2) \qquad K = k_E/(-2\omega)$$

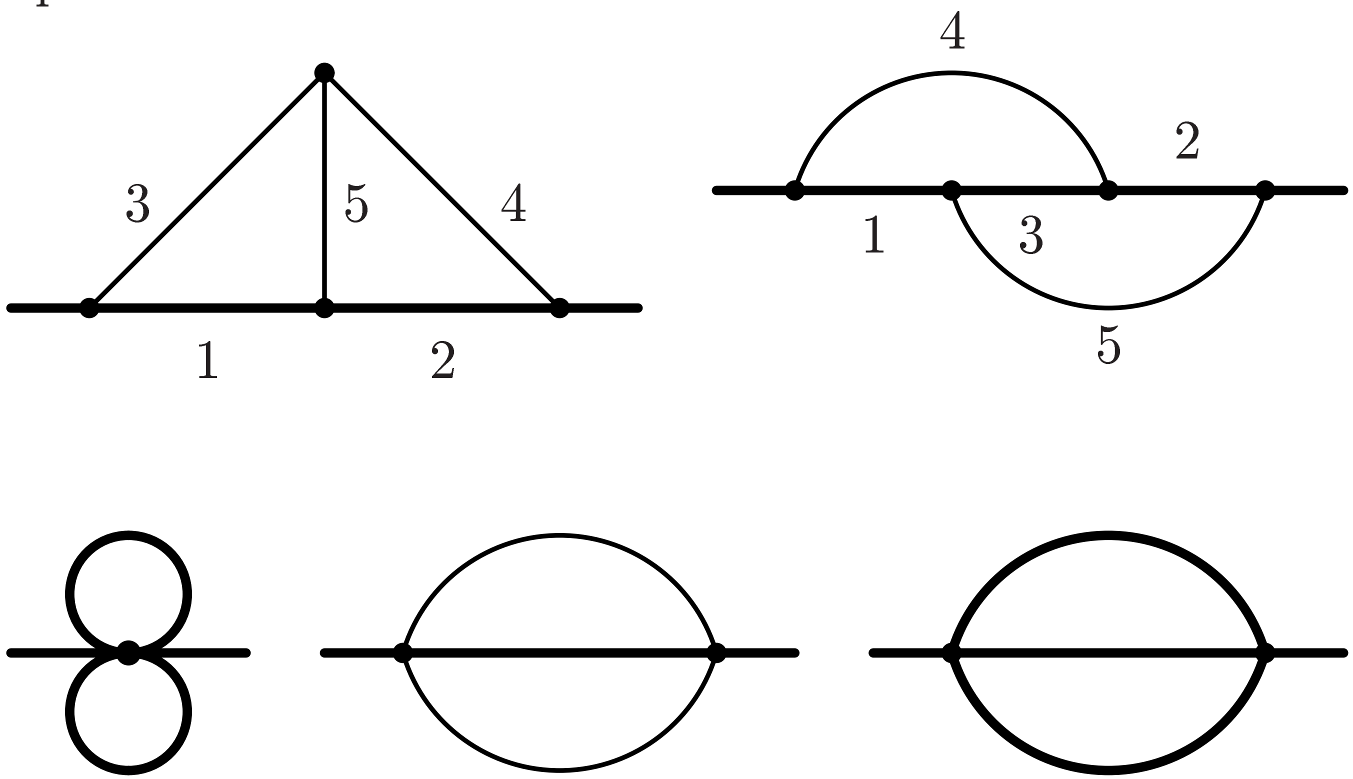
Inversion $K = K'/K'^2 \qquad K^2 - 2iK_0 = \left(1 - 2iK'_0\right)/K'^2$

$d^d K = K^{d-1} dK d\Omega = d^d K' / \left(K'^2\right)^d$

$$M(n_1,n_2) = I(n_1, d-n_1-n_2) = \frac{\Gamma(d-n_1-2n_2)\Gamma(-d/2+n_1+n_2)}{\Gamma(n_1)\Gamma(d-n_1-n_2)}$$

UV: $\Gamma(-d/2+n_1+n_2)$ IR: $\Gamma(d-n_1-2n_2)$

Two loops



$$\int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = -\pi^d m^{2(d-\sum n_i)} M(n_1, n_2, n_3, n_4, n_5)$$

$$D_1 = m^2 - (k_1 + mv)^2 \qquad D_2 = m^2 - (k_2 + mv)^2$$

$$D_3 = -k_1^2 \qquad D_4 = -k_2^2 \qquad D_5 = -(k_1 - k_2)^2$$

Inversion $(K_1 - K_2)^2 = (K_1' - K_2')^2 / (K_1'^2 K_2'^2)$

$M(n_1, n_2, n_3, n_4, n_5) = I(n_1, n_2, d - n_1 - n_3 - n_5, d - n_2 - n_4 - n_5, n_5)$

Integration by parts: linear combinations of M_1^2, M_2

$$M_n = \frac{\Gamma(1 + (n-1)\epsilon)\Gamma(1 + n\epsilon)\Gamma(1 - 2n\epsilon)\Gamma^n(1-\epsilon)}{\Gamma(1 - n\epsilon)\Gamma(1 - (n+1)\epsilon)}$$

$$\int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = -\pi^d m^{2(d-\sum n_i)} N(n_1, n_2, n_3, n_4, n_5)$$

$$D_1 = m^2 - (k_1 + mv)^2 \qquad D_2 = m^2 - (k_2 + mv)^2$$

$$D_3 = m^2 - (k_1 + k_2 + mv)^2$$

$$D_4 = -k_1^2 \qquad D_4 = -k_2^2$$

Integration by parts: M_1^2, M_2, $N_2 = N(1,1,1,0,0)$

$$N(1,1,1,1,1) = \pi^2 \log 2 - \tfrac{3}{2}\zeta(3) + \mathcal{O}(\epsilon)$$

On-shell renormalization

$$L = \overline{Q}_0 (i\not{D}_0 - m) Q_0 + \delta m\, \overline{Q}_0 Q_0 + \cdots$$

$$\delta m = m - m_0 \qquad m_0 = Z_m^{\mathrm{os}} m \qquad Q_0 = Z_Q^{\mathrm{os}} Q$$

$$\Sigma(p) = m\Sigma_1(p^2) + (\not{p} - m)\,\Sigma_2(p^2)$$

$$S_0(p) = \frac{1}{(1 - \Sigma_2(p^2))\,(\not{p} - m) + \delta m - m\Sigma_1(p^2)}$$

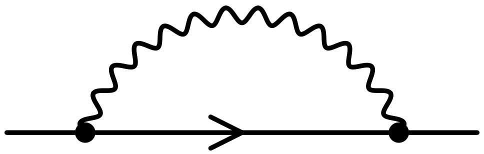
$$\delta m = m\Sigma_1(m^2)$$

$$\Sigma_1(p^2) - \delta m/m = \Sigma_1'(m^2)\,(p^2 - m^2) + \cdots$$

$$S_0(p) = \frac{1}{1 - \Sigma_2(m^2) - 2m^2\Sigma_1'(m^2)}\,\frac{\not{p} + m}{p^2 - m^2} + \cdots$$

$$Z_Q^{\mathrm{os}} = \left[1 - \Sigma_2(m^2) - 2m^2\Sigma_1'(m^2)\right]^{-1}$$

$$T(t) = \frac{1}{4m}\,\mathrm{Tr}(\not{v} + 1)\Sigma(mv(1+t))$$
$$= \Sigma_1(m^2) + \left[\Sigma_2(m^2) + 2m^2\Sigma_1'(m^2)\right] t + \mathcal{O}(t^2)$$
$$Z_m^{\mathrm{os}} = 1 - T(0) \qquad Z_Q^{\mathrm{os}} = \left[1 - T'(0)\right]^{-1}$$



$$T(t) = -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{D_1(t) D_2} \frac{1}{4m} \operatorname{Tr}(\rlap{/}v + 1)\gamma^\mu (\rlap{/}p + \rlap{/}k + m)\gamma^\nu$$

$$\times \left[g_{\mu\nu} + (1 - a_0) \frac{k_\mu k_\nu}{D_2} \right]$$

$$= -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{D_1(t)} \left[\frac{2}{D_2} - \frac{d-2}{2m^2}(1-t) \right]$$

$$= C_F \frac{g_0^2 m^{-2\epsilon}}{(4\pi)^{d/2}} \Gamma(\epsilon) \frac{d-1}{d-3}(1-t) + \cdots$$

$$D_1(t) = D_1 + (D_1 - D_2 - 2m^2)t + \cdots$$

$$Z_m^{\mathrm{os}} = Z_Q^{\mathrm{os}} = 1 - C_F \frac{g_0^2 m^{-2\epsilon}}{(4\pi)^{d/2}} \Gamma(\epsilon) \frac{d-1}{d-3}$$

Scattering of on-shell heavy quark in external gluon field

$$\overline{u}(p')\left(F_1(q^2)\frac{(p+p')^\mu}{2m}+G_m(q^2)\frac{[\not{q},\gamma^\mu]}{4m}\right)t^a u(p)=\overline{u}(p')A^\mu(q)t^a u(p)$$

$$p=mv \qquad p'=mv+q$$

$$F_1(q^2)=1+\mathcal{O}(q^2) \qquad G_m(q^2)=\mu_g+\mathcal{O}(q^2)$$

$$F_1(q^2) = -\frac{1}{2(d-2)\left(1-\frac{q^2}{4m^2}\right)}\frac{1}{4}\,\mathrm{Tr}\,A_\mu(q)(\not{v}+1)\left(\gamma^\mu - \frac{(d-1)v^\mu}{1-\frac{q^2}{4m^2}}\right)$$
$$\times\left(\not{v}+\frac{\not{q}}{m}+1\right)$$

$$G_m(q^2) = \frac{2m^2}{(d-2)q^2}\frac{1}{4}\,\mathrm{Tr}\,A_\mu(q)(\not{v}+1)\left(\gamma^\mu - \frac{v^\mu}{1-\frac{q^2}{4m^2}}\right)\left(\not{v}+\frac{\not{q}}{m}+1\right)$$

$$A_\mu(q) = A_\mu + A_{\mu\nu} q^\nu + \cdots$$

$$F_1(0) = \frac{1}{4} \operatorname{Tr} A_\mu (\not{v} + 1) v^\mu$$

$$\overline{q^\alpha q^\beta} = \frac{q^2}{d-1} \left[\left(1 - \frac{q^2}{4m^2} \right) g^{\alpha\beta} - \left(1 - d \frac{q^2}{4m^2} \right) v^\alpha v^\beta \right]$$

$$G_m(0) = \frac{1}{d-1} \frac{1}{4} \operatorname{Tr} A_\mu (\not{v} + 1) \big(\gamma^\mu - v^\mu \big)$$
$$+ \frac{2}{(d-1)(d-2)} \frac{1}{4} \operatorname{Tr} A_{\mu\nu} (\not{v} + 1) \big(\gamma^\mu \gamma^\nu - g^{\mu\nu} + \gamma^\mu v^\nu - \gamma^\nu v^\mu \big)$$

HQET scattering amplitude

$$\overline{u}_v(q)\left(v^\mu + \frac{q^\mu}{2m} + C_m(\mu)\widetilde{Z}^{-1}(\mu)\widetilde{\mu}_{g0}\frac{[\not{q},\gamma^\mu]}{4m} + \cdots\right) t^a u_v(0)$$

Foldy-Wouthuysen transformation

$$u(mv+k) = \left(1 + \frac{\not{k}}{2m} + \mathcal{O}\left(\frac{k^2}{m^2}\right)\right) u_v(k)$$

Matching

$$C_m(\mu)\widetilde{Z}^{-1}(\mu)\widetilde{\mu}_{g0} = \mu_g$$

If the other flavours are massless, $\widetilde{\mu}_{g0} = 1$

$$C_m(\mu) = C_m(m) \exp\left(- \int\limits_{\alpha_s(m)}^{\alpha_s(\mu)} \frac{\widetilde{\gamma}_m(\alpha)}{2\beta(\alpha)} \frac{d\alpha}{\alpha} \right)$$

$$C_m(m) = 1 + C_1 \frac{\alpha_s(m)}{4\pi} + C_2 \left(\frac{\alpha_s(m)}{4\pi} \right)^2 + \cdots$$

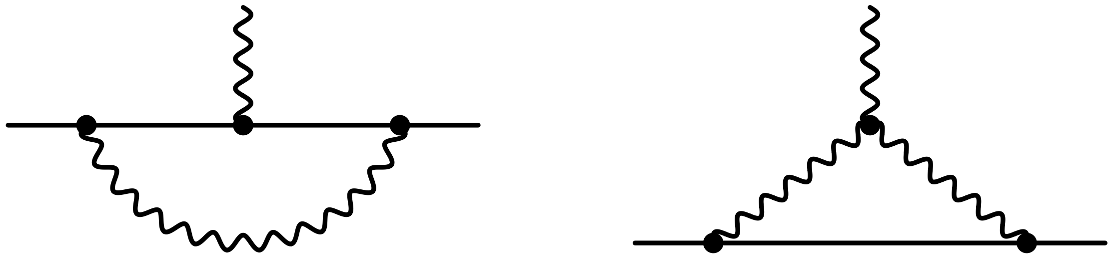
$$\widetilde{\gamma}_m(\alpha_s) = \frac{d \log \widetilde{Z}_m}{d \log \mu} = \widetilde{\gamma}_0 \frac{\alpha_s}{4\pi} + \widetilde{\gamma}_1 \left(\frac{\alpha_s}{4\pi} \right)^2 + \cdots$$

$$C_m(\mu) = 1 + (C_1 - \widetilde{\gamma}_0 L)\,\frac{\alpha_s(m)}{4\pi} \qquad\qquad L = \log\frac{m}{\mu}$$

$$+ \left[C_2 - (C_1\widetilde{\gamma}_0 + \widetilde{\gamma}_1)L + \widetilde{\gamma}_0(\widetilde{\gamma}_0 - \beta_0)L^2\right]\left(\frac{\alpha_s(m)}{4\pi}\right)^2 + \cdots$$

$$= \left(\frac{\alpha_s(\mu)}{\alpha_s(m)}\right)^{-\frac{\widetilde{\gamma}_0}{2\beta_0}}\left[1 + C_1\frac{\alpha_s(m)}{4\pi} - \frac{\beta_0\widetilde{\gamma}_1 - \beta_1\widetilde{\gamma}_0}{2\beta_0^2}\,\frac{\alpha_s(\mu) - \alpha_s(m)}{4\pi} + \cdots\right]$$

One loop



$$\mu_g = 1 + \frac{g_0^2 m^{-2\epsilon}}{(4\pi)^{d/2}} \frac{\Gamma(\epsilon)}{2(d-3)} \left[2(d-4)(d-5)C_F + (d^2 - 8d + 14)C_A\right] + \cdots$$

$$\widetilde{\gamma}_m = 2C_A \frac{\alpha_s}{4\pi} + \frac{4}{9} C_A \bigl(17C_A - 13T_F n_l\bigr) \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

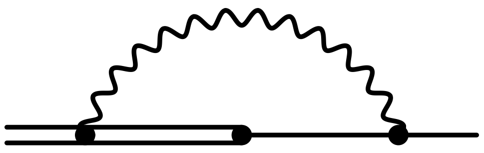
$$C_m(m) = 1 + 2(C_F + C_A) \frac{\alpha_s(m)}{4\pi} + \cdots$$

Heavy-light currents

$\widetilde{\jmath}_0 = \overline{q}_0 \Gamma \widetilde{Q}_0 = \widetilde{Z}_j(\mu)\widetilde{\jmath}(\mu)$

$<q|\widetilde{\jmath}_0|\widetilde{Q}> = Z_q^{1/2}\widetilde{Z}_Q^{1/2}\Gamma_0$, $\Gamma_0 = \widetilde{Z}_\Gamma \Gamma$, $\widetilde{Z}_j = \widetilde{Z}_\Gamma Z_q^{1/2}\widetilde{Z}_Q^{1/2}$

UV divergence of



$$\Gamma_0 = 1 - iC_F g^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu \not{k} v^\nu \left(g_{\mu\nu} - (1-a)k_\mu k_\nu / k^2\right)}{(k^2)^2 (k\cdot v)}$$

$$= 1 - iC_F g^2 \int \frac{d^d k}{(2\pi)^d} \frac{\not{v}\not{k} - (1-a) k\cdot v}{(k^2)^2 (k\cdot v)} \qquad \not{k} = k\cdot v \not{v} + \not{k}_\perp$$

$$= 1 - iC_F g^2 a \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2}\bigg|_{\mathrm{UV}} = i\frac{2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int_\lambda^\infty k^{-1-2\epsilon} dk$$

$$= -\frac{i}{(4\pi)^2 \epsilon} \, k^{-2\epsilon}\Big|_\lambda^\infty = \frac{i}{(4\pi)^2 \epsilon}$$

$$\widetilde{Z}_\Gamma = 1 + aC_F \frac{\alpha_s}{4\pi\epsilon}$$

$$Z_q = 1 - aC_F \frac{\alpha_s}{4\pi\epsilon} \qquad \widetilde{Z}_Q = 1 - (a-3)C_F \frac{\alpha_s}{4\pi\epsilon}$$

$$\widetilde{Z}_j = 1 + \frac{3}{2} C_F \frac{\alpha_s}{4\pi\epsilon}$$

$$\widetilde{\gamma}_j = -3C_F \frac{\alpha_s}{4\pi}$$

Two loops

$$
\begin{aligned}
\widetilde{\gamma}_j &= -3C_F \frac{\alpha_s}{4\pi} \\
&+ C_F \left[\left(-\frac{8}{3}\pi^2 + \frac{5}{2} \right) C_F + \left(\frac{2}{3}\pi^2 - \frac{49}{6} \right) C_A + \frac{10}{3} T_F n_l \right] \left(\frac{\alpha_s}{4\pi} \right)^2 + \cdots \\
&= \frac{1}{2}\gamma_{\overline{q}q} + 4C_F \left[\left(-\frac{2}{3}\pi^2 + 1 \right) C_F + \left(\frac{\pi^2}{6} + 2 \right) C_A \right] \left(\frac{\alpha_s}{4\pi} \right)^2 + \cdots
\end{aligned}
$$

QCD current $j(\mu) = Z_j^{-1}(\mu) j_0, \; j_0 = \overline{q}\Gamma Q$

$$
j(\mu) = C_\Gamma(\mu)\widetilde{\jmath}(\mu) + \frac{1}{m}\sum_i B_i(\mu)\widetilde{O}_i(\mu) + \mathcal{O}\left(\frac{1}{m^2} \right)
$$

Renormalization group

$$\exp\left(-\int_{\alpha_s(m)}^{\alpha_s(\mu)} \frac{\gamma_j(\alpha)}{2\beta(\alpha)}\frac{d\alpha}{\alpha}\right) j(\mu) = C_\Gamma(m)\exp\left(-\int_{\alpha'_s(m)}^{\alpha'_s(\widetilde{\mu})} \frac{\widetilde{\gamma}_j(\alpha)}{2\beta'(\alpha)}\frac{d\alpha}{\alpha}\right)\widetilde{\jmath}(\widetilde{\mu})$$
$$+\mathcal{O}\left(\frac{1}{m}\right)$$

One loop

$$C_\Gamma(m) = 1 + C_F\frac{\alpha_s(m)}{4\pi}\left[3(n-2)^2 + (2-\eta)(n-2) - 4\right]$$

for n antisymmetrized γ matrices

$\eta = -\sigma(-1)^n \qquad\qquad \gamma_0\Gamma = \sigma\Gamma\gamma_0$

γ_5 prescriptions

$$\overline{q}\gamma_5^{\mathrm{AC}}\widetilde{Q} = \overline{q}\gamma_5^{\mathrm{HV}}\widetilde{Q} \qquad \cdots$$

$$\overline{q}\gamma_5^{\mathrm{AC}}Q = Z_P\overline{q}\gamma_5^{\mathrm{HV}}Q \qquad \overline{q}\gamma^\mu\gamma_5^{\mathrm{AC}}Q = Z_A\overline{q}\gamma^\mu\gamma_5^{\mathrm{HV}}Q$$

$$Z_P(\mu) = \frac{C_1(\mu)}{C_{\gamma_0\gamma_1\gamma_2\gamma_3}(\mu)} = 1 - 8C_F\frac{\alpha_s(\mu)}{4\pi} + \cdots$$

$$Z_A(\mu) = \frac{C_{\gamma_0}(\mu)}{C_{\gamma_1\gamma_2\gamma_3}(\mu)} = \frac{C_{\gamma_3}(\mu)}{C_{\gamma_0\gamma_1\gamma_2}(\mu)} = 1 - 4C_F\frac{\alpha_s(\mu)}{4\pi}$$

$$Z_T(\mu) = \frac{C_{\gamma_0\gamma_1}(\mu)}{C_{\gamma_2\gamma_3}(\mu)} = \frac{C_{\gamma_2\gamma_3}(\mu)}{C_{\gamma_0\gamma_1}(\mu)} = 1$$

Ratios of meson matrix elements

$$<0|\overline{q}\gamma_5 Q|B> = -im_B f_B^P(\mu) \qquad <0|\overline{q}\gamma^\mu\gamma_5 Q|B> = if_B p^\mu$$

$$<0|\overline{q}\gamma^\mu Q|B^*> = im_{B^*} f_{B^*} e^\mu \qquad <0|\overline{q}\sigma^{\mu\nu} Q|B^*> = f_{B^*}^T(\mu)(p^\mu e^\nu - p^\nu e^\mu)$$

$$\frac{f_B^P(\mu)}{f_B} = \frac{<0|\overline{q}\gamma_5 Q|B>}{<0|\overline{q}\gamma_0\gamma_5 Q|B>} = \frac{m_B}{\overline{m}(\mu)}$$

$$\frac{f_B^P(m)}{f_B} = \frac{m}{\overline{m}(m)} = \frac{C_1(m)}{C_{\gamma_0}(m)} = 1 + 4C_F\frac{\alpha_s(\mu)}{4\pi} + \cdots$$

$$\frac{f_{B^*}}{f_B} = \frac{C_{\gamma_1}(m)}{C_{\gamma_0}(m)} = 1 - 2C_F\frac{\alpha_s(\mu)}{4\pi} + \cdots$$

$$\frac{f_{B^*}^T(m)}{f_{B^*}} = \frac{C_{\gamma_0\gamma_1}(m)}{C_{\gamma_1}(m)} = 1 + \mathcal{O}\left(\alpha_s^2(m)\right)$$