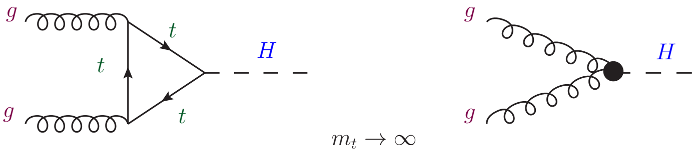
# Higgs Decay, Z Decay and the QCD Beta Function

J. H. Kühn
(KIT, Karlsruhe)
with P. Baikov, K. Chetyrkin

1. Higgs Decay to $gg$ and $b\bar{b}$
2. Quark Mass and Field Anomalous Dimension
3. Z Decay to $\mathcal{O}\left(\alpha_s^4\right)$
4. Beta Function to Five Loops

# I. Higgs decay to $gg$ and $b\bar{b}$: the two most important modes

## 1) $H \rightarrow gg$




$$\mathcal{L}_{eff} = -2^{1/4} G_F^{1/2} H\, C_1\left(\alpha_s, \ln\frac{\mu^2}{M_t^2}\right) G^a_{\mu\nu} G^{a\mu\nu}$$

$\mathcal{L}_{eff}$ counts number of heavy quark species

Born approximation $\sim \alpha_s^2$

$$\Gamma_{Born}(H \to gg) = \frac{G_F M_H^3}{36\pi\sqrt{2}} \left( \frac{\alpha_s(\mu)}{\pi} \right)^2$$

$\to$ strong scale dependence

$\to$ important role of higher orders

$\to$ corrections for decay

NLO: Inami, Kugota, Okada (1983)

NNLO: Chetyrkin, Kniehl, Steinhauser (1997)

$\to$ corrections for production (gluon fusion)

NLO: Inami, Kubota, Okada (1983); Dawson (1991);
Djouadi, Spira, Zerwas (1991)

NNLO: Harlander, Kilgore (2002); Anastasiou, Melnikov (2002);
Ravindran, Smith, van Neerven (2003)

# Important effects of higher orders

- NLO increases both $\sigma\,(pp \to H + X)$ and $\Gamma\,(H \to gg)$ by 60-70% (!!)
- NNLO adds about 20% (for both production and decay)
- residual scale dependence of NNLO result: 15-20%
- similarity of corrections for production and decay $\Rightarrow \sigma_{gg}^{SM}/\Gamma_{gg}^{SM}$ is significantly more stable (Melnikov and Petriello)

# $N^3LO$ correction

Optical theorem

$$\Gamma\left(H \to gg\right) = \frac{\sqrt{2}G_F}{M_H} C_1^2 Im\Pi^{GG}\left(q^2 = M_H^2\right)$$

where

$$\Pi^{GG}\left(q^2\right) = \int \mathrm{d}x e^{iqx} \langle 0| T\left(\left[O_1'\right](x)\left[O_1'\right](0)\right)|0\rangle$$

$[O_1']$ = renormalized counterpart of bare operator $O_1' = G^{0'}_{a\mu\nu} G_a^{0'\,\mu\nu}$

$C_1$ from massive tadpoles in order $\alpha_s^3$ (4 loops):
Chetyrkin, Kniehl, Steinhauser (1997)

# Result: (Baikov, Chetyrkin)

$$
\begin{aligned}
Im\Pi^{gg} &= \frac{2q^4}{\pi}\left(1 + 12.4167 a_s + 68.6482 a_s^2 - 212.447 a_s^3\right) \\
\Gamma\left(H \to gg\right) &= \Gamma_{Born}\left(H \to gg\right) \cdot K; \text{with } \mu = M_H \\
K &= 1 + 17.9167 a_s' + 152.5 {a_s'}^2 + 381.5 {a_s'}^3 \\
&= 1 + 0.65575 + 0.2043 + 0.0187 \\
&= 1.87875
\end{aligned}
$$

with $a_s = \alpha_s/\pi$
Residual scale dependence $\delta\Gamma/\Gamma$
LO: $\pm 24\%$, NLO: $\pm 22\%$, NNLO: $\pm 10\%$, NNNLO: $\pm 3\%$
result in NNNLO $\approx$ 1.9 result in LO

recent further improvement to $\mathcal{O}(\alpha_s^4)$ by F. Herzog, B. Ruijl, T. Ueda, J. Vermaseren, A. Vogt

# 2) Scalar Correlator in 5 Loops and Higgs Decay into $b$-quarks

Higgs boson decays into quark-antiquark pair ($f\bar{f}$)

$$\Gamma\left(H \to f\bar{f}\right) = \frac{G_F M_H}{4\sqrt{2}\pi} m_f^2 \widetilde{R}\left(s = M_H^2\right)$$

where $\widetilde{R}\left(s\right) = Im\widetilde{\Pi}(-s - i\epsilon)/(2\pi s)$ is the absorptive part of the scalar two-point correlator

$$\widetilde{\Pi}\left(Q^2\right) = (4\pi)^2 i \int \mathrm{d}x e^{iqx} \left\langle 0 \right| T\left(\left[J_f^S(x)\right]\left[J_f^S(0)\right]\right) \left|0\right\rangle$$

$$\widetilde{R}\left(s\right) = 1 + \sum \widetilde{r}_i a_s^i(s)$$

Strong cancellations between ”kinematical” and ”dynamical” terms

$$\widetilde{R} = 1 + ... + a_s^4\big[(9470.8 - \underline{9431.4}) - n_f(1454.3 - \underline{1233.4}) \\ + n_f^2(54.78 - \underline{45.10}) - n_f^3(0.454 - \underline{0.433})\big]$$

Underlined terms from analytic continuation from spacelike to timelike region!

Remarkable cancellations in all $n_f$ powers, nice ”convergence”:

$$\begin{aligned}\widetilde{R} &= 1 + 5.6667 a_s + 29.147 a_s^2 + 41.76 a_s^3 - 825.7 a_s^4 \\ &= 1 + 0.2075 + 0.0391 + 0.0020 - 0.0018 \\ &= 1.2504\end{aligned}$$

(with $a_s = \frac{\alpha_s}{\pi} = 0.0366$)

dominant contribution from $H \to b\bar{b}$!
$\mathrm{Br}\big(H \to b\bar{b}\big) = 58.4 \pm 3.3\%$

$m_b =$ running mass at scale $m_H$
$m_b(m_H) = 2771 \pm 8\big|_{m_b} \pm 15\big|_{\alpha_s}$ MeV
(Chetyrkin et al.; Herren, Steinhauser)

Complete hadronic Higgs boson decay at order $\alpha_s^4$:
mixed terms between $H \to gg$ and $H \to b\bar{b}$
$\Rightarrow$ Davies, Steinhauser, Wellmann: arXiv:1703.02988

## $\Rightarrow$ dominant Higgs decay modes very well under control

# II. Quark Mass and Field Anomalous Dimension

"Running" quark masses depend on renormalization scale

$$\mu^2 \frac{d}{d\mu^2} m \Big|_{g^0, m^0} = m \gamma_m(a_s) \equiv -m \sum_{i \geq 0} \gamma_i a_s^{i+1}$$

(with $a_s = \alpha_s/\pi$ and $\gamma_i$ known from $\gamma_0$ to $\gamma_4$)
in numerical form for SU(3) (Baikov, Chetyrkin, Kühn, JHEP 10 (2014) 76, [1402.6611]):

$$\gamma_m \underset{n_f=4}{=} -a_s - 3.65278 a_s^2 - 9.94704 a_s^3 - 27.3029 a_s^4 - 111.59 a_s^5$$

$$\gamma_m \underset{n_f=5}{=} -a_s - 3.51389 a_s^2 - 7.41986 a_s^3 - 11.0343 a_s^4 - 41.821 a_s^5$$

| $n_f$ | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| $(\gamma_m)_4^{exact}$ | 198.899 | 111.579 | 41.807 | -9.777 |
| $(\gamma_m)_4^{APAP}$ [Ellis et al] | 162.0 | 67.1 | -13.7 | -80 |
| $(\gamma_m)_4^{APAP}$ [Elias et al] | 163.0 | 75.2 | 12.6 | 12.2 |
| $(\gamma_m)_4^{APAP}$ [Kataev et al] | 164.0 | 71.6 | -4.8 | -64.6 |

exact values for $(\gamma_m)_4$ and predictions based on APAP (”Asymptotic Páde Approximants”)

general result for arbitrary gauge group:

Luthe, Maier, Marquard, Schroeder: 1612.05512; JHEP

and Baikov, Chetyrkin, Kühn: 1702.01458; JHEP

# III. Z Decay to $\mathcal{O}(\alpha_s^4)$

drastic variation of QCD coupling $\alpha_s$ between $M_\tau$ and $M_Z$ or $M_H$
example:

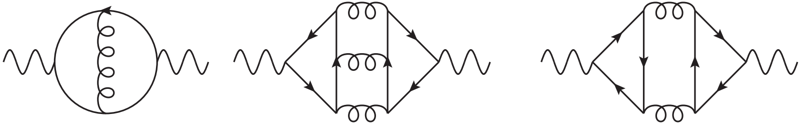
$$\alpha_s\,(M_\tau) = 0.332 \pm 0.005_{exp} \pm 0.015_{th}$$

four loop running and matching:

$$\Rightarrow \alpha_s\,(M_Z) = 0.1202 \pm 0.0006_{exp} \pm 0.0018_{th} \pm 0.0003_{evol}$$

(evolution error receives contributions from $c$ and $b$ mass, matching scale, four loop truncation of RG equation) (Baikov, Chetyrkin, JK)

$\alpha_s(M_Z)$ **from** $\tau$ **decay** in excellent agreement with direct determination in Z decays (Baikov, Chetyrkin, JK, Rittinger: 1201.5804)



non-singlet & singlet, vector & axial correlators

$$R^{nc} = 3\left[\sum_f v_f^2 r_{NS} + \left(\sum_f v_f\right)^2 r_S^V + \sum_f a_f^2 r_{NS} + r^A_{S;t,b}\right] ,$$

with

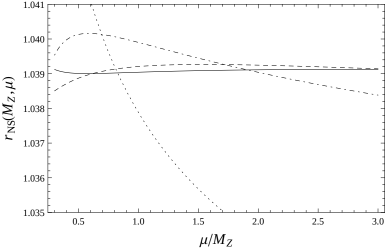
$$r_{NS} = 1 + a_s + 1.40923 a_s^2 - 12.7671 a_s^3 - 79.9806 a_s^4 \,,$$

$$r_S^V = -0.41318 a_s^3 - 4.9841 a_s^4 \,,$$

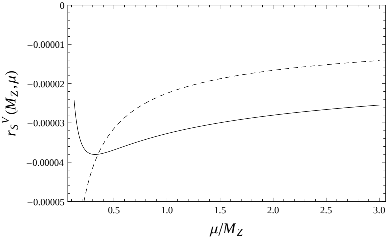
$$r_{S;t,b}^A = (-3.08333 + l_t) a_s^2 + (-15.9877 + 3.72222 l_t + 1.91667 l_t^2) a_s^3 + (49.162 - 17.6822 l_t + 14.7153 l_t^2 + 3.67361 l_t^3) a_s^4 \,,$$

where $l_t = \ln \frac{M_Z^2}{M_t^2}$ and $\Gamma_Z = \Gamma_0 R^{nc} = \frac{G_F M_Z^3}{24\pi\sqrt{2}} R^{nc}$

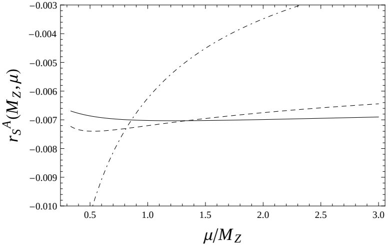
$\Rightarrow \alpha_s(M_Z) = 0.1190 \pm 0.0026_{exp}$ and small theory error!


1.041
1.040
1.039
1.038
1.037
1.036
1.035
0.5
1.0
1.5
2.0
2.5
3.0
$r_{NS}(M_Z, \mu)$
$\mu/M_Z$

(a)


0
−0.00001
−0.00002
−0.00003
−0.00004
−0.00005
0.5
1.0
1.5
2.0
2.5
3.0
$r_S{}^V(M_Z, \mu)$
$\mu/M_Z$

(b)


−0.003
−0.004
−0.005
−0.006
−0.007
−0.008
−0.009
−0.010
0.5
1.0
1.5
2.0
2.5
3.0
$r_S{}^A(M_Z, \mu)$
$\mu/M_Z$

(c)

- non-singlet term starts in order $\alpha_s^0$ (Born) and is identical for $\tau$-decay, $\sigma(e^+e^- \to \mathit{hadrons})$ through vector current (= virtual photon) and $\Gamma(Z \to \mathit{hadrons})$ through vector and axial correlator
- singlet axial term starts in order $\alpha_s^2$, is present in $Z \to \mathit{hadrons}$ and depends on $\ln \frac{M_Z^2}{M_t^2}$ (origin: imbalance between top and bottom quark)
- singlet vector term is present in $\gamma^* \to \mathit{hadrons}$ and $Z \to \mathit{hadrons}$ and starts in order $\alpha_s^3$
- all three terms are known up to order $\alpha_s^4$

recent confirmation: F. Herzog, B. Ruijl, T. Ueda, J. Vermaseren, A. Vogt

# IV. Five-loop $\beta$-function

$$\beta(a_s) = \mu^2 \frac{d}{d\mu^2} a_s(\mu) = -\sum_{i\geq 0} \beta_i a_s^{i+2}$$

$$\beta_0 = \frac{1}{4}\left\{11 - \frac{2}{3} n_f\right\},$$

Gross + Wilczek, Politzer

$$\beta_1 = \frac{1}{4^2}\left\{102 - \frac{38}{3} n_f\right\},$$

Caswell, Jones

$$\beta_2 = \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2\right\},$$

Tarasov + Vladimirov + Zharkov, Larin + Vermaseren

$$\beta_3 = \frac{1}{4^4}\left\{\frac{149753}{6} + 3564\zeta_3 - \left[\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right] n_f + \left[\frac{50065}{162} + \frac{6472}{81}\zeta_3\right] n_f^2 + \frac{1093}{729} n_f^3\right\},$$

van Ritbergen + Vermaseren + Larin, Czakon

$$\beta_4 = \frac{1}{4^5}\Big\{\frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5\Big]$$
$$+ n_f\Big[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\Big]$$
$$+ n_f^2\Big[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\Big]$$
$$+ n_f^3\Big[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\Big]$$
$$+ n_f^4\Big[\frac{1205}{2916} - \frac{152}{81}\zeta_3\Big]\Big\}$$

Baikov, Chetyrkin, JK: PRL 118 (2017) no.8, 082002; [1606.08659]

Absence of $\zeta_4$ and $\zeta_5$ in $\beta_3$ term!
Absence of $\zeta_6$ and $\zeta_7$ in $\beta_4$ term!
Numerically the terms are surprisingly small!
Consider $\overline{\beta} \equiv \frac{\beta}{-\beta_0 a_s^2} = 1 + \sum_{i \geq 1} \overline{\beta_i} a_s^i$

$$\overline{\beta}(n_f = 3) = 1 + 1.78 a_s + 4.47 a_s^2 + 20.99 a_s^3 + 56.59 a_s^4 \,,$$
$$\overline{\beta}(n_f = 4) = 1 + 1.54 a_s + 3.05 a_s^2 + 15.07 a_s^3 + 27.33 a_s^4 \,,$$
$$\overline{\beta}(n_f = 5) = 1 + 1.26 a_s + 1.47 a_s^2 + 9.83 a_s^3 + 7.88 a_s^4 \,,$$
$$\overline{\beta}(n_f = 6) = 1 + 0.93 a_s - 0.29 a_s^2 + 5.52 a_s^3 + 0.15 a_s^4$$

very modest growth of coefficients!

qualitative agreement with $\beta_4$, as calculated with Asymptotic Páde Approximant (APAP)

$$\beta_4^{APAP} = 740 - 213 n_f + 20 n_f^2 - 0.0486 n_f^3 - \boxed{0.0017993 n_f^4} \leftarrow \mathrm{input}$$

$$\beta_4^{exact} = 524.56 - 181.8 n_f + 17.16 n_f^2 - 0.22586 n_f^3 - 0.0017993 n_f^4$$

but large cancellations $\Rightarrow$ numerical disagreement

| $n_f$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $(\beta)_4^{exact}$ | 525 | 360 | 228 | 127 | 57 | 15 | 0.27 |
| $(\beta)_4^{APAP}$ | 741 | 548 | 395 | 281 | 205 | 169 | 170 |

excellent agreement between $\alpha_s(M_Z)$ from $\tau$ decays (+ running and matching) and direct measurement

$$\alpha_s(m_\tau) = 0.33 \pm 0.014 \Rightarrow \alpha_s(M_Z) = 0.1198 \pm 0.0015$$
$$\textbf{vs } \alpha_s(M_Z) = 0.1197 \pm 0.0028 \textbf{ directly from Z} - \textbf{decay}$$

(result confirmed and generalized to arbitrary gauge group:
F. Herzog, B. Ruijl, T. Ueda, J. Vermaseren, A. Vogt, arXiv:1701.01404
T. Luthe, A. Maier, P. Marquard, Y. Schroeder)

# Summary

Karlsruhe Institute of Technology

- QCD corrections for
  Higgs decay to $f\bar{f}$, Higgs decay to gluons, $\tau$ decay to $\nu + had$,
  $\mathbf{R} = \frac{\sigma_{tot}(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$, $Z$ decay to $f\bar{f}$. All are availabe to $\mathcal{O}(\alpha_s^4)$
  corresponding to 4 loops
- matched by QCD $\beta$-function in 5-loops
- excellent agreement between theory and experiment
- theory prediction significantly ahead of experiment