The Polarization Function, the QED Beta Function and the Muon Anomalous Magnetic Moment

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I. Four loop polarization function

II. QED beta function at five loops

III. Anomalous magnetic moment of the muon:
selected five- and six-loop terms

based on

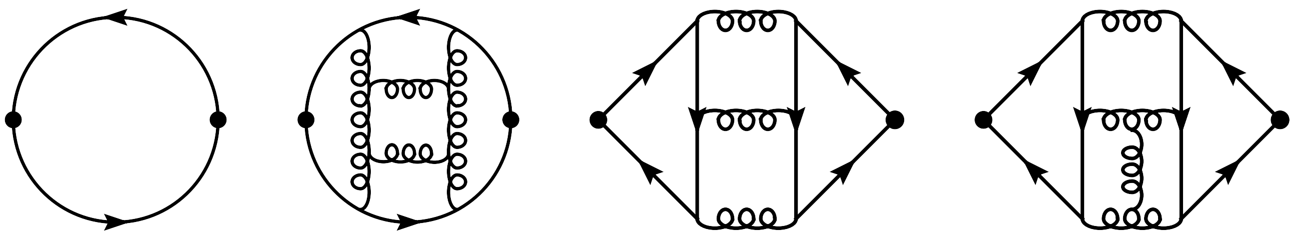
Baikov, Chetyrkin, JHK, J. Rittinger, arxiv: 1206.1284, JHEP 1207(2012)017

Baikov, Chetyrkin, JHK, C. Sturm, arxiv: 1207.2199

I. The Polarization Function

$$(-g_{\alpha\beta}q^2 + q_\alpha q_\beta)\,\Pi(L, a_s) = i\int \mathrm{d}^4 x e^{iq\cdot x}\langle 0|\mathrm{T} j_\alpha(x) j_\beta(0)|0\rangle$$

available in 4 loops (including constant piece)



Examples of two non-singlet and two singlet diagrams contributing to the vector correlator.

using

$$D(L, a_s) = 12\pi^2 \left(\gamma(a_s) - \left(\beta(a_s)\frac{\partial}{\partial a_s}\right)\Pi(L, a_s)\right)$$

with Π in $\mathcal{O}(\alpha_s^3)$ and anomalous dimension γ in 5 loops ($\mathcal{O}(\alpha_s^4)$)

$\Rightarrow R \equiv \sigma(\mathrm{had}/\sigma(\mu^+\mu^-)$ in $\mathcal{O}(\alpha_s^4)$)

Results:

$$\Pi^{NS} = \frac{d_R}{16\pi^2}\left(\sum_{i\geq 0} p_i^{NS}\, a_s^i\right), \quad \Pi^{SI} = \frac{d_R}{16\pi^2}\left(\sum_{i\geq 3} p_i^{SI}\, a_s^i\right),$$

$$\begin{aligned}
p_0^{NS} &= \frac{20}{9}, \\
p_1^{NS} &= C_F\left[\frac{55}{12} - 4\zeta_3\right], \\
p_2^{NS} &= C_F^2\left[-\frac{143}{72} - \frac{37}{6}\zeta_3 + 10\zeta_5\right] + C_F\,C_A\left[\frac{44215}{2592} - \frac{227}{18}\zeta_3 - \frac{5}{3}\zeta_5\right] \\
&\quad + C_F\,T_F\,n_f\left[-\frac{3701}{648} + \frac{38}{9}\zeta_3\right], \\
p_3^{NS} &= C_F^3\left[-\frac{31}{192} + \frac{13}{8}\zeta_3 + \frac{245}{8}\zeta_5 - 35\zeta_7\right] + T^2\,n_f^2\,C_F\left[\frac{196513}{23328} - \frac{809}{162}\zeta_3 - \frac{20}{9}\zeta_5\right] \\
&\quad + T\,n_f\,C_F^2\left[-\frac{7505}{10368} + \frac{1553}{54}\zeta_3 - 4\,\zeta_3^2 + \frac{11}{24}\zeta_4 - \frac{250}{9}\zeta_5\right] \\
&\quad + T\,n_f\,C_F\,C_A\left[-\frac{5559937}{93312} + \frac{41575}{1296}\zeta_3 + \frac{2}{3}\,\zeta_3^2 - \frac{11}{24}\zeta_4 + \frac{515}{27}\zeta_5\right] \\
&\quad + C_F^2\,C_A\left[-\frac{382033}{20736} - \frac{46219}{864}\zeta_3 - \frac{11}{48}\zeta_4 + \frac{9305}{144}\zeta_5 + \frac{35}{2}\zeta_7\right] \\
&\quad + C_F\,C_A^2\left[\frac{34499767}{373248} - \frac{147473}{2592}\zeta_3 + \frac{55}{6}\,\zeta_3^2 + \frac{11}{48}\zeta_4 - \frac{28295}{864}\zeta_5 - \frac{35}{12}\zeta_7\right], \\
p_3^{SI} &= \frac{d^{abc}\,d^{abc}}{d_R}\left\{\frac{431}{1728} - \frac{21}{64}\zeta_3 - \frac{1}{6}\,\zeta_3^2 - \frac{1}{16}\zeta_4 + \frac{5}{16}\zeta_5\right\}.
\end{aligned}$$

Can be applied for QCD (corresponding to quark loops) or to pure QED (lepton loops) with properly chosen colour factors.

II. QED Beta Function

- The QED beta function receives contributions from non-singlet (starting from 1-loop) and from singlet (starting from 4-loop) terms.

- RG-equation: perturbative QCD contribution to

$$\mu^2 \frac{d}{d\mu^2} A = \beta^{EM}(A, a_s) = 16\pi^2 A^2 \gamma^{EM}(a_s)$$

with $\gamma^{EM} = (\sum q_i^2)\gamma^{NS} + (\sum q_i^2)\gamma^{SI}$
and $A = \alpha/4\pi$; $\quad a_s = \alpha_s/4\pi$
$\gamma =$ anomalous dimension, evaluated in 5 loops (with the help of massless 4-loop propagator integrals)
$\Rightarrow$ result in $\overline{\text{MS}}$ scheme

- conversion: MOM-scheme

$\Pi^{MOM}(Q^2, \mu^2)$ vanishes at $Q^2 = \mu^2$ (with $\mu^2 \neq 0!$)

$$\Rightarrow \tilde{A}(\mu) = \frac{A(\mu)}{1 + (4\pi)^2 \, A(\mu) \, \Pi(L = 0, a_s(\mu))}.$$

with $L \equiv \ln \frac{\mu^2}{Q^2}$ and

$$\beta^{EM}_{MOM}(\tilde{A}, a_s) = 16\pi^2 \, \tilde{A}^2 \left[\gamma^{EM}(a_s) - \beta^{QCD}(a_s) \, \frac{\partial}{\partial a_s} \, \Pi^{EM}(L = 0, a_s) \right]$$

No new calculation needed.

- application: pure QED, $\overline{\text{MS}}$ scheme

$$
\begin{aligned}
\beta^{QED}(A) = {}& n_f \left[\frac{4\ A^2}{3}\right] + 4\, n_f A^3 - A^4 \left[2\, n_f + \frac{44}{9}\, n_f^2\right] \\
+ {}& A^5 \left[-46\, n_f + \frac{760}{27}\, n_f^2 - \frac{832}{9}\, \zeta_3\, n_f^2 - \frac{1232}{243}\, n_f^3\right] \\
+ {}& A^6 \left(n_f \left[\frac{4157}{6} + 128\zeta_3\right] + n_f^2 \left[-\frac{7462}{9} - 992\zeta_3 + 2720\zeta_5\right] \right. \\
+ {}& \left. n_f^3 \left[-\frac{21758}{81} + \frac{16000}{27}\zeta_3 - \frac{416}{3}\zeta_4 - \frac{1280}{3}\zeta_5\right] + n_f^4 \left[\frac{856}{243} + \frac{128}{27}\zeta_3\right] \right).
\end{aligned}
$$

- conversion to MOM-scheme: as before

- conversion: on-shell scheme

  $\Pi^{OS}(Q^2, M^2)$ vanishes at $Q^2 = 0$ ($M^2 \neq 0$!)

  $\Rightarrow$ $\Pi^{\overline{\text{MS}}}(Q^2 = 0, m^2, \mu^2)$ is required: 4-loop tadpoles!

  conversion of coupling constant (4-loop)
  conversion of mass (3-loop)

  $\Rightarrow$ $\Pi^{OS}(Q^2, M^2)$ Q^2-dependent (logarithmic) part at 5 loop.
  (μ^2 disappears, M^2 appears)

Results: term of order α^5, Q^2 dependent part

$$
\begin{aligned}
\Pi^{(5)}(\ell_{MQ}) = N\ell_{MQ}\Bigg\{ & \frac{4157}{6144} + \frac{1}{8}\zeta_3 \\
+ \quad & N\left[\frac{55}{96} + \frac{5}{96}\pi^2 + \frac{179}{256}\zeta_3 - \frac{115}{12}\zeta_5 + \frac{35}{4}\zeta_7 + \frac{13}{128}\ell_{MQ} - \frac{1}{12}\pi^2 \ln(2)\right] \\
+ \quad & N\,\mathbf{si}\left[-\frac{13}{12} - \frac{4}{3}\zeta_3 + \frac{10}{3}\zeta_5\right] \\
+ \quad & N^2\left[-\frac{11}{432} + \frac{1}{36}\pi^2 - \frac{17089}{2304}\zeta_3 + \zeta_3^2 + \frac{125}{18}\zeta_5 + \frac{35}{288}\ell_{MQ} \right. \\
& \left. \quad - \frac{7}{8}\zeta_3\ell_{MQ} + \frac{5}{6}\zeta_5\ell_{MQ} + \frac{1}{72}\ell_{MQ}^2\right] \\
+ \quad & N^2\,\mathbf{si}\left[-\frac{149}{108} + \frac{13}{6}\zeta_3 + \frac{2}{3}\zeta_3^2 - \frac{5}{3}\zeta_5 - \frac{11}{72}\ell_{MQ} + \frac{1}{3}\zeta_3\ell_{MQ}\right] \\
+ \quad & N^3\left[-\frac{6131}{2916} + \frac{203}{162}\zeta_3 + \frac{5}{9}\zeta_5 - \frac{151}{324}\ell_{MQ} + \frac{19}{54}\zeta_3\ell_{MQ} - \frac{11}{216}\ell_{MQ}^2 \right. \\
& \left. \quad + \frac{1}{27}\zeta_3\ell_{MQ}^2 - \frac{1}{432}\ell_{MQ}^3\right]\Bigg\}.
\end{aligned} \tag{1}
$$

N = number of leptons; $\ell_{MQ} = \ln M^2/Q^2$

$\Rightarrow$ β-function in OS scheme at 5 loops

III. Anomalous magnetic moment of the muon

Recall (numbers from Kinoshita et al. 1205.5370)

$$a_\mu(exp) = 116592089(63) \times 10^{-11}$$
$$\delta_{exp} = 63 \times 10^{-11}$$

Theory: dominant errors (hadronic)

$$\delta_{vacpol} = (37.2)_{exp} + (21.0)_{rad} \times 10^{-11}$$
$$\delta_{ll} = 40 \times 10^{-11}$$

QED: 2 loop, 3 loop: exact, analytic
4 loop, 5 loop (recently): numerical (Kinoshita).

3 loop: $a_\mu^{(6)} = (1.181\ldots + 22.868\ldots)\left(\frac{\alpha}{\pi}\right)^3 \approx 3{\cdot}10^{-7}$

const $\quad [\log(m_\mu/m_e)]^n$

4 loop: $a_\mu^{(8)} = (-1.9106(20) + 132.6852(60))\left(\frac{\alpha}{\pi}\right)^4 \approx 382{\cdot}10^{-11}$

const $\quad [\log(m_\mu/m_e)]^n$

(theory error: $1.7{\cdot}10^{-13}$)

5 loop: $a_\mu^{(10)} = (9.168(571) + 742.18(87) + \ldots)\left(\frac{\alpha}{\pi}\right)^5 \approx 5{\cdot}10^{-11}$

const $\quad [\log(m_\mu/m_e)]^n$

factor 10 below experimental uncertainty.

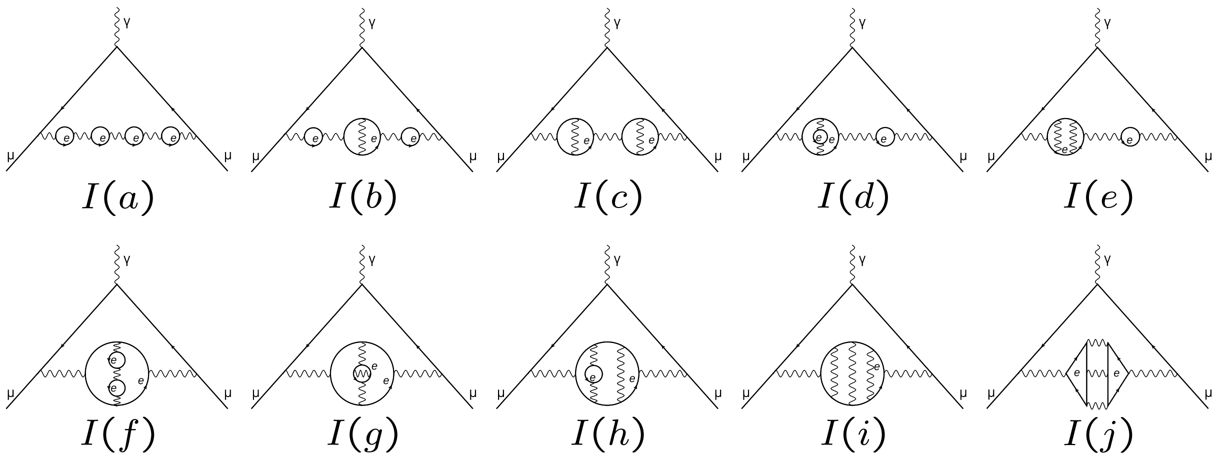
Nevertheless: should be checked:

master formulae

$$a_\mu^{\text{asymp}} = \frac{\alpha}{\pi}\int_0^1 dx(1-x)\left[d_R^{\text{asymp}}\left(\frac{x^2}{1-x}\frac{M_\mu^2}{M_e^2},\alpha\right)-1\right],$$

$$d_R^{\text{asymp}}(Q^2/M^2,\alpha) = \frac{1}{1+\Pi^{\text{asymp}}(Q^2/M^2,\alpha)}.$$

with Π evaluated in the OS scheme



The ten gauge invariant subsets contributing to the muon anomaly which originate from inserting the vacuum polarization up to four-loop order into the first order QED vertex. For each diagram class only one typical representative is shown. Wavy lines denote photons(γ), solid lines denote electrons(e) or muons (μ). The last five diagrams $\{I(f),\ I(g),\ I(h),\ I(i),\ I(j)\}$ are non-factorizable insertions of the vacuum polarization function; the first five diagrams $\{I(a),\ I(b),\ I(c),\ I(d),\ I(e)\}$ are factorizable ones.

Result for coefficients:

Subset	analytical	numerical	Ref.	num.-ana.
$I(a)$	20.1832 $+ \mathcal{O}(M_e/M_\mu)$	20.14293(23)	(Kinoshita)	$\approx$ -0.04
$I(b)$	27.7188 $+ \mathcal{O}(M_e/M_\mu)$	27.69038(30)	(Kinoshita)	$\approx$ -0.03
$I(c)$	4.81759 $+ \mathcal{O}(M_e/M_\mu)$	4.74212(14)	(Kinoshita)	$\approx$ -0.08
$I(d)$	7.44918 $+ \mathcal{O}(M_e/M_\mu)$	7.45173(101)	(Kinoshita)	$\approx$ 0.003
$I(e)$	-1.33141 $+ \mathcal{O}(M_e/M_\mu)$	-1.20841(70)	(Kinoshita)	$\approx$ 0.12
$I(f)$	2.89019 $+ \mathcal{O}(M_e/M_\mu)$	2.88598(9)	(Kinoshita)	$\approx$ -0.004
$I(g)+I(h)$	1.50112 $+ \mathcal{O}(M_e/M_\mu)$	1.56070(64)	(Kinoshita)	$\approx$ 0.06
$I(i)$	0.25237 $+ \mathcal{O}(M_e/M_\mu)$	0.0871(59)	(Kinoshita)	$\approx$ -0.17
$I(j)$	-1.21429 $+ \mathcal{O}(M_e/M_\mu)$	-1.24726(12)	(Kinoshita)	$\approx$ -0.03

The first column shows the different gauge invariant subsets of diagrams. The second column contains the corresponding results evaluated numerically, where we have used for the mass ratio $M_\mu/M_e = 206.7682843(52)$. This result is correct only up to power corrections in the small mass ratio M_e/M_μ. The third column contains the numerical result obtained by Kinoshita et al. . The last column shows the difference between the numerical and asymptotic analytical results. The subsets $\{I(a), I(b), I(c), I(d), I(e)\}$ originate from Feynman diagrams with factorizable vacuum polarization insertions, whereas the subsets $\{I(f), I(g), I(h), I(i), I(j)\}$ are non-factorizable.

good overall agreement!

sum: vacpol= $\sum I = 62.26675$

to be compared with 751.35 for the total

lessons from 5-loop

logarithmically enhanced terms and factorizable terms dominate:

$$\sum I = 62.26675 = \underbrace{58.8374}_{\text{factorizable}} + \underbrace{1.915}_{\substack{\text{irreducible}\\ \text{4 loop vacpol}\\ \text{logs}}} + \underbrace{1.514}_{\substack{\text{irreducible}\\ \text{4 loop vacpol}\\ \text{const}}}$$

prediction for 6 loops (vacpol-subset)

$$\sum I = \underbrace{246.381}_{\text{factorizable}} + \underbrace{10.8647}_{\substack{\text{irreducible}\\ \text{5 loop vacpol}\\ \text{logs}}} + \substack{\text{small irreducible}\\ \text{5 loop vacpol}\\ \text{const}} \approx 257$$

still missing (and dominant): light by light!