

The three-loop polarized pure singlet operator matrix element with two different masses

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Abstract

We present the two-mass QCD contributions to the polarized pure singlet operator matrix element at three loop order in x -space. These terms are relevant for calculating the polarized structure function $g_1(x, Q^2)$ at $O(\alpha_s^3)$ as well as for the matching relations in the variable flavor number scheme and the polarized heavy quark distribution functions at the same order. The result for the operator matrix element is given in terms of generalized iterated integrals. These integrals depend on the mass ratio through the main argument, and the alphabet includes square-root valued letters.

1 Introduction

Massive operator matrix elements (OMEs) are essential building blocks for the massive Wilson coefficients in deep-inelastic scattering in the limit $Q^2 \gg m^2$, and they are the transition matrix elements in the variable flavor number scheme (VFNS) [1,2]. Here Q^2 denotes the virtuality of the deep-inelastic process and m is the heavy quark mass. From 2-loop order onward these matrix elements A_{ij} receive two-mass corrections. In the unpolarized case the two-mass corrections have been calculated for all OMEs to three-loop order in Refs. [3–7]. For the OME $A_{Qq}^{(3),\text{tm}}$ a large set of Mellin moments for even values of the Mellin variable $N \in \mathbb{N}$ has been derived, expanding in the mass ratio

$$\eta = \frac{m_c^2}{m_b^2}, \quad (1.1)$$

to a finite power, where $m_{c(b)}$ denote the charm and bottom quark mass, respectively.

In the polarized case, the flavor non-singlet three loop OME $A_{qq,Q}^{(3),\text{NS},\text{tm}}$ [4] and $A_{gq,Q}^{(3),\text{tm}}$ [7] have been calculated. In the present paper we compute the two mass contributions to the pure singlet massive OME $A_{Qq}^{(3),\text{PS},\text{tm}}$. Like in the unpolarized case, the calculation cannot be performed in N space, transforming to momentum fraction x space later, because the associated recurrences do not factorize to first order. This, however, is the case for the corresponding differential equations in x space. In the result we obtain iterative integrals, partly with limited support in $x \in [0, 1]$. This has also been observed in the single mass pure singlet case [8].

Since we will use dimensional regularization in the calculation, a consistent description of the Dirac matrix γ_5 is necessary. For this we use the Larin scheme [9]. The polarized massive OME $A_{Qq}^{(3),\text{PS},\text{tm}}$ contributes to the polarized three-loop massive Wilson coefficient $H_{Qq}^{(3)}(z, Q^2)$ and is one of the contributions of the two-mass variable flavor number scheme [4, 10] in the polarized case, describing the respective transitions of the polarized parton densities in the case the heavy quarks become light. They contribute in particular also to the charm- and bottom quark distributions.

The paper is organized as follows. In Section 2 we present the renormalized pure-singlet OME in the 2-mass case. Details of the calculation are given in Section 3. In many steps of the calculation we follow the computation performed in Ref. [5] in the unpolarized case, to be able to use the function space and the integral relations which have been developed there. In Section 4 the result of the calculation is presented and numerical results are given in Section 5. Section 6 contains the conclusions. In the appendix we provide complete analytic expressions for a number of Mellin moments $N \in \mathbb{N}$.

2 The renormalized 2-mass pure singlet OME

The generic pole structure of the polarized PS three-loop two-mass contribution to the massive OME is given by [4]

$$\begin{aligned} \hat{A}_{Qq}^{(3),\text{PS},\text{tm}} &= \frac{8}{3\varepsilon^3} \gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(0)} \beta_{0,Q} + \frac{1}{\varepsilon^2} \left[2\gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(0)} \beta_{0,Q} (L_1 + L_2) + \frac{1}{6} \hat{\gamma}_{qg}^{(0)} \hat{\gamma}_{gq}^{(1)} - \frac{4}{3} \beta_{0,Q} \hat{\gamma}_{qg}^{\text{PS},(1)} \right] \\ &+ \frac{1}{\varepsilon} \left[\gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(0)} \beta_{0,Q} (L_1^2 + L_1 L_2 + L_2^2) + \left\{ \frac{1}{8} \hat{\gamma}_{qg}^{(0)} \hat{\gamma}_{gq}^{(1)} - \beta_{0,Q} \hat{\gamma}_{qg}^{\text{PS},(1)} \right\} (L_2 + L_1) \right] \\ &+ \frac{1}{3} \hat{\gamma}_{qg}^{(2),\text{PS}} - 8a_{Qq}^{(2),\text{PS}} \beta_{0,Q} + \hat{\gamma}_{qg}^{(0)} a_{gq}^{(2)} + \tilde{a}_{Qq}^{(3),\text{PS}} (m_1^2, m_2^2, \mu^2), \end{aligned} \quad (2.1)$$

where we used the short hand notation

$$\hat{\gamma}_{ij} = \gamma_{ij}(N_F + 2) - \gamma_{ij}(N_F), \quad (2.2)$$

$$\hat{\hat{\gamma}}_{ij} = \frac{\gamma_{ij}(N_F + 2)}{N_F + 2} - \frac{\gamma_{ij}(N_F)}{N_F}. \quad (2.3)$$

The tilde in $\tilde{\hat{A}}_{Qq}^{(3),\text{PS}}$ indicates that we are considering only the genuine two-mass contributions, and the double hat is used to denote a completely unrenormalized OME. Here the $\gamma_{ij}^{(l)}$'s are anomalous dimensions at $l + 1$ loops [11–14], $\beta_{0,Q} = -\frac{4}{3}T_F$, and

$$L_1 = \ln\left(\frac{m_1^2}{\mu^2}\right), \quad L_2 = \ln\left(\frac{m_2^2}{\mu^2}\right), \quad (2.4)$$

where m_1 and m_2 are the masses of the heavy quarks, and μ is the renormalization scale. Our goal is to compute the $O(\varepsilon^0)$ term $\tilde{a}_{Qq}^{(3),\text{PS}}(m_1^2, m_2^2, \mu^2)$.

We renormalize the heavy masses on-shell and the coupling constant in the $\overline{\text{MS}}$ scheme. The polarized OME in the Larin scheme is given by

$$\begin{aligned} \tilde{A}_{Qq}^{(3),\overline{\text{MS}},\text{PS},\text{tm}} &= -\gamma_{gq}^{(0)}\hat{\gamma}_{gq}^{(0)}\beta_{0,Q} \left(\frac{1}{4}L_2^2L_1 + \frac{1}{4}L_1^2L_2 + \frac{1}{3}L_1^3 + \frac{1}{3}L_2^3 \right) \\ &+ \left\{ -\frac{1}{16}\hat{\gamma}_{gq}^{(0)}\hat{\gamma}_{gq}^{(1)} + \frac{1}{2}\beta_{0,Q}\hat{\gamma}_{gq}^{\text{PS},(1)} \right\} (L_2^2 + L_1^2) \\ &+ \left\{ 4a_{Qq}^{(2),\text{PS}}\beta_{0,Q} - \frac{1}{2}\hat{\gamma}_{gq}^{(0)}a_{gq}^{(2)} - \frac{1}{4}\beta_{0,Q}\zeta_2\gamma_{gq}^{(0)}\hat{\gamma}_{gq}^{(0)} \right\} (L_1 + L_2) \\ &+ 8\bar{a}_{Qq}^{(2),\text{PS}}\beta_{0,Q} - \hat{\gamma}_{gq}^{(0)}\bar{a}_{gq}^{(2)} + \tilde{a}_{Qq}^{(3),\text{PS}}(m_1^2, m_2^2, \mu^2). \end{aligned} \quad (2.5)$$

The transition relations for the renormalization of the heavy quarks in the $\overline{\text{MS}}$ -scheme is given in [8], Eq. (5.100), but it only applies to the equal mass case since for the unequal mass case the first contributions emerge at 3-loop order. In Eqs. (2.1) and (2.5), $a_{Qq}^{(2),\text{PS}}$ and $a_{gq}^{(2)}$ represent the $O(\varepsilon^0)$ terms of the two-loop OMEs $\hat{A}_{Qq}^{(2),\text{PS}}$ and $\hat{A}_{gq}^{(2)}$, respectively, while $\bar{a}_{Qq}^{(2),\text{PS}}$ and $\bar{a}_{gq}^{(2)}$ represent the corresponding $O(\varepsilon)$ terms, cf. Refs. [7, 15–18]. Here and in what follows, ζ_k , $k \in \mathbb{N}, k \geq 2$ denotes the Riemann ζ -function at integer argument.

3 Details of the calculation

There are sixteen irreducible diagrams for $\tilde{A}_{Qq}^{(3),\text{PS},\text{tm}}$, which are shown in Figure 1. The unrenormalized operator matrix element is obtained by adding all the diagrams and applying the quarkonic projector P_q to the corresponding Green function \hat{G}_Q^{ij} , cf. [14],

$$P_q \hat{G}_l^{ij} = -\delta_{ij} \frac{i(\Delta \cdot p)^{-N-1}}{4N_c(D-2)(D-3)} \varepsilon_{\mu\nu\rho\Delta} \text{tr} \left[\not{p} \gamma^\mu \gamma^\nu \hat{G}_l^{ij} \right], \quad (3.1)$$

where p is the momentum of the on-shell external massless quark ($p^2 = 0$), Δ is a light-like D -vector, with $D = 4 + \varepsilon$, the dimension of space-time in which we work, i and j are the color indices of each external leg, and N_c is the number of colors. Note that the projector (3.1) is different from that in the unpolarized case [5]. The diagrams, D_1, \dots, D_{16} , are calculated directly within dimensional regularization. The Dirac algebra is performed using FORM [19]. Diagrams

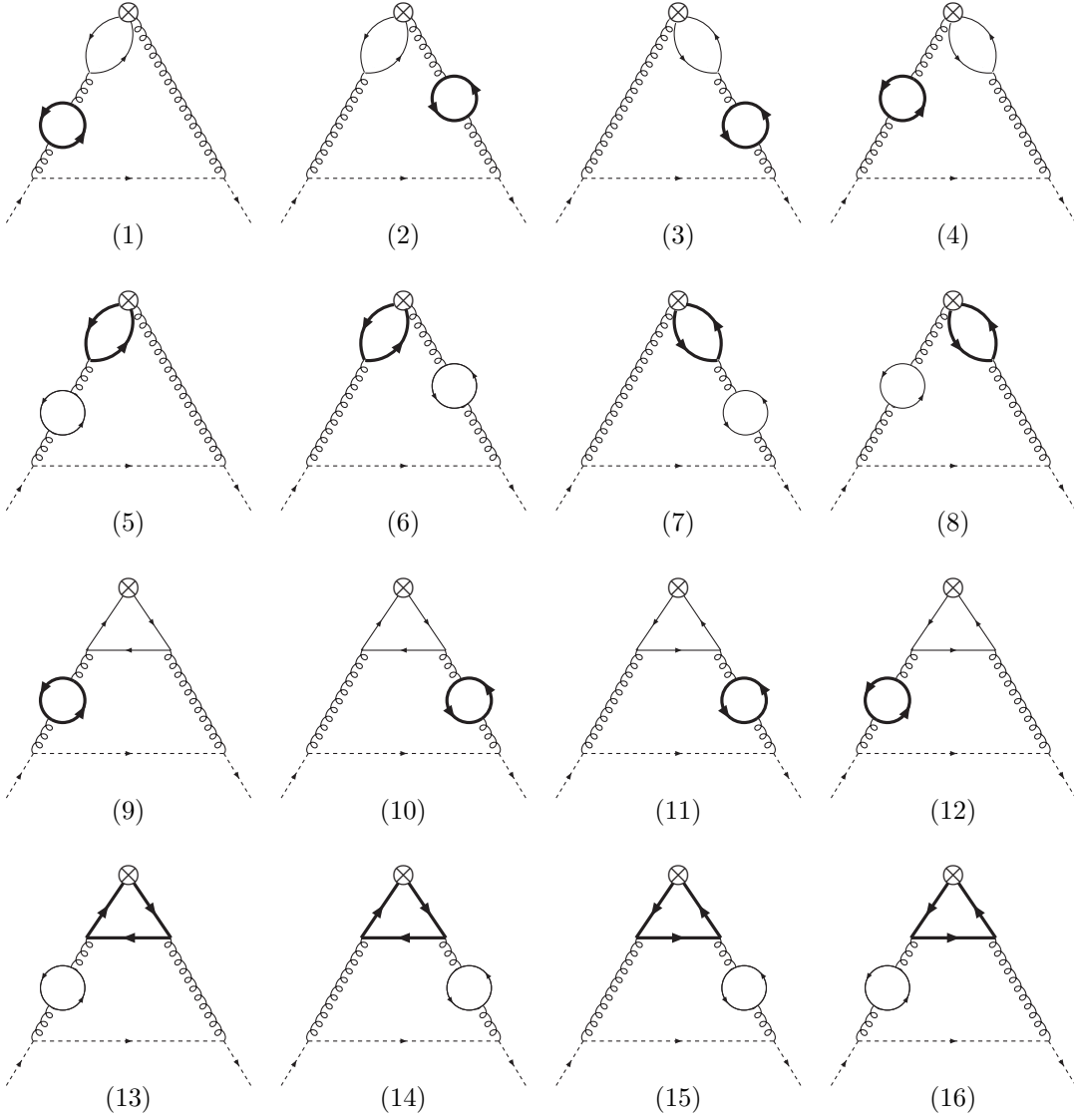


Figure 1: The diagrams for the two-mass contributions to $\tilde{A}_{Qq}^{(3),\text{PS}}$. The dashed arrow line represents the external massless quarks, while the thick solid arrow line represents a quark of mass m_1 , and the thin arrow line a quark of mass m_2 . We assume $m_1 > m_2$.

1–8 turn out to vanish. Diagrams 9–12 and 13–16 can be mapped by symmetry relations to each other, respectively. These two classes are furthermore related by exchanging $\eta \Leftrightarrow 1/\eta$.

One therefore obtains

$$A_{Qq}^{(3),\text{PS,tm}}(N) = 2 [1 + (-1)^{N-1}] D_9(m_1, m_2, N) + 2 [1 + (-1)^{N-1}] D_9(m_2, m_1, N), \quad (3.2)$$

where N is the Mellin variable appearing in the Feynman rules for the operator insertions, cf. [11, 14]. In the following we use the variable

$$\eta = \frac{m_2^2}{m_1^2}, \quad (3.3)$$

with $m_2 < m_1$, i.e. $\eta < 1$, which we will assume in what follows.

While in other calculations one could derive the results working either in Mellin N or x -space, cf. e.g. [4, 6], this is not the case here, see also [5]. We will, therefore, present our result only in

x -space, which is anyway all we need in order to obtain the corresponding contribution to the structure function $g_1(x, Q^2)$ for large values of Q^2 , as well as the contribution to the variable flavor number scheme. In most of the applications one finally works in x -space.

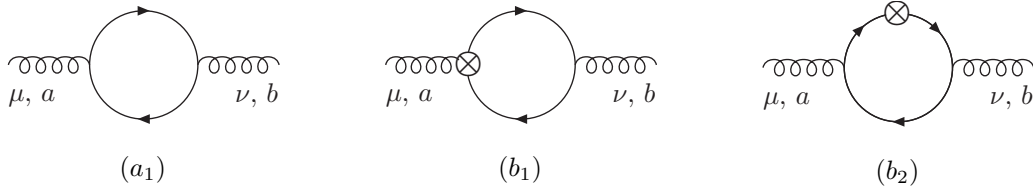


Figure 2: Massive bubbles appearing in the Feynman diagrams shown in Figure 1.

All the diagrams contain a massive fermion loop with an operator insertion (Figures 2b₁ and 2b₂) and a massive bubble without the operator (Figure 2a₁). The latter can be rendered effectively massless by using a Mellin–Barnes integral [20–24]. One obtains

$$I_{a_1}^{\mu\nu,ab}(k) = -\frac{8iT_F g_s^2}{(4\pi)^{D/2}} \delta_{ab} (k^2 g^{\mu\nu} - k^\mu k^\nu) \int_0^1 dx \frac{\Gamma(2 - D/2) (x(1-x))^{D/2-1}}{\left(-k^2 + \frac{m^2}{x(1-x)}\right)^{2-D/2}}, \quad (3.4)$$

$$\begin{aligned} I_{b_2}^{\mu\nu,ab}(k) &= \alpha_s T_F i e^{-\gamma_E \varepsilon/2} (k \cdot \Delta)^{N-1} (\mu^2)^{-\varepsilon/2} S_\varepsilon \varepsilon^{\Delta k \mu \nu} \int_0^1 dx x^{N+D/2-1} (1-x)^{D/2-1} \\ &\times \left\{ \left(-k^2 + \frac{m^2}{x(1-x)}\right)^{-2+D/2} 2\Gamma(2 - D/2) [(D-6)x^{-2} + (D+2N)x^{-1}] \right. \\ &+ \left(-k^2 + \frac{m^2}{x(1-x)}\right)^{-3+D/2} 4\Gamma(3 - D/2) (1-x)^{-1} [m^2(x^{-3} + x^{-2}) \\ &\left. + (-k^2)(1-x^{-1}) \right\}, \quad (3.6) \end{aligned}$$

where μ and ν are the respective Lorentz indices of the external legs, a and b are the color indices, k is the external momentum, m is the mass of the fermion, which can be either m_1 or m_2 , $g_s = \sqrt{4\pi\alpha_s}$ is the strong coupling constant, and $T_F = 1/2$ in $SU(N_c)$, with N_c the number of colors. The other color factors are $C_F = (N_c^2 - 1)/(2N_c)$ and $C_A = N_c$. The term $I_{b_1}^{\mu\nu,ab}(k)$ only appears in diagrams which vanish and is not displayed here.

For diagram 9 we obtain the representation

$$\begin{aligned} D_9(m_1, m_2, N) &= C_F T_F^2 \alpha_s^3 S_\varepsilon^3 \frac{16}{2+\varepsilon} \left\{ 4(2-\varepsilon)J_1 - 8\eta J_2 - 8(N+3)J_3 + 8J_4 + 8\left(2 + \frac{\varepsilon}{2} + N\right) \right. \\ &\times J_5 - 8J_6 - (\varepsilon-2)^2 J_7 + 2(2-\varepsilon)\eta J_8 + 2(2-\varepsilon)(3+N)J_9 - 2(2-\varepsilon)J_{10} \\ &\left. - 2(2-\varepsilon)\left(2 + \frac{\varepsilon}{2} + N\right)J_{11} + 2(2-\varepsilon)J_{12} - 8\eta J_{13} + 2(2-\varepsilon)\eta J_{14} \right\}, \quad (3.7) \end{aligned}$$

with

$$J_1 = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N)}{\Gamma(1 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{-1 + \frac{\varepsilon}{2} + N} B_1\left(\frac{\eta}{x(1-x)}\right), \quad (3.8)$$

$$J_2 = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N)}{\Gamma(1 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{-1 + \frac{\varepsilon}{2} + N} B_3\left(\frac{\eta}{x(1-x)}\right), \quad (3.9)$$

$$J_3 = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N)}{\Gamma(1 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{\frac{\varepsilon}{2} + N} B_1\left(\frac{\eta}{x(1-x)}\right), \quad (3.10)$$

$$J_4 = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N)}{\Gamma(1 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{\frac{\varepsilon}{2} + N} B_2\left(\frac{\eta}{x(1-x)}\right), \quad (3.11)$$

$$J_5 = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N)}{\Gamma(1 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{1 + \frac{\varepsilon}{2} + N} B_1\left(\frac{\eta}{x(1-x)}\right), \quad (3.12)$$

$$J_6 = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N)}{\Gamma(1 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{1 + \frac{\varepsilon}{2} + N} B_2\left(\frac{\eta}{x(1-x)}\right), \quad (3.13)$$

$$J_7 = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N+1)}{\Gamma(2 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{-1 + \frac{\varepsilon}{2} + N} B_1\left(\frac{\eta}{x(1-x)}\right), \quad (3.14)$$

$$J_8 = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N+1)}{\Gamma(2 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{-1 + \frac{\varepsilon}{2} + N} B_3\left(\frac{\eta}{x(1-x)}\right), \quad (3.15)$$

$$J_9 = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N+1)}{\Gamma(2 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{\frac{\varepsilon}{2} + N} B_1\left(\frac{\eta}{x(1-x)}\right), \quad (3.16)$$

$$J_{10} = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N+1)}{\Gamma(2 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{\frac{\varepsilon}{2} + N} B_2\left(\frac{\eta}{x(1-x)}\right), \quad (3.17)$$

$$J_{11} = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N+1)}{\Gamma(2 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{1 + \frac{\varepsilon}{2} + N} B_1\left(\frac{\eta}{x(1-x)}\right), \quad (3.18)$$

$$J_{12} = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N+1)}{\Gamma(2 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{1 + \frac{\varepsilon}{2} + N} B_2\left(\frac{\eta}{x(1-x)}\right), \quad (3.19)$$

$$J_{13} = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N)}{\Gamma(1 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{-2 + \frac{\varepsilon}{2} + N} B_3\left(\frac{\eta}{x(1-x)}\right), \quad (3.20)$$

$$J_{14} = \left(\frac{m_1^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \frac{\Gamma(N+1)}{\Gamma(2 + \frac{\varepsilon}{2} + N)} \int_0^1 dx (1-x)^{\frac{\varepsilon}{2}} x^{-2 + \frac{\varepsilon}{2} + N} B_3\left(\frac{\eta}{x(1-x)}\right). \quad (3.21)$$

The functions B_i are given by

$$B_1(\xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\sigma \xi^\sigma \Gamma(-\sigma) \Gamma(-\sigma + \varepsilon) \Gamma\left(\sigma - \frac{3\varepsilon}{2}\right) \Gamma\left(\sigma - \frac{\varepsilon}{2}\right) \frac{\Gamma^2(\sigma + 2 - \varepsilon)}{\Gamma(2\sigma + 4 - 2\varepsilon)}, \quad (3.22)$$

$$B_2(\xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\sigma \xi^\sigma \Gamma(-\sigma) \Gamma(-\sigma + \varepsilon) \Gamma\left(\sigma - \frac{3\varepsilon}{2}\right) \Gamma\left(\sigma + 1 - \frac{\varepsilon}{2}\right) \frac{\Gamma^2(\sigma + 2 - \varepsilon)}{\Gamma(2\sigma + 4 - 2\varepsilon)}, \quad (3.23)$$

$$B_3(\xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\sigma \xi^\sigma \Gamma(-\sigma) \Gamma(-\sigma - 1 + \varepsilon) \Gamma\left(\sigma + 1 - \frac{3\varepsilon}{2}\right) \Gamma\left(\sigma + 1 - \frac{\varepsilon}{2}\right) \times \frac{\Gamma^2(\sigma + 3 - \varepsilon)}{\Gamma(2\sigma + 6 - 2\varepsilon)}. \quad (3.24)$$

Expanding in the dimensional parameter ε the pre-factors of the functions J_i reduce to the

following denominators

$$\frac{1}{N+l}, \quad \text{with } l \in \{0, 1\}, \quad (3.25)$$

after partial fractioning. These factors have still to be absorbed under the integral, which can be achieved by applying the following relations

$$\frac{1}{N+l} \int_a^b dx x^{N-1} f(x) = \frac{b^{N+l}}{N+l} \int_a^b dy \frac{f(y)}{y^{l+1}} - \int_a^b dx x^{N+l-1} \int_a^x dy \frac{f(y)}{y^{l+1}} \quad (3.26)$$

$$= \frac{a^{N+l}}{N+l} \int_a^b dy \frac{f(y)}{y^{l+1}} + \int_a^b dx x^{N+l-1} \int_x^b dy \frac{f(y)}{y^{l+1}}. \quad (3.27)$$

One has still to perform the contour integral in the functions B_i . To do this, the range in x is split into the intervals

$$[0, \eta_-], \quad [\eta_-, \eta_+], \quad [\eta_+, 1], \quad \text{with } \eta_{\pm} = \frac{1}{2} \left(1 \pm \sqrt{1 - \eta} \right). \quad (3.28)$$

For the second region the integral contour is closed to the right, and for the two other regions to the left. One obtains then the functions B_i for both regions in terms of infinite sum representations, cf. [5], over rational expressions and harmonic sums [25, 26]

$$S_{b, \bar{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\bar{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z} \setminus \{0\}. \quad (3.29)$$

In the expressions ratios of Γ -functions are related to special binomial coefficients, like

$$\frac{\Gamma^2(k+1)}{\Gamma(2k+2)} = \frac{1}{2k} \frac{1}{\binom{2k}{k}}. \quad (3.30)$$

All of the above sums can be performed using the Mathematica packages **Sigma** [27, 28], **HarmonicSums** [29–31], **EvaluateMultiSums** and **SumProduction** [32]. We have performed numerical checks on these steps using the package **MB** and **MBresolve** [33, 34].

The results can be expressed in terms of the following generalized iterated integrals.

$$G(\{f_1(\tau), f_2(\tau), \dots, f_n(\tau)\}, z) = \int_0^z d\tau_1 f_1(\tau_1) G(\{f_2(\tau), \dots, f_n(\tau)\}, \tau_1), \quad (3.31)$$

with

$$G\left(\underbrace{\left\{\frac{1}{\tau}, \frac{1}{\tau}, \dots, \frac{1}{\tau}\right\}}_{n \text{ times}}, z\right) \equiv \frac{1}{n!} \ln^n(z). \quad (3.32)$$

In principle, the letters in the alphabet of these iterated integrals, i.e., the functions $f_k(\tau)$, can be any function (or distribution), for which the iterated integral exists. In the particular case where the letters are restricted to $\frac{1}{\tau}$, $\frac{1}{1-\tau}$ and $\frac{1}{1+\tau}$, these integrals correspond to the harmonic polylogarithms [35], which are defined by

$$H_{b, \bar{a}}(x) = \int_0^x dy f_b(y) H_{\bar{a}}(y), \quad H_{\emptyset} = 1, \quad a_i, b \in \{0, 1, -1\}, \quad (3.33)$$

with

$$f_0(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{1-x}, \quad f_{-1}(x) = \frac{1}{1+x}, \quad (3.34)$$

and

$$H_{\underbrace{0, \dots, 0}_{n \text{ times}}}(x) = G\left(\underbrace{\left\{\frac{1}{\tau}, \frac{1}{\tau}, \dots, \frac{1}{\tau}\right\}}_{n \text{ times}}, x\right) = \frac{1}{n!} \ln^n(x), \quad (3.35)$$

see the appendix of Ref. [5] for details. Square-root valued letters usually play a role in two mass OMEs but also for some single mass OMEs starting from three-loop order and related quantities [5, 6, 10, 36–39].

4 The massive operator matrix element

We obtain the following expression for the $O(\varepsilon^0)$ term of the unrenormalized 3-loop two-mass pure singlet operator matrix element

$$\begin{aligned} \tilde{a}_{Q_q}^{(3),\text{PS}}(x) &= C_F T_F^2 \left\{ R_0(m_1, m_2, x) + (\theta(\eta_- - x) + \theta(x - \eta_+)) x g_0(\eta, x) \right. \\ &\quad + \theta(\eta_+ - x) \theta(x - \eta_-) \left[x f_0(\eta, x) - \int_{\eta_-}^x dy \left(f_1(\eta, y) + \frac{x}{y} f_3(\eta, y) \right) \right] \\ &\quad + \theta(\eta_- - x) \int_x^{\eta_-} dy \left(g_1(\eta, y) + \frac{x}{y} g_3(\eta, y) \right) \\ &\quad - \theta(x - \eta_+) \int_{\eta_+}^x dy \left(g_1(\eta, y) + \frac{x}{y} g_3(\eta, y) \right) \\ &\quad + x h_0(\eta, x) + \int_x^1 dy \left(h_1(\eta, y) + \frac{x}{y} h_3(\eta, y) \right) \\ &\quad + \theta(\eta_+ - x) \int_{\eta_-}^{\eta_+} dy \left(f_1(\eta, y) + \frac{x}{y} f_3(\eta, y) \right) \\ &\quad \left. + \int_{\eta_+}^1 dy \left(g_1(\eta, y) + \frac{x}{y} g_3(\eta, y) \right) \right\}. \quad (4.1) \end{aligned}$$

Here we follow the notation used in Ref. [5]. In the present case no functions carrying the index 2 occur. The functions $g_i(\eta, x)$ in Eq. (4.1) shall not be confounded with polarized structure functions, also often denoted by g_i . Here $\theta(z)$ denotes the Heaviside function

$$\theta(z) = \begin{cases} 1 & z \geq 0 \\ 0 & z < 0. \end{cases} \quad (4.2)$$

The pole terms are obtained in analytic form in terms of harmonic polylogarithms, cf. (2.1). For convenience we define the auxiliary functions u and v as

$$u = \frac{x(1-x)}{\eta}, \quad v = \frac{\eta}{x(1-x)}. \quad (4.3)$$

If in the following expressions the harmonic polylogarithms $H_{\bar{a}}$ are given without argument it is understood that their argument is x . The functions appearing in Eq. (4.1) are given by

$$R_0(m_1, m_2, x) = 32 \left(L_1^3 + L_1 L_2 (L_1 + L_2) + L_2^3 \right) \left[5(-1+x) - 2(1+x) H_0 \right]$$

$$\begin{aligned}
& +128L_1L_2 \left[(x+1) \left(\frac{2}{3}H_{0,1} - \frac{10}{9}H_0 - \frac{2}{3}\zeta_2 \right) + (x-1) \left(\frac{10}{9} - \frac{5}{3}H_1 \right) \right] \\
& +32(L_1^2 + L_2^2) \left[(x+1) \left(\frac{2}{3}H_{0,1} + H_0^2 - \frac{2}{3}\zeta_2 \right) + (x-1) \left(\frac{1}{9} - \frac{5}{3}H_1 \right) \right. \\
& \left. + \frac{1}{9}(17 - 37x)H_0 \right] \\
& +64(L_1 + L_2) \left[(1+x) \left(\left(2H_{0,1} - \frac{8\zeta_2}{3} \right) H_0 - \frac{2}{9}H_0^3 - \frac{10}{3}H_{0,0,1} \right. \right. \\
& \left. \left. - \frac{4}{3}H_{0,1,1} + \frac{14}{3}\zeta_3 \right) + (x-1) \left(\frac{442}{27} + \frac{5}{3}H_1^2 - \frac{5}{9}H_1(1+9H_0) \right) \right. \\
& \left. - \frac{2}{27}(56 + 137x)H_0 + \frac{1}{9}(-5 + 4x)H_0^2 + \frac{2}{9}(-17 + 28x)H_{0,1} \right. \\
& \left. + \frac{2}{9}(-28 + 17x)\zeta_2 \right] + \frac{64}{1215} \left[(1+x) \left((3240H_{0,0,1} + 1620H_{0,1,1})H_0 \right. \right. \\
& \left. \left. + (-1620H_{0,1} + 945\zeta_2)H_0^2 + 90H_0^4 - 1080H_{0,0,0,1} \right. \right. \\
& \left. \left. - 2700H_{0,0,1,1} + 540H_{0,1,1,1} + 1296\zeta_2^2 \right) + (-1+x) \left(20(437 + 54x) + \right. \right. \\
& \left. \left(1080H_0 + 4050H_0^2 + 2025\zeta_2 \right) H_1 - 225H_1^3 \right. \\
& \left. - 45H_1^2(11 + 45H_0) \right) + (-10(-842 + 1111x + 81x^2) \\
& - 540(-7 + 11x)H_{0,1} - 45(-53 + 73x)\zeta_2 - 4860(1+x)\zeta_3) H_0 \\
& + 165(19 + 37x)H_0^2 - 30(-19 + 8x)H_0^3 + 30(-1+x)(157 + 27x)H_1 \\
& + (-30(61 + 169x) - 810(1+x)\zeta_2)H_{0,1} + 180(-11 + 25x)H_{0,0,1} \\
& \left. + 180(-14 + 13x)H_{0,1,1} + 15(131 + 329x)\zeta_2 + 90(-55 + 29x)\zeta_3 \right], \tag{4.4}
\end{aligned}$$

$$\begin{aligned}
g_0(\eta, x) &= -\frac{32(1-x)}{9} \left[-\frac{16(-1+x)x}{\eta} + 18 \left(-\frac{2(\eta - 4(-1+x)x)^2}{9\eta^2} + \frac{1}{3}\zeta_2 \right) H_0(u) \right. \\
& \left. + 5H_0^2(u) + 2 \left(-1 + \frac{(\eta - 4(1-x)x)^{3/2}}{\eta^{3/2}} \right) \zeta_2 + H_0^3(u) \right] \\
& - \frac{64(1-x)}{9} \left[\left(2\frac{(\eta - 4(1-x)x)^{3/2}}{\eta^{3/2}} - 3\zeta_2 \right) G \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau} \right\}, u \right) \right] \\
& + \frac{64(1-x)}{3} \left[G^2 \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau} \right\}, u \right) + G \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau} \right\}, u \right) \right] \\
& - \frac{64(1-x)}{9} G \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau} \right\}, u \right) \frac{(\eta - 4(1-x)x)^{3/2}}{\eta^{3/2}}, \tag{4.5}
\end{aligned}$$

$$\begin{aligned}
g_1(\eta, x) &= \frac{64}{27\eta^2x} \left[-6(1-x)H_0(u)P_2 - 8\eta(-1+x)(1+x)(7\eta + 24(1-x)x) \right. \\
& \left. + 3\eta^2(-1+x)(-5 + 13x)H_0^2(u) - 3\eta^2(1-x)(-1 + 2x)H_0^3(u) \right]
\end{aligned}$$

$$\begin{aligned}
& -(6(1-x)) \left((1+x)\eta^{3/2} - 4\eta(1+x)\sqrt{\eta-4(1-x)x} + 2(-1+x)x(1+10x) \right. \\
& \times \left. \sqrt{\eta-4(1-x)x} \right) \sqrt{\eta}\zeta_2 \Big] + \frac{128(-1+x)}{9x} \left[\left(-4 \frac{\sqrt{\eta-4(1-x)x}}{\eta^{3/2}} P_1 \right. \right. \\
& \left. \left. - 3(-1+2x)\zeta_2 \right) G \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau} \right\}, u \right) \right] \\
& - \frac{128(-1+x)(-1+2x)}{3x} G^2 \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau} \right\}, u \right) \\
& - \frac{128(-1+x)(-1+2x)}{3x} G \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau} \right\}, u \right) \\
& - \frac{256(-1+x)P_1}{9x} G \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau} \right\}, u \right) \frac{\sqrt{\eta-4(1-x)x}}{\eta^{3/2}}, \tag{4.6}
\end{aligned}$$

$$\begin{aligned}
g_3(\eta, x) &= -\frac{32}{27\eta^2 x} \left[8\eta(1-x)P_4 - 6(1-x)H_0(u)P_5 + 3\eta^2(-1+x)(-5+8x) \right. \\
& \times H_0^2(u) + 3\eta^2(-1+x)^2 H_0^3(u) - (6(1-x)) \left((1+2x)\eta^{3/2} \right. \\
& \left. \left. - \eta(4+5x)\sqrt{\eta-4(1-x)x} + 2(-1+x)x(1+8x)\sqrt{\eta-4(1-x)x} \right) \sqrt{\eta}\zeta_2 \right] \\
& - \frac{64(1-x)}{9x} \left[\left(2 \frac{\sqrt{\eta-4(1-x)x}}{\eta^{3/2}} P_3 + 3(-1+x)\zeta_2 \right) G \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau} \right\}, u \right) \right] \\
& + \frac{64(-1+x)^2}{3x} G^2 \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau} \right\}, u \right) \\
& + \frac{64(-1+x)^2}{3x} G \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau} \right\}, u \right) \\
& - \frac{64(1-x)P_3}{9x} G \left(\left\{ \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau} \right\}, u \right) \frac{\sqrt{\eta-4(1-x)x}}{\eta^{3/2}}, \tag{4.7}
\end{aligned}$$

with the polynomials

$$P_1 = 2\eta(x+1) - 10x^3 + 9x^2 + x, \tag{4.8}$$

$$P_2 = 3\eta^2(2x\zeta_2 + x - \zeta_2 + 1) + 8\eta x(10x^2 - 9x - 1) - 16(1-x)^2 x^2(10x+1), \tag{4.9}$$

$$P_3 = \eta(5x+4) + 2x(-8x^2 + 7x + 1), \tag{4.10}$$

$$P_4 = 7\eta(x+1) + 6x(-5x^2 + x + 4), \tag{4.11}$$

$$P_5 = \eta^2(x(3\zeta_2 + 5) - 3\zeta_2 + 3) + 8\eta x(8x^2 - 7x - 1) - 16(x-1)^2 x^2(8x+1), \tag{4.12}$$

and

$$\begin{aligned}
f_0(\eta, x) &= \left[-\frac{16(1-x)}{3} G \left(\left\{ \frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau} \right\}, v \right) - \frac{4P_6}{9(-1+x)x^2} \left[-1 + 2H_0(v) \right] \right. \\
& \times \left. \frac{(-\eta + 4(1-x)x)^{3/2}}{\eta^{3/2}} \right] G \left(\left\{ \sqrt{4-\tau}\sqrt{\tau} \right\}, v \right) \\
& + \frac{4(1-x)}{3} \left[\left[-1 + 2H_0(v) \right] G^2 \left(\left\{ \sqrt{4-\tau}\sqrt{\tau} \right\}, v \right) \right] \\
& + \frac{16(1-x)}{3} G \left(\left\{ \frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}, \sqrt{4-\tau}\sqrt{\tau} \right\}, v \right) + \frac{1}{18} \left[-1536(1-x) \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{9\eta^4}{(-1+x)^3x^4} - \frac{80\eta^3}{(-1+x)^2x^3} - \frac{104\eta^2}{(-1+x)x^2} + \frac{576\eta}{x} + \frac{4P_7}{(-1+x)^3x^4}H_0(v) \\
& -320(1-x)H_0^2(v) + 64(1-x)H_0^3(v) - \left(128(1-x)\right)\left(5-3H_0(v)\right)\zeta_2 \\
& +768(1-x)\zeta_3 \left] - \frac{8P_6}{9(1-x)x^2}G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, v\right) \right. \\
& \left. \times \frac{(-\eta+4(1-x)x)^{3/2}}{\eta^{3/2}}, \tag{4.13}
\end{aligned}$$

$$\begin{aligned}
f_1(\eta, x) = & \frac{1}{27x^5} \left[-\frac{1}{(1-x)^3} \left[6912\eta(-1+x)^3x^4 - 27\eta^4(-1+2x) - 240\eta^3(-1+x)x \right. \right. \\
& (-1+2x) + 512(-1+x)^4x^4(-16+11x) + 12H_0(v)P_9 \\
& -24\eta^2(-1+x)^2x^2(-25+2x) + 192(-1+x)^4x^4(-5+13x)H_0^2(v) \\
& \left. -192(-1+x)^4x^4(-1+2x)H_0^3(v) \right] + 384(1-x)x^4 \left(5-13x \right. \\
& \left. +3(-1+2x)H_0(v) \right) \zeta_2 - 2304(-1+x)x^4(-1+2x)\zeta_3 \left] + \right. \\
& \left[\frac{32(-1+x)(-1+2x)}{3x} G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, v\right) \right. \\
& \left. - \frac{8P_8}{9(1-x)x^3} \left(-1+2H_0(v) \right) \frac{\sqrt{-\eta+4(1-x)x}}{\eta^{3/2}} \right] \\
& \times G\left(\left\{\sqrt{4-\tau}\sqrt{\tau}\right\}, v\right) - \frac{8(-1+x)(-1+2x)}{3x} \left[\left(-1+2H_0(v) \right) \right. \\
& \left. \times G^2\left(\left\{\sqrt{4-\tau}\sqrt{\tau}\right\}, v\right) \right] + \frac{32(1-x)(-1+2x)}{3x} \\
& \times G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, v\right) - \frac{16P_8}{9(-1+x)x^3} \\
& \left. \times G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, v\right) \frac{\sqrt{-\eta+4(1-x)x}}{\eta^{3/2}}, \tag{4.14}
\end{aligned}$$

$$\begin{aligned}
f_3(\eta, x) = & \frac{1}{54(-1+x)^2x^5} \left[27\eta^4 - 240\eta^3(1-x)x - 5184\eta(-1+x)^2x^4 - 1024(-8+x) \right. \\
& \times (-1+x)^3x^4 - 12H_0(v)P_{10} - 24\eta^2(-1+x)x^2(25+11x) \\
& -192(-1+x)^3x^4(-5+8x)H_0^2(v) + 192(-1+x)^4x^4H_0^3(v) \\
& \left. +384(-1+x)^3x^4 \left(5-8x-3(1-x)H_0(v) \right) \zeta_2 + 2304(-1+x)^4x^4\zeta_3 \right] \\
& + \left[-\frac{16(-1+x)^2}{3x} G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, v\right) - \frac{4P_{11}}{9x^3} \left(-1 \right. \right. \\
& \left. \left. +2H_0(v) \right) \frac{\sqrt{-\eta+4(1-x)x}}{\eta^{3/2}} \right] G\left(\left\{\sqrt{4-\tau}\sqrt{\tau}\right\}, v\right) \\
& + \frac{4(-1+x)^2}{3x} \left[\left(-1+2H_0(v) \right) G\left(\left\{\sqrt{4-\tau}\sqrt{\tau}\right\}, v\right)^2 \right] \\
& + \frac{16(-1+x)^2}{3x} G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, v\right) + \frac{8P_{11}}{9x^3}
\end{aligned}$$

$$\times G \left(\left\{ \frac{1}{\tau}, \sqrt{4 - \tau} \sqrt{\tau} \right\}, v \right) \frac{\sqrt{-\eta + 4(1 - x)x}}{\eta^{3/2}}, \quad (4.15)$$

with

$$P_6 = 3\eta^2 + 6\eta(1 - x)x + 4(x - 1)^2x^2, \quad (4.16)$$

$$P_7 = 3\eta^4 - 24\eta^3(1 - x)x + 20\eta^2(x - 1)^2x^2 - 160\eta(x - 1)^3x^3 - 128(x - 1)^4x^4, \quad (4.17)$$

$$P_8 = \eta^3(6x - 3) + 6\eta^2x(2x^2 - 3x + 1) - 8\eta(x - 1)^2x^2(8x - 1) + 8(x - 1)^3x^3(10x + 1), \quad (4.18)$$

$$P_9 = \eta^4(6x - 3) + 24\eta^3x(2x^2 - 3x + 1) - 4\eta^2(x - 1)^2x^2(2x + 1) - 64\eta(x - 1)^3x^3(8x - 1) + 96(x - 1)^4x^4(x + 4), \quad (4.19)$$

$$P_{10} = 3\eta^4 - 24\eta^3(1 - x)x - 4\eta^2x^2(7x^2 + 4x - 11) - 32\eta(x - 1)^2x^3(11x - 2) + 32(x - 1)^3x^4(7x + 12), \quad (4.20)$$

$$P_{11} = 3\eta^3 - 6\eta^2(1 - x)x - 4\eta x^2(11x^2 - 13x + 2) + 8(1 - x)^2x^3(8x + 1). \quad (4.21)$$

The functions h_i are defined as follows

$$h_i(\eta, x) = g_i \left(\frac{1}{\eta}, x \right), \quad i = 0, 1, 3. \quad (4.22)$$

In deriving the expressions given above, we have used shuffle algebra relations wherever possible, cf. [40]. The function $R_0(m_1, m_2, x)$ arises from the residues taken in order to resolve the singularities in ε of the contour integrals in the functions B_i . The functions $f_i(\eta, x)$, $g_i(\eta, x)$, with $i = 0, 1, 3$, arise from the sum of residues of the contour integrals that remain after the ε expansion, as described in the previous section. The functions with $i = 0$ are those where no additional factor depending on N occurs. The functions with $i = 1$, and $i = 3$ are those where a factor of $1/N$ and $1/(N + 1)$ was absorbed, respectively, see Eqs. (3.26, 3.27). The different Heaviside functions restrict the corresponding values of x to the appropriate regions.

Since no contour integral needs to be performed in the case $R_0(m_1, m_2, x)$, the easiest way to compute this function is to integrate in x and then perform the Mellin inversion using `HarmonicSums`. The expressions of the G -functions in the above equations can all be given in terms of harmonic polylogarithms, cf. Appendix of Ref. [5], containing square-root valued arguments.

We see that iterated integrals of up to weight three appear in our result. The alphabet of these integrals is given in terms of just three letters:

$$\frac{1}{\tau}, \quad \sqrt{4 - \tau} \sqrt{\tau}, \quad \frac{\sqrt{1 - 4\tau}}{\tau}. \quad (4.23)$$

One may try to integrate the remaining integrals in Eq. (4.1) over y into iterated integrals of higher weight. However, the numerical representation needs to be done in addition, unlike the case of harmonic polylogarithms [41, 42].

In order to compute the corresponding contribution to the structure function $g_1(x, Q^2)$ or for the transition rate in the VFNS, we have to perform the convolution with parton distribution functions, which can be obtained straightforwardly.

5 Numerical Results

We compare the polarized pure singlet 2-mass contributions to the complete $O(T_F^2 C_F)$ term as a function of x and μ^2 in Figure 3. Typical virtualities are $\mu^2 \in [30, 1000]$ GeV². The ratio of

the 2-mass contributions to the complete term of $O(T_F^2 C_F)$ has a singularity around $x \sim 0.1$. At lower virtualities the corrections are nearly constant in the small x region and grow with μ^2 rising from negative to positive values. In the large x region the ratio falls and rises once again towards $x \rightarrow 1$. At $\mu^2 = 1000 \text{ GeV}^2$ the corrections are comparatively large and positive due to the large logarithms, except in the pole region. In size the corrections are comparable to those found in the unpolarized case and do majorly range between -0.1 to 0.4 .

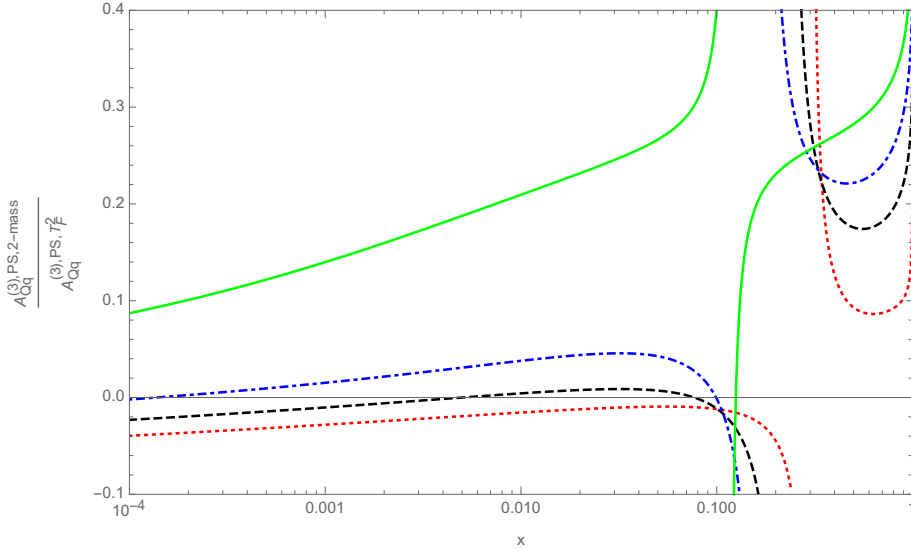


Figure 3: The ratio of the 2-mass (tm) contributions to the massive OME $A_{Qq}^{\text{PS},(3)}$ to all contributions to $A_{Qq}^{\text{PS},(3)}$ of $O(T_F^2)$ as a function of x and μ^2 . Dotted line (red): $\mu^2 = 30 \text{ GeV}^2$. Dashed line (black): $\mu^2 = 50 \text{ GeV}^2$. Dash-dotted line (blue): $\mu^2 = 100 \text{ GeV}^2$. Full line (green): $\mu^2 = 1000 \text{ GeV}^2$. Here the on-shell heavy quark masses $m_c = 1.59 \text{ GeV}$ and $m_b = 4.78 \text{ GeV}$ [43, 44] have been used.

6 Conclusions

We have calculated the two-mass 3-loop contributions to the polarized massive OME $A_{Qq}^{\text{PS},(3)}$ in analytic form in x -space for a general mass-ratio η in the Larin scheme. It contributes to the massive 3-loop Wilson coefficient of the deep-inelastic structure function $g_1(x, Q^2)$ in the region $m^2 \ll Q^2$ and is, as well, one of the polarized OMEs in the two-mass 3-loop VFNS, needed to describe the process of heavy quarks becoming massless at large virtualities. As a function of x , its relative contribution to the $O(T_F^2 C_{A,F})$ terms of the whole matrix element $A_{Qq}^{\text{PS},\text{TF}^2,(3)}$ lay in the region of about $[-0.1, 0.4]$ and exhibit a pole at $x \sim 0.1$. the two-mass contribution is not negligible against the single mass contributions.

We applied Mellin-Barnes techniques to obtain the x -space result by factoring out the N -dependence in terms of the kernel x^N , and used integration by parts to absorb the N -dependent polynomial pre-factors. The result can be written as single limited integrals within the range $x \in [0, 1]$ over iterated integrals containing also square-root valued letters. These integrals can be turned into polylogarithms of involved root-valued arguments, depending on the real parameter η . The odd Mellin moments of the OME exhibit a growing number of polynomial terms in η with growing values of N . Due to this structural property and the arbitrariness of η , which enters the ground field, the method of arbitrarily large moments [45] cannot be used to find the result in the present case. The set of necessary integrals for these representations has already been derived

in the unpolarized case in Ref. [5]. The concept of square root-valued iterated integrals turned out to be of central importance in deriving the present results. Moreover, their weights are such that one can still relate them to harmonic polylogarithms of more complicated arguments.¹

The Larin scheme is one of the valid schemes to perform calculations in the polarized case. At present the massless polarized three-loop Wilson coefficients are not yet available [48]. They will also be calculated in the Larin scheme first. Together with parton distribution functions, evolved in the Larin scheme, one can then form observables like $g_1(x, Q^2)$ and related quantities [49]. The anomalous dimensions for the Larin scheme are available to three-loop order [13, 14].

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¹This has been a principle also in early Mellin-representations of harmonic sums [26] and [46], limiting the alphabet to that of Nielsen integrals [47].

A Fixed Moments of $\tilde{A}_{Qq}^{\text{PS, tm, (3)}}$

For fixed values of $N = 2k + 1, k \in \mathbb{N}$, the difference equations factorize to first order and we find the following moments, unexpanded in the mass ratio η . In the following we will use the notation

$$H_{\bar{w}}(\sqrt{\eta}) \equiv H_{\bar{w}}. \quad (\text{A.1})$$

The fixed moments are given by

$$\begin{aligned} \hat{A}_{Qq}^{\text{PS, tm, (3)}}(N=1) = & \\ & C_F T_F^2 \left\{ \frac{256}{3\varepsilon^2} + \frac{1}{\varepsilon} \left[\frac{320}{9} + 64(L_1 + L_2) \right] - \frac{3}{2\eta^{3/2}} \left[-2Q_1 + Q_2(H_1 + H_{-1}) \right] \right. \\ & \times \left(L_1^2 + L_2^2 \right) + \frac{3}{\eta^{3/2}} L_1 L_2 \left[-2Q_3 + Q_2(H_1 + H_{-1}) \right] \\ & - \frac{2}{3\eta^{3/2}} \left[2Q_4 + 9Q_2(H_{0,1} + H_{0,-1}) \right] L_1 \\ & + \frac{2}{3\eta^{3/2}} \left[2Q_5 + 9Q_2(H_{0,1} + H_{0,-1}) \right] L_2 \\ & \left. - \frac{4}{27\eta^{3/2}} \left[-2Q_6 + 81Q_2(H_{0,0,1} + H_{0,0,-1}) \right] + 32\zeta_2, \right. \end{aligned} \quad (\text{A.2})$$

$$Q_1 = \sqrt{\eta}(1 + 10\eta + \eta^2), \quad (\text{A.3})$$

$$Q_2 = (1 - \eta)^2(1 + \eta), \quad (\text{A.4})$$

$$Q_3 = \sqrt{\eta}(1 - 6\eta + \eta^2), \quad (\text{A.5})$$

$$Q_4 = \sqrt{\eta}(-9 - 20\eta + 9\eta^2), \quad (\text{A.6})$$

$$Q_5 = \sqrt{\eta}(-9 + 20\eta + 9\eta^2), \quad (\text{A.7})$$

$$Q_6 = \sqrt{\eta}(81 + 284\eta + 81\eta^2), \quad (\text{A.8})$$

$$\begin{aligned} \hat{A}_{Qq}^{\text{PS, tm, (3)}}(N=3) = & \\ & C_F T_F^2 \left\{ -\frac{1280}{81\varepsilon^3} + \frac{1}{\varepsilon^2} \left[\frac{1760}{243} - \frac{320}{27}(L_1 + L_2) \right] + \frac{1}{\varepsilon} \left[-\frac{12820}{729} \right. \right. \\ & - \frac{160}{27}(L_1^2 + L_1 L_2 + L_2^2) + \frac{440}{81}(L_1 + L_2) - \frac{160}{27}\zeta_2 \left. \right] \\ & - \frac{200}{81}L_1^3 - \frac{160}{81}L_2^3 - \frac{80}{27}L_1 L_2^2 - \frac{40}{27}L_2 L_1^2 + (L_1^2 + L_2^2) \left[\frac{5Q_7}{324\eta} \right. \\ & - \frac{5Q_{12}}{216} \frac{H_1}{\eta^{3/2}} - \frac{5Q_{13}}{216} \frac{H_{-1}}{\eta^{3/2}} \left. \right] + L_1 L_2 \left[-\frac{5Q_8}{162\eta} \right. \\ & + \frac{5Q_{12}}{108} \frac{H_1}{\eta^{3/2}} + \frac{5Q_{13}}{108} \frac{H_{-1}}{\eta^{3/2}} \left. \right] + L_1 \left[-\frac{5Q_9}{243\eta} \right. \\ & \left. - \frac{5Q_{12}}{54} \frac{H_{0,1}}{\eta^{3/2}} - \frac{5Q_{13}}{54} \frac{H_{0,-1}}{\eta^{3/2}} - \frac{40}{9}\zeta_2 \right] + \frac{5Q_{10}}{4374\eta} \end{aligned}$$

$$\begin{aligned}
& +L_2 \left[\frac{5Q_{11}}{243\eta} + \frac{5Q_{12}}{54} \frac{H_{0,1}}{\eta^{3/2}} + \frac{5Q_{13}}{54} \frac{H_{0,-1}}{\eta^{3/2}} - \frac{40}{9} \zeta_2 \right] \\
& - \left. \frac{5Q_{12}}{27} \frac{H_{0,0,1}}{\eta^{3/2}} - \frac{5Q_{13}}{27} \frac{H_{0,0,-1}}{\eta^{3/2}} + \frac{220}{81} \zeta_2 + \frac{160}{81} \zeta_3 \right\} \tag{A.9}
\end{aligned}$$

$$Q_7 = 15 + 262\eta + 15\eta^2, \tag{A.10}$$

$$Q_8 = 15 - 2\eta + 15\eta^2, \tag{A.11}$$

$$Q_9 = -45 + 641\eta + 45\eta^2, \tag{A.12}$$

$$Q_{10} = 1620 + 5659\eta + 1620\eta^2, \tag{A.13}$$

$$Q_{11} = -45 - 641\eta + 45\eta^2, \tag{A.14}$$

$$Q_{12} = 5 + 27\eta + 27\eta^2 + 5\eta^3 - 64\eta^{3/2}, \tag{A.15}$$

$$Q_{13} = 5 + 27\eta + 27\eta^2 + 5\eta^3 + 64\eta^{3/2}, \tag{A.16}$$

$$\hat{A}_{Qq}^{\text{PS, tm, (3)}}(N=5) =$$

$$\begin{aligned}
& C_F T_F^2 \left\{ -\frac{14336}{2025\varepsilon^3} + \frac{1}{\varepsilon^2} \left[\frac{20608}{6075} - \frac{3584}{675} (L_1 + L_2) \right] + \frac{1}{\varepsilon} \left[-\frac{3724784}{455625} \right. \right. \\
& - \frac{1792}{675} (L_1^2 + L_1 L_2 + L_2^2) + \frac{5152}{2025} (L_1 + L_2) - \frac{1792}{675} \zeta_2 \left. \right] - \frac{448}{405} L_1^3 \\
& - \frac{1792}{2025} L_2^3 - \frac{896}{675} L_1 L_2^2 - \frac{448}{675} L_1^2 L_2 + \frac{7Q_{16}}{583200\eta^2} \\
& + (L_1^2 + L_2^2) \left[\frac{7Q_{17}}{864000\eta^2} - \frac{7Q_{14}}{345600\eta^{5/2}} H_1 - \frac{7Q_{15}}{345600\eta^{5/2}} H_{-1} \right] \\
& + L_1 L_2 \left[-\frac{7Q_{18}}{432000\eta^2} + \frac{7Q_{14}}{172800\eta^{5/2}} H_1 + \frac{7Q_{15}}{172800\eta^{5/2}} H_{-1} \right] \\
& + L_1 \left[-\frac{7Q_{19}}{9720000\eta^2} - \frac{7Q_{14}}{86400\eta^{5/2}} H_{0,1} - \frac{7Q_{15}}{86400\eta^{5/2}} H_{0,-1} - \frac{448}{225} \zeta_2 \right] \\
& + L_2 \left[\frac{7Q_{20}}{9720000\eta^2} + \frac{7Q_{14}}{86400\eta^{5/2}} H_{0,1} + \frac{7Q_{15}}{86400\eta^{5/2}} H_{0,-1} - \frac{448}{225} \zeta_2 \right] \\
& \left. - \frac{7Q_{14}}{43200\eta^{5/2}} H_{0,0,1} - \frac{7Q_{15}}{43200\eta^{5/2}} H_{0,0,-1} + \frac{2576\zeta_2}{2025} + \frac{1792\zeta_3}{2025} \right\}, \tag{A.17}
\end{aligned}$$

$$Q_{14} = 189 + 2425\eta + 13770\eta^2 + 13770\eta^3 + 2425\eta^4 + 189\eta^5 - 32768\eta^{5/2}, \tag{A.18}$$

$$Q_{15} = 189 + 2425\eta + 13770\eta^2 + 13770\eta^3 + 2425\eta^4 + 189\eta^5 + 32768\eta^{5/2}, \tag{A.19}$$

$$Q_{16} = 5103 + 65664\eta + 260834\eta^2 + 65664\eta^3 + 5103\eta^4, \tag{A.20}$$

$$Q_{17} = 945 + 12440\eta + 230094\eta^2 + 12440\eta^3 + 945\eta^4, \tag{A.21}$$

$$Q_{18} = 945 + 12440\eta - 5426\eta^2 + 12440\eta^3 + 945\eta^4, \tag{A.22}$$

$$Q_{19} = -42525 - 550350\eta + 8513792\eta^2 + 550350\eta^3 + 42525\eta^4, \tag{A.23}$$

$$Q_{20} = -42525 - 550350\eta - 8513792\eta^2 + 550350\eta^3 + 42525\eta^4, \tag{A.24}$$

$$\hat{A}_{Qq}^{\text{PS, tm, (3)}}(N=7) =$$

$$\begin{aligned}
& C_F T_F^2 \left\{ -\frac{192}{49\varepsilon^3} + \frac{1}{\varepsilon^2} \left[\frac{508}{245} - \frac{144}{49} (L_1 + L_2) \right] + \frac{1}{\varepsilon} \left[-\frac{45155}{9604} \right. \right. \\
& - \frac{72}{49} (L_1^2 + L_1 L_2 + L_2^2) + \frac{381}{245} (L_1 + L_2) - \frac{72}{49} \zeta_2 \left. \right] - \frac{30}{49} L_1^3 - \frac{24}{49} L_2^3 \\
& - \frac{Q_{21}}{2765952000\eta^3} - \frac{36}{49} L_1 L_2^2 - \frac{18}{49} L_1^2 L_2 + (L_1^2 + L_2^2) \left[-\frac{3Q_{22}}{14049280\eta^3} \right. \\
& + \frac{9Q_{23}}{802816\eta^{7/2}} H_1 + \frac{9Q_{24}}{802816\eta^{7/2}} H_{-1} \left. \right] + L_1 L_2 \left[\frac{3Q_{27}}{1404928\eta^3} \right. \\
& - \frac{9Q_{23}}{401408\eta^{7/2}} H_1 - \frac{9Q_{24}}{401408\eta^{7/2}} H_{-1} \left. \right] + L_2 \left[\frac{Q_{25}}{24586240\eta^3} \right. \\
& - \frac{9Q_{23}}{200704\eta^{7/2}} H_{0,1} - \frac{9Q_{24}}{200704\eta^{7/2}} H_{0,-1} - \frac{54}{49} \zeta_2 \left. \right] \\
& + L_1 \left[-\frac{Q_{26}}{24586240\eta^3} + \frac{9Q_{23}}{200704\eta^{7/2}} H_{0,1} + \frac{9Q_{24}}{200704\eta^{7/2}} H_{0,-1} - \frac{54}{49} \zeta_2 \right] \\
& \left. + \frac{9Q_{23}}{100352\eta^{7/2}} H_{0,0,1} + \frac{9Q_{24}}{100352\eta^{7/2}} H_{0,0,-1} + \frac{381}{490} \zeta_2 + \frac{24}{49} \zeta_3 \right\}, \tag{A.25}
\end{aligned}$$

$$\begin{aligned}
Q_{21} = & 22325625 + 170997750\eta - 1400033145\eta^2 - 5593159388\eta^3 \\
& - 1400033145\eta^4 + 170997750\eta^5 + 22325625\eta^6, \tag{A.26}
\end{aligned}$$

$$\begin{aligned}
Q_{22} = & 4725 + 37590\eta - 284725\eta^2 - 5196076\eta^3 - 284725\eta^4 \\
& + 37590\eta^5 + 4725\eta^6, \tag{A.27}
\end{aligned}$$

$$\begin{aligned}
Q_{23} = & 45 + 343\eta - 2835\eta^2 - 13937\eta^3 - 13937\eta^4 - 2835\eta^5 + 343\eta^6 \\
& + 45\eta^7 + 32768\eta^{7/2}, \tag{A.28}
\end{aligned}$$

$$\begin{aligned}
Q_{24} = & 45 + 343\eta - 2835\eta^2 - 13937\eta^3 - 13937\eta^4 - 2835\eta^5 + 343\eta^6 \\
& + 45\eta^7 - 32768\eta^{7/2}, \tag{A.29}
\end{aligned}$$

$$\begin{aligned}
Q_{25} = & 99225 + 767340\eta - 6163171\eta^2 - 86697600\eta^3 + 6163171\eta^4 \\
& - 767340\eta^5 - 99225\eta^6, \tag{A.30}
\end{aligned}$$

$$\begin{aligned}
Q_{26} = & 99225 + 767340\eta - 6163171\eta^2 + 86697600\eta^3 + 6163171\eta^4 \\
& - 767340\eta^5 - 99225\eta^6, \tag{A.31}
\end{aligned}$$

$$\begin{aligned}
Q_{27} = & 945 + 7518\eta - 56945\eta^2 + 53188\eta^3 - 56945\eta^4 + 7518\eta^5 + 945\eta^6, \tag{A.32}
\end{aligned}$$

$$\hat{A}_{Qq}^{\text{PS, tm, (3)}}(N=9) =$$

$$\begin{aligned}
& C_F T_F^2 \left\{ -\frac{45056}{18225\varepsilon^3} + \frac{1}{\varepsilon^2} \left[\frac{2721664}{1913625} - \frac{11264}{6075} (L_1 + L_2) \right] \right. \\
& + \frac{1}{\varepsilon} \left[-\frac{5568605768}{1808375625} - \frac{5632}{6075} (L_1^2 + L_1 L_2 + L_2^2) + \frac{680416}{637875} (L_1 + L_2) \right. \\
& - \left. \frac{5632\zeta_2}{6075} \right] - \frac{1408}{3645} L_1^3 - \frac{5632}{18225} L_2^3 - \frac{2816}{6075} L_1 L_2^2 - \frac{1408}{6075} L_1^2 L_2 \\
& + \frac{11Q_{28}}{77774618880000\eta^4} + (L_1^2 + L_2^2) \left[\frac{11Q_{29}}{62705664000\eta^4} - \frac{11Q_{30}}{398131200\eta^{9/2}} H_1 \right. \\
& - \left. \frac{11Q_{31}}{398131200\eta^{9/2}} H_{-1} \right] + L_1 L_2 \left[-\frac{11Q_{32}}{4478976000\eta^4} + \frac{11Q_{30}}{199065600\eta^{9/2}} H_1 \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{11Q_{31}}{199065600\eta^{9/2}} H_{-1} \Big] + L_1 \left[-\frac{11Q_{33}}{4938071040000\eta^4} - \frac{11Q_{30}}{99532800\eta^{9/2}} H_{0,1} \right. \\
& - \frac{11Q_{31}}{99532800\eta^{9/2}} H_{0,-1} - \frac{1408\zeta_2}{2025} \Big] + L_2 \left[\frac{11Q_{34}}{4938071040000\eta^4} + \frac{11Q_{30}}{99532800\eta^{9/2}} H_{0,1} \right. \\
& + \frac{11Q_{31}}{99532800\eta^{9/2}} H_{0,-1} - \frac{1408\zeta_2}{2025} \Big] - \frac{11Q_{30}}{49766400\eta^{9/2}} H_{0,0,1} \\
& \left. - \frac{11Q_{31}}{49766400\eta^{9/2}} H_{0,0,-1} + \frac{340208\zeta_2}{637875} + \frac{5632\zeta_3}{18225} \right\}, \tag{A.33}
\end{aligned}$$

$$\begin{aligned}
Q_{28} = & 84234583125 + 1142396443125\eta - 12238532911710\eta^2 + 27996557899275\eta^3 \\
& + 103736391394322\eta^4 + 27996557899275\eta^5 - 12238532911710\eta^6 \\
& + 1142396443125\eta^7 + 84234583125\eta^8, \tag{A.34}
\end{aligned}$$

$$\begin{aligned}
Q_{29} = & 848925 + 11764725\eta - 119776230\eta^2 + 247801995\eta^3 + 4258879634\eta^4 \\
& + 247801995\eta^5 - 119776230\eta^6 + 11764725\eta^7 + 848925\eta^8, \tag{A.35}
\end{aligned}$$

$$\begin{aligned}
Q_{30} = & 2695 + 36450\eta - 392931\eta^2 + 909975\eta^3 + 3638115\eta^4 + 3638115\eta^5 + 909975\eta^6 \\
& - 392931\eta^7 + 36450\eta^8 + 2695\eta^9 - 8388608\eta^{9/2}, \tag{A.36}
\end{aligned}$$

$$\begin{aligned}
Q_{31} = & 2695 + 36450\eta - 392931\eta^2 + 909975\eta^3 + 3638115\eta^4 + 3638115\eta^5 + 909975\eta^6 \\
& - 392931\eta^7 + 36450\eta^8 + 2695\eta^9 + 8388608\eta^{9/2}, \tag{A.37}
\end{aligned}$$

$$\begin{aligned}
Q_{32} = & 121275 + 1680675\eta - 17110890\eta^2 + 35400285\eta^3 - 43091362\eta^4 + 35400285\eta^5 \\
& - 17110890\eta^6 + 1680675\eta^7 + 121275\eta^8, \tag{A.38}
\end{aligned}$$

$$\begin{aligned}
Q_{33} = & -267411375 - 3646463625\eta + 38576020770\eta^2 - 86110332525\eta^3 \\
& + 1036773146624\eta^4 + 86110332525\eta^5 - 38576020770\eta^6 \\
& + 3646463625\eta^7 + 267411375\eta^8, \tag{A.39}
\end{aligned}$$

$$\begin{aligned}
Q_{34} = & -267411375 - 3646463625\eta + 38576020770\eta^2 - 86110332525\eta^3 \\
& - 1036773146624\eta^4 + 86110332525\eta^5 - 38576020770\eta^6 \\
& + 3646463625\eta^7 + 267411375\eta^8. \tag{A.40}
\end{aligned}$$

The above expressions depend on η only, but not on $\sqrt{\eta}$, and they are symmetric under $\eta \leftrightarrow \eta^{-1}$. The expansions of the $O(\varepsilon^0)$ terms for $N = 1, 3, 5$ up to $O(\eta^3 \ln^3(\eta))$ for $\eta < 1$ agree with the results we have found in an independent calculation using Q2e [50, 51]. With growing values of N , these expressions exhibit a growing degree of the polynomials in η . We have found that the general N formula cannot be expressed as a sum-product solution by means of difference field theory. This means that the corresponding solution will be given by a higher transcendental function depending on N and η . Forming the corresponding numerical Mellin moments for the complete x -space expressions, we agree with the analytic Mellin moments given in this appendix.

References

- [1] M. Buza, Y. Matiounine, J. Smith, R. Migneron and W.L. van Neerven, Nucl. Phys. B **472** (1996) 611–658 [hep-ph/9601302].
- [2] M. Buza, Y. Matiounine, J. Smith and W.L. van Neerven, Eur. Phys. J. C **1** (1998) 301–320 [hep-ph/9612398].
- [3] J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock, arXiv:1106.5937 [hep-ph]; J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider and F. Wißbrock, PoS (RADCOR2011) 031 [arXiv:1202.2700 [hep-ph]].
- [4] J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, C. Schneider and F. Wißbrock, Nucl. Phys. B **921** (2017) 585–688 [arXiv:1705.07030[hep-ph]].
- [5] J. Ablinger, J. Blümlein, A. De Freitas, C. Schneider and K. Schönwald, Nucl. Phys. B **927** (2018) 339–367 [arXiv:1711.06717 [hep-ph]].
- [6] J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, C. Schneider and K. Schönwald, Nucl. Phys. B **932** (2018) 129–240 [arXiv:1804.02226 [hep-ph]].
- [7] K. Schönwald, *Massive Two- and Three-loop Calculations in QED and QCD*, PhD Thesis, July 2019, TU Dortmund.
- [8] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, Nucl. Phys. B **890** (2014) 48–151 [arXiv:1409.1135 [hep-ph]].
- [9] S.A. Larin, Phys. Lett. B **303** (1993) 113–118 [hep-ph/9302240].
- [10] J. Blümlein, A. De Freitas, C. Schneider and K. Schönwald, Phys. Lett. B **782** (2018) 362–366 [arXiv:1804.03129 [hep-ph]].
- [11] R. Mertig and W.L. van Neerven, Z. Phys. C **70** (1996) 637–654 [hep-ph/9506451].
- [12] W. Vogelsang, Phys. Rev. D **54** (1996) 2023–2029 [hep-ph/9512218]; Nucl. Phys. B **475** (1996) 47–72 [hep-ph/9603366].
- [13] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B **889** (2014) 351–400 [arXiv:1409.5131 [hep-ph]].
- [14] A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, S. Klein, A. von Manteuffel, C. Schneider and K. Schönwald, Nucl. Phys. B **948** (2019) 114753 [arXiv:1908.03779 [hep-ph]].
- [15] M. Buza, Y. Matiounine, J. Smith and W.L. van Neerven, Nucl. Phys. B **485** (1997) 420–456 [hep-ph/9608342].
- [16] I. Bierenbaum J. Blümlein, A. De Freitas, S. Klein, and K. Schönwald, DESY 15–004, DO–TH 15/01.
- [17] A. Hasselhuhn, *3-Loop Contributions to Heavy Flavor Wilson Coefficients of Neutral and Charged Current DIS*, DESY-THESIS-2013-050.
- [18] J. Blümlein, C. Raab and K. Schönwald, Nucl. Phys. B **948** (2019) 114736 [arXiv:1904.08911 [hep-ph]].

- [19] J.A.M. Vermaseren, *New features of FORM*, math-ph/0010025.
- [20] E.W. Barnes, Proc. Lond. Math. Soc. (2) **6** (1908) 141–177.
- [21] E.W. Barnes, Quarterly Journal of Mathematics **41** (1910) 136–140.
- [22] H. Mellin, Math. Ann. **68**, no. 3 (1910) 305–337.
- [23] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, (Cambridge University Press, Cambridge, 1927; reprinted 1996).
- [24] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, (Clarendon Press, Oxford, 1937; 2nd Edition 1948).
- [25] J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037–2076 [hep-ph/9806280].
- [26] J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241].
- [27] C. Schneider, Sémin. Lothar. Combin. **56** (2007) 1–36 article B56b.
- [28] C. Schneider, *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions* Texts and Monographs in Symbolic Computation eds. C. Schneider and J. Blümlein (Springer, Wien, 2013) 325–360 arXiv:1304.4134 [cs.SC].
- [29] J. Ablinger, PoS (LL2014) 019; *Computer Algebra Algorithms for Special Functions in Particle Physics*, Ph.D. Thesis, J. Kepler University Linz, 2012, arXiv:1305.0687 [math-ph]; *A Computer Algebra Toolbox for Harmonic Sums Related to Particle Physics*, Diploma Thesis, J. Kepler University Linz, 2009, arXiv:1011.1176 [math-ph].
- [30] J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063 [math-ph]].
- [31] J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378 [math-ph]].
- [32] J. Ablinger, J. Blümlein, S. Klein and C. Schneider, Nucl. Phys. Proc. Suppl. **205-206** (2010) 110–115 [arXiv:1006.4797 [math-ph]]; J. Blümlein, A. Hasselhuhn and C. Schneider, PoS (RADCOR 2011) 032 [arXiv:1202.4303 [math-ph]]; C. Schneider, J. Phys. Conf. Ser. **523** (2014) 012037 [arXiv:1310.0160 [cs.SC]].
- [33] M. Czakon, Comput. Phys. Commun. **175** (2006) 559–571 [hep-ph/0511200].
- [34] A.V. Smirnov and V.A. Smirnov, Eur. Phys. J. C **62** (2009) 445–449 [arXiv:0901.0386 [hep-ph]].
- [35] E. Remiddi and J.A.M. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725–754 [hep-ph/9905237].
- [36] J. Ablinger, J. Blümlein, C. G. Raab and C. Schneider, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822 [hep-th]].
- [37] J. Ablinger, J. Blümlein, C. Raab, C. Schneider and F. Wißbrock, Nucl. Phys. B **885** (2014) 409–447 [arXiv:1403.1137 [hep-ph]].

- [38] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, *Comput. Phys. Commun.* **202** (2016) 33–112 [arXiv:1509.08324 [hep-ph]].
- [39] J. Blümlein, A. De Freitas, C.G. Raab and K. Schönwald, *Phys. Lett. B* **791** (2019) 206–209 [arXiv:1901.08018 [hep-ph]].
- [40] J. Blümlein, *Comput. Phys. Commun.* **159** (2004) 19–54 [hep-ph/0311046].
- [41] T. Gehrmann and E. Remiddi, *Comput. Phys. Commun.* **141** (2001) 296–312 [hep-ph/0107173].
- [42] J. Ablinger, J. Blümlein, M. Round and C. Schneider, *Comput. Phys. Commun.* **240** (2019) 189–201 [arXiv:1809.07084 [hep-ph]].
- [43] S. Alekhin, J. Blümlein, K. Daum, K. Lipka and S. Moch, *Phys. Lett. B* **720** (2013) 172–176 [arXiv:1212.2355 [hep-ph]].
- [44] K.A. Olive *et al.* [Particle Data Group], *Chin. Phys. C* **38** (2014) 090001.
- [45] J. Blümlein and C. Schneider, *Phys. Lett. B* **771** (2017) 31–36 [arXiv:1701.04614 [hep-ph]].
- [46] J. Blümlein, *Comput. Phys. Commun.* **180** (2009) 2218–2249 [arXiv:0901.3106 [hep-ph]].
- [47] N. Nielsen, *Der Eulersche Dilogarithmus und seine Verallgemeinerungen*, Nova Acta Leopoldina, **XC**, Nr. 3, Halle, 1909 125–211;
K.S. Kölbig, *SIAM J. Math. Anal.* **16** (1986) 1232–1258.
- [48] S. Moch, J.A.M. Vermaseren, and A. Vogt, private communication.
- [49] J. Blümlein and A. Guffanti, *Nucl. Phys. Proc. Suppl.* **152** (2006) 87–91 [hep-ph/0411110].
- [50] R. Harlander, T. Seidensticker and M. Steinhauser, *Phys. Lett. B* **426** (1998) 125–132 [hep-ph/9712228].
- [51] T. Seidensticker, *Automatic application of successive asymptotic expansions of Feynman diagrams*, hep-ph/9905298.