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# Power corrections to the production of a prompt photon in association with a jet in the $N$-jettiness slicing scheme at NLO QCD

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Abstract: We compute the next-to-leading-power corrections in the $N$-jettiness variable to the production of a prompt photon and a jet at next-to-leading order in perturbative QCD in the $q\bar{q}$ annihilation channel. We employ the $k_\perp$ jet algorithm and assume that the $N$-jettiness value divided by the jet transverse momentum is the smallest parameter in the problem; in particular it should be small compared to the jet radius $R$.

## Contents



## 1 Introduction

A robust description of hard scattering processes at the LHC requires significant theoretical innovations. Such innovations encompass many different aspects of collider theory including computation of scattering amplitudes at high orders of QCD and electroweak perturbation theory [1–30], development of subtraction and slicing schemes for real-radiation contributions [31–53], refinement of parton shower programs [54–61] and their interfaces to fixed-order calculations [62–65], as well as advances in understanding non-perturbative hadronization effects [66, 67].

*Power corrections* in the slicing schemes are one aspect of the theoretical description of hard scattering processes at the LHC where further progress is desirable. Such corrections appear because of the very nature of slicing computations, where one separates the phase space for a process with $N$ final-state particles or jets, into phase-space regions where *all* $N$ partons are resolved, and regions where only a smaller number of partons or jets are resolved.

This separation requires a resolution variable. If the resolution variable is taken to be very small, dependencies of cross sections on the resolution variable, originating from resolved and unresolved regions, follow the double-logarithmic pattern, which is typical to radiative corrections in QCD. Such dependencies on resolution variables are nearly universal and are very well understood. However, for a better matching between the resolved and unresolved contributions to cross sections, it is beneficial to expose the dependence of the unresolved contribution on the resolution variable *beyond* the leading terms. Such subleading terms are referred to as power corrections.

Power corrections for different resolution variables were investigated in a large number of publications in recent years [68–90]. The majority of these papers focused on contributions that are either logarithmically enhanced for small values of a resolution variable, or originate from emissions of soft gluon only. It is therefore unclear how to extend these studies to arbitrary collider processes and, in particular, to go beyond the soft limit in a general way. Amusingly, this problem exists even at the next-to-leading order in perturbative QCD, where one might have thought that everything is well-understood by now.

In Ref. [91] we presented a methodology to compute subleading power corrections to *a*rbitrary colorless final states at NLO QCD using the $N$-jettiness resolution variable [45]. The next natural step is to remove the restriction on the final states, and design a way to compute power corrections in the $N$-jettiness variable for arbitrary processes at colliders. To simplify this step, in this paper we consider such corrections to the process where a prompt photon is produced in association with a jet. Furthermore, since very little is known about power corrections to processes with jets,[1] we decided to first consider a single partonic channel $q\bar{q} \to \gamma + g$. For this channel, the so-called photon isolation [92] is not needed, and we can focus on the central question that we want to discuss, namely how the presence of a jet algorithm affects the computation of power corrections. The process $q\bar{q} \to \gamma + g$ is well suited to study this question, since it is sufficiently simple, and we can directly work with the relevant matrix elements to understand power corrections to the partonic cross section.

We note that power-suppressed corrections arise also from observables or the selection cuts that define fiducial cross sections. In this paper, we assume that observables are such that their dependence on the $N$-jettiness variable is analytic.[2] It is known, however, that this is not always the case and that observables exist which induce a non-analytic dependences of fiducial cross sections on the resolution variable [65, 78, 95–98] which enhances the power-suppressed contributions.

The rest of the paper is organized as follows. In Section 2 we explain how the presence of a jet in the final state affects the computation of power corrections, and define quantities that we use in the remainder of the paper. We also summarize the method for calculating power corrections developed in Ref. [91], that we employ in this paper. In Section 3 we compute the power corrections in the $N$-jettiness variable to the process $q\bar{q} \to \gamma + \text{jet}$. We investigate various soft and collinear contributions, as well as subtleties related to differences between cases when partons are clustered into a jet and cases when they are not. We also discuss the validation of our results in Section 3. We conclude in Section 4.

[1]We are aware of a single paper [75] where power corrections to the production of a vector boson in association with a jet are studied. However, in that paper only logarithmically-enhanced contributions to power corrections have been computed, and an unconventional jet algorithm has been employed, to simplify the computation.

[2]As we demonstrate below, this class of observables includes the fully-realistic inclusive sequential $k_\perp$ jet algorithm [93, 94].

Some technical details are discussed in appendices.

## 2 General remarks

We study the production of a prompt photon and a jet in hadron collisions, $pp \to \gamma + j$. We focus on the $q\bar{q} \to \gamma + g$ partonic channel, and do not consider any other partonic channels in this paper. We imagine that the $N$-jettiness slicing scheme [41, 45, 48] is employed for computing the NLO QCD corrections. An important ingredient for calculations in this scheme is the differential cross section for final states with $\mathcal{T}_1 < \tau_{\rm cut}$, where $\mathcal{T}_1$ is the $N$-jettiness variable, and $\tau_{\rm cut}$ is a small quantity. To allow the choice of somewhat larger values of $\tau_{\rm cut}$ in practical computations, we need to construct an expansion of the $q\bar{q} \to \gamma + j$ cross section through first *subleading* power in the one-jettiness variable $\mathcal{T}_1 \sim \tau_{\rm cut}$.

To define a final-state jet, we require a jet algorithm. We will consider the so-called inclusive sequential $k_\perp$ jet algorithm [93, 94].[3] We will describe how it works before addressing the question of how it impacts the calculation of power corrections.

To this end, we introduce two phase-space "distances"

$$d_{ij} = \min(k_{\perp,i}^2, k_{\perp,j}^2)\, \frac{R_{ij}^2}{R^2}, \quad d_{iB} = k_{\perp,i}^2, \tag{2.1}$$

where $k_{\perp,x}$ is the transverse momentum of a parton $x \in (i,j)$ defined with respect to the collision axis. We note that $d_{ij}$ and $d_{iB}$ measure distances between the final-state partons $i$ and $j$, and between the final-state parton $i$ and the beam axis, respectively. For the quantity $R_{ij}$, one typically takes

$$R_{ij}^2 = (\eta_i - \eta_j)^2 + f_\varphi^2(\varphi_i - \varphi_j), \tag{2.2}$$

where $\eta_{i,j}$ are pseudo-rapidities of the two partons $i$ and $j$, and $f_\varphi$ is a function of their azimuthal angles $\varphi_{i,j}$. We choose

$$f_\varphi(\varphi_i - \varphi_j) = \arccos(\cos(\varphi_i - \varphi_j)), \tag{2.3}$$

since this maps the difference of two azimuthal angles onto the $[0,\pi]$ interval, independent of how azimuthal angles are parametrized.

To apply the jet algorithm to a set of final-state partons $P_N = \{1,2,3,..,N\}$, we start by computing two lists. One of them is composed of $d_{ij}$'s calculated for each $(ij)$ pair from $P_N$, and the second – of $d_{iB}$ for each $i \in P_N$. We then compare the minimal values of the two lists

$$d_{\rm min} = \min\left[\min\{d_{ij}\}, \min\{d_{iB}\}\right]. \tag{2.4}$$

If $d_{\rm min}$ is the minimum of the $\{d_{ij}\}$ list, the two partons $i$ and $j$ are removed from $P_N$ and replaced there by a new parton $ij$ whose momentum is $p_{ij} = p_i + p_j$. If, however, the minimum is provided by the $\{d_{iB}\}$ list, the parton $i$ is removed from the list $P_N$ and added to the list of *jets* $P_J$ that is empty at the start of this procedure. We continue this process

[3] We note that our results can be used, without any modification, for the anti-$k_\perp$ jet clustering algorithm as well.

until no partons are left in the list $P_N$. Finally, all jets with the transverse momentum lower than a pre-selected value $p_{\perp,\rm cut}$ are removed from the list of jets. Once this is done, we associate a definite number of jets with the partonic final state described by the original list $P_N$.

We continue with the discussion of what this algorithm implies for the computation of the power corrections. At leading order, we apply it to the partonic process

$$q_a + \bar{q}_b \to \gamma + g_{\mathfrak{m}}. \tag{2.5}$$

Therefore, the list of partons consists of a single gluon $g_{\mathfrak{m}}$. This parton is moved to the list of jets immediately, and if its transverse momentum exceeds $p_{\perp,\rm cut}$, it is identified with a jet. Once the jet is identified, we compute the one-jettiness and find

$$\mathcal{T}_1 = \min\left\{\frac{2p_{\mathfrak{m}}p_a}{P_a}, \frac{2p_{\mathfrak{m}}p_b}{P_b}, \frac{2p_{\mathfrak{m}}p_J}{P_J}\right\} = 0, \tag{2.6}$$

because in this case $p_J = p_{\mathfrak{m}}$.

At next-to-leading order, we have to consider the process[4]

$$q_a + \bar{q}_b \to \gamma + g_{\mathfrak{m}} + g_{\mathfrak{n}}, \tag{2.7}$$

and apply the jet algorithm to two gluons $g_{\mathfrak{m}}$ and $g_{\mathfrak{n}}$. To simplify further steps, it is convenient to order the two gluons in the transverse momenta, and label them in such a way that $p_{\perp,\mathfrak{m}} > p_{\perp,\mathfrak{n}}$. The starting point for the application of the jet algorithm is the list $P_2 = \{\mathfrak{m}, \mathfrak{n}\}$. We have to find the minimum of $\{d_{\mathfrak{mn}}, d_{\mathfrak{m}B}, d_{\mathfrak{n}B}\}$. Thanks to the transverse momentum ordering, this minimum is $d_{\mathfrak{mn}}$ if $R_{\mathfrak{mn}} < R$, and $d_{\mathfrak{n}B}$ if $R_{\mathfrak{mn}} > R$. Then, in the first (clustered) case, we have a one-jet event with the jet momentum $p_J = p_{\mathfrak{m}} + p_{\mathfrak{n}}$ provided that $p_{\perp,J} > p_{\perp,\rm cut}$, and in the second (unclustered) case we have, potentially, two jets with the transverse momenta $p_{\perp,\mathfrak{n}}$ and $p_{\perp,\mathfrak{m}}$. To have a one-jet event we require $p_{\perp,\mathfrak{n}} < p_{\perp,\rm cut}$ and $p_{\perp,\mathfrak{m}} > p_{\perp,\rm cut}$.

Following this discussion, we can make the NLO QCD real-emission contribution to the $\gamma + j$ production explicit. To simplify the notation, we write the one-jettiness variable with an argument which refers to the momentum of the jet used in its definition, i.e.

$$\mathcal{T}_1(p_J) = \sum_{i\in\{\mathfrak{m},\mathfrak{n}\}} \min\left\{\frac{2p_i p_a}{P_a}, \frac{2p_i p_b}{P_b}, \frac{2p_i p_J}{P_J}\right\}. \tag{2.8}$$

As we just described, the jet momentum depends on whether gluons are clustered into a jet or not. We find

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\mathcal{O}}}{\mathrm{d}\tau} = \mathcal{N}^{-1} \int \mathrm{d}\Phi(p_a, p_b|p_{\mathfrak{m}}, p_{\mathfrak{n}}, p_\gamma)\, |\mathcal{M}|^2(p_a, p_b; p_{\mathfrak{m}}, p_{\mathfrak{n}}, p_\gamma)\, \theta(p_{\perp,\mathfrak{m}} - p_{\perp,\mathfrak{n}}) \\ \times \Big\{\theta(R_{\mathfrak{mn}} - R)\, \theta(p_{\perp,\mathfrak{m}} - p_{\perp,\rm cut})\, \theta(p_{\perp,\rm cut} - p_{\perp,\mathfrak{n}})\, \delta(\tau - \mathcal{T}_1(p_{\mathfrak{m}}))\, \mathcal{O}(p_{\mathfrak{m}}, p_\gamma) \\ + \theta(R - R_{\mathfrak{mn}})\, \theta(p_{\perp,[\mathfrak{mn}]} - p_{\perp,\rm cut})\, \delta(\tau - \mathcal{T}_1(p_{[\mathfrak{mn}]}))\, \mathcal{O}(p_{[\mathfrak{mn}]}, p_\gamma)\Big\},\end{aligned} \tag{2.9}$$

[4]We do not need to consider the virtual corrections to the process in Eq. (2.5) because they will only contribute at $\mathcal{T}_1 = 0$.

where $\mathcal{N}$ is the normalization factor, $\mathcal{O}$ is an observable that depends on the jet momentum and the momentum of the photon, and $\mathrm{d}\Phi$ is the phase space that will be defined in the next section. We use $p_{[\mathfrak{mn}]}$ to denote the sum of the gluon momenta, $p_{[\mathfrak{mn}]} = p_{\mathfrak{m}} + p_{\mathfrak{n}}$.

To simplify the notation, we will absorb the $\theta$-functions, that ensure that the transverse momentum of the jet is larger than the transverse-momentum cut, into the definition of the observable $\mathcal{O}$. Hence, from now on, we will only write explicitly the $R$-dependent $\theta$-functions from the jet algorithm, as well as the $\theta$-function that ensures that the transverse momentum of the parton $\mathfrak{n}$ is small.

There is an important difference in the one-jettiness functions that appear in the two terms in the integrand in Eq. (2.9). In the first term, $p_J = p_{\mathfrak{m}}$ and therefore

$$\mathcal{T}_1(p_{\mathfrak{m}}) = \min\left\{ \frac{2p_{\mathfrak{n}}p_a}{P_a}, \frac{2p_{\mathfrak{n}}p_b}{P_b}, \frac{2p_{\mathfrak{n}}p_{\mathfrak{m}}}{P_J} \right\}. \tag{2.10}$$

In the second term, $p_J = p_{[\mathfrak{mn}]}$, and we find a more complicated expression for the one-jettiness function

$$\begin{aligned} \mathcal{T}_1(p_{[\mathfrak{mn}]}) &= \sum_{i\in\{\mathfrak{m},\mathfrak{n}\}} \min\left\{ \frac{2p_i p_a}{P_a}, \frac{2p_i p_b}{P_b}, \frac{2p_i p_{[\mathfrak{mn}]}}{P_J} \right\} \\ &= \min\left\{ \frac{2p_{\mathfrak{n}}p_a}{P_a}, \frac{2p_{\mathfrak{n}}p_b}{P_b}, \frac{2p_{\mathfrak{n}}p_{\mathfrak{m}}}{P_J} \right\} + \min\left\{ \frac{2p_{\mathfrak{m}}p_a}{P_a}, \frac{2p_{\mathfrak{m}}p_b}{P_b}, \frac{2p_{\mathfrak{m}}p_{\mathfrak{n}}}{P_J} \right\}. \end{aligned} \tag{2.11}$$

To compute power-suppressed one-jettiness corrections, we need to analyze different contributions to Eq. (2.9), finding kinematic regions where the one-jettiness function defined in Eqs (2.10,2.11) is small.

If the two partons $\mathfrak{m}$ and $\mathfrak{n}$ have generic momenta, $\mathcal{T}_1$ cannot be small. For this to occur, partons $\mathfrak{m}$ and $\mathfrak{n}$ should have special, singular kinematics. In general, given that $p_{\perp,\mathfrak{m}} > p_{\perp,\mathfrak{n}}$ and at least one jet is required, there are four options for the parton $\mathfrak{n}$:

- $\mathfrak{n}$ is collinear to $a$, $b$ or $\mathfrak{m}$;
- $\mathfrak{n}$ is soft.

We will continue with the discussion of these cases separately. We choose the reference frame where momenta of partons $a$ and $b$ are along the $z$ axis, and we denote a polar angle of a parton $x$ by $\theta_x$.

**The collinear case $\mathfrak{n}||a$** In this case, the energy of the parton $\mathfrak{n}$ is large, $E_{\mathfrak{n}} \sim E_a \sim E_b$, but the polar angle is small, $\theta_{\mathfrak{n}} \sim \sqrt{\tau P_J/E_a^2}$. At the same time the four-momentum of the parton $\mathfrak{m}$ is generic, i.e. $E_{\mathfrak{m}} \sim E_a$ and $\theta_{\mathfrak{m}} \sim 1$. Clustering of $\mathfrak{m}$ and $\mathfrak{n}$ into a jet is impossible because the rapidity of the parton $\mathfrak{n}$ is very large

$$\eta_{\mathfrak{n}} = \frac{1}{2}\ln\frac{1+\cos\theta_{\mathfrak{n}}}{1-\cos\theta_{\mathfrak{n}}} \sim -\frac{1}{2}\ln\frac{\tau P_J}{E_a^2}. \tag{2.12}$$

Therefore,

$$R_{\mathfrak{mn}}^2 \sim \ln^2\left(\frac{\tau P_J}{E_a^2}\right), \tag{2.13}$$

and, as long as

$$\ln\left(\frac{E_a^2}{\tau P_J}\right) \gg R, \tag{2.14}$$

clustering of partons $\mathfrak{m}$ and $\mathfrak{n}$ into a single jet does not occur.[5] Hence, in this case we can write

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\mathcal{O}}^{ca}}{\mathrm{d}\tau} =& \mathcal{N}^{-1}\int \mathrm{d}\Phi(p_a,p_b|p_{\mathfrak{m}},p_{\mathfrak{n}},p_\gamma)|\mathcal{M}|^2(p_a,p_b;p_{\mathfrak{m}},p_{\mathfrak{n}},p_\gamma)\\ &\times\delta\left(\tau-\frac{2p_ap_{\mathfrak{n}}}{P_a}\right)\mathcal{O}(p_{\mathfrak{m}},p_\gamma).\end{aligned} \tag{2.15}$$

Note that we have dropped the constraint on the transverse momentum of the parton $\mathfrak{n}$ because $p_{\perp,\mathfrak{n}} \sim \sqrt{\tau P_a}$ and, as long as $\tau$ is small and $p_{\perp,\mathrm{cut}} \sim \mathcal{O}(E_a)$, the transverse momentum of the parton $\mathfrak{n}$ cannot exceed the cut value.

**The collinear case $\mathfrak{n}||b$** This case is analogous to the $\mathfrak{n}||a$ case. Hence, without further discussion, we write

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\mathcal{O}}^{cb}}{\mathrm{d}\tau} =& \mathcal{N}^{-1}\int \mathrm{d}\Phi(p_a,p_b|p_{\mathfrak{m}},p_{\mathfrak{n}},p_\gamma)\;|\mathcal{M}|^2(p_a,p_b;p_{\mathfrak{m}},p_{\mathfrak{n}},p_\gamma)\\ &\times\delta\left(\tau-\frac{2p_bp_{\mathfrak{n}}}{P_b}\right)\mathcal{O}(p_{\mathfrak{m}},p_\gamma).\end{aligned} \tag{2.16}$$

**The collinear case $\mathfrak{m}||\mathfrak{n}$** This case corresponds to the final-state collinear configuration. Computing the invariant mass of partons $\mathfrak{m}$ and $\mathfrak{n}$ in the collinear approximation, we obtain

$$s_{\mathfrak{mn}} \approx E_{\mathfrak{m}}E_{\mathfrak{n}}\left((\theta_{\mathfrak{m}}-\theta_{\mathfrak{n}})^2+\sin^2\theta_{\mathfrak{m}}\,(\varphi_{\mathfrak{m}}-\varphi_{\mathfrak{n}})^2\right) \approx p_{\perp,\mathfrak{m}}p_{\perp,\mathfrak{n}}R_{\mathfrak{mn}}^2. \tag{2.17}$$

Using the jettiness constraint, we estimate that in the collinear $\mathfrak{m}||\mathfrak{n}$ case, the $s_{\mathfrak{mn}}$ invariant mass becomes

$$s_{\mathfrak{mn}} \sim \tau P_J. \tag{2.18}$$

Hence,

$$R_{\mathfrak{mn}}^2 \sim \frac{\tau P_J}{p_{\perp,\mathfrak{m}}p_{\perp,\mathfrak{n}}} \sim \frac{\tau P_J}{p_{\perp,\mathrm{cut}}^2} \ll R^2. \tag{2.19}$$

and the two partons are clustered into a single jet.[6] The expression for the cross section reads

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\mathcal{O}}^{\mathfrak{mn}}}{\mathrm{d}\tau} =& \frac{\mathcal{N}^{-1}}{2}\int \mathrm{d}\Phi(p_a,p_b|p_{\mathfrak{m}},p_{\mathfrak{n}},p_\gamma)\;|\mathcal{M}|^2(p_a,p_b;p_{\mathfrak{m}},p_{\mathfrak{n}},p_\gamma)\\ &\times\delta\left(\tau-\frac{4p_{\mathfrak{m}}p_{\mathfrak{n}}}{P_J}\right)\mathcal{O}(p_{[\mathfrak{mn}]},p_\gamma),\end{aligned} \tag{2.20}$$

We note that we have introduced the factor 1/2, and used the $\mathfrak{m}\leftrightarrow\mathfrak{n}$ replacement symmetry in this kinematic configuration, to remove the $p_{\mathfrak{m},\perp} > p_{\mathfrak{n},\perp}$ condition from the integrand.

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[5] Another condition on the jet radius that restricts it from *above*, is derived later.

[6] We note that, under the assumption that $p_{\perp,\mathrm{cut}} \sim E_a \sim P_J$, Eqs (2.14) and (2.19) *together* imply that the jet radius should satisfy the following constraint $\sqrt{\tau/p_\perp} \ll R \ll \ln p_\perp/\tau$.

**The soft case $E_\mathfrak{n} \to 0$** Finally, we need to discuss the soft case where $E_\mathfrak{n} \sim \tau$. Then $p_{\perp,\mathfrak{n}} \ll p_{\perp,\mathfrak{m}} \sim p_{\perp,\text{cut}}$, and conditions that ensure these requirements can be dropped. At the same time, since the soft gluon $\mathfrak{n}$ can be emitted at an arbitrary angle, it is impossible to say a priori whether it will be clustered into a jet together with $\mathfrak{m}$, or not. Because of this, we write the soft contribution in such a way, that both clustered and non-clustered cases can be described. The soft contribution reads

$$\begin{aligned}\frac{\mathrm{d}\sigma^s_{\mathcal{O}}}{\mathrm{d}\tau} = \mathcal{N}^{-1} &\int \mathrm{d}\Phi(p_a, p_b | p_\mathfrak{m}, p_\mathfrak{n}, p_\gamma) |\mathcal{M}|^2(p_a, p_b; p_\mathfrak{m}, p_\mathfrak{n}, p_\gamma) \\ &\times \Theta(R_{\mathfrak{mn}}, q_j)\, \delta(\tau - \mathcal{T}_1(q_j))\, \mathcal{O}(q_j, p_\gamma),\end{aligned} \tag{2.21}$$

where $q_j$ is the momentum of the identified jet, and $\Theta(R_{\mathfrak{mn}}, q_j)$ is the remnant of the angular distance of the jet algorithms defined as follows

$$\Theta(R_{\mathfrak{mn}}, q_j) = \theta(R_{\mathfrak{mn}} - R)\, \delta_{q_j, p_\mathfrak{m}} + \theta(R - R_{\mathfrak{mn}})\, \delta_{q_j, p_{[\mathfrak{mn}]}}. \tag{2.22}$$

The one-jettiness function reads

$$\mathcal{T}_1(q_j) = \min\left\{ \frac{2 p_a p_\mathfrak{n}}{P_a}, \frac{2 p_b p_\mathfrak{n}}{P_b}, \frac{2 p_\mathfrak{m} p_\mathfrak{n}}{P_J} \right\} + \delta_{q_j, p_{[\mathfrak{mn}]}} \frac{2 p_\mathfrak{m} p_\mathfrak{n}}{P_J}. \tag{2.23}$$

The last term distinguishes between the clustered and the non-clustered cases.

**Computational strategy** To compute the various contributions, we adopt the strategy discussed in Ref. [91], where we constructed Lorentz transformations for different cases, used them to factorize the phase space for the photon and the two partons with the power accuracy, and expanded the squared matrix element and the observable functions around soft and collinear limits. In Ref. [91] we developed a process-independent procedure to expand the matrix element for the production of a color-singlet final state. For simplicity, in this paper we make use of the explicit form of the matrix element for the $q\bar{q} \to \gamma + g + g$ process, to construct an expansion in the soft and collinear limits.

The required Lorentz transformations for the cases $\mathfrak{n}||a$ and $\mathfrak{n}||b$, as well as for the case when the parton $\mathfrak{n}$ is soft, are discussed in detail in Ref. [91]. The new technical element required here is the momenta mappings for the collinear $\mathfrak{m}||\mathfrak{n}$ case. We describe these mappings in Appendix A.

## 3 Power corrections to the $\gamma$+jet production in the $q\bar{q} \to g\gamma$ channel

### 3.1 Leading order

We consider the partonic process

$$q(p_a) + \bar{q}(p_b) \to \gamma(p_\gamma) + g(p_j), \tag{3.1}$$

and associate the final-state gluon with a jet. The differential cross section of the process in Eq. (3.1) reads

$$\mathrm{d}\sigma_0 = \frac{N_c Q_q^2 (e g_s)^2}{2 s N_c^2} \int \mathrm{d}\Phi^{ab}_{\gamma j} \sum_{\text{col,pol}} \frac{|\mathcal{M}_0|^2(p_a, p_b, p_j, p_\gamma)}{4 N_c (Q_q e g_s)^2}\, \mathcal{O}(p_j, p_\gamma), \tag{3.2}$$

where $Q_q$ is the quark electric charge in units of the positron charge $e$, $g_s$ is the (bare) strong coupling constant, $N_c = 3$ is the number of colors, $s = 2p_a \cdot p_b$ and $\mathcal{O}(p_j, p_\gamma)$ is the infrared-safe observable that depends on the momenta of the jet and the photon. Furthermore,

$$\mathrm{d}\Phi^{ab}_{\gamma j} = [dp_j][dp_\gamma](2\pi)^d \delta^{(d)}(p_a + p_b - p_\gamma - p_j), \tag{3.3}$$

is the phase space[7] with

$$[\mathrm{d}p_x] = \frac{\mathrm{d}^d p_x}{(2\pi)^{d-1}}\, \delta_+(p_x^2). \tag{3.4}$$

We employ the Sudakov decomposition of the photon and jet momenta to parametrize the Born phase space. We use momenta of the incoming partons to define the basis vectors. Then,

$$\begin{aligned} p_j &= \beta p_a + (1-\beta)p_b - \sqrt{s\beta(1-\beta)}\, n_\perp, \\ p_\gamma &= (1-\beta)p_a + \beta p_b + \sqrt{s\beta(1-\beta)}\, n_\perp, \end{aligned} \tag{3.5}$$

with $\beta \in [0,1]$, $p_{a,b} \cdot n_\perp = 0$ and $n_\perp^2 = -1$. The transverse momentum of the jet reads

$$|p_{\perp,j}| = \sqrt{s\beta(1-\beta)}. \tag{3.6}$$

We use Eq. (3.5), to write the Born phase space as follows

$$\mathrm{d}\Phi^{ab}_{\gamma j} = \frac{s^{-\epsilon}\Omega^{(d-2)}}{4(2\pi)^{d-2}}\, \mathrm{d}\beta\, [\mathrm{d}\Omega^{(d-2)}]\, \beta^{-\epsilon}(1-\beta)^{-\epsilon} = \frac{1}{8\pi}\mathrm{d}\beta\, \frac{\mathrm{d}\varphi}{2\pi} + \mathcal{O}(\epsilon), \tag{3.7}$$

where $\Omega^{(d-2)}$ is the solid angle in $d-2$ dimensions and $[\mathrm{d}\Omega^{(d-2)}] = \mathrm{d}\Omega^{(d-2)}/\Omega^{(d-2)}$. The azimuthal angles in $\mathrm{d}\Omega^{(d-2)}$ parametrize the direction of the vector $n_\perp$ in the $(d-2)$-dimensional space orthogonal to $p_{a,b}$.

The appropriately normalized squared matrix element for the process in Eq. (3.1), summed over polarizations and colors reads

$$\sum_{\text{pol,col}} \frac{|\mathcal{M}_0|^2(p_b, p_a; p_{\mathfrak{m}}, p_\gamma)}{4N_c(Q_q e g_s)^2} = C_F\left[\frac{(1-\epsilon)}{2}\left(\frac{t}{u} + \frac{u}{t}\right) - \epsilon\right] = C_F\frac{(1 - 2\beta + 2\beta^2 - \epsilon)}{2\beta(1-\beta)}, \tag{3.8}$$

where $t = -2p_a \cdot p_\gamma$, $u = -2p_b \cdot p_\gamma$. Finally, using the above ingredients, we write the leading order differential cross section as

$$\mathrm{d}\sigma_0 = \bar{\sigma}_0\, \mathrm{d}\Phi^{ab}_{\gamma j}\, \frac{(1 - 2\beta + 2\beta^2 - \epsilon)}{2\beta(1-\beta)}, \tag{3.9}$$

where

$$\bar{\sigma}_0 = \frac{16\pi^3 C_F Q_q^2\, \alpha_{\text{QED}}\, [\alpha_s]}{sN_c}, \tag{3.10}$$

with

$$[\alpha_s] = \frac{g_s^2 \Omega^{(d-2)}}{2(2\pi)^{d-1}} = \frac{\alpha_s}{2\pi} + \mathcal{O}(\epsilon). \tag{3.11}$$

Having discussed the cross section for the Born process, we proceed with the computation of the power corrections in the one-jettiness variable. As pointed out in Section 2, several contributions need to be considered. We will start with the discussion of the soft case, and continue with the collinear ones.

[7] Throughout the paper, we employ dimensional regularization and work in $d = 4 - 2\epsilon$ dimensions.

### 3.2 The soft contribution

We consider the case when the parton $\mathfrak{n}$ is soft, which means that its energy is of order $\tau$. We note that this kinematic configuration has to be considered for the case when partons $\mathfrak{m}$ and $\mathfrak{n}$ are clustered into a jet, and for the case when they are not. Our starting point is Eq. (2.21) that we repeat here for convenience

$$\begin{aligned}\frac{\mathrm{d}\sigma^s_{\mathcal{O}}}{\mathrm{d}\tau} = \mathcal{N}^{-1} \int \mathrm{d}\Phi(p_a, p_b | p_{\mathfrak{m}}, p_{\mathfrak{n}}, \tilde{p}_\gamma) |\mathcal{M}|^2(p_a, p_b; p_{\mathfrak{m}}, p_{\mathfrak{n}}, \tilde{p}_\gamma) \\ \times \Theta(R_{\mathfrak{mn}}, q_j)\, \delta(\tau - \mathcal{T}_1(q_j))\, \mathcal{O}(q_j, \tilde{p}_\gamma).\end{aligned} \tag{3.12}$$

We note that the normalization factor in Eq. (3.12) coincides with the one for the leading order process $q\bar{q} \to g + \gamma$. This means that

$$\mathcal{N}^{-1}\, \mathrm{d}\Phi^{ab}_{\gamma j}\, |\mathcal{M}_0(p_a, p_b, p_j, p_\gamma)|^2 = \bar{\sigma}_0\, \mathrm{d}\Phi^{ab}_{\gamma j}\, \frac{(1 - 2\beta + 2\beta^2 - \epsilon)}{2\beta(1-\beta)}, \tag{3.13}$$

and we will use this equation for simplifying some computations in what follows.

We also note that we have written the photon momentum in Eq. (3.12) as $\tilde{p}_\gamma$. This is done on purpose since, because of Lorentz transformations, this momentum will be redefined as we proceed with the calculation, and we would like to reserve the notation $p_\gamma$ for the photon momentum appearing in the final equations.

The soft contribution corresponds to the scaling $p_{\mathfrak{n}} \sim \tau$. For the sake of convenience, in what follows we will refer to $p_{\mathfrak{n}}$ as $k$. To construct the expansion around the soft limit, we define the four-momentum

$$P_{ab} = p_a + p_b, \tag{3.14}$$

and perform a boost and a rescaling to remove the momentum $k$ from the energy-momentum conservation constraint $p_a + p_b = p_{\mathfrak{m}} + k + p_\gamma$, which is implicitly present in Eq. (3.12). We write

$$\lambda P^\mu_{ab} = [\Lambda_s]^\mu_\nu\, (P_{ab} - k)^\nu. \tag{3.15}$$

The matrix $\Lambda_s$ in the above equation is the Lorentz boost. The rescaling parameter $\lambda$, computed through first order in $k \sim \tau$, reads

$$\lambda \approx 1 - \frac{P_{ab} \cdot k}{P^2_{ab}}. \tag{3.16}$$

Performing the boost, and using the phase-space modification in the soft limit computed in Ref. [91], we find

$$\begin{aligned}\frac{\mathrm{d}\sigma^s_{\mathcal{O}}}{\mathrm{d}\tau} = \mathcal{N}^{-1} \int \mathrm{d}\Phi^{ab}_{j\gamma}\, [\mathrm{d}k] \left(1 + 2\epsilon \frac{P_{ab}\cdot k}{P^2_{ab}}\right)\, \Theta(R_{\mathfrak{mn}}, q_j) \\ \times |\mathcal{M}|^2(p_a, p_b; \lambda\Lambda_s^{-1} p_j, k, \lambda\Lambda_s^{-1} p_\gamma)\, \delta(\tau - \mathcal{T}_1(q_j))\, \mathcal{O}(q_j, \lambda\Lambda_s^{-1} p_\gamma),\end{aligned} \tag{3.17}$$

where $p_{\mathfrak{m}} = \lambda\Lambda_s^{-1} p_j$, $\tilde{p}_\gamma = \lambda\Lambda_s^{-1} p_\gamma$ and $p_j^2 = p_\gamma^2 = 0$. Furthermore,

$$[\mathrm{d}k] = \mathrm{d}\omega_k\, \omega_k^{1-2\epsilon} \frac{\Omega^{d-2}}{2(2\pi)^{d-1}} [\mathrm{d}\Omega_k], \quad \left[\Lambda_s^{-1}\right]_{\mu\nu} = g_{\mu\nu} - \frac{k^\mu P^\nu_{ab} - P^\mu_{ab} k^\nu}{P^2_{ab}} + \mathcal{O}(k^2). \tag{3.18}$$

Since the jet momentum $q_j$ depends on whether partons are clustered into a jet or not, there are two possible ways for $q_j$ to transform under the soft boost and rescaling. They are

$$\begin{aligned} q_j &= p_{\mathfrak{m}} + p_{\mathfrak{n}} = \lambda\Lambda_s^{-1} p_j + k, \quad \text{if clustered,} \\ q_j &= p_{\mathfrak{m}} = \lambda\Lambda_s^{-1} p_j, \quad \text{if not clustered.} \end{aligned} \tag{3.19}$$

The one-jettiness function also differs for the two cases. However, since

$$p_{\mathfrak{m}} \cdot p_{\mathfrak{n}} = (\lambda\Lambda_s^{-1} p_j) \cdot k = p_j \cdot k + \mathcal{O}(k^3), \tag{3.20}$$

we find that the following equation holds

$$\mathcal{T}_1(q_j) = \min\left\{ \frac{2p_a k}{P_a}, \frac{2p_b k}{P_b}, \frac{2p_j k}{P_J} \right\} + \delta_{q_j, p_{[\mathfrak{mn}]}} \frac{2p_j k}{P_J}, \tag{3.21}$$

and no $\mathcal{O}(k)$ corrections appear in the expression for the one-jettiness.

To compute $\mathrm{d}\sigma^s/\mathrm{d}\tau$ through first subleading correction in $\tau$, we need to expand all the relevant quantities in the integrand in Eq. (3.17) to first subleading order in the gluon energy $\omega_k$. This includes the expansion of the matrix element squared, the observable and also the function $\Theta(R_{\mathfrak{mn}}, q_j)$ which gets modified because the angular distance between partons $j$ and $k$, and $\mathfrak{m}$ and $k$ is not the same. We will start with the discussion of the matrix element.

The next-to-soft correction to the squared matrix element can be obtained from the extension of the Burnett-Kroll-Low theorem [99, 100] to QCD. For the process $q\bar{q} \to \gamma + j$, such a study was performed in Ref. [76], where it was shown that, with the required accuracy, the squared matrix element for this process can be written in the following way

$$\begin{aligned} &g_s^{-2} |\mathcal{M}|^2(p_a, p_b, p_{\mathfrak{m}}, k, \tilde{p}_\gamma) \approx \\ &\left( C_F - \frac{C_A}{2} \right) \frac{2p_a \cdot p_b}{p_a \cdot k \; p_b \cdot k} |\mathcal{M}_0(p_a + \delta p_{a,b}, p_b + \delta p_{b,a}, p_{\mathfrak{m}}, \tilde{p}_\gamma)|^2 \\ &+ \frac{C_A}{2} \frac{2p_a \cdot p_{\mathfrak{m}}}{p_a \cdot k \; p_{\mathfrak{m}} \cdot k} |\mathcal{M}_0(p_a + \delta p_{a,j}, p_b, p_{\mathfrak{m}} - \delta p_{\mathfrak{m},a}, \tilde{p}_\gamma)|^2 \\ &+ \frac{C_A}{2} \frac{2p_b \cdot p_{\mathfrak{m}}}{p_b \cdot k \; p_{\mathfrak{m}} \cdot k} |\mathcal{M}_0|^2(p_a, p_b + \delta p_{b,\mathfrak{m}}, p_{\mathfrak{m}} - \delta p_{\mathfrak{m},b}, \tilde{p}_\gamma)|^2. \end{aligned} \tag{3.22}$$

The momenta shifts in Eq. (3.22) read

$$\delta p_{l,m} = -\frac{1}{2} \left( k + \frac{p_m \cdot k}{p_l \cdot p_m} \, p_l - \frac{p_l \cdot k}{p_l \cdot p_m} p_m \right). \tag{3.23}$$

They satisfy the following equations

$$\delta p_{l,m} + \delta p_{m,l} = -k, \quad (p_l \pm \delta p_{l,m})^2 = \pm 2 p_l \cdot \delta p_{l,m} + \mathcal{O}(k^2) = \mathcal{O}(k^2). \tag{3.24}$$

These equations ensure that with the next-to-soft accuracy, all momenta that appear in the matrix element $\mathcal{M}_0$ in Eq. (3.22) are on-shell, and that the momentum conservation is satisfied, provided that equation

$$p_a + p_b = p_{\mathfrak{m}} + \tilde{p}_\gamma + k, \tag{3.25}$$

holds.

According to Eq. (3.17), we need to compute the matrix element for boosted and rescaled momenta. We will make use of the fact that the mass dimension of the $|\mathcal{M}_0|^2$ is zero (see Eq. (3.8)), and that it is boost-invariant. Then, the following equation holds

$$|\mathcal{M}_0(p_a, p_b, \lambda\Lambda_s^{-1}p_j, \lambda\Lambda_s^{-1}p_\gamma)|^2 = |\mathcal{M}_0(\lambda^{-1}\Lambda_s p_a, \lambda^{-1}\Lambda_s p_b, p_j, p_\gamma)|^2. \tag{3.26}$$

It is easy to check, using explicit formula for the boost and the rescaling that

$$\lambda^{-1}\Lambda_s p_a = p_a - \delta p_{a,b}, \quad \lambda^{-1}\Lambda_s p_b = p_b - \delta p_{b,a}, \tag{3.27}$$

where $\delta p_{a,b}$ and $\delta p_{b,a}$ are defined in Eq. (3.23). Then, through next-to-soft terms, Eq. (3.22) becomes

$$\begin{aligned}
&g_s^{-2}|\mathcal{M}|^2(p_a, p_b, \lambda\Lambda_s^{-1}p_j, k, \lambda\Lambda_s^{-1}p_\gamma) \approx \\
&\left(C_F - \frac{C_A}{2}\right)\frac{2p_a\cdot p_b}{p_a\cdot k\, p_b\cdot k}|\mathcal{M}_0(p_a,p_b,p_j,p_\gamma)|^2 \\
&+\frac{C_A}{2}\frac{2p_a\cdot\lambda\Lambda_s^{-1}p_j}{p_a\cdot k\, p_j\cdot k}|\mathcal{M}_0(p_a-\delta p_{a,b}+\delta p_{a,j}, p_b-\delta p_{b,a}, p_j-\delta p_{j,a}, p_\gamma)|^2 \\
&+\frac{C_A}{2}\frac{2p_b\cdot\lambda\Lambda_s^{-1}p_j}{p_b\cdot k\, p_j\cdot k}|\mathcal{M}_0|^2(p_a-\delta p_{a,b}, p_b-\delta p_{b,a}+\delta p_{b,j}, p_j-\delta p_{j,b}, p_\gamma)|^2,
\end{aligned} \tag{3.28}$$

where we have used the fact that $k\cdot(\lambda\Lambda_s^{-1}p_j) = k\cdot p_j$ with the required accuracy, c.f. Eq. (3.20). We stress that momenta in Eq. (3.28) satisfy the leading-order energy-momentum conservation equation

$$p_a + p_b = p_j + p_\gamma. \tag{3.29}$$

Furthermore, we note a peculiar fact that the momenta transformations removed the next-to-soft correction from the $(a,b)$ dipole, whereas such corrections do remain in the $(a,j)$ and $(b,j)$ dipoles.

Eq. (3.28) provides a suitable starting point for computing the required expansion of the real-emission squared matrix element in the soft limit through subleading power. We use explicit form of the leading-order matrix element squared Eq. (3.8) and find

$$\frac{\omega_k^2}{g_s^2}\,|\mathcal{M}|^2(p_a,p_b,\lambda\Lambda_s^{-1}p_j,k,\lambda\Lambda_s^{-1}p_\gamma) \approx |\mathcal{M}_0|^2(p_a,p_b,p_j,p_\gamma)\left(S_1(\vec{n}_k) + \frac{\omega_k}{\sqrt{s}}\,S_2(\vec{n}_k)\right), \tag{3.30}$$

where

$$\begin{aligned}
S_1(\vec{n}_k) &= \left(C_F - \frac{C_A}{2}\right)\frac{2\rho_{ab}}{\rho_{ak}\rho_{bk}} + \frac{C_A}{2}\left(\frac{2\rho_{aj}}{\rho_{ak}\rho_{jk}} + \frac{2\rho_{bj}}{\rho_{bk}\rho_{jk}}\right), \\
S_2(\vec{n}_k) &= C_A\left(\frac{\rho_{ab}}{\rho_{ak}\rho_{bk}} - \frac{\rho_{aj}}{\rho_{ak}\rho_{jk}} - \frac{\rho_{bj}}{\rho_{bk}\rho_{jk}} - \frac{2}{\rho_{jk}}\right),
\end{aligned} \tag{3.31}$$

and $\rho_{xy} = 1 - \vec{n}_x\cdot\vec{n}_y$. We stress that in deriving Eq. (3.30) no $\epsilon$-dependent terms have been neglected.

We continue with the discussion of a generic observable $\mathcal{O}$. We remind the reader that $\mathcal{O}$ contains the constraint on the jet transverse momentum, according to our convention. To find how the observable is affected by the momenta transformation, we note that according to Eq. (3.22), to compute the jet momentum we always need to transform the harder gluon $\mathfrak{m}$, and then either combine it with a softer gluon or not. Since

$$\lambda\Lambda_s^{-1}p_j = \left(1 - \frac{P_{ab}\cdot k}{P_{ab}^2}\right)p_j - k\frac{P_{ab}\cdot p_j}{P_{ab}^2} + P_{ab}\frac{k\cdot p_j}{P_{ab}^2}, \tag{3.32}$$

and $P_{ab}\cdot p_j/P_{ab}^2 = 1/2$, we find the following result for the two cases

$$\begin{aligned} &\bullet \text{ clustered}: \quad q_j = p_{\mathfrak{m}} + p_{\mathfrak{n}} = \lambda\Lambda_s^{-1}p_j + k = \left(1 - \frac{P_{ab}\cdot k}{P_{ab}^2}\right)p_j + \frac{1}{2}k + P_{ab}\frac{k\cdot p_j}{P_{ab}^2}, \\ &\bullet \text{ not clustered}: \quad q_j = p_{\mathfrak{m}} = \lambda\Lambda_s^{-1}p_j = \left(1 - \frac{P_{ab}\cdot k}{P_{ab}^2}\right)p_j - \frac{1}{2}k + P_{ab}\frac{k\cdot p_j}{P_{ab}^2}. \end{aligned} \tag{3.33}$$

We can now expand the observable to the desired order in the soft approximation

$$\begin{aligned} \mathcal{O}(q_j,\tilde{p}_\gamma) = \mathcal{O}(p_j,p_\gamma) &+ \sum_{x\in j,\gamma}\left(-\frac{P_{ab}\cdot k}{P_{ab}^2}p_x^\mu - \frac{1}{2}k^\mu + \frac{k\cdot p_x}{P_{ab}^2}P_{ab}^\mu\right)\partial_{p_x,\mu}\mathcal{O}(p_j,p_\gamma) \\ &+ \theta(R - R_{jk})\, k^\mu\partial_{p_j,\mu}\mathcal{O}(p_j,p_\gamma) + \mathcal{O}(k^2). \end{aligned} \tag{3.34}$$

We emphasize that when gluons are clustered, the square of the jet momentum $q_j^2 \neq 0$ whereas $p_j^2 = 0$. Hence, when computing the derivative $\partial_{p_j}$ on the right-hand side of the above equation, one should write the definition of the observable *without assuming* $p_j^2 = 0$, calculate the derivative, and take the $p_j^2 \to 0$ limit after that. This remark concerns, in particular, the dependence of the observable $\mathcal{O}$ on the transverse momentum of the jet.[8]

It is useful to rewrite Eq. (3.34) separating the energy of the gluon $\omega_k$ from its direction. We will work in the center-of mass frame of the partonic collision, where partons $a, b$ are back-to-back and have equal energies. We find

$$\begin{aligned} \mathcal{O}(q_j,\tilde{p}_\gamma) = \mathcal{O}(p_j,p_\gamma) + \frac{\omega_k}{\sqrt{s}}\Bigg[&\sum_{x\in j,\gamma}\left(-p_x^\mu - \frac{\sqrt{s}}{2}\hat{k}^\mu + \frac{\rho_{kx}}{2}\,P_{ab}^\mu\right)\partial_{p_x,\mu} \\ &+ \theta(R - R_{jk})\,\sqrt{s}\,\hat{k}^\mu\partial_{p_j,\mu}\Bigg]\mathcal{O}(p_j,p_\gamma), \end{aligned} \tag{3.35}$$

where $\hat{k}^\mu = (1,\vec{n}_k)$.

**Modification of the angular distance in the jet algorithm** Similarly to the matrix element and the observable, in Eq. (3.17) we need to expand the $\Theta(R_{\mathfrak{mn}}, q_j)$-function that determines whether the gluons are clustered into a jet or not. The power correction in this case is actually finite; for this reason it is useful to consider it separately.

[8] The transverse momentum of the jet can be defined through the following equation $p_{j,\perp} = \sqrt{2(p_ap_j)(p_bp_j)/(p_ap_b) - p_j^2}$.

The original $\eta - \varphi$ distance refers to partons $\mathfrak{m}$ and $\mathfrak{n}$. We have identified $\mathfrak{n}$ with $k$, but the momentum of $\mathfrak{m}$ is expressed through the (large) momentum $p_j$ and additional $k$-dependent terms. Hence, as explained in Appendix B, in the center-of-mass frame of the colliding partons $p_{a,b}$, the following relation holds

$$R_{\mathfrak{m}\mathfrak{n}} = R_{jk} + \frac{\omega_k}{\sqrt{s}}\, \mathcal{R}_{jka}, \tag{3.36}$$

where $\omega_k \sim \tau$ is the (small) energy of the gluon $k$. Indices of the function $\mathcal{R}$ indicate that it depends on $\vec{n}_j, \vec{n}_k$ and $\vec{n}_a$. Explicitly, this function reads

$$\mathcal{R}_{jka} = \frac{1}{\sin^2\theta_j}[\vec{n}_k \times \vec{n}_j] \cdot \left( \frac{\partial R_{jk}}{\partial \varphi_j}\, \vec{n}_a - \frac{\partial R_{jk}}{\partial \eta_j}\, [\vec{n}_a \times \vec{n}_j] \right), \tag{3.37}$$

where $\theta_j$ and $\varphi_j$ are the polar and azimuthal angles of the parton $j$. The derivation of Eq. (3.37) is provided in Appendix B. It follows that

$$\Theta(R_{\mathfrak{m}\mathfrak{n}}, q_j) = \Theta(R_{jk}, q_j) + \frac{\omega_k}{\sqrt{s}}\, R_{jka}\, \delta(R - R_{jk}) \left( \delta_{q_j,p_j} - \delta_{q_j,p_{[\mathfrak{m}\mathfrak{n}]}} \right) + \mathcal{O}(\tau^2). \tag{3.38}$$

The first term on the the right-hand side of Eq. (3.38) is not power-suppressed; it will have to be combined with corrections to the matrix element, the observable and the phase space. Therefore, this term will contribute both at leading and at next-to-leading order in the expansion in $\tau$.

On the contrary, the $\mathcal{O}(\omega_k/\sqrt{s})$ term in Eq. (3.38) is already power-suppressed; it involves two contributions with opposite signs which depend on whether the two gluons are clustered into a jet or not. Since this is a power-suppressed contribution already, the clustering issue is only relevant for the *jettiness function*, where the difference between the two cases in the soft limit is a *leading order* effect.

Therefore, the power correction that originates from the expansion of $R_{\mathfrak{m}\mathfrak{n}}$ in Eq. (3.38) reads

$$\begin{aligned} \frac{\mathrm{d}\sigma_{\mathcal{O}}^{s,R}}{\mathrm{d}\tau} &= g_s^2\, \mathcal{N}^{-1} \int \mathrm{d}\Phi_{j\gamma}^{ab}\, [\mathrm{d}k]\, |\mathcal{M}_0|^2(p_a, p_b; p_j, p_\gamma)\, \mathcal{O}(p_j, p_\gamma)\, \delta(R - R_{jk}) \\ &\times S_1(\vec{n}_k)\, \frac{1}{\omega_k \sqrt{s}}\, \mathcal{R}_{jka}\, \left( \delta(\tau - \omega_k \psi_{\mathrm{nc}}(\vec{n}_k)) - \delta(\tau - \omega_k \psi_{\mathrm{c}}(\vec{n}_k)) \right), \end{aligned} \tag{3.39}$$

where the functions $\psi_{\mathrm{nc,c}}$ refer to non-clustered and clustered definitions of the one-jettiness function, respectively. They read

$$\psi_{\mathrm{nc}} = \min\left\{ \frac{2E_a\rho_{ak}}{P_a}, \frac{2E_b\rho_{bk}}{P_b}, \frac{2E_j\rho_{jk}}{P_J} \right\}, \quad \psi_{\mathrm{c}} = \psi_{\mathrm{nc}} + \frac{2E_j\rho_{jk}}{P_J}. \tag{3.40}$$

Integrating over $\omega_k$ in Eq. (3.39), taking the $\epsilon \to 0$ limit and separating the integration over directions of the vector $\vec{k}$, we obtain

$$\frac{\mathrm{d}\sigma_{\mathcal{O}}^{s,R}}{\mathrm{d}\tau} = \frac{\alpha_s}{2\pi} \bar{\sigma}_0 \int \mathrm{d}\Phi_{j\gamma}^{ab}\, \frac{1 - 2\beta + 2\beta^2}{2\beta(1-\beta)}\, \mathcal{O}(p_j, p_\gamma)\, \mathcal{F}_R(\vec{n}_a, \vec{n}_b, \vec{n}_j), \tag{3.41}$$

where

$$\mathcal{F}_R = \frac{1}{\sqrt{s}} \int \frac{\mathrm{d}\Omega_k}{2\pi}\, \delta(R - R_{jk})\, S_1(\vec{n}_k)\, \mathcal{R}_{jka} \left( \frac{1}{\psi_{\rm nc}(\vec{n}_k)} - \frac{1}{\psi_{\rm c}(\vec{n}_k)} \right). \tag{3.42}$$

We have mentioned above that, when writing Eq. (3.41), we have taken the $\epsilon \to 0$ limit. To justify this step, we note that the $\delta$-function $\delta(R - R_{jk})$ in Eq. (3.42) depends on the polar and azimuthal angles of the gluon $k$, and the integration over directions of the gluon momentum cannot produce collinear singularities. To perform it, we integrate first over the gluon azimuthal angle $\varphi_k$ and find

$$\begin{aligned} \mathcal{F}_R = \frac{R}{2\pi\sqrt{s}} \int\limits_{-1}^{1} \frac{\mathrm{d}\cos\theta_k}{\sqrt{R^2 - (\eta_k - \eta_j)^2}} \sum_{\alpha=1}^{2} S_1(\vec{n}_k^{(\alpha)}) \\ \times \mathcal{R}_{jka}(\vec{n}_a, \vec{n}_k^{(\alpha)}, \vec{n}_j) \left( \frac{1}{\psi_{\rm nc}(\vec{n}_k^{(\alpha)})} - \frac{1}{\psi_{\rm c}(\vec{n}_k^{(\alpha)})} \right) \theta\left( \pi - \sqrt{R^2 - (\eta_k - \eta_j)^2} \right). \end{aligned} \tag{3.43}$$

The sum runs over two solutions for the azimuthal angle of the vector $\vec{n}_k$. The parametrization reads

$$\vec{n}_k^{(\alpha)} = (\sin\theta_k \cos\varphi_k^{(\alpha)}, \sin\theta_k \sin\varphi_k^{(\alpha)}, \cos\theta_k), \tag{3.44}$$

where

$$\varphi_k^{(\alpha)} = \mathrm{mod}(\varphi_j \pm \sqrt{R^2 - (\eta_k - \eta_j)^2}, 2\pi). \tag{3.45}$$

**Remaining contributions** The remaining soft contributions involve modifications of the matrix element, the phase space and the observables, but not the angular measure of the clustering algorithm. Putting everything together, we write the result in the following way

$$\begin{aligned} \frac{\mathrm{d}\sigma^s_{\mathcal{O}}}{\mathrm{d}\tau} = \mathcal{N}^{-1}[\alpha_s] \int \mathrm{d}\Phi^{ab}_{\gamma j}\, \frac{\mathrm{d}\omega_k}{\omega_k^{1+2\epsilon}}\, [\mathrm{d}\Omega_k]\, |\mathcal{M}_0|^2(p_a, p_b; p_j, p_\gamma)\, \Theta(R_{jk}, q_j)\, \delta(\tau - \mathcal{T}(q_j)) \\ \times \Bigg[ S_1(\vec{n}_k) \Big( 1 + \frac{\omega_k}{\sqrt{s}} \Big\{ 2\epsilon - \sum_{x \in j,\gamma} (p_x^\mu + \frac{\sqrt{s}}{2}\hat{k}^\mu - \frac{\rho_{kx}}{2} P^\mu_{ab}) \partial_{p_x,\mu} \\ + \theta(R - R_{jk}) \sqrt{s} \hat{k}^\mu\, \partial_{p_j,\mu} \Big\} \Big) \mathcal{O}(p_j, p_\gamma) + \frac{\omega_k}{\sqrt{s}} S_2(\vec{n}_k) \mathcal{O}(p_j, p_\gamma) \Bigg], \end{aligned} \tag{3.46}$$

where all terms beyond next-to-leading-power corrections in $\tau$ have been omitted. Similarly to what has been discussed earlier, we integrate over the gluon energy $\omega_k$ using the fact that the one-jettiness function is linear in it, c.f. Eq. (3.39). However, instead of employing functions $\psi_{\rm c,nc}$ introduced earlier, we write

$$\mathcal{T}(q_j) = \omega_k \psi_k(\vec{n}_k), \tag{3.47}$$

which allows us to proceed without indicating whether we deal with the clustered or the unclustered case, until later. Integrating over $\omega_k$, we find

$$\begin{aligned}\frac{\mathrm{d}\sigma^s_{\mathcal{O}}}{\mathrm{d}\tau} &= \mathcal{N}^{-1}\frac{[\alpha_s]}{\tau^{1+2\epsilon}}\int \mathrm{d}\Phi^{ab}_{\gamma j}[\mathrm{d}\Omega_k]\ \psi_k^{2\epsilon}(\vec{n}_k)|\mathcal{M}_0|^2(p_a,p_b;p_j,p_\gamma)\ \Theta(R_{jk},q_j)\\ &\times\Bigg[S_1(\vec{n}_k)\Big(1+\frac{\tau}{\sqrt{s}\ \psi_k(\vec{n}_k)}\ \Big\{2\epsilon\ -\sum_{x\in j,\gamma}\ (p_x^\mu+\frac{\sqrt{s}}{2}\hat{k}^\mu-\frac{\rho_{kx}}{2}P^\mu_{ab})\partial_{p_x,\mu}\\ &+\theta(R-R_{jk})\sqrt{s}\hat{k}^\mu\ \partial_{p_j,\mu}\Big\}\Big)\mathcal{O}(p_j,p_\gamma)+\frac{\tau}{\sqrt{s}\ \psi_k(\vec{n}_k)}S_2(\vec{n}_k)\mathcal{O}(p_j,p_\gamma)\Bigg].\end{aligned} \tag{3.48}$$

It remains to integrate Eq. (3.48) over $\vec{n}_k$. This step is non-trivial because the integration is divergent, and the structure of divergences depends on whether the gluons have been clustered in a jet or not.

Indeed, if the clustering happens, the function $\Theta(R_{jk},q_j)$ restricts the integration over the gluon angle to the region around $\vec{n}_k||\vec{n}_j$, which implies that this is the only direction that can cause collinear singularities in this case. On the contrary, if no clustering occurs, the only possible collinear configurations are $\vec{n}_k||\vec{n}_a$ and $\vec{n}_k||\vec{n}_b$.

To extract the singularities, and to re-write the integral in Eq. (3.48) in such a way that non-trivial integrations can be performed in three dimensions, we employ the methodology of local subtractions. Although it is fairly straightforward to construct the subtraction terms, the present case is somewhat special because in the subleading terms the collinear singularities are *power-like*. To see this, we note that in Eq. (3.48) functions $S_{1,2}(\vec{n}_k)$ have linear singularities in the collinear limits (c.f. Eq. (3.31)), leading to usual logarithmic singularities when the integration over directions of $\vec{n}_k$ is performed. However, in the subleading terms these singularities are further amplified by singularities caused by the presence of the function $1/\psi_k$ in the integrand. We explain below how suitable subtraction terms can be constructed in this case as well.

**The clustered case** We begin by considering the clustered case where the only singular direction is $\vec{n}_k||\vec{n}_j$. To subtract this singularity, we need to expand the integrand in Eq. (3.48) around this limit. Since the one-jettiness function, as well as the function $\theta(R-R_{jk})$ do not change in the vicinity of the collinear limit, we can replace them with their limiting values, $\psi_k\to 2E_j\rho_{jk}/P_J$ and $\theta(R-R_{jk})\to 1$, for the purpose of subtracting *both* leading- and next-to-leading collinear singularities. It remains to expand $S_{1,2}(\vec{n}_k)$ through *constant* terms in the $\rho_{jk}\to 0$ limit. We find

$$\begin{aligned}S_1(\vec{n}_k) &= \frac{2C_A}{\rho_{kj}}+\frac{C_F}{\beta(1-\beta)}+\frac{C_A\epsilon}{1-\epsilon}\frac{(1-2\beta+2\beta^2)}{\beta(1-\beta)}+\mathcal{O}(\rho_{jk}),\\ S_2(\vec{n}_k) &= -\frac{4C_A}{\rho_{kj}}-\frac{C_A\epsilon}{1-\epsilon}\ \frac{(1-2\beta+2\beta^2)}{\beta(1-\beta)}+\mathcal{O}(\rho_{jk}).\end{aligned} \tag{3.49}$$

To obtain these expressions, we constructed the expansion of $S_{1,2}$ in $\rho_{jk}$, and integrated the obtained expressions over components of $\vec{n}_k$, which are transversal to the direction of $\vec{n}_j$. While one can perform this integration in the subtraction term, when the difference of the integrand in Eq. (3.48) and the subtraction term is constructed, one needs to keep the dependence on the transverse components of $\vec{n}_k$ to make sure that all singularities of the integrand are removed *locally*.

Similarly, we need to write the vector $\hat{k}$ by separating its component along $\vec{n}_j$ from the transversal ones. We find

$$\frac{\sqrt{s}}{2}\hat{k}^\mu = (1-\rho_{jk})p_j^\mu + \frac{\rho_{jk}}{2}P_{ab}^\mu + \frac{\sqrt{s\rho_{jk}(2-\rho_{jk})}}{2}\hat{k}_\perp^\mu, \tag{3.50}$$

where $\hat{k}_\perp^\mu$ is a four-dimensional unit vector ( $\hat{k}_\perp^\mu \hat{k}_{\perp,\mu} = -1$), orthogonal to $\vec{n}_j$. Using this decomposition, we obtain

$$\begin{aligned} &S_1(\vec{n}_k)\Big(\sum_{x\in j,\gamma}(p_x^\mu + \frac{\sqrt{s}}{2}\hat{k}^\mu - \frac{\rho_{kx}}{2}P_{ab}^\mu)\partial_{p_x,\mu} - \sqrt{s}\hat{k}^\mu\,\partial_{p_j,\mu}\Big) = \\ &-C_A\left\{2p_\gamma^\mu - \frac{1-2\beta}{2(1-\epsilon)}\left[\frac{p_a^\mu}{\beta} - \frac{p_b^\mu}{(1-\beta)} - \frac{(1-2\beta)}{\beta(1-\beta)}\,p_\gamma^\mu\right]\right\}(\partial_{p_j,\mu} - \partial_{p_\gamma,\mu}) + \mathcal{O}(\rho_{jk}), \end{aligned} \tag{3.51}$$

where we have expanded $S_1(\vec{n}_k)$ around the $\vec{n}_k||\vec{n}_j$ limit and have averaged over directions of $k_\perp^\mu$.

We use Eqs (3.49,3.51) to write the linear power correction, that originates from the clustered case, in the following way

$$\begin{aligned} \frac{\mathrm{d}\sigma_{\mathcal{O}}^{s,\mathrm{NLP}}}{\mathrm{d}\tau}\Big|_{\mathrm{cl}} =& \frac{\mathcal{N}^{-1}[\alpha_s]}{\sqrt{s}\tau^{2\epsilon}}\left(\frac{P_J}{2\sqrt{s}}\right)^{1-2\epsilon}\mathrm{d}\Phi_{\gamma j}^{ab}\,|\mathcal{M}_0(p_a,p_b,p_j,p_\gamma)|^2 \\ &\times\left(S_c + \int[\mathrm{d}\Omega_k]\,\mathcal{F}_k^{\mathrm{cl}}\right)\mathcal{O}(p_j,p_\gamma). \end{aligned} \tag{3.52}$$

In Eq. (3.52) $S_c$ is the integrated subtraction term given by

$$\begin{aligned} S_c =& \frac{2C_F}{\beta(1-\beta)} - \frac{C_A(1-2\beta+2\beta^2)}{\beta(1-\beta)} \\ &+ \frac{C_A}{\epsilon}\left\{2p_\gamma^\mu - \frac{1-2\beta}{2(1-\epsilon)}\left[\frac{p_a^\mu}{\beta} - \frac{p_b^\mu}{(1-\beta)} - \frac{(1-2\beta)}{\beta(1-\beta)}\,p_\gamma^\mu\right]\right\}(\partial_{p_j,\mu} - \partial_{p_\gamma,\mu})\,, \end{aligned} \tag{3.53}$$

and

$$\begin{aligned} \mathcal{F}_k^{\mathrm{cl}} =& \frac{2\sqrt{s}\,\theta(R-R_{jk})}{P_J\,\psi_k}\Bigg[S_2(\vec{n}_k) \\ &- S_1(\vec{n}_k)\left((p_j^\mu - \frac{\sqrt{s}}{2}\hat{k}^\mu - \frac{\rho_{jk}}{2}P_{ab}^\mu)\partial_{p_j,\mu} + (p_\gamma^\mu + \frac{\sqrt{s}}{2}\hat{k}^\mu - \frac{\rho_{\gamma k}}{2}P_{ab}^\mu)\partial_{p_\gamma,\mu}\right)\Bigg] \\ &- \frac{C_A}{\rho_{jk}}\left[-\frac{4}{\rho_{jk}} + \frac{(1-2\beta)}{2\beta(1-\beta)}\frac{\sqrt{2}}{\sqrt{\rho_{jk}}}(\vec{n}_{k,\perp}\cdot\vec{n}_a) + \frac{(1-2\beta+2\beta^2)}{\beta(1-\beta)}\left(1 - \frac{(\vec{n}_{k,\perp}\cdot\vec{n}_a)^2}{2\beta(1-\beta)}\right)\right] \\ &- \frac{C_A}{\rho_{jk}}\left[2p_\gamma^\mu - \frac{(1-2\beta)}{2\beta(1-\beta)}(\vec{n}_{k,\perp}\cdot\vec{n}_a)\sqrt{s}\,\hat{k}_\perp^\mu + \frac{\sqrt{2s}}{\sqrt{\rho_{jk}}}\hat{k}_\perp^\mu\right](\partial_{p_j,\mu} - \partial_{p_\gamma,\mu})\,. \end{aligned} \tag{3.54}$$

The transverse vector $\vec{n}_{k,\perp}$, which parametrizes spatial components of the four-vector $k_\perp$, is defined by the following equation

$$\vec{n}_k = (1-\rho_{jk})\vec{n}_j + \sqrt{\rho_{jk}(2-\rho_{jk})}\,\vec{n}_{k,\perp}, \tag{3.55}$$

with $\vec{n}_{k,\perp}\cdot\vec{n}_j = 0$. Integration over directions of the vector $\vec{n}_k$ in Eq. (3.52) is finite and, therefore, is performed in the three-dimensional space.

**The non-clustered case.** We continue with the non-clustered case, where the singularities arise if $\vec{n}_k||\vec{n}_a$ or $\vec{n}_k||\vec{n}_b$. We construct the expansion of the integrand in Eq. (3.48) around these limits, closely following what has been done in the clustered case. To this end, we again replace $\theta(R-R_{jk}) \to 1$, and $\psi_k(\vec{n}_k) \to 2E_x\rho_{xk}/P_x$ where $x \in (a,b)$, depending on the collinear limit that we consider.

We first construct the expansion of the integrand for the $\vec{n}_k||\vec{n}_a$ case. The expansion of the functions $S_{1,2}$ for $\vec{n}_k||\vec{n}_a$ reads

$$\begin{aligned} S_1(\vec{n}_k) &= \frac{2C_F}{\rho_{ak}} + C_F + \frac{C_A}{1-\epsilon}\,\frac{\beta}{1-\beta} + \mathcal{O}(\rho_{ak}), \\ S_2(\vec{n}_k) &= -\frac{C_A}{1-\epsilon}\,\frac{(1-\epsilon+\beta)}{1-\beta} + \mathcal{O}(\rho_{ak}), \end{aligned} \tag{3.56}$$

where we have averaged over directions of the vector $\vec{n}_k$ which are orthogonal to $\vec{n}_a$. It is peculiar that $S_2(\vec{n}_k)$ does not have the corresponding collinear singularity.

We also need to re-write the vector $\hat{k}$ that appears in Eq. (3.48), separating its components orthogonal to the collision axis, and averaging over their directions. To this end, we write

$$\frac{\sqrt{s}}{2}\hat{k}^\mu = (1-\rho_{ak})p_a^\mu + \frac{\rho_{ak}}{2}P_{ab}^\mu + \frac{\sqrt{s\rho_{ak}(2-\rho_{ak})}}{2}k_\perp^\mu, \tag{3.57}$$

where $k_\perp^\mu$ is related to the spatial direction perpendicular to $\vec{n}_a$. With this we get

$$\begin{aligned} &S_1(\vec{n}_k)\sum_{x\in j,\gamma}\left(p_x^\mu + \frac{\sqrt{s}}{2}\hat{k}^\mu - \frac{\rho_{kx}}{2}P_{ab}^\mu\right)\partial_{p_x,\mu} = \\ &\frac{2C_F}{\rho_{ak}}\left[\left(p_j^\mu + \beta p_a^\mu - \bar\beta p_b^\mu\right)\partial_{p_j\mu} + \left(p_\gamma^\mu + \bar\beta p_a^\mu - \beta p_b^\mu\right)\partial_{p_\gamma\mu}\right] \\ &+\left\{\left[\frac{C_A}{2(1-\epsilon)\bar\beta}\left(p_j^\mu(1+2\beta) + \beta p_a^\mu - \bar\beta p_b^\mu\right) + C_F\left(p_j^\mu - \beta p_a^\mu + \bar\beta p_b^\mu\right)\right]\partial_{p_j\mu}\right. \\ &\left.+\left[C_F\left(p_\gamma^\mu - \bar\beta p_a^\mu + \beta p_b^\mu\right) - \frac{C_A}{2(1-\epsilon)\bar\beta}\left(p_\gamma^\mu(1-2\beta) - \bar\beta p_a^\mu + \beta p_b^\mu\right)\right]\partial_{p_\gamma\mu}\right\} + \mathcal{O}(\rho_{ak}), \end{aligned} \tag{3.58}$$

where we introduced the short-hand notation $\bar\beta = 1-\beta$.

The last potential collinear singularity to consider is $\vec{n}_k||\vec{n}_b$. The analysis in this case follows steps discussed in connection with $\vec{n}_k||\vec{n}_j$ and $\vec{n}_k||\vec{n}_a$ singularities. We find

$$\begin{aligned} S_1(\vec{n}_k) &= \frac{2C_F}{\rho_{bk}} + C_F + \frac{C_A}{1-\epsilon}\,\frac{1-\beta}{\beta} + \mathcal{O}(\rho_{bk}), \\ S_2(\vec{n}_k) &= -\frac{C_A}{1-\epsilon}\,\frac{(2-\beta-\epsilon)}{\beta} + \mathcal{O}(\rho_{bk}), \end{aligned} \tag{3.59}$$

where we have averaged over directions of the vector $k$ orthogonal to the collision axis. Writing momentum $\hat{k}$ in terms of $p_b$ and the transverse component,

$$\frac{\sqrt{s}}{2}\hat{k}^\mu = (1-\rho_{bk})p_b^\mu + \frac{\rho_{bk}}{2}P_{ab}^\mu + \frac{\sqrt{s\rho_{bk}(2-\rho_{bk})}}{2}k_\perp^\mu, \tag{3.60}$$

and averaging over directions of $k_\perp$, we find

$$\begin{aligned}
&S_1(\vec{n}_k)\sum_{x\in j,\gamma}\left(p_x^\mu + \frac{\sqrt{s}}{2}\hat{k}^\mu - \frac{\rho_{kx}}{2}P_{ab}^\mu\right)\partial_{p_x,\mu} = \\
&\frac{2C_F}{\rho_{bk}}\left[\left(p_j^\mu - \beta p_a^\mu + \bar{\beta}p_b^\mu\right)\partial_{p_j,\mu} + \left(p_\gamma^\mu - \bar{\beta}p_a^\mu + \beta p_b^\mu\right)\partial_{p_\gamma,\mu}\right] \\
&+\left\{\left[\frac{C_A}{2(1-\epsilon)\beta}\left(p_j^\mu(3-2\beta) - \beta p_a^\mu + \bar{\beta}p_b^\mu\right) + C_F\left(\beta p_a^\mu - \bar{\beta}p_b^\mu + p_j^\mu\right)\right]\partial_{p_j,\mu}\right. \\
&\left.+\left[C_F\left(\bar{\beta}p_a^\mu - \beta p_b^\mu + p_\gamma^\mu\right) + \frac{C_A}{2(1-\epsilon)\beta}\left(p_\gamma^\mu(1+2\beta) - \bar{\beta}p_a^\mu + \beta p_b^\mu\right)\right]\partial_{p_\gamma,\mu}\right\} + \mathcal{O}(\rho_{bk}).
\end{aligned} \tag{3.61}$$

We use the above results to write the linear power correction to the non-clustered contribution in the following way

$$\begin{aligned}
\frac{\mathrm{d}\sigma_{\mathcal{O}}^{s,\mathrm{NLP}}}{\mathrm{d}\tau}\Big|_{\mathrm{nc}} &= \frac{\mathcal{N}^{-1}[\alpha_s]}{\sqrt{s}\tau^{2\epsilon}}\left(\frac{1}{\sqrt{s}}\right)^{1-2\epsilon}\mathrm{d}\Phi_{\gamma j}^{ab}\,|\mathcal{M}_0(p_a,p_b,p_j,p_\gamma)|^2 \\
&\quad\times\left(P_a^{1-2\epsilon}S_{ca} + P_b^{1-2\epsilon}S_{cb} + \int[\mathrm{d}\Omega_k]\,\mathcal{F}_k^{\mathrm{nc}}\right)\mathcal{O}(p_j,p_\gamma).
\end{aligned} \tag{3.62}$$

The terms $S_{ca}$ and $S_{cb}$ describe the integrated subtraction term for the limit $\vec{n}_k||\vec{n}_a$ and $\vec{n}_k||\vec{n}_b$ respectively; all the $1/\epsilon$ divergences are collected there. These terms read

$$\begin{aligned}
S_{ca} &= 2C_F + C_A\frac{\beta}{\bar{\beta}} - \frac{C_A}{\epsilon}\frac{(1+\beta)}{\bar{\beta}} \\
&-\frac{1}{\epsilon}\left\{\left[\frac{C_A}{2(1-\epsilon)\bar{\beta}}\left(p_j^\mu(1+2\beta) + \beta p_a^\mu - \bar{\beta}p_b^\mu\right) + C_F\left(p_j^\mu - \beta p_a^\mu + \bar{\beta}p_b^\mu\right)\right]\partial_{p_j,\mu}\right. \\
&\left.+\left[C_F\left(p_\gamma^\mu - \bar{\beta}p_a^\mu + \beta p_b^\mu\right) - \frac{C_A}{2(1-\epsilon)\bar{\beta}}\left(p_\gamma^\mu(1-2\beta) - \bar{\beta}p_a^\mu + \beta p_b^\mu\right)\right]\partial_{p_\gamma,\mu}\right\},
\end{aligned} \tag{3.63}$$

and

$$\begin{aligned}
S_{cb} &= 2C_F + C_A\frac{\bar{\beta}}{\beta} - \frac{C_A}{\epsilon}\frac{(2-\beta)}{\beta} \\
&-\frac{1}{\epsilon}\left\{\left[\frac{C_A}{2(1-\epsilon)\beta}\left(p_j^\mu(3-2\beta) - \beta p_a^\mu + \bar{\beta}p_b^\mu\right) + C_F\left(p_j^\mu + \beta p_a^\mu - \bar{\beta}p_b^\mu\right)\right]\partial_{p_j,\mu}\right. \\
&\left.+\left[C_F\left(p_\gamma^\mu + \bar{\beta}p_a^\mu - \beta p_b^\mu\right) + \frac{C_A}{2(1-\epsilon)\beta}\left(p_\gamma^\mu(1-2\beta) - \bar{\beta}p_a^\mu + \beta p_b^\mu\right)\right]\partial_{p_\gamma,\mu}\right\}.
\end{aligned} \tag{3.64}$$

The remaining integral over directions of the vector $\vec{k}$ in Eq. (3.62) is finite, and can be performed in three dimensions. The function $\mathcal{F}_k^{\rm nc}$ reads

$$\begin{aligned}\mathcal{F}_k^{\rm nc} = &\frac{\sqrt{s}\,\theta(R_{jk}-R)}{\psi_k}\Bigg[S_2(\vec{n}_k) - S_1(\vec{n}_k)\sum_{x\in j,\gamma}(p_x^\mu + \frac{\sqrt{s}}{2}\hat{k}^\mu - \frac{\rho_{xk}}{2}P_{ab}^\mu)\,\partial_{p_x,\mu}\Bigg]\\ &-\Bigg\{\frac{P_a C_A}{\rho_{ak}}\Bigg[-\frac{1}{2\bar{\beta}}\frac{\sqrt{2}}{\sqrt{\rho_{ak}}}(\vec{n}_{k,\perp a}\cdot\vec{n}_j) - \frac{1}{\bar{\beta}}\left(1+\frac{(\vec{n}_{k,\perp a}\cdot\vec{n}_j)^2}{2\bar{\beta}}\right)\Bigg]\\ &+\frac{P_b C_A}{\rho_{bk}}\Bigg[-\frac{1}{2\beta}\frac{\sqrt{2}}{\sqrt{\rho_{bk}}}(\vec{n}_{k,\perp b}\cdot\vec{n}_j) - \frac{1}{\beta}\left(1+\frac{(\vec{n}_{k,\perp b}\cdot\vec{n}_j)^2}{2\beta}\right)\Bigg]\Bigg\}\\ &+\frac{P_a}{\rho_{ak}}\left\{S^\mu_{a,p_j}\partial_{p_j,\mu} + S^\mu_{a,p_\gamma}\partial_{p_\gamma,\mu}\right\} + \frac{P_b}{\rho_{bk}}\left\{S^\mu_{a,p_j}\partial_{p_j,\mu} + S^\mu_{a,p_\gamma}\partial_{p_\gamma,\mu}\right\}_{\substack{\beta\leftrightarrow\bar{\beta}\\ p_a\leftrightarrow p_b}},\end{aligned} \tag{3.65}$$

where

$$\begin{aligned}S^\mu_{a,p_j} = &\frac{C_A}{2\bar{\beta}}\Bigg[(p_j^\mu + p_a^\mu)\frac{(\vec{n}_{k,\perp a}\cdot\vec{n}_j)^2}{\bar{\beta}} + (\vec{n}_{k,\perp a}\cdot\vec{n}_j)\sqrt{s}\hat{k}_\perp^\mu\\ &+\frac{\sqrt{2}\,(\vec{n}_{k,\perp a}\cdot\vec{n}_j)}{\sqrt{\rho_{ak}}}\left(p_j^\mu + \beta p_a^\mu - \bar{\beta}p_b^\mu\right)\Bigg] + C_F\Bigg[2\,\frac{p_j^\mu + \beta p_a^\mu - \bar{\beta}p_b^\mu}{\rho_{ak}}\\ &+p_j^\mu - \beta p_a^\mu + \bar{\beta}p_b^\mu + \frac{\sqrt{2}}{\sqrt{\rho_{ak}}}\left((\vec{n}_{k,\perp a}\cdot\vec{n}_j)P_{ab}^\mu + \sqrt{s}\,\hat{k}_\perp^\mu\right)\Bigg],\end{aligned} \tag{3.66}$$

$$\begin{aligned}S^\mu_{a,p_\gamma} = &-\frac{C_A}{2\bar{\beta}}\Bigg[(p_b^\mu - p_\gamma^\mu)\frac{(\vec{n}_{k,\perp a}\cdot\vec{n}_j)^2}{\bar{\beta}} - (\vec{n}_{k,\perp a}\cdot\vec{n}_j)\sqrt{s}\hat{k}_\perp^\mu\\ &-\frac{\sqrt{2}\,(\vec{n}_{k,\perp a}\cdot\vec{n}_j)}{\sqrt{\rho_{ak}}}\left(p_\gamma^\mu + \bar{\beta}p_a^\mu - \beta p_b^\mu\right)\Bigg] + C_F\Bigg[2\,\frac{p_\gamma^\mu + \bar{\beta}p_a^\mu - \beta p_b^\mu}{\rho_{ak}}\\ &+p_\gamma^\mu - \bar{\beta}p_a^\mu + \beta p_b^\mu + \frac{\sqrt{2}}{\sqrt{\rho_{ak}}}\left(-(\vec{n}_{k,\perp a}\cdot\vec{n}_j)P_{ab}^\mu + \sqrt{s}\,\hat{k}_\perp^\mu\right)\Bigg].\end{aligned} \tag{3.67}$$

In the above equations we again have used $\bar{\beta} = 1-\beta$, and we introduced the unit transverse vectors $\vec{n}_{k,\perp a}$, $\vec{n}_{k,\perp b}$ defined as follows

$$\vec{n}_{kx} = (1-\rho_{xk})\vec{n}_x + \sqrt{\rho_{xk}(2-\rho_{xk})}\,\vec{n}_{k,\perp x}, \quad x\in(a,b). \tag{3.68}$$

This completes our discussion of the soft contribution to next-to-leading-power corrections. The final result is obtained as a sum of Eqs (3.41,3.52,3.62). These results still contain $1/\epsilon$ poles which, however, are only present in the integrated subtraction terms. These $1/\epsilon$ poles cancel against the ones from the collinear contributions to power corrections that we will now discuss.

### 3.3 The case $\mathfrak{n}||a$

We consider the collinear $\mathfrak{n}||a$ case. Our starting point is Eq. (2.15) which we repeat here for convenience

$$\frac{{\rm d}\sigma^{ca}_{\mathcal{O}}}{{\rm d}\tau} = \mathcal{N}^{-1}\int{\rm d}\Phi(p_a,p_b|\tilde{p}_j,p_\mathfrak{n},\tilde{p}_\gamma)|\mathcal{M}|^2(p_a,p_b;\tilde{p}_j,p_\mathfrak{n},\tilde{p}_\gamma)\delta\left(\tau - \frac{2p_ap_\mathfrak{n}}{P_a}\right)\mathcal{O}(\tilde{p}_j,\tilde{p}_\gamma). \tag{3.69}$$

We note that we renamed $p_{\mathfrak{m}} \to \tilde{p}_j$ for reasons explained earlier.

To extract the subleading one-jettiness contribution, we proceed in the same way as in the case of the color-singlet production [91]. To align the current notations with that reference, we will re-name $p_{\mathfrak{n}} \to k$. Following the discussion in Ref. [91], we decompose the momentum $k$ as

$$k = (1-x)p_a + \tilde{k}_a, \tag{3.70}$$

where $(1-x) = k \cdot P_{ab}/p_a \cdot p_b$. The momentum conservation becomes

$$xp_a + p_b = \tilde{p}_j + \tilde{p}_\gamma + \tilde{k}_a. \tag{3.71}$$

Since the invariant masses of vectors $\tilde{Q} = \tilde{p}_j + \tilde{p}_\gamma$ and $\tilde{Q} + \tilde{k}_a$ are the same [91], one can obtain the latter by performing the Lorentz boost of the former. We denote the required Lorentz boost by $\Lambda_a$,[9] and write

$$\tilde{p}_j + \tilde{p}_\gamma + \tilde{k}_a = p_j + p_\gamma, \tag{3.72}$$

with

$$p_j = \Lambda_a \tilde{p}_j, \quad p_\gamma = \Lambda_a \tilde{p}_\gamma. \tag{3.73}$$

Using the boost and the parametrization of the phase space from Ref. [91], we find

$$\begin{aligned} \frac{\mathrm{d}\sigma^{ca}}{\mathrm{d}\tau} = & \frac{C_F[\alpha_s]P_a^{1-\epsilon}}{2\tau^{1+\epsilon}}\mathcal{N}^{-1} \int\limits_0^1 \mathrm{d}x\, \mathrm{d}\Phi_{\gamma j}^{xa,b} \left[\mathrm{d}\Omega_n^{(d-2)}\right] (1-x)^{-\epsilon}\left(1+\frac{\epsilon\rho_{ak}^*}{2}\right) \\ & \times \mathcal{O}(\Lambda_a^{-1}p_j, \Lambda_a^{-1}p_\gamma) \sum_{\text{pol,col}} C_F^{-1} g_s^{-2}\tau\, |\mathcal{M}(p_b, p_a, k, \Lambda_a^{-1}p_j, \Lambda_a^{-1}p_\gamma)|^2. \end{aligned} \tag{3.74}$$

In Eq. (3.74) $\mathrm{d}\Phi_{\gamma j}^{xa,b}$ denotes the phase space of partons with momenta $p_{j,\gamma}$, produced in a collision of a parton with momentum $xp_a$ and $p_b$, and

$$\rho_{ak}^* = \frac{2P_a\tau}{s(1-x)}. \tag{3.75}$$

The boost matrix $\Lambda_a$ depends on the four-vector $\tilde{k}_a$, which can be parametrized as

$$\tilde{k}_a = \frac{2kp_a}{s}(p_b - p_a) + k_{\perp,a}. \tag{3.76}$$

The vector $k_{\perp,a}$ is orthogonal to $p_{a,b}$. Since the emission angle of the gluon with the momentum $k$ relative to the collision axis scales as $\theta \sim \sqrt{\tau/\sqrt{s}}$, the transverse momentum $k_{\perp,a}$ scales as $k_{\perp,a} \sim \sqrt{\tau}$. This implies that $kp_a \sim \tau$.

As follows from Eq. (3.74), to compute the power corrections, we need to expand both the matrix element and the observable in $\tau$. Since $\Lambda_a^{-1}p_j$ and $\Lambda_a^{-1}p_\gamma$ deviate from $p_{j,\gamma}$ by terms proportional to $\tilde{k}_a \sim \sqrt{\tau}$, we need to expand the observable $\mathcal{O}$ up to the second order in $\tilde{k}_a$. Furthermore, the expansion around collinear limits introduces soft $(x \to 1)$

[9] This matrix is given explicitly in Appendix A of Ref. [91].

singularities in the expansion terms. These singularities need to be extracted, and we discuss below how we deal with this problem.

We note in this respect that in the current paper we work with the particular matrix element squared, and we do not attempt to repeat a more general approach described in Ref. [91] for colorless final states. Hence, we use the explicit form of the matrix element squared of the process $\bar{q} + q \to gg\gamma$, and the explicit expression for the boost, to construct the expansion of the matrix element squared and the observable through next-to-leading power. Collecting terms that become singular in the $x \to 1$ limit, we find

$$
\begin{aligned}
\frac{\mathrm{d}\sigma^{ca,\mathrm{NLP}}}{\mathrm{d}\tau}\bigg|_{x\to 1} = {} & \frac{[\alpha_s]}{\tau^{1+\epsilon}} \frac{P_a^{1-\epsilon}\tau}{2s}\tilde{\sigma}_0 \\
& \times \int\limits_0^1 \mathrm{d}x\, \mathrm{d}\Phi_{\gamma j}^{xa,b}\, \frac{(1-2\beta+2\beta^2-\epsilon)}{\beta\bar{\beta}}\,(1-x)^{-1-\epsilon} \\
& \times \Bigg\{ \frac{2(1+\epsilon)C_F}{(1-x)} - \frac{(1+\beta-\epsilon)C_A}{(1-\epsilon)\bar{\beta}} + \frac{2\beta C_A}{(1-x)(1-\epsilon)\bar{\beta}} \\
& - C_A \left[ \frac{\beta(1-2\beta)p_a^\mu - \bar{\beta}(1-2\beta)p_b^\mu + p_j^\mu}{2\bar{\beta}(1-\epsilon)} \right] \partial_{p_j,\mu} \\
& + C_A \left[ \frac{\beta(1-2\beta)p_b^\mu - \bar{\beta}(1-2\beta)p_a^\mu + p_\gamma^\mu}{2\bar{\beta}(1-\epsilon)} \right] \partial_{p_\gamma,\mu} \\
& + 2C_F \left[ \left(\beta p_a^\mu - \bar{\beta} p_b^\mu\right) \partial_{p_j,\mu} + \left(\bar{\beta} p_a^\mu - \beta p_b^\mu\right) \partial_{p_\gamma,\mu} \right] \Bigg\}\, \mathcal{O}(p_j, p_\gamma).
\end{aligned}
\tag{3.77}
$$

We note that the parameter $\beta$ in this case refers to the Sudakov parametrization of momenta $p_{j,\gamma}$, and $\bar{\beta} = 1-\beta$. The required parametrization can be obtained from Eq. (3.5) provided that one replaces there $p_a \to x p_a$ and $s \to xs$. The same applies to the phase space $\mathrm{d}\Phi_{\gamma j}^{xa,b}$ – one can use Eq. (3.7) provided that $s$ is replaced with $xs$ there.

It follows from Eq. (3.77) that there is a logarithmic and a power-like singularity in the term that contains an observable $\mathcal{O}(p_j, p_\gamma)$, and a logarithmic singularity in the terms with derivatives of the observable $\mathcal{O}$. The logarithmic singularities are standard; we deal with them by expressing $(1-x)^{-1-\epsilon}$ in Eq. (3.77) through $\delta(1-x)$ and plus-distributions.

On the contrary, power-like singularities are unusual, and the easiest way to deal with them is to integrate by parts. We write

$$
\int\limits_0^1 \mathrm{d}x\, \frac{F(x)}{(1-x)^{2+\epsilon}} = -\frac{1}{1+\epsilon} \int\limits_0^1 \mathrm{d}x\, (1-x)^{-1-\epsilon}\, \frac{\partial F}{\partial x},
\tag{3.78}
$$

where we made use of the fact that the boundary term at $x = 0$ drops out because it corresponds to a collision where the parton $a$ has a vanishing four-momentum.

In the context of Eq. (3.77), the function $F(x)$ is a product of the phase-space element that contains the factor $x^{-\epsilon}$, and the observable $\mathcal{O}$. In fact, $\mathcal{O}$ is the only quantity where computation of the derivative with respect to $x$ requires further discussion. The dependence of the observable $\mathcal{O}$ on $x$ arises through the dependences of $p_j$ and $p_\gamma$ on this variable. We find

$$\partial_x \mathcal{O}(p_j, p_\gamma) = \frac{1}{2}\left(\beta p_a^\mu + \frac{1}{x}(p_j^\mu - \bar{\beta} p_b^\mu)\right)\partial_{p_{j,\mu}}\mathcal{O} + \frac{1}{2}\left(\bar{\beta} p_a^\mu + \frac{1}{x}(p_\gamma^\mu - \beta p_b^\mu)\right)\partial_{p_{\gamma,\mu}}\mathcal{O}. \quad (3.79)$$

Since at this point all the divergences are logarithmic, it is straightforward to extract them by rewriting $1/(1-x)^{1+\epsilon}$ through the plus-distributions and the function $\delta(1-x)$.

Putting everything together, we find that the divergent contribution reads

$$\begin{aligned}
\frac{\mathrm{d}\sigma^{ca,\mathrm{NLP}}}{\mathrm{d}\tau}\bigg|_{\mathrm{div}} &= \frac{[\alpha_s]}{\epsilon}\,\frac{P_a}{2s}\,(\tau P_a)^{-\epsilon}\,\tilde{\sigma}_0\,\mathrm{d}\Phi_{\gamma j}^{ab}\,\frac{(1-2\beta+2\beta^2-\epsilon)}{\beta\bar{\beta}} \\
&\times\Bigg\{C_F\bigg[\left(p_j^\mu - \beta p_a^\mu + \bar{\beta} p_b^\mu\right)\partial_{p_{j,\mu}} + \left(p_\gamma^\mu - \bar{\beta} p_a^\mu + \beta p_b^\mu\right)\partial_{p_{\gamma,\mu}}\bigg] \\
&+ C_A\bigg[\frac{(1+\beta)}{\bar{\beta}} + \frac{1}{2\bar{\beta}}\left(p_j^\mu(1+2\beta) + \beta p_a^\mu - \bar{\beta} p_b^\mu\right)\partial_{p_{j,\mu}} \\
&\quad - \frac{1}{2\bar{\beta}}\left(p_\gamma^\mu(1-2\beta) - \bar{\beta} p_a^\mu + \beta p_b^\mu\right)\partial_{p_{\gamma,\mu}}\bigg]\Bigg\}\,\mathcal{O}(p_j, p_\gamma).
\end{aligned} \quad (3.80)$$

We note that all momenta in the above expression are evaluated at $x = 1$; this is indicated, in particular, by the fact that it contains the phase-space element $\mathrm{d}\Phi_{\gamma j}^{ab}$. The finite contribution to the NLP cross section evaluates to

$$\begin{aligned}
\frac{\mathrm{d}\sigma^{ca,\mathrm{NLP}}}{\mathrm{d}\tau}\bigg|_{\mathrm{fin}} &= \frac{[\alpha_s]P_a}{s}\,\tilde{\sigma}_0\,\frac{(1-2\beta+2\beta^2)}{2\beta\bar{\beta}} \\
&\times\int_0^1 \mathrm{d}x\,\mathrm{d}\Phi_{\gamma j}^{xa,b}\Bigg\{-4\delta(1-x)\left(C_F + \frac{3C_A\beta}{4\bar{\beta}}\right) - C_A\,\mathcal{L}_0(1-x)\left(\frac{1+\beta}{\bar{\beta}}\right) \\
&+ C_F\,\mathcal{L}_0(1-x)\Big[(\bar{\beta} p_a^\mu - \beta p_b^\mu - p_\gamma^\mu)\partial_{p_{\gamma,\mu}} + (\beta p_a^\mu - \bar{\beta} p_b^\mu - p_j^\mu)\partial_{p_{j,\mu}}\Big] \\
&+ C_A\,\frac{\delta(1-x)}{2\bar{\beta}}\Big[(\bar{\beta}(1-2\beta)p_a^\mu - \beta(1-2\beta)p_b^\mu - {p_\gamma}^\mu)\partial_{p_{\gamma,\mu}} \\
&+ (\beta(1-2\beta)p_a^\mu - \bar{\beta}(1-2\beta)p_b^\mu + {p_j}^\mu)\partial_{p_{j,\mu}}\Big] \\
&+ C_A\,\frac{\mathcal{L}_0(1-x)}{2\bar{\beta}}\Big[(\beta p_b^\mu - \bar{\beta} p_a^\mu + (1-2\beta){p_\gamma}^\mu)\partial_{p_{\gamma,\mu}} \\
&+ (\bar{\beta} p_b^\mu - \beta p_a^\mu - (1+2\beta){p_j}^\mu)\partial_{p_{j,\mu}}\Big] + \frac{\beta\bar{\beta}}{4(1-2\beta+2\beta^2)}\,R_{\mathrm{ca}}(\beta, x, p_a, p_b)\Bigg\}\,\mathcal{O}(p_j, p_\gamma),
\end{aligned} \quad (3.81)$$

where $\mathcal{L}_0(1-x) = 1/(1-x)_+$ and $R_{\mathrm{ca}}$ is given by the following expression

$$\begin{aligned}
R_{\mathrm{ca}}(\beta, x, p_a, p_b) = &\frac{C_F}{\beta\bar{\beta}}\bigg[\frac{16\beta^4 - 32\beta^3 + 18\beta^2 - 2\beta + 5}{\beta\bar{\beta}}\left(1 + \frac{1}{x^2}\right) \\
&+ \frac{4(2\beta^4 - 4\beta^3 + 3\beta^2 - \beta - 2)}{\beta\bar{\beta}x} + g_2(x,\beta)\,p_\gamma^\mu\partial_{p_{\gamma,\mu}} + g_2(x,\bar{\beta})\,p_j^\mu\partial_{p_{j,\mu}}
\end{aligned}$$

$$
\begin{aligned}
&+g_1(x,\bar\beta)\ p_a^\mu\partial_{p_\gamma,\mu}+\frac{g_1\left(x_1,\bar\beta\right)}{x}\ p_b^\mu\partial_{p_j,\mu}+g_1(x,\beta)p_a^\mu\partial_{p_j,\mu}+\frac{g_1\left(x_1,\beta\right)}{x}p_b^\mu\partial_{p_\gamma,\mu}\\
&+4f_0(\beta)p_a^\mu\left(\bar\beta\partial_{p_\gamma,\mu}+\beta\partial_{p_j,\mu}\right)+\frac{(1+x^2)f_0(\beta)}{2x^2}\Bigg\{\Big[-2\bar\beta^2x^2\ p_a^\mu p_a^\nu-2\beta^2\ p_b^\mu p_b^\nu\\
&-x\ (p_ap_b)\ g^{\mu\nu}+2\left(x\ p_a^\mu p_\gamma^\nu+p_b^\mu p_\gamma^\nu\right)+4x\beta\bar\beta\ p_a^\mu p_b^\nu\Big]\partial_{p_\gamma,\nu}\partial_{p_\gamma,\mu}\\
&+\Big[\left(f_0(\beta)-2\right)\left(x^2\ p_a^\mu p_a^\nu+p_b^\mu p_b^\nu\right)+\left(x\ p_\gamma^\mu p_a^\nu+p_\gamma^\mu p_b^\nu\right)+\left(x\ p_a^\mu p_j^\nu+p_b^\mu p_j^\nu\right)\\
&-x\ (p_ap_b)\ g^{\mu\nu}-x\left(1-2\beta^2\right)p_a^\mu p_b^\nu+x\left(1-4\beta+2\beta^2\right)p_b^\mu p_a^\nu\Big]\partial_{p_j,\nu}\partial_{p_\gamma,\mu}\\
&+\left(p_\gamma\leftrightarrow p_j,\ \beta\leftrightarrow\bar\beta\right)\Bigg\}\Bigg]\\
&-\frac{C_A}{\bar\beta}\Bigg[\frac{2\beta f_0(\beta)+4(1-2\beta)}{\bar\beta\beta x}+\frac{2f_0(\beta)-8\beta}{\bar\beta}+\frac{f_0(\beta)}{\beta\bar\beta}\Big[g_3(x,\beta)\ p_a^\mu\partial_{p_\gamma,\mu}\\
&-g_3(x_1,\bar\beta)\ p_b^\mu\partial_{p_\gamma,\mu}-g_3(x,\bar\beta)\ p_a^\mu\partial_{p_j,\mu}+g_3(x_1,\beta)\ p_b^\mu\partial_{p_j,\mu}\\
&+(1-x_1)(p_\gamma^\mu\partial_{p_\gamma,\mu}-p_j^\mu\partial_{p_j,\mu})-2p_a^\mu(\bar\beta\partial_{p_\gamma,\mu}+\beta\partial_{p_j,\mu})\Big]\Bigg],
\end{aligned}
\tag{3.82}
$$

with $\bar\beta=1-\beta$, $x_1=1/x$,

$$
\begin{aligned}
&g_1(x,\beta)=-4\beta f_0(\beta)+f_1(\beta)x+\frac{f_2(\beta)}{x},\\
&g_2(x,\beta)=f_3(1-\beta)+\frac{f_3(\beta)}{x^2},\quad g_3(x,\beta)=(1-\beta)(1-2\beta)(1-x),
\end{aligned}
\tag{3.83}
$$

and

$$
\begin{aligned}
&f_0(\beta)=1-2\beta+2\beta^2,\quad f_1(\beta)=\frac{\beta(14\beta^3-30\beta^2+19\beta-4)}{1-\beta},\\
&f_2(\beta)=\frac{-10\beta^4+18\beta^3-5\beta^2-4\beta+2}{1-\beta},\quad f_3(\beta)=\frac{-2\beta^4+3\beta^2+\beta-1}{\beta(1-\beta)}.
\end{aligned}
\tag{3.84}
$$

Similar to the case of the color-singlet production studied in Ref. [91], the complexity of this result stems from the fact that we keep the observable arbitrary; for any specific observable, the above expression significantly simplifies.

### 3.4 The case $\mathfrak{n}||b$

This case is completely analogous to the $\mathfrak{n}||a$ one described in the previous section. Hence, we discuss it only very briefly. The starting point is the following expression (c.f. Eq. (2.16))

$$
\frac{\mathrm{d}\sigma^{cb}_{\mathcal{O}}}{\mathrm{d}\tau}=\mathcal{N}^{-1}\int\mathrm{d}\Phi(p_a,p_b|\tilde p_j,p_\mathfrak{n},\tilde p_\gamma)|\mathcal{M}|^2(p_a,p_b;\tilde p_j,p_\mathfrak{n},\tilde p_\gamma)\delta\left(\tau-\frac{2p_bp_\mathfrak{n}}{P_b}\right)\mathcal{O}(\tilde p_j,\tilde p_\gamma). \tag{3.85}
$$

After performing the boost, the momentum conservation becomes

$$
p_a+xp_b=p_j+p_\gamma. \tag{3.86}
$$

The jet and photon momenta are parametrized using the Sudakov decomposition as in Eq. (3.5) but with $p_b$ replaced with $xp_b$, and $s$ replaced with $xs$.

Similar to the $\mathfrak{n}||a$ case, the collinear expansion generates power divergences in the $x \to 1$ limit; these divergences are dealt with using integration by parts, as discussed in the preceding section. Hence, without further ado, we just note that results for the $\mathfrak{n}||b$ case can be obtained from Eqs (3.80) and (3.81) by applying the following replacements

$$p_a \leftrightarrow p_b, \quad \beta \leftrightarrow 1-\beta, \quad P_a \leftrightarrow P_b, \quad \mathrm{d}\Phi^{xa,b} \leftrightarrow \mathrm{d}\Phi^{a,xb}. \tag{3.87}$$

### 3.5 The case $\mathfrak{n}||\mathfrak{m}$

As explained in Section 2, we assume that the relation between the one-jettiness value $\tau$ and the jet radius $R$ is such, that when the smallest scalar product is $p_\mathfrak{m} \cdot p_\mathfrak{n}$, partons $\mathfrak{m}$ and $\mathfrak{n}$ are clustered into a jet. Hence, in this case, the expression for the cross section reads

$$\begin{aligned}\frac{\mathrm{d}\sigma^{\mathfrak{mn}}_{\mathcal{O}}}{\mathrm{d}\tau} = \mathcal{N}^{-1}\int \mathrm{d}\Phi(p_a,p_b|p_\mathfrak{m},p_\mathfrak{n},\tilde{p}_\gamma)|\mathcal{M}|^2(p_a,p_b;p_\mathfrak{m},p_\mathfrak{n},\tilde{p}_\gamma)\theta(p_{\perp,\mathfrak{m}}-p_{\perp,\mathfrak{n}})\\ \times\,\delta\left(\tau - \frac{4p_\mathfrak{m}p_\mathfrak{n}}{P_J}\right)\mathcal{O}(p_{[\mathfrak{mn}]},\tilde{p}_\gamma),\end{aligned} \tag{3.88}$$

where $p_{[\mathfrak{mn}]} = p_\mathfrak{m} + p_\mathfrak{n}$ is the jet four-momentum.

To simplify Eq. (3.88), we use the symmetry of the integrand with respect to $\mathfrak{m} \leftrightarrow \mathfrak{n}$ exchange, to remove the transverse momentum ordering $\theta(p_{\perp,\mathfrak{m}} - p_{\perp,\mathfrak{n}})$, and divide the cross section by two. We then use the momentum mapping described in Appendix A to write Eq. (3.88) in the following way

$$\begin{aligned}\frac{\mathrm{d}\sigma^{\mathfrak{mn}}_{\mathcal{O}}}{\mathrm{d}\tau} &= \frac{\mathcal{N}^{-1}}{2}\frac{\Omega_\perp^{d-2}}{4(2\pi)^{d-2}}\int \mathrm{d}\Phi^{ab}_{\gamma j}\,\frac{\mathrm{d}s_\mathfrak{mn}}{2\pi}\,s_\mathfrak{mn}^{-\epsilon}\,\lambda^{1-2\epsilon}\,\mathrm{d}\alpha_\mathfrak{m}\,[\mathrm{d}\Omega_\perp^{d-2}]\,(\alpha_\mathfrak{m}(1-\alpha_\mathfrak{m}))^{-\epsilon}\\ &\times|\mathcal{M}|^2(p_a,p_b;p_\mathfrak{m},p_\mathfrak{n},\lambda p_\gamma)\,\delta\left(\tau-\frac{2s_\mathfrak{mn}}{P_J}\right)\mathcal{O}(p_j+(1-\lambda)p_\gamma,\lambda p_\gamma),\end{aligned} \tag{3.89}$$

where $s_\mathfrak{mn} = 2p_\mathfrak{m}\cdot p_\mathfrak{n}$ and $\lambda = 1 - s_\mathfrak{mn}/s$. The vectors $p_j$ and $p_\gamma$ are light-like. We emphasize that $p_j$ is *not* the jet momentum as follows from the first argument of the observable function $\mathcal{O}$. Furthermore, the "original" photon momentum $\tilde{p}_\gamma$ and the "final" photon momentum $p_\gamma$ are proportional, but not equal, to each other, i.e. $\tilde{p}_\gamma = \lambda p_\gamma$.

The rest of the computation involves the expansion of the matrix element squared, and the observable around the collinear limit. To perform it, we use the Sudakov decomposition of $p_{\mathfrak{m},\mathfrak{n}}$ in terms of $p_{j,\gamma}$. As shown in Appendix A, the following equations hold

$$\begin{aligned}p_\mathfrak{m} &= \alpha_m p_j + \frac{s_\mathfrak{mn}}{s}(1-\alpha_\mathfrak{m})p_\gamma + \sqrt{s_\mathfrak{mn}\alpha_\mathfrak{m}(1-\alpha_\mathfrak{m})}\,n_\perp,\\ p_\mathfrak{n} &= (1-\alpha_\mathfrak{m})p_j + \frac{s_\mathfrak{mn}}{s}\alpha_\mathfrak{m}p_\gamma - \sqrt{s_\mathfrak{mn}\alpha_\mathfrak{m}(1-\alpha_\mathfrak{m})}\,n_\perp,\end{aligned} \tag{3.90}$$

where $n_\perp \cdot p_{j,\gamma} = 0$.

We have to use this decomposition in the matrix element squared in Eq. (3.89) and expand it in $s_\mathfrak{mn} \sim \tau$. Such expansion generates terms of the form

$$p_{a,b}\cdot n_\perp, \quad (p_{a,b}\cdot n_\perp)^2. \tag{3.91}$$

We note that integration over directions of $n_\perp$ is possible because both the constraint and the observable do not depend on $n_\perp$. Hence,

$$p_x \cdot n_\perp \to 0, \quad (p_x \cdot n_\perp)^2 \to -p_x^\mu p_x^\nu \frac{g_{\perp,\mu\nu}}{2(1-\epsilon)}, \tag{3.92}$$

where $x = a, b$ and

$$g_\perp^{\mu\nu} = g^{\mu\nu} - \frac{p_j^\mu p_\gamma^\nu + p_j^\nu p_\gamma^\mu}{p_j \cdot p_\gamma}. \tag{3.93}$$

Using Eqs (3.90,3.5), we easily find

$$(p_x \cdot n_\perp)^2 \to \frac{s\beta(1-\beta)}{d-2}, \quad x = a, b. \tag{3.94}$$

To present the final result for the collinear $\mathfrak{m}||\mathfrak{n}$ contribution to the cross section, we also need to expand the observable $\mathcal{O}$. With the required accuracy, we find[10]

$$\mathcal{O}(p_j + (1-\lambda)p_\gamma, \lambda p_\gamma) = \left[1 + \frac{s_{\mathfrak{mn}}}{s}\, p_\gamma^\mu \left(\partial_{p_j\mu} - \partial_{p_\gamma\mu}\right)\right] \mathcal{O}(p_j, p_\gamma). \tag{3.95}$$

We are now in a position to write the result for the $\mathfrak{m}||\mathfrak{n}$ collinear contribution. While it is straightforward to do so, there is one peculiar aspect of the outcome of such a calculation that we would like to discuss.

Computing the expansion of the observable and the matrix element squared, and integrating over $s_{\mathfrak{mn}}$, $\alpha_{\mathfrak{m}}$ and $\mathrm{d}\Omega_k$ in Eq. (3.89), we find leading- and subleading contributions to the cross section in the expansion in $\tau$

$$\frac{\mathrm{d}\sigma^{c_{\mathfrak{mn}},\mathrm{LP}}}{\mathrm{d}\tau} = \frac{[\alpha_s]\bar\sigma_0\, 2^\epsilon\, P_J^{-\epsilon} C_A}{8\tau^{1+\epsilon}} \mathrm{d}\Phi_{\gamma j}^{ab}\left[-8\frac{(1-2\beta+2\beta^2)}{\beta(1-\beta)\epsilon} + \cdots\right] \mathcal{O}(p_j, p_\gamma), \tag{3.96}$$

$$\frac{\mathrm{d}\sigma^{c_{\mathfrak{mn}},\mathrm{NLP}}}{\mathrm{d}\tau} = \frac{[\alpha_s]\bar\sigma_0\, C_A\, 2^\epsilon P_J^{1-\epsilon}}{8s\tau^\epsilon} \mathrm{d}\Phi_{\gamma j}^{ab}\Bigg[-2\frac{(4\beta^4 - 8\beta^3 + 2\beta^2 + 2\beta - 1)}{\beta^2(1-\beta)^2\epsilon} - 4\frac{(1-2\beta+2\beta^2)}{\beta(1-\beta)\epsilon}\, p_\gamma^\mu \left(\partial_{p_j\mu} - \partial_{p_\gamma\mu}\right) + \cdots\Bigg]\mathcal{O}(p_j, p_\gamma), \tag{3.97}$$

where ellipses stand for terms without the $\epsilon \to 0$ poles.

A peculiar aspect of the above result is that if we combine the subleading (NLP) contribution in Eq. (3.97) with the collinear and soft contributions discussed earlier, we do not immediately observe the cancellation of $1/\epsilon$ poles. In fact, it only happens *after* one integrates by parts over $\beta$ in Eq. (3.97). This integration is particularly simple, because the term without derivatives of the observable in Eq. (3.97) can be written as

$$\frac{(4\beta^4 - 8\beta^3 + 2\beta^2 + 2\beta - 1)}{\beta^2(1-\beta)^2} = \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{(1-2\beta+2\beta^2)(1-2\beta)}{\beta(1-\beta)}\right). \tag{3.98}$$

Since $\mathrm{d}\Phi_{\gamma j}^{ab} \sim \beta^{-\epsilon}(1-\beta)^{-\epsilon}\,\mathrm{d}\beta$, integration by parts over $\beta$ is straightforward. We find

$$\frac{\mathrm{d}\sigma^{c_{\mathfrak{mn}},\mathrm{NLP}}}{\mathrm{d}\tau} = \frac{[\alpha_s]\bar\sigma_0\, C_A\, 2^\epsilon P_J^{1-\epsilon}}{4s\tau^\epsilon\epsilon}\, \mathrm{d}\Phi_{\gamma j}^{ab}\, \frac{(1-2\beta+2\beta^2)}{2\beta(1-\beta)} \times \left\{-4p_{\gamma,\mu}\left(\partial_{p_j}^\mu - \partial_{p_\gamma}^\mu\right) + 2(1-2\beta)\frac{\mathrm{d}}{\mathrm{d}\beta}\right\}\mathcal{O}(p_j, p_\gamma) + \cdots. \tag{3.99}$$

[10] We remind the reader that since in this case partons are clustered into a jet, one needs to write the observable without assuming $p_j^2 = 0$, compute the derivative and take the limit $p_j^2 = 0$ only after that.

Writing the derivative of the observable with respect to $\beta$ as derivatives with respect to $p_j$ and $p_\gamma$, we obtain

$$\frac{\mathrm{d}\sigma^{\mathfrak{cmn},\mathrm{NLP}}}{\mathrm{d}\tau} = \frac{[\alpha_s]\bar{\sigma}_0\, C_A\, 2^\epsilon P_J^{1-\epsilon}}{4s\tau^\epsilon \epsilon}\, \mathrm{d}\Phi_{\gamma j}^{ab}\, \frac{(1-2\beta+2\beta^2)}{2\beta(1-\beta)} \Bigg\{ -4p_{\gamma,\mu} + (1-2\beta)\left[\frac{p_{a,\mu}}{\beta} - \frac{p_{b,\mu}}{(1-\beta)} - \frac{(1-2\beta)}{\beta(1-\beta)}\, p_{\gamma,\mu}\right]\Bigg\} \left(\partial^\mu_{p_j} - \partial^\mu_{p_\gamma}\right) \mathcal{O}(p_j,p_\gamma) + \cdots . \tag{3.100}$$

The representation of the divergent contribution in Eq. (3.100) turns out to be suitable for establishing the cancellation of the $1/\epsilon$ poles among all next-to-leading-power contributions.

We quote here the result for the remaining finite terms in the $\mathfrak{m}||\mathfrak{n}$ configuration, that we obtain in addition to the divergent ones in Eq. (3.100)

$$\left.\frac{\mathrm{d}\sigma^{\mathfrak{cmn},\mathrm{NLP}}}{\mathrm{d}\tau}\right|_{\mathrm{fin}} = \frac{[\alpha_s]\bar{\sigma}_0\, P_J}{8s}\, \mathrm{d}\Phi_{\gamma j}^{ab} \Bigg\{ \frac{C_A}{3}\Bigg[\frac{6(1-2\beta+2\beta^2)}{\beta^2(1-\beta)^2} - \frac{11}{\beta(1-\beta)} - 22 + \left(\frac{1}{\beta(1-\beta)} + 22\right) p_{\gamma,\mu}\left(\partial^\mu_{p_j} - \partial^\mu_{p_\gamma}\right)\Bigg] - 6C_F\frac{(1-2\beta+2\beta^2)}{\beta^2(1-\beta)^2}\Bigg\}\, \mathcal{O}(p_j,p_\gamma). \tag{3.101}$$

We note that we have taken the $\epsilon \to 0$ limit in the above equation.

### 3.6 The final result for the power corrections to $q\bar{q} \to \gamma + j$ at NLO QCD

Having calculated all the contributions required to obtain the next-to-leading-power corrections to the production of a photon and a jet in the $q\bar{q}$ annihilation channel, we combine them into the final result. The cancellation of the $1/\epsilon$ poles occurs separately for the clustered and non-clustered cases, leaving the $\ln\tau$ terms behind. The final result is obtained by combining

- the *clustered* contributions given in Eqs (3.52,3.100,3.101);
- the *unclustered* ones from Eqs (3.62,3.80,3.81);
- the contribution in Eq. (3.41) that arises because of the modification of the angular distance of the jet algorithm due to the soft recoil.

We therefore write

$$\frac{\mathrm{d}\sigma_{\mathcal{O}}^{\mathrm{NLP}}}{\mathrm{d}\tau} = \left.\frac{\mathrm{d}\sigma_{\mathcal{O}}^{\mathrm{NLP}}}{\mathrm{d}\tau}\right|_{\mathrm{cl}} + \left.\frac{\mathrm{d}\sigma_{\mathcal{O}}^{\mathrm{NLP}}}{\mathrm{d}\tau}\right|_{\mathrm{nc}} + \frac{\mathrm{d}\sigma_{\mathcal{O}}^{s,R}}{\mathrm{d}\tau}, \tag{3.102}$$

where the last term can be found in Eq. (3.41) and the two other contributions read

$$\left.\frac{\mathrm{d}\sigma_{\mathcal{O}}^{\mathrm{NLP}}}{\mathrm{d}\tau}\right|_{\mathrm{cl}} = \frac{[\alpha_s]\tilde{\sigma}_0 P_J}{2s}\mathrm{d}\Phi_{\gamma j}^{ab}\, \frac{(1-2\beta+2\beta^2)}{2\beta\bar{\beta}} \Bigg\{ -\frac{C_F}{\beta\bar{\beta}} + \frac{C_A}{6}\left(23 - \frac{22}{1-2\beta+2\beta^2}\right) - \frac{C_A}{2}\ln\left(\frac{\tau P_J}{2s}\right)\left[4p_\gamma^\mu - (1-2\beta)\left(\frac{p_a^\mu}{\beta} - \frac{p_b^\mu}{\bar{\beta}} - \frac{(1-2\beta)}{\beta\bar{\beta}}\, p_\gamma^\mu\right)\right]\left(\partial_{p_j\mu} - \partial_{p_\gamma\mu}\right) - \frac{C_A}{2}\left[\frac{11}{3}p_\gamma^\mu - \frac{2(1-2\beta)\beta\bar{\beta}}{(1-2\beta+2\beta^2)}\left(\frac{p_a^\mu}{\beta} - \frac{p_b^\mu}{\bar{\beta}} - \frac{(1-2\beta)}{\beta\bar{\beta}}\, p_\gamma^\mu\right)\right]\left(\partial_{p_j\mu} - \partial_{p_\gamma\mu}\right) + \int [\mathrm{d}\Omega_k]\, \mathcal{F}_k^{\mathrm{cl}}\Bigg\}\, \mathcal{O}(p_j,p_\gamma), \tag{3.103}$$

and

$$\frac{\mathrm{d}\sigma_{\mathcal{O}}^{\mathrm{NLP}}}{\mathrm{d}\tau}\bigg|_{\mathrm{nc}} = \frac{[\alpha_s]\tilde{\sigma}_0}{s}\ \frac{(1-2\beta+2\beta^2)}{2\beta\bar{\beta}}\Bigg\{P_a\int\limits_0^1 \mathrm{d}x\ \mathrm{d}\Phi_{\gamma j}^{xa,b}\ \mathcal{C}_a(x,\beta,P_a,p_a,p_b,p_j,p_\gamma)$$
$$+ P_b\int\limits_0^1 \mathrm{d}x\ \mathrm{d}\Phi_{\gamma j}^{a,xb}\ \mathcal{C}_a(x,\bar{\beta},P_b,p_b,p_a,p_j,p_\gamma) + \mathrm{d}\Phi_{\gamma j}^{ab}\ \int [\mathrm{d}\Omega_k]\ \mathcal{F}_k^{\mathrm{nc}}\Bigg\}\ \mathcal{O}(p_j,p_\gamma). \tag{3.104}$$

In the above equation, we have introduced the function $\mathcal{C}_a$ defined as follows

$$\mathcal{C}_a(x,\beta,P_a,p_a,p_b,p_j,p_\gamma) = \left[\delta(1-x)\ln\left(\frac{\tau P_a}{s}\right) - \mathcal{L}_0(1-x)\right]\Bigg\{\frac{1+\beta}{\bar{\beta}}C_A$$
$$+\left(C_F\left(p_j^\mu - \beta p_a^\mu + \bar{\beta}p_b^\mu\right) + \frac{C_A}{2}\frac{(1+2\beta)p_j^\mu + \beta p_a^\mu - \bar{\beta}p_b^\mu}{\bar{\beta}}\right)\partial_{p_j\mu}$$
$$+\left(C_F\left(p_\gamma^\mu - \bar{\beta}p_a^\mu + \beta p_b^\mu\right) - \frac{C_A}{2}\frac{(1-2\beta)p_\gamma^\mu - \bar{\beta}p_a^\mu + \beta p_b^\mu}{\bar{\beta}}\right)\partial_{p_\gamma\mu}\Bigg\} \tag{3.105}$$
$$-\delta(1-x)\Bigg[\frac{C_A\beta}{\bar{\beta}}\left[\left(p_j^\mu + \beta p_a^\mu - \bar{\beta}p_b^\mu\right)\partial_{p_j\mu} + \left(p_\gamma^\mu + \bar{\beta}p_a^\mu - \beta p_b^\mu\right)\partial_{p_\gamma\mu}\right]$$
$$+2C_F + \frac{2\beta}{\bar{\beta}}C_A\Bigg] + \frac{\beta\bar{\beta}}{4(1-2\beta+2\beta^2)}\ R_{\mathrm{ca}}(\beta,x,p_a,p_b)\Bigg\}.$$

Functions $\mathcal{F}_k^{\mathrm{cl}}$, $\mathcal{F}_k^{\mathrm{nc}}$ and $R_{\mathrm{ca}}(\beta,x,p_a,p_b)$ have been already introduced in Eqs (3.54,3.65,3.82), respectively.

### 3.7 Numerical checks

In this section, we provide a numerical validation of the next-to-leading-power corrections presented in Section 3.6, focusing on partonic cross sections for various transverse-momenta cuts. Hence, we choose

$$\mathcal{O}(p_j,p_\gamma) = \theta(p_{\perp,j} - p_{\perp,\mathrm{cut}}), \tag{3.106}$$

and compute the cross section

$$\sigma_{\mathrm{num}}(\tau_{\max},\tau_{\min}) = \int\limits_{\tau_{\min}}^{\tau_{\max}} \mathrm{d}\tau\frac{\mathrm{d}\sigma_{\gamma j}}{\mathrm{d}\tau}\ \mathcal{O}(p_j), \tag{3.107}$$

for several small values of $\tau_{\min}$ and $\tau_{\max}$, using the exact matrix element for $q\bar{q}\to\gamma+gg$, and the phase space for the three-particle final state. To this end, we implement the expression shown in Eq. (2.9) in a numerical code. Since we work at small but non-vanishing $\tau$, dimensional regularization is not needed, as one-jettiness provides an infra-red cutoff. We use $\sqrt{s} = 200$ GeV, $P_a = P_b = P_j = \sqrt{s}/2$, $R = 0.4$, and set $[\alpha_s]$ and $\tilde{\sigma}_o$ to one.

At the same time, for (sufficiently) small values of $\tau_{\min}$, $\tau_{\max}$, the same integrated cross section can be computed using leading and next-to-leading-power contributions derived in this paper,

$$\frac{\mathrm{d}\sigma_{\gamma j}}{\mathrm{d}\tau} = \frac{\mathrm{d}\sigma_{\gamma j}^{\mathrm{LP}}}{\mathrm{d}\tau} + \frac{\mathrm{d}\sigma_{\gamma j}^{\mathrm{NLP}}}{\mathrm{d}\tau} + \ldots \tag{3.108}$$

| $p_{\perp,\mathrm{cut}}$(GeV) | $C_{\mathrm{NLP,LL}}$ | | $C_{\mathrm{NLP,NLL}}$ | |
|---|---|---|---|---|
| | analytic | fitted | analytic | fitted |
| 20 | 32.00 | 32.0(3) | 69.27(6) | 71(1) |
| 25 | 20.25 | 20.1(3) | 31.29(3) | 31.9(9) |
| 30 | 13.88 | 13.9(3) | 14.13(3) | 14.5(8) |

**Table 1**. Comparison of the subleading coefficients $C_{\mathrm{NLP,LL}}$ and $C_{\mathrm{NLP,NLL}}$ obtained using the fit against the analytic calculation for different cuts on the jet's transverse momenta. See text for further details.

Verifying that the two results actually agree provides a check on the next-to-leading-power corrections reported in this paper.

We note that it is challenging to check the correctness of the next-to-leading-power corrections with decent accuracy; the reason is that the integral in Eq. (3.107) is dominated by the double- and single-logarithmic, leading-power contributions. Our strategy is to subtract them from $\sigma_{\mathrm{num}}(\tau_{\mathrm{max}}, \tau_{\mathrm{min}})$ by considering

$$\bar{\sigma}_{\mathrm{num}}(\tau_{\mathrm{max}}, \tau_{\mathrm{min}}) = \sigma_{\mathrm{num}}(\tau_{\mathrm{max}}, \tau_{\mathrm{min}}) - \int\limits_{\tau_{\mathrm{min}}}^{\tau_{\mathrm{max}}} \mathrm{d}\tau \frac{\mathrm{d}\sigma_{\gamma j}^{\mathrm{LP}}}{\mathrm{d}\tau}\, \mathcal{O}(p_j), \tag{3.109}$$

and fit $\bar{\sigma}$ which receives contributions from the subleading terms only. To present the results, we write the higher-order power corrections in the following form

$$\sqrt{s}\left(\frac{\mathrm{d}\sigma_{\gamma j}}{\mathrm{d}\tau} - \frac{\mathrm{d}\sigma_{\gamma j}^{\mathrm{LP}}}{\mathrm{d}\tau}\right) = \ln\nu\, C_{\mathrm{NLP,LL}} + C_{\mathrm{NLP,NLL}} + \nu\ln\nu\, C_{\mathrm{NNLP,LL}} + \nu\, C_{\mathrm{NNLP,NLL}} + \cdots\,, \tag{3.110}$$

where $\nu = \tau/\sqrt{s}$ and the ellipses indicate the neglected power corrections at higher orders in the expansion in $\nu$.

We determine the $C$-coefficients in Eq. (3.110) by fitting $\bar{\sigma}_{\mathrm{num}}$ computed for $\nu_{\mathrm{min}} = 10^{-5}$ and choosing $\mathcal{O}(40)$ points for $\nu_{\mathrm{max}}$ from the interval $\nu_{\mathrm{max}} \in [5\times10^{-5}, 5\times10^{-3}]$. We note that we do not fit all the $C$-coefficients in Eq. (3.110) simultaneously. Instead, we first extract the leading-log coefficient $C_{\mathrm{NLP,LL}}$ from data and verify its consistency with the analytic result. Once this is accomplished, we assume that the $C_{\mathrm{NLP,LL}}$ is correct, subtract it from $\bar{\sigma}_{\mathrm{num}}(\tau_{\mathrm{max}}, \tau_{\mathrm{min}})$ and fit the difference for the coefficient $C_{\mathrm{NLP,NLL}}$. The value of the obtained coefficient $C_{\mathrm{NLP,NLL}}$ is then compared to the analytic results derived this paper, c.f. Eqs (3.103) and (3.104).

The comparison of the numerical and analytic results for the power corrections is shown in Table 1 for different values of the transverse-momentum cut. It follows from Table 1 that the agreement is quite impressive, especially given the smallness of the sub-leading contributions in the region of the fit.

Another useful illustration of the correctness of the next-to-leading-power corrections computed in this paper is provided in Figure 1. There we plot ratios of analytic and numerical NLO cross sections $\sigma_{\rm num}(\tau_{\rm max}, \tau_{\rm min})$; the important point is that different number of terms in the $\tau$-expansion are retained in the analytic results shown there. It is clear from the plot that the inclusion of full next-to-leading-power corrections extends the region of the $\nu$ values, where the numerical and analytical results agree, indicating the correctness of the latter.

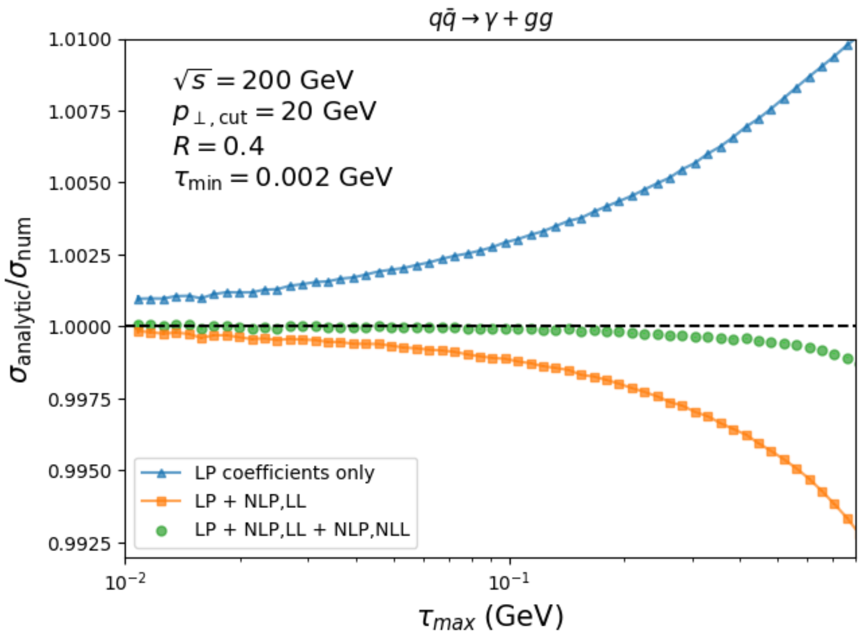


**Figure 1**. The comparison of the "exact" cross section $\sigma_{\rm num}$ (c.f. Eq. (3.107) and its various approximations obtained by different truncations of the expansion in small $N$-jettiness. The analytic approximations including the leading-power (LP) contributions, the LP + leading-logarithmic (LL) next-to-leading-power (NLP) correction, and the LP + full NLP corrections. The three curves in the plot become indistinguishable for $\tau < 10^{-3}$.

## 4 Conclusions

In this paper we have derived, for the very first time, the subleading power corrections in the one-jettiness variable to a process with the final-state jet. We focused on the partonic process $q\bar{q} \to \gamma + j$ since it is sufficiently simple to directly work with the relevant matrix elements, and it does not require the photon-isolation procedure to get a physical result. We employed a fully-realistic $k_\perp$ jet algorithm in this study.

We have shown that the method for computing power corrections developed by us in Ref. [91] to describe production of arbitrary color-singlet final states in hadron collisions, remains effective also for processes with final-state jets. Key elements of this approach are momenta redefinitions and Lorentz transformations; they are familiar from the discussion of general subtraction schemes at NLO and NNLO (see [101] and references therein).

Our study of power corrections can be extended in several ways in the future. First, in this paper we have relied on the explicit form of the matrix element and did not attempt to design a process-independent framework similar to what has been done in Ref. [91] for the color-singlet final states. It will be interesting to understand how to generalize this approach to final states with arbitrary number of jets, where the analytic expressions for relevant matrix elements cannot be used.

Second, it is worthwhile to extend the current analysis to processes with an on-shell vector boson in the final state. Although such an extension should be straightforward, the gauge-boson on-shell constraint may require some care with Lorentz transformations and momenta redefinitions.

Third, the major reason for the complicated analytic expressions for the power corrections is the derivatives of observables. For this reason, it will be useful to design a framework that will allow one to treat them as changes in kinematics of observable quantities in a more universal and easy-to-handle way.

Finally, it would be interesting to extend the analysis of power corrections in the $N$-jettiness variable to next-to-next-to-leading order. Although the complexity of this task remains outstanding, we hope that the improved understanding of the power corrections provided by this paper and also by Ref. [91] constitutes a good starting point for attempting it.

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## A Phase-space parametrization for the final-state collinear limit

In this appendix, we derive the momenta mapping and the phase-space parametrization that is suitable for describing the final-state collinear limit. The goal is to map the momentum conservation condition

$$p_a + p_b = p_{\mathfrak{m}} + p_{\mathfrak{n}} + \tilde{p}_\gamma, \tag{A.1}$$

onto

$$p_a + p_b = p_j + p_\gamma, \tag{A.2}$$

where $p_j^2 = 0$ and $p_\gamma^2 = \tilde{p}_\gamma^2 = 0$. The momentum $p_j$ is related to the momentum of the final-state jet, but it is not identical to it.

To construct the momentum $p_j$, we write it as a linear combination of two vectors $p_{\mathfrak{mn}} = p_{\mathfrak{m}} + p_{\mathfrak{n}}$ and $P_{ab} = p_a + p_b$,

$$p_j = \frac{1}{\lambda}\left(p_{\mathfrak{mn}} - \frac{p_{\mathfrak{mn}} \cdot P_{ab}}{P_{ab}^2} P_{ab}\right) + x P_{ab}. \tag{A.3}$$

Since the four-vector in brackets is orthogonal to $P_{ab}$, we find

$$x = \frac{p_j \cdot P_{ab}}{P_{ab}^2} = \frac{1}{2}. \tag{A.4}$$

Furthermore, using $p_\gamma^2 = \tilde{p}_\gamma^2 = 0$, we obtain

$$P_{ab} \cdot p_j = P_{ab} \cdot p_{\mathfrak{mn}} - \frac{1}{2} s_{\mathfrak{mn}}, \tag{A.5}$$

where $s_{\mathfrak{mn}} = 2p_{\mathfrak{m}} \cdot p_{\mathfrak{n}}$.

The parameter $\lambda$ in Eq. (A.3) is adjusted to ensure that $p_j^2 = 0$. We find

$$\lambda = 1 - \frac{s_{\mathfrak{mn}}}{P_{ab}^2}. \tag{A.6}$$

Finally, using Eq. (A.3), we express $p_{\mathfrak{mn}}$ in terms of $p_j$

$$p_{\mathfrak{mn}} = \lambda p_j + (1-\lambda) P_{ab}, \tag{A.7}$$

which immediately implies the following relation for the photon momenta

$$\tilde{p}_\gamma = \lambda p_\gamma. \tag{A.8}$$

Our goal is to rewrite the phase space for $\mathfrak{m}, \mathfrak{n}$ and $\tilde{\gamma}$ in such a way that expansion in $s_{\mathfrak{mn}} \sim \tau$ at fixed $p_{j,\gamma}$ becomes possible. Below we sketch the derivation of the relevant formula; its detailed discussion in a broader context can be found in Ref. [102].

We begin by writing

$$\begin{aligned} &\int [\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}][\mathrm{d}\tilde{p}_\gamma](2\pi)^d \delta(P_{ab} - p_{\mathfrak{m}} - p_{\mathfrak{n}} - \tilde{p}_\gamma) = \\ &\int \frac{\mathrm{d}s_{\mathfrak{mn}}}{2\pi} [\mathrm{d}p_{\mathfrak{mn}}][\mathrm{d}\tilde{p}_\gamma](2\pi)^d \delta(P_{ab} - p_{\mathfrak{mn}} - \tilde{p}_\gamma) \int [\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}](2\pi)^d \delta(p_{\mathfrak{mn}} - p_{\mathfrak{m}} - p_{\mathfrak{n}}), \end{aligned} \tag{A.9}$$

As the next step, we consider the integral over $[\mathrm{d}p_{\mathfrak{mn}}][\mathrm{d}\tilde{p}_\gamma]$ in the rest frame of $P_{ab}$ and find

$$[\mathrm{d}p_{\mathfrak{mn}}][\mathrm{d}\tilde{p}_\gamma](2\pi)^d \delta(P_{ab} - p_{\mathfrak{mn}} - \tilde{p}_\gamma) = \mathrm{d}\Omega_{\tilde{\gamma}}\, \mathcal{N} \left(1 - \frac{s_{\mathfrak{mn}}}{P_{ab}^2}\right)^{d-3}, \tag{A.10}$$

where $\mathcal{N}$ is a function of $P_{ab}^2$ only, and $\mathrm{d}\Omega_{\tilde{\gamma}}$ is the solid angle that parametrizes the direction of the photon momentum $\vec{p}_{\tilde{\gamma}}$ or, equivalently, of $\vec{p}_{\mathfrak{mn}}$. To relate this result to the phase-space of $p_j$ and $p_\gamma$, we use Eq. (A.3). It follows from that equation that in the rest frame of $P_{ab}$, the directions of $\vec{p}_j$ and $\vec{p}_{\mathfrak{mn}}$ coincide. Thus,

$$[\mathrm{d}p_{\mathfrak{mn}}][\mathrm{d}\tilde{p}_\gamma](2\pi)^d \delta(P_{ab} - p_{\mathfrak{mn}} - \tilde{p}_\gamma) = \lambda^{d-3} [\mathrm{d}p_j][\mathrm{d}p_\gamma](2\pi)^d \delta(P_{ab} - p_j - p_\gamma). \tag{A.11}$$

The relation between $p_\gamma$ and $\tilde{p}_\gamma$ is given in Eq. (A.8). Putting everything together, we arrive at the final formula for the phase space that is suitable for describing the collinear limit

$$\begin{aligned} \int [\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}][\mathrm{d}\tilde{p}_\gamma](2\pi)^d \delta(P_{ab} - p_{\mathfrak{m}} - p_{\mathfrak{n}} - \tilde{p}_\gamma) &= \int [\mathrm{d}p_j][\mathrm{d}p_\gamma](2\pi)^d \delta(P_{ab} - p_j - p_\gamma) \\ &\times \int_0^s \frac{\mathrm{d}s_{\mathfrak{mn}}}{2\pi} \lambda^{d-3} \int [\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}](2\pi)^d \delta(p_{\mathfrak{mn}} - p_{\mathfrak{m}} - p_{\mathfrak{n}}). \end{aligned} \tag{A.12}$$

To use this formula for computing power corrections, we need to understand how to integrate over $p_{\mathfrak{m}}$ and $p_{\mathfrak{n}}$ in the vicinity of the collinear $\mathfrak{m}||\mathfrak{n}$ limit. To this end, we use $p_j$ and $p_\gamma$ as basis vectors for the Sudakov decomposition of $p_{\mathfrak{m}}$ and $p_{\mathfrak{n}}$. We find

$$\begin{aligned} p_{\mathfrak{m}} &= \alpha_{\mathfrak{m}} p_j + \beta_{\mathfrak{m}} p_\gamma + p_\perp, \\ p_{\mathfrak{n}} &= \alpha_{\mathfrak{n}} p_j + \beta_{\mathfrak{n}} p_\gamma - p_\perp. \end{aligned} \tag{A.13}$$

If the invariant mass $s_{\mathfrak{mn}} = 2p_{\mathfrak{m}} \cdot p_{\mathfrak{n}}$ is small, and $p_j$ is the collinear direction, then $\alpha_{\mathfrak{m}} \sim \alpha_n \sim 1$, $\beta_{\mathfrak{m},\mathfrak{n}} \sim s_{\mathfrak{mn}}/P_{ab}^2$ and $|p_\perp| \sim \sqrt{s_{\mathfrak{mn}}/P_{ab}^2}$. Using the Sudakov decomposition, we easily find the following parametrization of the $(\mathfrak{mn})$ phase space

$$\begin{aligned} &\int [\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}](2\pi)^d \delta(p_{\mathfrak{mn}} - p_{\mathfrak{m}} - p_{\mathfrak{n}}) \\ &= \frac{s_{\mathfrak{mn}}^{-\epsilon}\, \Omega_\perp^{(d-2)}}{4(2\pi)^{d-2}} \int\limits_0^1 \mathrm{d}\alpha_{\mathfrak{m}}\, [\mathrm{d}\Omega_\perp^{(d-2)}]\, (\alpha_{\mathfrak{m}}(1-\alpha_{\mathfrak{m}}))^{-\epsilon}, \end{aligned} \tag{A.14}$$

where the azimuthal angle describes directions of the vector $p_\perp$ in Eq. (A.13). With this parametrization, it is possible to expand the explicit matrix element squared for $q\bar{q} \to \gamma g g$ in the collinear $\mathfrak{m}||\mathfrak{n}$ kinematics. Indeed, since

$$\begin{aligned} p_{\mathfrak{m}} &= \alpha_m p_j + \frac{s_{\mathfrak{mn}}}{s}(1-\alpha_m) p_\gamma + \sqrt{s_{\mathfrak{mn}} \alpha_m (1-\alpha_m)}\, n_\perp, \\ p_{\mathfrak{n}} &= (1-\alpha_m) p_j + \frac{s_{\mathfrak{mn}}}{s} \alpha_m p_\gamma - \sqrt{s_{\mathfrak{mn}} \alpha_m (1-\alpha_m)}\, n_\perp, \end{aligned} \tag{A.15}$$

and $s_{\mathfrak{mn}} \sim \tau$, it is straightforward to construct the expansion through next-to-leading power.

## B A shift in $R_{\mathfrak{mn}}$

In this appendix, we discuss the change in $R_{\mathfrak{mn}}$ induced by the soft boost. According to our notation, the momentum of the parton $\mathfrak{m}$ is a boosted and rescaled $p_j$, whereas the parton $\mathfrak{n}$ is assigned the momentum $k$. Then, the following relation between $p_{\mathfrak{m}}$, $p_j$ and $k$ holds

$$p_{\mathfrak{m}}^\mu = \left(1 - \frac{\omega_k}{2E_j}\right) p_j^\mu - \frac{k^\mu}{2} + \frac{k \cdot p_j}{2E_j} t^\mu, \tag{B.1}$$

where $t^\mu = (1, \vec{0})$, we work in the center-of-mass frame of colliding partons $a$ and $b$, $E_j$ is the energy of $j, \gamma, a, b$, and $\omega_k$ is the energy of the parton $\mathfrak{m}$. We can rewrite the above formula in the following way

$$p_{\mathfrak{m}} = \left(1 - \frac{\omega_k}{2E_j}(1 + \cos\theta_{kj})\right) p_j - \frac{\omega_k}{2}\,(0, \vec{n}_k - \cos\theta_{kj} \vec{n}_j)\,. \tag{B.2}$$

We can also write $p_{\mathfrak{m}}$ as follows

$$p_{\mathfrak{m}} = \frac{E_{\mathfrak{m}}}{E_j} p_j + E_j\,(0, \vec{n}_{\mathfrak{m}} - \vec{n}_j)\,, \tag{B.3}$$

where we work to first order in the difference between $p_{\mathfrak{m}}$ and $p_j$ caused by the emission of a gluon.

We can match the two equations if we choose

$$E_{\mathfrak{m}} = E_j \left(1 - \frac{\omega_k}{2E_j}(1+\cos\theta_{kj})\right), \tag{B.4}$$

and

$$\vec{n}_{\mathfrak{m}} = \vec{n}_j - \frac{\omega_k}{2E_j}\left(\vec{n}_k - \cos\theta_{kj}\vec{n}_j\right). \tag{B.5}$$

We assume that vectors $\vec{n}_{\mathfrak{m}}$ and $\vec{n}_j$ are parametrized as follows

$$\vec{n}_x = (\sin\theta_x\cos\varphi_x, \sin\theta_x\sin\varphi_x, \cos\theta_x), \tag{B.6}$$

where $x = \mathfrak{m}, j$, and the $z$-axis is aligned with the vector $\vec{n}_a$. Then,

$$[\vec{n}_{\mathfrak{m}} \times \vec{n}_j]\cdot\vec{n}_a = \sin\theta_m\sin\theta_j\sin\left(\varphi_j - \varphi_m\right). \tag{B.7}$$

At the same time,

$$[\vec{n}_{\mathfrak{m}} \times \vec{n}_j]\cdot\vec{n}_a = -\frac{\omega_k}{2E_j}[\vec{n}_k \times \vec{n}_j]\cdot\vec{n}_a. \tag{B.8}$$

Since $\theta_m \sim \theta_j$, we easily find

$$\varphi_{\mathfrak{m}} - \varphi_j \approx \frac{\omega_k}{2E_j\sin^2\theta_j}\,[\vec{n}_k \times \vec{n}_j]\cdot\vec{n}_a + \mathcal{O}(\omega_k^2). \tag{B.9}$$

Similarly,

$$\theta_m - \theta_j \approx \frac{\omega_k}{2E_j\sin\theta_j}[\vec{n}_k \times \vec{n}_j]\cdot[\vec{n}_a \times \vec{n}_j]. \tag{B.10}$$

We can use these results to derive the difference between $R_{\mathfrak{m}\mathfrak{n}}$ and $R_{j\gamma}$. Expanding in Taylor series, we obtain

$$R_{\mathfrak{m}\mathfrak{n}} = R_{jk} + \frac{\omega_k}{2E_j\sin^2\theta_j}[\vec{n}_k \times \vec{n}_j]\cdot\left(\frac{\partial R_{jk}}{\partial\varphi_j}\vec{n}_a - \frac{\partial R_{jk}}{\partial\eta_j}[\vec{n}_a \times \vec{n}_j]\right) + \mathcal{O}(\omega_k^2). \tag{B.11}$$

For the jet algorithm in Eq. (2.2), we find

$$-\frac{\partial R_{jk}}{\partial\varphi_j} = \frac{f_\varphi(\varphi_{jk})\,\mathrm{sgn}(\sin\varphi_{jk})}{R_{jk}}, \tag{B.12}$$

where $\varphi_{jk} = \varphi_j - \varphi_k$.

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