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Towards understanding fermion masses and mixings

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Abstract

The Standard Model does not constrain the form of the Yukawa matrices and thus the origin of fermion mass hierarchies and mixing pattern remains puzzling. On the other hand, there are intriguing relations between the quark masses and their weak mixing angles, such as the well-known one $\tan\theta_C = \sqrt{m_d/m_s}$ for the Cabibbo angle, which may point towards specific textures of Yukawa matrices hypothesized by Harald Fritzsch at the end of the 70's. Though the original ansatz of Fritzsch is excluded by the experimental data, one can consider its minimal modification which consists in introducing an asymmetry between the 23 and 32 entries in the down-quark Yukawa matrix. We show that this structure is perfectly compatible with the present precision data on quark masses and CKM mixing matrix, and theoretically it can be obtained in the context of $SU(5)$ model with inter-family $SU(3)_H$ symmetry. We also discuss some alternative approaches which could give a natural description of the fermion mass spectrum and weak mixing pattern.

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1 Introduction

The replication of fermion families is one of the main puzzles of particle physics. Three fermion families are in identical representations of the Standard Model (SM) gauge symmetry $SU(3) \times SU(2) \times U(1)$. The left-handed (LH) quarks $q_{Li} = (u_L, d_L)_i$ and leptons $\ell_{Li} = (\nu_L, e_L)_i$ transform as weak doublets while right-handed (RH) ones u_{Ri}, d_{Ri}, e_{Ri} are weak singlets, $i = 1, 2, 3$ being the family index. The SM contains the unique order parameter – vacuum expectation value (VEV) of the Higgs doublet ϕ, $\langle \phi^0 \rangle = v_{\rm w} = 174$ GeV, which spontaneously breaks the electroweak symmetry $SU(2) \times U(1)$. It determines the mass scale of the weak bosons $W^\pm, Z$ as well as fermion masses which emerge via the Yukawa couplings

$$Y_u^{ij} u_i^c q_j \phi + Y_d^{ij} d_i^c q_j \tilde{\phi} + Y_e^{ij} e_i^c \ell_j \tilde{\phi} + \text{h.c.} \tag{1}$$

where $Y_{e,u,d}$ are the Yukawa coupling matrices, and $\tilde{\phi} = i\tau_2 \phi^*$. Here we use instead of the RH fermion fields their complex conjugates as $u_L^c = C\overline{u_R}^T$ (anti-fields) and omit in the following the subscript L for q, u^c, d^c etc., all being LH Weyl spinors.[3] The Yukawa couplings (1), after substituting the Higgs VEV v_w, induce the fermion mass matrices $M_f = Y_f v_{\rm w}$ $(f = u, d, e)$ which are generically non-diagonal. They can be brought to the diagonal form (i.e. to the mass eigenstate basis) via bi-unitary transformations:

$$U_f^\dagger M_f V_f = M_f^{\rm diag} \tag{2}$$

so that the quark masses m_u, m_c, m_t and m_d, m_s, m_b are the eigenvalues of the mass matrices M_u and M_d, as the charged lepton masses m_e, m_μ, m_τ are the eigenvalues of M_e (we shall not discuss the neutrino masses in this paper).

In the SM context, the matrices $U_{u,d}$ which rotate the RH fermions have no physical meaning while the "left" ones $V_{u,d}$ give rise to the mixing in the quark charged currents coupled to weak $W^\pm$ bosons. This mixing is described by the unitary Cabibbo-Kobayashi-Maskawa (CKM) $V_{\rm CKM}$ [1, 2]:

$$V_{\rm CKM} = V_u^\dagger V_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{3}$$

By rotating away the irrelevant phases, the unitary matrix $V_{\rm CKM}$ can be conveniently parameterized in terms of four parameters, three mixing angles and a CP-violating phase [2]. In the standard parameterization adopted by Particle Data Group (PDG) [3], the angles are chosen as θ_{12}, θ_{23}, θ_{13}, and the CKM matrix reads

$$V_{\rm CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{4}$$

[3] With these notations, the description can be conveniently extended for a supersymmetric version of the SM, in which case the terms (1) correspond to the Yukawa superpotential, with $\phi = \phi_u$ and $\tilde{\phi} = \phi_d$ being two Higgs doublets taken as left-chiral superfields.

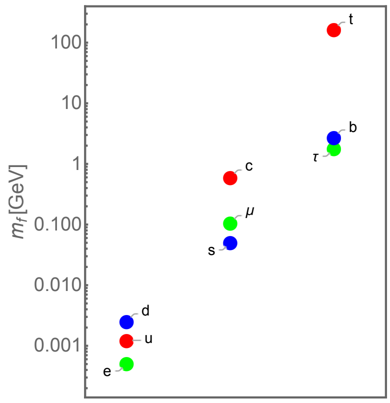

Figure 1: The running masses of quarks and leptons, normalized at the electroweak scale $\mu = v_w$.

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$ and θ_{ij} are chosen so that $s_{ij}, c_{ij} \geq 0$. As a measure of CP violation, the rephasing-invariant quantity $J \sum_{m,n} \epsilon_{ikm}\epsilon_{jln} = \mathrm{Im}[V_{ij}V_{kl}V^*_{il}V^*_{kj}]$ (the Jarlskog invariant) [4] in the standard parameterization reads:

$$J = \sin\delta\, s_{12}s_{23}s_{13}\, c_{12}c_{23}c_{13}^2 \tag{5}$$

The SM has a remarkable feature: the natural flavor conservation in neutral currents, namely in the fermion couplings with the Higgs and Z bosons [5, 6, 7]. However, it contains no theoretical input that could explain the fermion mass spectrum and the weak mixing pattern. In a sense, the SM is technically natural since it can tolerate any pattern of the Yukawa matrices $Y_{u,d,e}$ but it can tell nothing on the origin of the strong hierarchies between their eigenvalues as well as of the nearly aligned structures of the matrices Y_u and Y_d. This remains true also in the context of supersymmetric and/or grand unification theories (GUT). So, the origin of the fermion mass and mixing pattern remains a mystery.

The quark and lepton mass spectrum, schematically shown in fig. 1, exhibits a strong inter-family hierarchy. The mass hierarchy between the third and first families is $m_b/m_d \sim 10^3$ for the down quarks, and yet stronger for the up quarks, $m_t/m_u \sim 10^5$. Expressed in terms of the small parameters $m_d/m_s = \epsilon_d$, $m_s/m_b = \epsilon_s$ etc., these hierarchies approximately look as:

$$\begin{aligned}
& m_t : m_c : m_u = 1 : \epsilon_c : \epsilon_u\epsilon_c\,, \quad \epsilon_c \approx \frac{1}{300}, \quad \epsilon_u \approx \frac{1}{500} \;\to\; \epsilon_u\epsilon_c \approx \frac{1}{150000} \\
& m_b : m_s : m_d = 1 : \epsilon_s : \epsilon_d\epsilon_s\,, \quad \epsilon_s \approx \frac{1}{50}, \quad \epsilon_d \approx \frac{1}{20} \;\to\; \epsilon_d\epsilon_s \approx \frac{1}{1000} \\
& m_\tau : m_\mu : m_e = 1 : \epsilon_\mu : \epsilon_e\epsilon_\mu\,, \quad \epsilon_\mu \approx \frac{1}{17}, \quad \epsilon_e \approx \frac{1}{207} \;\to\; \epsilon_e\epsilon_\mu \approx \frac{1}{3500}
\end{aligned} \tag{6}$$

In a whole, these hierarchies do not exhibit any notable regularities, but only feature some order of magnitude connections. Namely, by comparing the up and down quarks, we see that $\epsilon_s \sim \epsilon_d$ while $\epsilon_u, \epsilon_c \sim \epsilon_d^2$. In other words, the up quark masses scale approximately as squares of the down quark masses. Comparing with leptons, we see that $\epsilon_\mu \sim \epsilon_d$ while $\epsilon_e \sim \epsilon_c$.

As for the CKM matrix, the quark mixing angles are small (unlike the case of neutrino mixing). Within the experimental uncertainties [3], they exhibit the following pattern

$$|V_{us}| = s_{12} \approx \lambda, \quad |V_{cb}| = s_{23} \approx a\lambda^2, \quad |V_{ub}| = s_{13} \approx 2a\lambda^4 \tag{7}$$

in terms of the small parameter $\lambda \approx \sqrt{1/20}$ (which is incidentally related as $\lambda^2 \approx \epsilon_d$ with the mass ratio $\epsilon_d = m_d/m_s$ in (6)), and the order one numerical factor $a \approx 0.8$. Then for the Jarlskog invariant one has $J \approx 2a^2\lambda^7 \sin\delta \approx 3.5 \times 10^{-5} \sin\delta$ which means that the smallness of observed CP-violation is originated from the small mixing angles rather than from a small CP-phase δ, and in fact $\sin\delta \simeq 1$.

2 Fritzsch Hypothesis

It is tempting to think that the fermion flavour structure is connected to some underlying theory which determines the pattern of the Yukawa matrices with a predictive power, and in particular that the well-known formula for the Cabibbo angle $V_{us} = \sqrt{m_d/m_s}$ is not accidental. Such relations between the fermion masses and mixing angles can be obtained by considering Yukawa matrix textures with reduced number of free parameters, and in particular, by assuming that certain elements in the fermion mass matrices are vanishing for some symmetry reasons. This *zero-texture* approach was originally thought to calculate the Cabibbo angle in the two-family framework [8, 9, 10], in fact before the discovery of b and t quarks. In the frame of six quarks, this picture was extended by Harald Fritzsch [11, 12] who suggested the following texture for the mass matrices:

$$M_{u,d} = \begin{pmatrix} 0 & M_{12}^{u,d} & 0 \\ M_{21}^{u,d} & 0 & M_{23}^{u,d} \\ 0 & M_{32}^{u,d} & M_{33}^{u,d} \end{pmatrix} \tag{8}$$

where the non-zero elements are generically complex, with the symmetricity condition $|M_{ij}^{u,d}| = |M_{ji}^{u,d}|$ which is motivated in the context of left-right symmetric models.[4]

By rotating the phases of the upper quarks: $u_k^c \to e^{i\alpha'_k} u_k^c$ and $u_k \to e^{i\alpha_k} u_k$, and similarly for down quarks, the complex phases in the matrices (8) can be removed and the non-zero entries can be rendered real. Namely, the matrices (8) can be parameterized as $M_{u,d} = F'_{u,d} \widetilde{M}_{u,d} F^*_{u,d}$, where $F_u = \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_2})$ etc. are the phase transformation matrices, and

$$F'^*_{u,d} M_{u,d} F_{u,d} = \widetilde{M}_{u,d} = v_w \widetilde{Y}_{u,d}\,, \qquad \widetilde{Y}_{u,d} = \begin{pmatrix} 0 & A_{u,d} & 0 \\ A_{u,d} & 0 & B_{u,d} \\ 0 & B_{u,d} & C_{u,d} \end{pmatrix} \tag{9}$$

[4] In the original works [11, 12], the 'zeros' in these matrices were obtained at the price of introducing several Higgs bi-doublets differently transforming under some discrete flavor symmetry. At present, this underlying theoretical construction looks rather obsolete. Namely, the need for several Higgs bi-doublets spoils the natural flavor conservation [5, 6, 7] and unavoidably leads to severe flavor-changing effects [13]. In a more natural way, the Fritzsch texture for the Yukawa matrices $Y_{u,d,e}$ (rather than for the mass matrices) was obtained in refs. [14, 15] in the context of models with horizontal $SU(3)_H$ gauge symmetry between the three fermion families [16, 17, 18].

are real symmetric matrices which can be further diagonalized by orthogonal transformations, $O^T_{u,d}\widetilde{M}_{u,d}O_{u,d} = M^{\rm diag}_{u,d}$. The three real parameters A_d, B_d, C_d can be expressed in terms of the three eigenvalues of $\widetilde{Y}_d$, i.e. in terms of the down quark masses m_d, m_s, m_b. Namely, one gets approximately, up to small corrections, $C_d \approx m_b/v_w$, $B_d \approx \sqrt{m_s m_b}/v_w$ and $A_d \approx \sqrt{m_d m_s}/v_w$, and similarly for up quarks and charged leptons (provided that the mass matrix of the latter has a structure similar to (9)). Therefore, we have:

$$\begin{aligned} C_u : B_u : A_u &\approx 1 : \epsilon_c^{1/2} : \epsilon_s \epsilon_d^{1/2} \approx 1 : \frac{1}{16} : \frac{1}{6000} \\ C_d : B_d : A_d &\approx 1 : \epsilon_s^{1/2} : \epsilon_s \epsilon_d^{1/2} ~\approx~ 1 : \frac{1}{7} : \frac{1}{240} \\ C_e : B_e : A_e &\approx 1 : \epsilon_\mu^{1/2} : \epsilon_\mu \epsilon_e^{1/2} ~\approx~ 1 : \frac{1}{4} : \frac{1}{240}\, . \end{aligned} \tag{10}$$

Hence, the fermion mass hierarchies (6) follow from somewhat milder hierarchies between the input parameters which are the non-zero entries in the matrices (9). Notice also the following relation between the down quark and leptons:

$$\frac{A_d}{C_d} \approx \frac{A_e}{C_e} \quad \longrightarrow \quad \frac{\sqrt{m_d m_s}}{m_b} \approx \frac{\sqrt{m_e m_\mu}}{m_\tau}\, . \tag{11}$$

Later on we shall explore its origin in the GUT context.

The three rotation angles in the orthogonal matrix O_d can be expressed in terms of the mass ratios m_d/m_s and m_s/m_b. Analogously, the three angles in O_u can be expressed in terms of the upper quarks mass ratios m_u/m_c and m_c/m_t. The CKM matrix (3) is obtained as $V_{\rm CKM} = O^T_u F^*_u F_d O_d$, where the diagonal matrix $F = F^*_u F_d$ can be parameterized by two phase parameters, $F = {\rm diag}(e^{i\alpha}, e^{i\beta}, 1)$. Then, the four physical elements of the CKM matrix, that is the three mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$ and the CP-phase δ, can be expressed in terms of the four mass ratios, m_d/m_s, m_s/m_b, m_u/m_c and m_c/m_t, and of two unknown phases β and $\gamma = \beta - \alpha$. Namely, in the leading approximation one has:

$$|V_{us}| \approx \left|\sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}}e^{i\gamma}\right|, \quad |V_{cb}| \approx \left|\sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}}e^{i\beta}\right|, \quad \left|\frac{V_{ub}}{V_{cb}}\right| \approx \sqrt{\frac{m_u}{m_c}}\, . \tag{12}$$

Besides reproducing the formula for the Cabibbo angle, this texture exhibits a remarkable feature in light of the interfamily hierarchies $m_d \ll m_s \ll m_b$ and $m_u \ll m_c \ll m_t$. Namely, in the limit $A_{u,d} \to 0$ the light quarks become massless, $m_{u,d} \to 0$, but at the same time the first family decouples in the CKM mixing, $s_{12}, s_{13} \to 0$. Next, in the limit $B_{u,d} \to 0$ which renders massless the second family, $m_{c,s} \to 0$, also its mixing with the third family disappears, $s_{23} \to 0$.

Generalizing these properties, Fritzsch suggested that in any kind of realistic flavor models the quark mixing pattern should be intimately related to the interfamily hierarchy. The mixing angles θ_{ij} should depend on quark mass ratios m_d/m_s, m_u/m_c etc. so that the smallness of the former stems from the smallness of the latter. He hypothesized the following analytic properties [19]:

- **Decoupling hypothesis:** in the limit of massless first family, $m_u, m_d \to 0$, its mixings with the heavier families disappear, i.e. $\theta_{12}, \theta_{13} \to 0$. At the next step, for massless second family, $m_s, m_c \to 0$, also the 2-3 mixing should disappear, i.e. $\theta_{23} \to 0$.

- **Scaling hypothesis:** in the limit when up and down quark masses become proportional, $m_u : m_c : m_t = m_d : m_s : m_b$, all mixing angles should vanish: $\theta_{12}, \theta_{23}, \theta_{13} \to 0$.

While the Fritzsch ansatz (9) has the first property (*decoupling* hypothesis), in general it does not meet the second one (scaling hypothesis), since the Yukawa matrices Y_u and Y_d do not necessarily become aligned in the limit $A_u : B_u : C_u = A_d : B_d : C_d$. In fact, in this case we have $m_u : m_c : m_t = m_d : m_s : m_b$ but the mixing angles do not generally vanish because of the arbitrary phases β and γ.

However, the original Fritzsch texture for quarks was excluded when the knowledge of quark masses and CKM parameters became accurate enough [20]. Given the present experimental and lattice results on quark masses and CKM elements, there is no parameter space in which the precision data can be reproduced. More concretely, the small enough value of $|V_{cb}|$ and large enough value of $|V_{ub}/V_{cb}|$ cannot be achieved for any values of the phase parameters β and γ in (12). A possibility to obtain viable textures is to extend the original Fritzsch texture by replacing one of the zero entries with a non-zero one, e.g. by introducing a non-zero 13 element [21] or a non-zero 22 element, as e.g. in refs. [22, 23] (for a review of different schemes, see [24]). However, the introduction of new parameters reduces the predictive power, and, in addition, these modifications generically do not satisfy the *decoupling* feature.

On the other hand, instead of decreasing the number of zero entries, one can think to break the symmetricity condition. Namely, an asymmetry in the 23 blocks, $|M_{23}^{u,d}| \neq |M_{32}^{u,d}|$ can be introduced in the mass matrices [25, 26, 27, 28]. In other words, one can consider the Yukawa textures of the form

$$Y_f = F_f' \widetilde{Y}_f F_f \,, \qquad \widetilde{Y}_f = \begin{pmatrix} 0 & A_f & 0 \\ A_f & 0 & x_f B_f \\ 0 & x_f^{-1} B_f & C_f \end{pmatrix}, \quad f = u, d, e \tag{13}$$

where the matrices F, F' contain only phases and $\widetilde{Y}$ are the real matrices, with $x_{u,d,e}$ being deformation parameters. It is worth noting that such a modification preserves the main properties of the original Fritzsch texture. In particular, it satisfies the *decoupling* property, and in addition the hierarchy relations (10) still remain valid.

As it was shown in a recent analysis in ref. [29], such a texture is perfectly compatible with the observed pattern of the CKM matrix. Namely, all defects of the original texture can be corrected at once provided that the upper quark matrix exactly has a 'symmetric' Fritzsch texture, that is $x_u = 1$, and only the down quark matrix is deformed by a factor $x_d = 3$ or so. In fact, such a modification leads to the correct values of the CKM mixing angles as well as of the CP-violating phase as functions of the fermion mass ratios and of the two phase factors [29].

In next section we describe how the Fritzsch texture can be obtained within the context of the inter-family gauge group $SU(3)_H$, and how it can be minimally deformed in the 2-3 blocks by using a scalar field in adjoint (octet) representation of $SU(3)_H$. In section 4 we analyse the predictions of Fritzsch textures in light of the present high precision determinations of quark masses and CKM matrix elements and show that this flavour structure predicts the mixing angles and the CP-violating phase in perfect agreement with the experimental results.

3 Fritzsch-like textures from horizontal symmetry $\mathbf{SU(3)_H}$

The key for understanding the replication of families, the fermion mass hierarchy and the mixing pattern may lie in symmetry principles. For example, one can assign to the fermion species different charges of an abelian global flavor symmetry $U(1)$ [30]. There are also models making use of an anomalous gauge symmetry $U(1)_A$ to explain the fermion mass hierarchy while also tackling other naturalness issues [31, 32, 33, 34, 35]. However, it is difficult to obtain the highly predictive quark mass matrices with the texture zeros within this approach.

One can point to a more complete picture by introducing a non-abelian horizontal gauge symmetry $SU(3)_H$ between the three families [14, 15, 16, 17, 36]. This symmetry should have a chiral character, with the LH and RH components of quarks (and leptons) transforming in different representations of the family symmetry. In particular, they can be arranged respectively as the triplet and anti-triplet representations of $SU(3)_H$ which in our notation means that LH fermions q, ℓ as well as anti-fermions u^c, d^c, e^c are $SU(3)_H$ triplets:

$$q_i,\, u_i^c, d_i^c \sim 3\,; \qquad \ell_i,\, e_i^c \sim 3 \tag{14}$$

with $i = 1, 2, 3$ being the family $SU(3)_H$ index. Clearly, in this case the fermion direct Yukawa couplings to the Higgs doublet is forbidden by $SU(3)_H$, which means that their masses cannot be induced without the breakdown of the horizontal symmetry. On the contrary, a vector-like $SU(3)_H$, with $q, \ell \sim 3$ and $u^c, d^c, e^c \sim \bar{3}$, would allow the Yukawa couplings inducing a degenerate fermion mass spectrum between the three families, without breaking $SU(3)_H$.

In addition, the vector-like $SU(3)_H$ would not allow the grand unified extensions of the SM as $SU(5)$ [37] or $SO(10)$ [38], while the chiral arrangement (14) is compatible with the GUT structures (see e.g. [44] for a review on fermion patterns in GUTs). In particular, in $SU(5)$ each family is represented by the LH spinors in $\bar{5}$ and 10 representations. Thus, in the context of $SU(5) \times SU(3)_H$ all fermion species in (14) are embedded in the following representations [15, 16, 17]:[5]

$$\bar{F}_i = (d^c,\, \ell)_i \sim (\bar{5}, 3), \qquad T_i = (u^c,\, q,\, e^c)_i \sim (10, 3)\,. \tag{15}$$

As for $SO(10)$ GUT, all fermions of one family can be packed into the 16-dimensional spinor representation of $SO(10)$, $\Psi = (\bar{F}, T, \nu^c)$, along with the "right-handed neutrino" $\nu_L^c = C\overline{\nu_R}^T$. Hence, in the context of $SO(10) \times SU(3)_H$ all three families compose the unique multiplet $\Psi_i = (\bar{F}, T, \nu^c)_i \sim (16, 3)$.[6] With this set of fermions, $SU(3)_H$ would have triangle anomalies. For their cancellation one can introduce additional chiral fermions which are SM singlets but transform nontrivially under $SU(3)_H$ [14, 15, 16], and they can be used for neutrino mass generation [40, 41, 42]. The easiest way to cancel the anomalies, suggested in ref. [25], is to share the $SU(3)_H$ symmetry with a parallel mirror sector of particles having exactly the same

[5] $SU(5) \times SU(3)_H$ can be embedded into $SU(8)$ group [18] but this possibility introduces in the particle spectrum some extra fermions in exotic representations.

[6] More generically, in the SM context the maximal chiral symmetry which can be achieved in the limit of vanishing Yukawa couplings (1) is $U(3)^5$ independently transforming the different species q, u^c, d^c, ℓ, e^c. In the context of $SU(5)$, the maximal flavor symmetry is reduced to $U(3)^2 = U(3)_{\bar{F}} \times U(3)_T$ independently transforming fermionic $\bar{5}$- and 10-plets, while the $SO(10)$ structure allows a unique flavor symmetry $U(3)_H$ from which we gauge a non-abelian $SU(3)_H$ part, with chiral species arranged as in (14).

physics as ordinary particles (for a review, see e.g. [43, 44, 45]). In this case $SU(3)_H$ anomalies will be cancelled between ordinary fermion species and their mirror counterparts of opposite chiralities.

The chiral character of the fermion representations (14) forbids their direct Yukawa couplings of with the Higgs doublet ϕ, so that the fermion masses cannot be induced without breaking $SU(3)_H$. As far as the fermion bilinears $u_i^c q_j$, $d_i^c q_j$ and $e^c \ell$ transform in representations $3 \times 3 = 6 + \bar{3}$, the fermion masses can be induced only via the higher order operators

$$\sum_n \left(C_n^{(u)} \frac{\chi_n^{ij}}{M} u_i^c q_j \phi + C_n^{(d)} \frac{\chi_n^{ij}}{M} d_i^c q_j \tilde{\phi} + C_n^{(e)} \frac{\chi_n^{ij}}{M} e_i^c \ell_j \tilde{\phi} \right) + \text{h.c.} \tag{16}$$

involving some amount n of horizontal scalars χ_n (coined as flavons) in symmetric (anti-sextets $\chi^{\{ij\}} \sim \bar{6}$) or antisymmetric (triplets $\chi^{[ij]} = \epsilon^{ijk}\chi_k \sim 3$) representations of $SU(3)_H$ which are gauge singlets of the SM. Here M is some effective scale and the Wilson coefficients $C_n^{(f)}$ ($f = u, d, e$) are generically complex. After inserting the flavon VEVs, the operators in eq. (16) give rise to the SM Yukawa couplings (1) as $Y_f^{ij} = \sum_n C_n^{(f)} \langle \chi_n^{ij} \rangle / M$. In a sense, these operators "project" the VEV pattern of the flavon fields χ onto the structure of the Yukawa matrices. In the UV-complete picture such operators can be induced in a seesaw-like manner, via integrating out some extra heavy scalars [16, 17] or extra heavy fermions in vector-like representations [14, 15] living at the mass scale $\sim M$.

Interestingly, besides being invariant under the local $SU(3)_H$ symmetry by construction, these operators (16) exhibit in fact a larger global symmetry $U(3)_H = SU(3)_H \times U(1)_H$. Namely, they are invariant also under an accidental global chiral $U(1)_H$ symmetry, implying the overall phase transformation of fermions $(u_i^c, d_i^c, q_i) \to e^{i\omega}(u_i^c, d_i^c, q_i)$ and flavon scalars $\chi_n \to e^{-2i\omega}\chi_n$. Hence, all families can become massive only if $U(3)_H$ symmetry is fully broken.

This feature allows to relate the fermion mass hierarchy and mixing pattern with the breaking steps of $U(3)_H$ symmetry, with a natural realization of the *decoupling* hypothesis. When $U(3)_H$ breaks down to $U(2)_H$, the fermions of the third fermion family get masses while the first two families remain massless and all mixing angles are vanishing. At the next step, when $U(2)_H$ breaks down to $U(1)_H$, the second family acquires masses and the CKM mixing angle θ_{23} can be non-zero, but the first family remains massless ($m_u, m_d = 0$) and unmixed with the heavier fermions ($\theta_{12}, \theta_{13} = 0$). Only at the last step, when $U(1)_H$ is broken, also the first family can acquire masses and its mixing with heavier families can emerge. In this way, the inter-family mass hierarchy can be related to the hierarchy of flavon VEVs inducing the horizontal symmetry breaking $U(3)_H \to U(2)_H \to U(1)_H \to$ *nothing*.

In the last step of this breaking chain, the chiral global $U(1)_H$ symmetry can be associated with the Peccei-Quinn symmetry provided that $U(1)_H$ is also respected by the Lagrangian of the flavon fields [15, 36]. This can be achieved by forbidding the trilinear terms between the χ-scalars by means of a discrete symmetry. Thus, in this framework, the Peccei-Quinn symmetry can be considered as an accidental symmetry emerging from the local symmetry $SU(3)_H$. In this case the axion will have non-diagonal couplings between the fermions of different families [14, 15, 36]. Phenomenological and cosmological implications of gauge family symmetry with such flavor-changing axion were discussed in refs. [46, 47, 48, 49, 50, 51].

In scenarios with horizontal $SU(3)_H$ symmetry the Fritzsch textures can be naturally obtained by a suitable choice of the representations and VEV configurations of the χ-flavons. As the simplest set, one can take two triplets χ_1, χ_2, and one anti-sextet χ_3, with their VEVs in the following form [15]:

$$\langle \chi_3^{\{ij\}} \rangle = \mathrm{diag}(0,0,V_3) \qquad \langle \chi_{2i} \rangle = \begin{pmatrix} V_2 \\ 0 \\ 0 \end{pmatrix} \qquad \langle \chi_{1i} \rangle = \begin{pmatrix} 0 \\ 0 \\ V_1 \end{pmatrix}, \tag{17}$$

i.e. the VEV of χ_3 is given by a symmetric rank-1 matrix which for a convenience can be directed towards the 3rd axis in the $SU(3)_H$ space. As for the VEVs of χ_1 and χ_2, they are respectively parallel and orthogonal to $\langle \chi_3 \rangle$ (so that $\langle \chi_3 \rangle$ can be oriented towards the 3rd axis without losing generality; the detailed analysis of the flavon scalar potential is given in ref. [16]). Thus, the total matrix of flavon VEVs has the Fritzsch texture:

$$\langle \chi^{ij} \rangle = \left\langle \chi_1^{[ij]} + \chi_2^{[ij]} + \chi_3^{\{ij\}} \right\rangle = \begin{pmatrix} 0 & V_1 & 0 \\ -V_1 & 0 & V_2 \\ 0 & -V_2 & V_3 \end{pmatrix}. \tag{18}$$

After the $SU(3)_H$ breaking, the theory reduces to the SM containing one standard Higgs doublet ϕ, and the 'projective' operators (16) transfer the Fritzsch-like pattern of the VEV matrix (18) to the Yukawa matrices $Y_f = \sum_n C_n^{(f)} \langle \chi_n \rangle / M \quad (f = u,d,e)$, modulo different constants $C_n^{(u)}$, $C_n^{(d)}$ and $C_n^{(e)}$ in the expansion (16).[7] (The $-$ signs between off-diagonal terms are irrelevant since they can be eliminated by quark phase transformations.) Namely, after rotating away the complex phases, the Yukawa matrices acquire the 'symmetric' Fritzsch forms $\widetilde{Y_f}$ as in eq. (9). The inter-family mass hierarchies can be related to a hierarchy in the horizontal symmetry breaking chain $U(3)_H \to U(2)_H \to U(1)_H \to$ *nothing*, that is $V_3 \gg V_2 \gg V_1$. In particular, the hierarchies (10) between the Yukawa entries can be originated from the hierarchy of the flavon VEVs which can be estimated as

$$V_3 : V_2 : V_1 \simeq 1 : 1/5 : 1/100 \implies U(3)_H \stackrel{V_3}{\to} U(2)_H \stackrel{V_2}{\to} U(1)_H \stackrel{V_1}{\to} \text{nothing}\,. \tag{19}$$

In the context of $SU(5) \times SU(3)_H$ theory [17, 16, 15], with the quarks and leptons in representations (15), one can consider the $SU(5)$-invariant effective operators

$$\sum_n \left(C_n^{(10)} \frac{\chi_n^{ij}}{M} T_i T_j H + C_n^{(5)} \frac{\chi_n^{ij}}{M} \bar{F}_i T_j \bar{H} \right) + \text{h.c.} \tag{20}$$

where H is the scalar 5-plet which contains the SM Higgs doublet ϕ. In fact, after $SU(5)$ breaking down to the SM, the last two terms in (16) emerge from the decomposition of the last term in (20). Then, if all flavons χ_n are $SU(5)$ singlets, the Yukawa constants of down quarks and charged leptons would be identical, $Y_e = Y_d^T$. Though $b - \tau$ unification is a successful prediction of $SU(5)$, for the first two families this would imply the wrong relations $m_b : m_s : m_d = m_\tau : m_\mu : m_e$, in obvious contradiction with the observed pattern given in (6). In addition, in the first term of (20) the antisymmetric flavons cannot contribute by symmetry.

[7] Thus, in this construction the flavor is naturally conserved in neutral currents, in difference from the Fritzsch's original model [11, 12] involved several Higgs doublets.

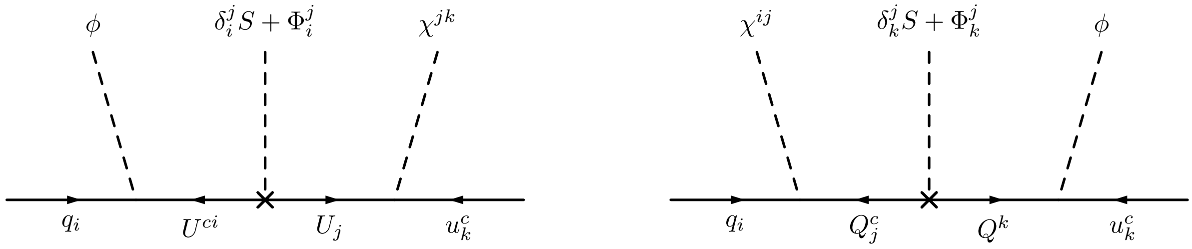

Figure 2: Seesaw diagrams inducing the Yukawa couplings of upper quarks via exchange of vector-like quarks U, U^c and Q, Q^c. Analogous diagrams with $u^c \to d^c$, $U, U^c \to D, D^c$ and $\phi \to \tilde{\phi}$ will work for down quarks.

Hence, one has $A_u, B_u = 0$ and so u and c quarks remain massless. However, this shortcoming can be avoided by assuming that some (at least one) of the flavon triplets transforms as 24-plet of $SU(5)$ [15].[8] This can avoid the wrong relation $Y_e = Y_d^T$ between the down quark and lepton Yukawas, and also induce the non-zero A_u and B_u in Y_u.

Let us turn back to the general case of the SM, with the quark and lepton species in $SU(3)_H$ representations (14). In the UV-complete pictures the operators (16) can be induced via renormalizable interactions after integrating out some extra heavy fields, scalars [16, 17], or verctor-like fermions [14, 15] (see also [36]). The latter possibility is more economic and natural, and we shall exploit it in our further considerations.[9] Namely, one can introduce the following vector-like set of chiral LH fermions of the up- and down-quark type in weak doublet and singlet representations

$$Q^i = (\mathcal{U}, \mathcal{D})^i,\, U^{ci},\, D^{ci} \sim \bar{3} \qquad Q_i^c = (\mathcal{U}^c, \mathcal{D}^c)_i,\, U_i\, D_i \sim 3\,. \tag{21}$$

Then the matrices of Yukawa couplings $Y_{u,d}$ in eq. (1) are induced after integrating out these fermion species with large mass terms. Analogously, the charged lepton Yukawa couplings can be induced by using the vector like lepton states, weak singlets $E_i \sim 3$, $E^{ci} \sim \bar{3}$ and weak doublets $L^i \sim \bar{3}$, $L_i^c \sim 3$. In particular, for the upper quarks this seesaw-like mechanism is illustrated by two diagrams in fig. 2 while the analogous diagrams will work for the down quarks and leptons. For the heavy vector-like species having $SU(3)_H$ invariant masses, this mechanism will induce the operators (16) which 'project' the Fritzsch-texture (18) of the flavon VEVs on the quark and lepton Yukawa matrices.

However, by using the diagrams of fig. 2, one can also achieve a realistic deformation of the Fritzsch ansatz in the following way. Let us assume that the the masses of vector-like fermions (21) are induced by spontaneous symmetry breaking from the VEVs of some scalars. Their mass terms transform as $\bar{3} \times 3 = 1 + 8$ and so one can introduce the Yukawa couplings of D, D^c

[8] More generally, the effective coefficients C_n in (20) can be functions of a scalar 24-plet Σ which breaks $SU(5)$ down to the SM $SU(3) \times SU(2) \times U(1)$ [39].

[9] In particular, in the context of supersymmetric theories with $SU(3)_H$ family symmetry this mechanism can lead to interesting relations between the fermion Yukawa couplings and the soft SUSY breaking terms which naturally realize the minimal flavor violation scenarios [25, 52, 53, 54].

fermions

$$(g_D S\delta^i_j + f_D \Phi^i_j) D_i D^{cj} + \text{h.c.} \tag{22}$$

and analogous couplings of U, U^c and Q^c, Q species, with the scalars in singlet and octet representations of $SU(3)_H$, $S \sim 1$ and $\Phi \sim 8$. In fact, one can introduce an adjoint scalar Φ of $SU(3)_H$ in analogy to the adjoint scalar of $SU(5)$, the 24-plet Σ, which possibility was discussed in refs. [25, 27]. We also assume that the cross-interaction terms of Φ with χ-flavons in the scalar potential align the VEV of Φ towards the largest VEV V_3 in (18), i.e. the VEV is proportional to the λ_8 generator of $SU(3)_H$: $\langle\Phi\rangle = V\,\text{diag}(1,1,-2)$, and it breaks the $SU(3)_H$ symmetry down to $SU(2)_H \times U(1)_H$. Then, the masses of the heavy species are contributed by both the singlet and the octet VEVs and their mass matrices have diagonal forms, namely $\mathrm{M}_D = g_D\langle S\rangle + f_D\langle\Phi\rangle = M_D\,\text{diag}(X_D, X_D, 1)$, and analogously for U and Q species, where generically $X_{U,D,Q} \neq 1$ are complex numbers. In this way, one obtains the Yukawa matrices with the minimally deformed Fitzsch textures (13), where the 'asymmetries' $x_{u,d}$ are introduced between 23 and 32 elements of $\widetilde{Y}_{u,d}$, without changing the amount of zero elements.

In the case of $SU(5) \times SU(3)_H$ [15], with ordinary quarks and leptons arranged as in (15), the vector-like fermions can be introduced in the representations

$$\mathcal{T}^i = (U^{ci}, Q^i, E^{ci}) \sim (10, \bar{3}), \qquad \overline{\mathcal{T}}_i = (U_i, Q^c_i, E_i) \sim (\overline{10}, 3)$$
$$\overline{\mathcal{F}}^i = (D^{ci}, L^i) \sim (\bar{5}, \bar{3}), \qquad \mathcal{F}_i = (D_i, L^c_i) \sim (5, 3) \tag{23}$$

which include all vector-like quarks in (21) together with lepton states L, L^c and E^c, E.

In principle, all Yukawa couplings $Y_{u,d,e}$ can be induced by diagrams similar to that of fig. 2 involving $\mathcal{T}, \overline{\mathcal{T}}$ states, while $Y_{d,e}$ can be contributed also by exchange of $\mathcal{F}, \overline{\mathcal{F}}$ states [15]. Using these tools, in ref. [29] the model was built which gives the Yukawa matrices (13) in a particularly constrained form. Namely, the (de-phased) up quark Yukawas is predicted to have a 'symmetric' Fritzsch texture, $x_u = 1$:

$$\widetilde{Y}_u = \begin{pmatrix} 0 & A_u & 0 \\ A_u & 0 & B_u \\ 0 & B_u & C_u \end{pmatrix} \tag{24}$$

while for the down quarks and leptons one has deformed patterns with $x_{d,e} \neq 1$:

$$\widetilde{Y}_d = \begin{pmatrix} 0 & A_d & 0 \\ A_d & 0 & x_d B_d \\ 0 & x_d^{-1} B_d & C_d \end{pmatrix}, \qquad \widetilde{Y}_e^T = \begin{pmatrix} 0 & A_e & 0 \\ A_e & 0 & x_e B_e \\ 0 & x_e^{-1} B_e & C_e \end{pmatrix} \tag{25}$$

We shall not describe the model of ref. [29] in details here, but only briefly mention its key ingredientts:
(i) flavon fields χ_1 and χ_3 are taken as $SU(5)$ singlets, i.e. $\chi_1 \sim (1,3)$ and $\chi_3 \sim (1,\bar{6})$, while χ_2 is taken in a mixed representation $\chi_2 \sim (24,3)$, with the horizontal directions of the VEVs as in (17) with hierarchy as in (19) between their values. This 'mixed' choice of χ_2 induces non-zero B_u in (24). It also introduces $SU(5)$ breaking Clebsch factors in 23 blocks of (25), so

that $B_d \neq B_e$, thus avoiding the wrong scaling relation $m_b : m_s : m_d = m_\tau : m_\mu : m_e$ between down quark and lepton masses. But $SU(5)$ invariance for the other two entries is kept, i.e. $C_d = C_e$ and $A_d = A_e$, which in turn implies the successful relation (11).
(ii) heavy 10-plets $\mathcal{T}, \overline{\mathcal{T}}$ get the dominant mass contribution from the Yukawa coupling with a field S which is singlet of $SU(5) \times SU(3)_H$, with a subdominant contribution from the $SU(5)$ adjoint Higgs Σ. This explains the smallness of A_u/C_u with respect to A_d/C_d (see eq. (10); in fact, without the latter contribution of Σ, we would have $A_u = 0$;
(iii) the masses of heavy 5-plets $\mathcal{F}, \overline{\mathcal{F}}$ are strongly contributed by the Yukawa couplings with the $SU(3)_H$ adjoint scalar Φ which induces the deformation factors $x_{d,e} \neq 1$.

In the next section we mainly concentrate on the implications of the textures in (24) and (25) for the quark sector, following the analysis performed in ref. [29], but having in mind that in the context of grand unification analogous considerations can be extended to leptons.

4 Deformed Fritzsch textures: CKM matrix vs. quark mass ratios

Let us consider the Yukawa matrices $\mathrm{Y}_{u,d}$ having the form (13). They can be brought to the diagonal forms

$$Y_u^{\rm diag} = {\rm diag}(y_u, y_c, y_t), \qquad Y_d^{\rm diag} = {\rm diag}(y_d, y_s, y_b) \tag{26}$$

via biunitary transformations as in eq. (2) with the unitary matrices $V_u = F_u O_u$ and $V_d = F_d O_d$, where $F_{u,d}$ are the diagonal matrices containing phases, and $O_{u,d}$ are the real orthogonal matrices. In other terms, by phase transformations $F'_{u,d} Y_{u,d} F_{u,d}$ with $F_{u,d} = {\rm diag}(e^{i\alpha_{u,d}}, e^{i\beta_{u,d}}, e^{i\gamma_{u,d}})$ (and similarly $F'_{u,d}$), the Yukawa matrices $Y_{u,d}$ are brought to the real forms $\widetilde{Y}_{u,d}$ which can be further diagonalized by bi-orthogonal transformations $O'_{u,d} \widetilde{Y}_{u,d} O_{u,d} = Y_{u,d}^{\rm diag}$.

Thus, for the CKM matrix of quark mixing we obtain

$$V_{\rm CKM} = V_u^\dagger V_d = O_u^T F O_d = \; O_u^T \begin{pmatrix} e^{i(\tilde\beta+\tilde\delta)} & 0 & 0 \\ 0 & e^{i\tilde\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix} O_d \,, \tag{27}$$

where the matrix $F = F_u^\dagger F_d$ without loss of generality can be parameterized by the two phases $\tilde\beta$ and $\tilde\delta$ while the orthogonal matrices $O_{u,d}$ can be parametrized as

$$O_d = O_{d23} O_{d13} O_{d12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23}^d & s_{23}^d \\ 0 & -s_{23}^d & c_{23}^d \end{pmatrix} \begin{pmatrix} c_{13}^d & 0 & s_{13}^d \\ 0 & 1 & 0 \\ -s_{13}^d & 0 & c_{13}^d \end{pmatrix} \begin{pmatrix} c_{12}^d & s_{12}^d & 0 \\ -s_{12}^d & c_{12}^d & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{28}$$

with $c_{ij}^d = \cos\theta_{ij}^d$ and $s_{ij}^d = \sin\theta_{ij}^d$. Analogously, for up-quarks we have $O_u = O_{u23} O_{u13} O_{u12}$, as well as for the rotations of right-handed states $O'_{u,d}$.

Hence, $\widetilde{Y}_d$ contains four parameters, A_d, B_d, C_d and x_d, which determine the three Yukawa eigenvalues $y_{d,s,b}$ and the three rotation angles in O_d. Analogously, the four parameters in $\widetilde{Y}_u$ determine the Yukawa eigenvalues $y_{u,c,t}$ and the three angles in O_u. Therefore, we have 10 real parameters $A_{u,d}, B_{u,d}, C_{u,d}, x_{u,d}$ and two phases $\tilde{\beta}, \tilde{\delta}$ which have to match 10 observables, the 6 Yukawa eigenvalues and 4 independent parameters of the CKM matrix (see eq. (4)).

The Yukawa eigenvalues and rotation matrices O and O' can be found by considering the "symmetric" squares respectively of the Yukawa matrices $\widetilde{Y}_f^T \widetilde{Y}_f$ and $\widetilde{Y}_f \widetilde{Y}_f^T$, $f = u, d$. In doing so, we obtain the following relations

$$\begin{aligned} &C^2 + (x^2 + x^{-2})B^2 + 2A^2 = Y_3^2 + Y_2^2 + Y_1^2 \\ &B^4 + 2C^2A^2 + (x^2 + x^{-2})B^2A^2 + A^4 = Y_3^2Y_2^2 + Y_3^2Y_1^2 + Y_2^2Y_1^2 \\ &A^2C = Y_1Y_2Y_3 \end{aligned} \tag{29}$$

where we omit the indices $f = u, d$ and imply $Y_{1,2,3} = y_{u,c,t}$ for the Yukawa eigenvalues of upper quarks and $Y_{1,2,3} = y_{d,s,b}$ for down quarks.

Having in mind the relations (6), we can use them in the synthetic form $y_t : y_c : y_u \sim 1 : \epsilon_u : \epsilon_u^2$ and $y_b : y_s : y_d \sim 1 : \epsilon_d : \epsilon_d^2$, noting that phenomenologically $\epsilon_u \sim \epsilon_d^2$. Then, in leading approximation (up to relative corrections of order $\epsilon \sim Y_2/Y_3 \sim Y_1/Y_2$) we have [26, 29]:

$$C \approx Y_3, \quad B \approx \sqrt{Y_2Y_3}, \quad A \approx \sqrt{Y_1Y_2} \tag{30}$$

so that $C_f : B_f : A_f \sim 1 : \epsilon_f^{1/2} : \epsilon_f^{3/2}$. Since the ratios $C_d : B_d : A_d \sim V_3 : V_2 : V_1$ in fact reflect the hierarchy in the horizontal symmetry breaking (19) (and also taking into account that the off-diagonal elements in Y_u have additional suppression in the context of the $SU(5) \times SU(3)_H$ model [29]), this means that the inter-family mass hierarchy can actually be induced by a milder hierarchy between the flavon VEVs. The matrix entries A_f, B_f and C_f depend on the deformation x_f only at higher orders in ϵ_f. As regards the rotation matrices, the angles in (28), up to small corrections of order ϵ_d, are given by

$$s_{23}^d \approx \frac{1}{x_d}\sqrt{\frac{y_s}{y_b}}, \quad s_{12}^d \approx \sqrt{\frac{y_d}{y_s}}, \quad s_{13}^d \approx x_d \, \frac{y_s}{y_b}\sqrt{\frac{y_d}{y_b}}\,. \tag{31}$$

The expressions for s'_{d23}, s'_{d12} and s'_{d13} are the same with the replacement $x_d \to x_d^{-1}$, Analogously, the up quark rotation angles in O_u can be expressed in terms of Yukawa ratios y_u/y_c and y_c/y_t. Then, at leading order, for the elements of we obtain:

$$\begin{aligned} &|V_{us}| \approx \left| s_{12}^d - s_{12}^u \, e^{i\tilde{\delta}} \right| \approx |V_{cd}|\,, \qquad |V_{cb}| \approx \left| s_{23}^d - s_{23}^u \, e^{i\tilde{\beta}} \right| \approx |V_{ts}| \\ &|V_{ub}| \approx \left| s_{13}^d \, e^{i\tilde{\delta}} - s_{12}^u \, \left(s_{23}^d - s_{23}^u \, e^{-i\tilde{\beta}} \right) \right| \,, \quad |V_{td}| \approx \left| s_{13}^d \, e^{i\tilde{\delta}} - s_{12}^d \, \left(s_{23}^d - s_{23}^u \, e^{i\tilde{\beta}} \right) \right| \,. \end{aligned} \tag{32}$$

It can be noticed that for fixed values of the asymmetries x_d, x_u, V_{us} depends on the phase $\tilde{\delta}$ while V_{cb} only on the phase $\tilde{\beta}$. It is also worth noting that for $x_d = 1$, the contribution of s_{13}^d in $|V_{ub}|$ is negligible and the Fritzsch texture implies the prediction $|V_{ub}/V_{cb}| \approx \sqrt{y_u/y_c}$. Similar considerations can be inferred for the other off-diagonal elements, with the prediction $|V_{td}/V_{ts}| \approx \sqrt{y_d/y_s}$ for $x_d = 1$. As regards the complex part of $V_{\rm CKM}$, we can consider the rephasing-invariant quantity $J = -{\rm Im}(V_{us}^* V_{cb}^* V_{ub} V_{cs})$, the Jarlskog invariant. In our scenario we have

$$\begin{aligned} J = &- \sin\tilde{\delta}\, s_{12}^u s_{12}^d \, \left[(s_{23}^d)^2 \, c_{23}^d c_{12}^d - 2\cos\tilde{\beta}\, s_{23}^d s_{23}^u + (s_{23}^u)^2 \right] + \\ &+ \left(\sin\tilde{\delta} s_{12}^u s_{23}^d + \sin\tilde{\beta} s_{12}^d s_{23}^u \right) s_{13}^d + \mathcal{O}(\epsilon_d^4)\,. \end{aligned} \tag{33}$$

In the following we are going to test the viability of asymmetric Fritzsch textures given the present precision of experimental data on moduli and phases of the mixing elements. The values of the CKM matrix parameters and their uncertainties are given in table 1.

Observable	Value	Observable	Value	Parameter	Global fit value [3]
$\lvert V_{ud}\rvert$	0.97373(31)	$\lvert V_{cs}\rvert$	0.975(6)	$\sin\theta_{12}$	0.22500 ± 0.00067
$\lvert V_{us}\rvert$	0.2243(8)	$\lvert V_{cd}\rvert$	0.221(4)	$\sin\theta_{23}$	$0.04182^{+0.00085}_{-0.00074}$
$\lvert V_{ub}\rvert$	0.00382(20)	$\lvert V_{td}\rvert$	0.0086(2)	$\sin\theta_{13}$	0.00369 ± 0.00011
$\lvert V_{cb}\rvert$	0.0408(14)	$\lvert V_{ts}\rvert$	0.0415(9)	J	$(3.08^{+0.15}_{-0.13}) \times 10^{-5}$
$\lvert V_{ub}/V_{cb}\rvert$	0.084(7)	$\lvert V_{td}/V_{ts}\rvert$	0.207(3)	δ	1.144 ± 0.027

Table 1: Independent determinations of the CKM elements and result of the global fit of four CKM parameters with the constraints implied by the unitarity of $V_{\rm CKM}$, as reported by PDG [3].

The input values in our analysis are the ratios of the Yukawa eigenvalues. More specifically, we want to reproduce the CKM elements as functions of the Yukawa couplings assuming that the matrices $Y_{u,d}$ acquire the Fritzsch form at some high energy scale of new physics. Thus, we need to consider the evolution of Yukawa matrices according to the renormalization group (RG) equations. For that purpose, first we bring the running masses of all quarks to the scale $\mu = m_t \approx v_w$, using the QCD renormalization factors (see e.g. in [55]) and determine their ratios (i.e. the Yukawa constant ratios) at this scale. In doing so we obtain:

$$\begin{aligned} m_d/m_s &= (20.17 \pm 0.27)^{-1}\,, \qquad & m_s/m_b &= (53.94 \pm 0.12)^{-1} \\ m_u/m_c &= (498 \pm 21)^{-1}\,, \qquad & m_c/m_t &= (272.3 \pm 2.6)^{-1} \end{aligned} \tag{34}$$

Here we used the PDG data on the top quark mass and the precision results of the lattice QCD computations for the ratios $m_b/m_c = 4.579(9)$, $m_b/m_s = 53.94(12)$, $m_c/m_s = 11.768(34)$, $m_u/m_d = 0.477(19)$ and $m_s/m_{ud} = 27.31(10)$, where $m_{ud} = (m_u+m_d)/2$ [3]. In particular, from the two latter results we obtain the mass ratios of light quarks, $m_s/m_d = m_s/m_{ud}\,(m_u/m_d+1)/2$ – see fig. 3.

The obtained pattern of the quark mass ratios (34) can be compared to that of the lepton masses known with much greater precision: $m_e/m_\mu = 206.76852^{-1}$ and $m_\mu/m_\tau = 16.817^{-1}$. Then we observe that the following combinations of down quark and lepton masses coincide with extremely good precision:

$$\frac{\sqrt{m_e m_\mu}}{m_\tau} = 241.819^{-1}\,, \qquad \frac{\sqrt{m_d m_s}}{m_b} = (242.2 \pm 1.7)^{-1} \tag{35}$$

In the context of the Fritzsch-like textures (25) this relation implies $A_d/C_d = A_e/C_e$ which in turn stems from the $SU(5)$ symmetry in the context of $SU(5) \times SU(3)_H$ model in ref. [29].

Now we are going to test the viability of the textures in eq. (13) when $x_u = 1$ but $x_d \neq 1$, i.e. when $\tilde{Y}_u$ has a 'symmetric' form (24) while $\tilde{Y}_d$ is deformed as in (25). In this case we have 9 real parameters $A_{d,u}$, $B_{d,u}$, $C_{d,u}$, $\tilde{\beta}$, $\tilde{\delta}$ and x_d, which have to match 10 observables, the 6 Yukawa eigenvalues and the 4 independent parameters of the CKM matrix.

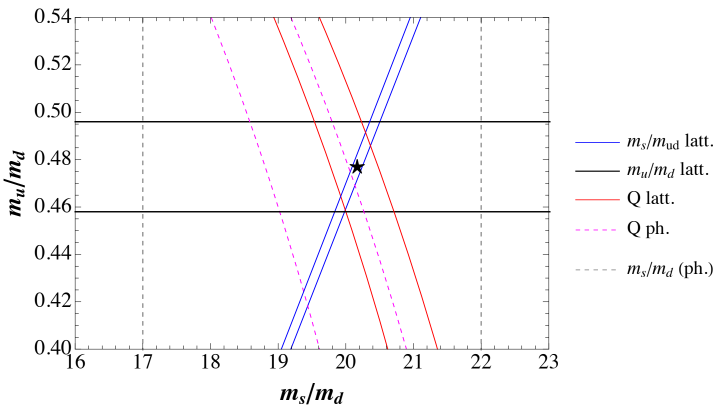

Figure 3: Light quarks mass ratios. Black solid lines show the average of the lattice determinations of m_u/m_d; blue lines represent the average of the lattice determinations of the ratio m_s/m_{ud}, $m_{ud} = (m_u + m_d)/2$; red lines are obtained from the relation $Q^2 = (m_s^2 - m_{ud}^2)/(m_d^2 - m_u^2)$, using lattice determinations of quark mass ratios. We also indicate the phenomenological result $Q = 22.1(7)$ [56] (dashed magenta) and 'old' limit $m_s/m_d = 17$–22 (dashed grey). The black star represents the central value $(m_u/m_d, m_s/m_d) = (0.477, 20.17)$.

In our numerical analysis the ratios of quark running masses at $\mu = m_t$ will be fixed as in eq. (6). However, we consider that the 'starting' scale (i.e. the mass scale of heavy fermions (21)) at which the Yukawa matrices acquire the Fritzsch form (13) can be much larger, say from 10^3 GeV to 10^{16} GeV.[10] For energy scales $\mu \gg m_t$, the quark mass ratios are no more given by (6) since their RG evolution will be influenced by additional contributions from the top Yukawa constant ~ 1. However, we anticipate that these effects are small and the obtained results are practically independent of the choice of the 'starting' scale μ [29].

The complete analysis was done in ref. [29]. In fig. 4 we show its results for a choice $x_d = 3.3$ which demonstrates that all CKM parameters can be obtained within their 1σ uncertainties for a proper choice of phases $\tilde{\delta}$ and $\tilde{\beta}$.

For having an insight of how it works, we can use the approximate relations in eqs. (32) and (31) (though in ref. [29] the compete analysis was done without any approximation). As it is apparent from eq. (32), the element $|V_{us}| = s_{12}$ (the Cabibbo angle) is fixed by the phase $\tilde{\delta}$ and has no dependence on $\tilde{\beta}$ in leading approximation. In other words, for the ratios y_d/y_s and y_u/y_c fixed as in (6), $\tilde{\delta} \approx \pm 2\pi/3$ is fixed by the value of s_{12} as shown in fig. 4(a). Conversely, $|V_{cb}| = s_{23}$ depends only on $\tilde{\beta}$, and thus, for the given x_d, the latter phase is fixed as $\tilde{\beta} \approx \pm\pi/3$ by the value of $|V_{cb}|$ – see fig. 4(b). The sign of $\tilde{\delta}$ and $\tilde{\beta}$ is fixed by the other observables, Namely, as one can see from the plots in fig. 4 the correct values of $|V_{ub}/V_{cb}|$ and J can be obtained only for both phases being on the negative side (see the green and yellow vertical bands in fig. 4 respectively for $\tilde{\delta}$ and $\tilde{\beta}$). In fact, the effect of the asymmetry parameter $x_d > 1$ is to decrease the rotation angle s_{23}^d while increasing s_{13}^d. This causes the prediction of $|V_{cb}|$ to shift towards lower values. In addition, the large value of s_{13}^d originates a dependence of $|V_{ub}|$

[10]The latter choice is natural in the context of the grand unified model $SU(5) \times SU(3)_H$, whereas the former is interesting since the quark mixing with the vector-like quarks (21) in the TeV mass range can be at the origin of the recently observed Cabibbo angle anomalies [57] and can also induce various flavor-changing phenomena accessible in the SM precision tests [58, 59, 60, 61, 62, 63] (see also ref. [64] for a review).

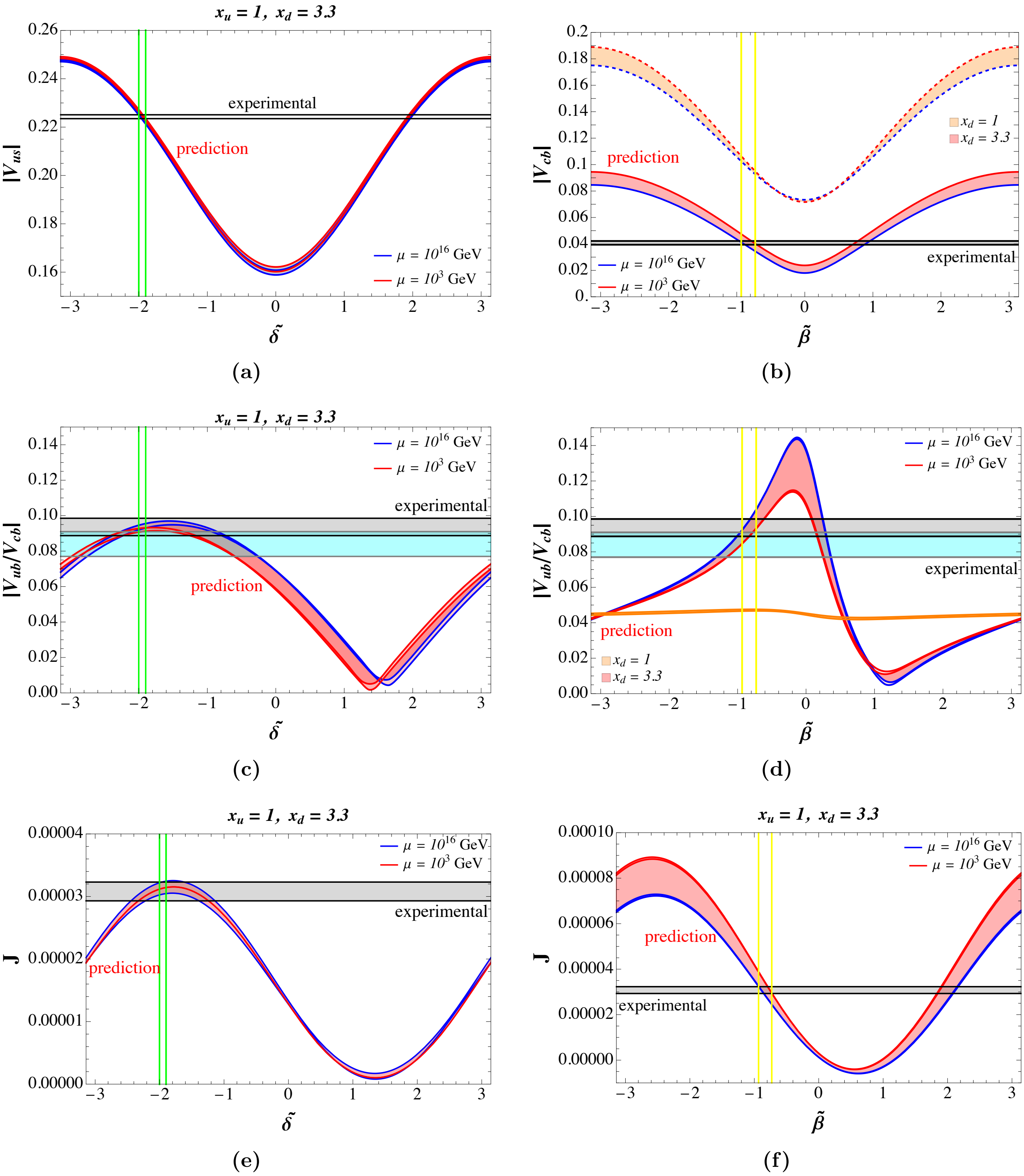

Figure 4: Predicted observables (red bands) for Fritzsch-like texture with $x_d = 3.3$, $x_u = 1$ confronted with experimental values (horizontal bands) at 1σ confidence level. The thickness of the bands of the prediction curves comprises the dependence on the starting scale $\mu = (10^3 \div 10^{16})$ GeV. In displaying the plots, the other variables are allowed to move inside the 1σ confidence region. We also indicate the 1σ interval of the phases $\tilde{\delta}$ and $\tilde{\beta}$ (green and yellow bands respectively). We also show the wrong predictions for $|V_{cb}|$ and $|V_{ub}/V_{cb}|$ with the symmetric Fritzsch textures (orange bands).

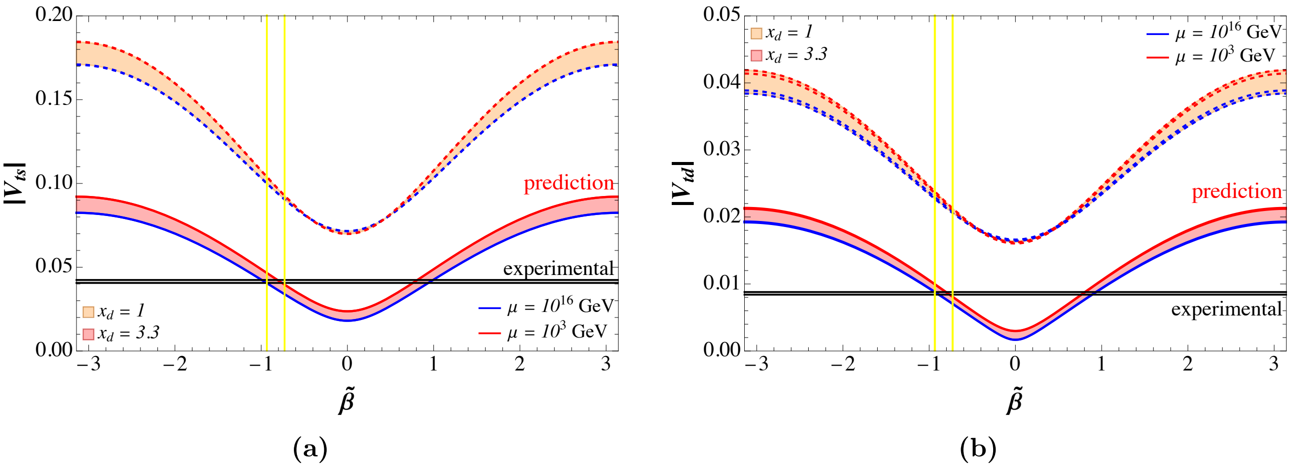

Figure 5: Predictions of V_{ts} and V_{td} with $x_d = 3.3$, $x_u = 1$. We also show the wrong predictions of the symmetric Fritzsch texture (orange region).

on both $\tilde{\delta}$ and $\tilde{\beta}$ so that for the 'negative' choices of these parameters the value of $|V_{ub}/V_{cb}|$ intercepts the experimental determination. We also show in fig. 5 how well the expectations on V_{td} and V_{ts} correspond to these phase parameters.

Concluding this discussion, we have shown that minimally deformed Fritzsch textures with $x_u = 1$ and $x_d \approx 3$ successfully predict all observables of the CKM matrix. The choice of $x_u = 1$ can be justified in the context of the grand unified picture $SU(5) \times SU(3)_H$, but x_d remains a free parameter [29]. It is tempting to think that $x_d = 3$ could also be originated from some more constrained scheme which is perhaps based on a larger GUT. Notice also that the results imply $\tilde{\delta} + \tilde{\beta} \approx -\pi$ which relation can be originated in the case of spontaneous CP-violation.

5 Discussion: variations on the theme

The hierarchy between the fermion masses and their weak mixing pattern remains a mystery. One can hope that some new physics beyond the SM can shed a light on these problems. However, a complete theory which could to be coined as quantum flavordynamics does not exist yet, and one can discuss some pieces of the puzzle. The approach of zero-textures put forward by Fritzsch [11, 12] is oriented towards the predictivity which in particular can relate the CKM mixing angles to the fermion mass ratios. Such predictive textures can be obtained within the models with the inter-family 'horizontal' symmetries as we described in this paper, but they do not address the origin of mass hierarchies: in fact, the hierarchies between the input parameters as in eq. (10) can be related to the hierarchy of the horizontal symmetry breaking as in (19), but the latter is in fact introduced at hands. The predictive power of the textures can be enhanced by a clever use of the grand unification theories as $SU(5)$ or $SO(10)$ unifying the fermions of one family. This can provide some successful relations between the quark and lepton masses as e.g. in eq. (11), but yet the explanation of the inter-family mass hierarchies remains beyond their reach.

The fermion mass spectrum schematically given in fig. 1 indeed looks very special. The only fermion with the mass of the order of electroweak scale $v_w = 174$ GeV is the top quark, implying the Yukawa coupling $y_t \simeq 1$. The other Yukawa couplings of the 3rd family are two orders of magnitude less, $y_b, y_\tau \sim 10^{-2}$ which indeed looks puzzling. Certainly, the coincidence of running couplings $y_b(\mu) \simeq y_\tau(\mu)$ at the GUT scale $\mu \simeq 10^{16}$ GeV, the so called $b-\tau$ unification, is a clear success of $SU(5)$, especially in its supersymmetric version, but the origin of $y_{b,\tau} \ll y_t$ still remains open. The question becomes more insistent in $SO(10)$ theory where t, b and τ are unified in one multiplet, implying the Yukawa unification $y_\tau = y_b = y_t$ at the GUT scale, and the only way to obtain the 'vertical' splitting in 3rd family masses, $m_{b,\tau} \sim 10^{-2} m_t$, is to introduce the large $\tan\beta$ at hands. The question is exacerbated by the fact that in the 2nd family the vertical splitting is much less, $m_{s,\mu} \sim 10^{-1} m_c$, Paradoxically, under the assumption of large $\tan\beta$, this would imply the Yukawa y_c an order of magnitude smaller than $y_{s,\mu}$, namely $y_s \sim y_\mu \sim 10 y_c$. This in turn implies $y_c \sim 10^{-2} y_{b,\tau}$, in obvious contrast with the mass values of c-quark (2nd family) and b-quark (3rd family) which in fact are of the same order, $m_c \sim m_{b,\tau}$ (see fig. 1). Moreover, at the normalization scale μ taken as the GUT scale, the masses of the first family become quasi-degenerate: $m_d/m_u \simeq 2$ while $m_e/m_u \simeq 1/2$, an additional difficulty for the large $\tan\beta$ scenario.

The overall pattern of fermion mass spectrum can be more naturally understood in the inverse hierarchy approach [14] which implies that the fermion mass hierarchies are inversely proportional to the hierarchy of the inter-family symmetry breaking (19). In fact, fig. 1 shows the following scaling laws for inverse masses:

$$m_u^{-1} : m_c^{-1} : m_t^{-1} \sim 1 : \epsilon_u : \epsilon_u^2, \qquad m_d^{-1} : m_d^{-1} : m_b^{-1} \sim 1 : \epsilon_d : \epsilon_d^2 \tag{36}$$

and similarly for leptons, with the near degeneracy $m_d \simeq m_u \simeq m_e$ which indicates that the Yukawa unification at the GUT scale may take place in the first rather than in the third family, i.e. $y_d = y_u = y_e$ instead of $y_b = y_t = y_\tau$. This in turn suggests the following pattern for the inverse Yukawa matrices [65, 66]:[11]

$$Y_f^{-1} = y^{-1}(P_1 + \epsilon_f P_2 + \epsilon_f^2 P_3), \quad f = u, d, e \tag{37}$$

where $\epsilon_f = \epsilon_u, \epsilon_d, \epsilon_e$ are small parameters and $P_{1,2,3}$ are symmetric rank-1 matrices with generically complex $O(1)$ elements. Without loosing generality, one can take $P_1 = (1,0,0)^T \bullet (1,0,0)$, $P_2 = (a,b,0)^T \bullet (a,b,0)$ and $P_3 = (x,y,z)^T \bullet (x,y,z)$. Therefore, the inverse Yukawa matrices have onion-like structures:

$$Y_f^{-1} = \frac{1}{y} \begin{pmatrix} 1 + a^2\epsilon_f & ab\epsilon_f & xz\epsilon_f^2 \\ ab\epsilon_f & b^2\epsilon_f & yz\epsilon_f^2 \\ xz\epsilon_f^2 & yz\epsilon_f^2 & z^2\epsilon_f^2 \end{pmatrix}, \qquad f = u, d, e \tag{38}$$

Such structure of the Yukawa matrices in the frame of $SO(10)$ symmetry was discussed in refs. [65, 66]. Three fermion families in representations 16^i, acquire masses via their seesaw-like mixing with extra vector-like fermions $16'_i + \overline{16}'_i$ while the latter get masses via couplings $16'_i\big[\chi_1^{ij} + (45/M)\chi_2^{ij} + (45/M)^2\chi_3^{ij}\big]\overline{16}'_j$. In the above 45 is a scalar in adjoint representation

[11] For quarks ($f = u, d$) such a pattern was first obtained in refs. [67, 68, 69] in the context of left-right symmetric models.

of $SO(10)$ having VEV $M_G \sim 10^{16}$ GeV (the GUT scale), $M > M_G$ is some large (string?) scale, and $\chi^{ij}_{1,2,3} \sim$ are flavon fields in symmetric (sextet) representations of $SU(3)_H$, having the rank-1 VEV configurations in family space (as $\langle\chi_3\rangle$ in (17)) disoriented by large angles [66]. In this way, after integrating out the heavy fermions, one obtains the quark and lepton Yukawa matrices of the form (37), with rank-1 projectors originated from the flavon VEVs: $P_n = \langle\chi_n\rangle/M$, and with small expansion parameters $\epsilon_f \sim M_G/M$. The VEV $\langle 45\rangle$ produces different Clebsch factors for different fermion species so that generically ϵ_u, ϵ_d and ϵ_e in (37) have different values. However, $SO(10)$ symmetry implies a remarkable relation between the expansion parameters:

$$\epsilon_e = -(\epsilon_d + 2\epsilon_u) \quad \Longrightarrow \quad \epsilon_e \approx -\epsilon_d \quad \text{since} \quad \epsilon_u \ll \epsilon_d \tag{39}$$

Eigenvalues of (38) reproduce the scaling pattern (36) implying that $\epsilon_u \ll \epsilon_d \ll 1$. However, the term $a^2\epsilon_d$ in Y_d cannot be neglected. Namely, for its eigenvalues we have $y_d \approx y/|1 + a^2\epsilon_d|$, $y_s \approx y|1 + a^2\epsilon_d|/|b^2\epsilon_d|$, $y_b \approx y/|z\epsilon_d|^2$ and similarly for leptons. However, eq. (39) implies that $y_d y_s \approx y_e y_\mu$ and $y_b \approx y_\tau$, and thus relation (35). As for up quarks, $a^2\epsilon_u$ is negligible, and we have $y_u \approx y$, $y_c \approx y/|b^2\epsilon_d|$, $y_t \approx y/|z\epsilon_u|^2$. In addition, the CKM mixing is dominated by the diagonalization of Y_d. Then, taking into account that $\epsilon_e \approx -\epsilon_d$, we get:

$$\frac{y_u}{y_d} \approx |1 + a^2\epsilon_d|, \quad \frac{y_u}{y_e} \approx |1 - a^2\epsilon_d|, \quad s_{12} \approx \frac{|ab\epsilon_d|}{|1 + a^2\epsilon_d|} \approx \sqrt{\frac{y_d}{y_u}}|a^2\epsilon_d| \tag{40}$$

Thus, the last relation for for the Cabibbo angle implies that $|a^2\epsilon_d| \simeq 1$ which allows to achieve mass splitting in first family, $m_u/m_d \approx 1/2$ and $m_u/m_e \approx 2$, by properly choosing the phase of $a^2\epsilon_d$. As for other mixing angles, one they naturally are $s_{23} \sim \epsilon_d$ and $s_{13} \sim \epsilon_d^2$ in a fine correspondence with (7).

In a completely different approach, the fermion mass pattern can be explained by exceptionally clever choice of the underlying GUT, even without introducing any type of flavor symmetry. Namely, supersymmetric grand unification based on $SU(6)$ [70] solves the fundamental problems of grand hierarchy and doublet-triplet splitting in elegant way, since the Higgs doublets emerge as pseudo-Goldstone modes of the accidental global symmetry $SU(6) \times SU(6)$ in the Higgs superpotential [70, 71, 72]. The mass parameters at low energies (soft supersymmetry breaking masses as well as supersymmetric μ-term) are all originated from the SUSY breaking terms. The pseudo-Goldstone nature of the Higgs has important consequences: it can have the renormalizable coupling in the superpotential, with the Yukawa constant $y_t \sim 1$, only to one of the three up-type quarks that can be identified as the top quark [73, 74]. For the rest of fermion species, the Yukawa couplings can emerge only from the higher order operators and thus are suppressed. Interestingly, the structure of these operators also leads to $y_c \sim y_{b,\tau} \ll y_t$, with a perfect match to small values of $\tan\beta$. As for the mass hierarchy between the 1st and 2nd families, it cannot be explained without help of flavor symmetries, but some discrete symmetries can be sufficient [74].

Postscriptum from Zurab Berezhiani

I was a Ph.D. student when I encountered the works of Harald Fritzsch [11, 12] which deeply influenced my further career. I started to think on the role of inter-family symmetry $SU(3)_H$ for understanding the fermion masses and mixings. The first time I met Harald personally was in 1990 in the last days of my one month visit to the Max-Planck Institute in Munich. During our conversations I discussed with him my papers [14, 15] in which I succeeded to realize the Fritzsch mass textures in a theoretically appealing way by relating them to the pattern of $SU(3)_H$ symmetry breaking. As a consequence, Harald invited me for a Humboldt fellowship to Ludwig-Maximilian University where I spent more than one year, and we never lost contacts after: he regularly invited me for seminars in Munich and for conferences that he organized worldwide, and many times he visited me in L'Aquila. Benedetta and me had discussions with Harald on the viable modifications of Fritzsch zero-texture during his last visit which work was completed later as ref. [29].

Harald often asked me about Georgia and he was eager to visit Tbilisi. After COVID era, we decided to organize a conference in Tbilisi on "Recent Advances in Fundamental Physics", planned for the fall 2022. But in August 2022 I received a sad news from Bigitte that Harald passed away.

I feel privileged to have had close scientific and personal ties with Harald Fritzsch. I remember him as a renowned physicist who authored many breakthrough works in different areas of particle physics but also as a man with a pleasant and exceptionally humble personality.

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References

[1] N. Cabibbo, Phys. Rev. Lett. **10** (1963) 531-533

[2] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49** (1973) 652-657

[3] R.L. Workman *et al.* [Particle Data Group], PTEP **2022** (2022) 083C01

[4] C. Jarlskog, Phys. Rev. Lett. **55** (1985) 1039

[5] S.L. Glashow, J. Iliopoulos and L. Maiani, Phys. Rev. D **2** (1970) 1285-1292

[6] S.L. Glashow and S. Weinberg, Phys. Rev. D **15** (1977) 1958

[7] E.A. Paschos, Phys. Rev. D **15** (1977) 1966

[8] S. Weinberg, Trans. New York Acad. Sci. **38** (1977) 185-201

[9] F. Wilczek and A. Zee, Phys. Lett. B **70** (1977) 418

[10] H. Fritzsch, Phys. Lett. B **70** (1977) 436-440

[11] H. Fritzsch, Phys. Lett. B **73** (1978) 317-322

[12] H. Fritzsch, Nucl. Phys. B **155** (1979) 189-207

[13] R. Gatto, G. Morchio and F. Strocchi, Phys. Lett. B **80** (1979) 265-268

[14] Z. Berezhiani, Phys. Lett. B **129** (1983) 99-102

[15] Z. Berezhiani, Phys. Lett. B **150** (1985) 177-181

[16] Z. Berezhiani and J.L. Chkareuli, Sov. J. Nucl. Phys. **37** (1983) 618-626

[17] Z. Berezhiani and J.L. Chkareuli, JETP Lett. **35** (1982) 612-615

[18] J.L. Chkareuli, JETP Lett. **32** (1980) 671

[19] H. Fritzsch, J. Phys. Colloq. **45** (1984) no.C3, 189-196

[20] K. Kang, J. Flanz and E. Paschos, Z. Phys. C **55** (1992) 75-82

[21] Z. Berezhiani and L. Lavoura, Phys. Rev. D **45** (1992) 934-945

[22] Z. z. Xing and Z. h. Zhao, Nucl. Phys. B **897** (2015) 302-325

[23] H. Fritzsch, Z. z. Xing and D. Zhang, Nucl. Phys. B **974** (2022) 115634

[24] H. Fritzsch and Z. z. Xing, Prog. Part. Nucl. Phys. **45** (2000) 1-81

[25] Z. Berezhiani, Phys. Lett. B **417** (1998) 287-296

[26] Z. Berezhiani and A. Rossi, JHEP **03** (1999) 002

[27] Z. Berezhiani and A. Rossi, Nucl. Phys. B **594** (2001) 113-168

[28] R.G. Roberts, A. Romanino, G.G. Ross and L. Velasco-Sevilla, Nucl. Phys. B **615** (2001) 358-384

[29] B. Belfatto and Z. Berezhiani, JHEP **08** (2023) 162

[30] C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B **147** (1979) 277-298

[31] L.E. Ibanez and G.G. Ross, Phys. Lett. B **332** (1994) 100-110

[32] P. Binetruy, S. Lavignac and P. Ramond, Nucl. Phys. B **477** (1996) 353-377

[33] E. Dudas, C. Grojean, S. Pokorski and C.A. Savoy, Nucl. Phys. B **481** (1996) 85-108

[34] Z. Berezhiani and Z. Tavartkiladze, Phys. Lett. B **396** (1997) 150-160

[35] Z. Berezhiani and Z. Tavartkiladze, Phys. Lett. B **409** (1997) 220-228

[36] Z. Berezhiani and M.Y. Khlopov, Sov. J. Nucl. Phys. **51** (1990) 739-746

[37] H. Georgi and S.L. Glashow, Phys. Rev. Lett. **32** (1974) 438-441

[38] H. Fritzsch and P. Minkowski, Annals Phys. **93** (1975) 193-266

[39] Z. Berezhiani, "Fermion masses and mixing in SUSY GUT," *Lectures in ICTP Summer School in High-energy Physics and Cosmology*, Edis E. Gava *et al.*, ICTP Series in Theoretical Physics, Vol. 12, p. 618 (1997) [arXiv:hep-ph/9602325]

[40] Z. Berezhiani and J.L. Chkareuli, JETP Lett. **37** (1983) 338-341

[41] Z. Berezhiani and J.L. Chkareuli, Sov. Phys. Usp. **28** (1985) 104-105

[42] A.A. Anselm and Z. Berezhiani, Phys. Lett. B **162** (1985) 349-353

[43] Z. Berezhiani, Int. J. Mod. Phys. A **19** (2004) 3775-3806

[44] Z. Berezhiani, "Through the looking-glass: Alice's adventures in mirror world," in *From Fields to Strings: Circumnavigating Theoretical Physics: A Conference in Tribute to I. Kogan,* Eds. M. Shifman *et al.* pp. 2147–2195 (2005) [arXiv:hep-ph/0508233]

[45] Z. Berezhiani, Eur. Phys. J. ST **163** (2008) 271-289

[46] Z. Berezhiani and M.Y. Khlopov, Z. Phys. C **49** (1991) 73-78

[47] Z. Berezhiani and M. Y. Khlopov, Sov. J. Nucl. Phys. **51** (1990) 935-942

[48] Z. Berezhiani and M.Y. Khlopov, Sov. J. Nucl. Phys. **52** (1990) 60-64

[49] Z. Berezhiani, M.Y. Khlopov and R. Khomeriki, Sov. J. Nucl. Phys. **52** (1990) 344-347

[50] Z. Berezhiani, M.Y. Khlopov and R. Khomeriki, Sov. J. Nucl. Phys. **52** (1990) 65-68

[51] Z. Berezhiani, A.S. Sakharov and M.Y. Khlopov, Sov. J. Nucl. Phys. **55** (1992) 1063-1071

[52] Z. Berezhiani, Nucl. Phys. B Proc. Suppl. **52** (1997) 153-158

[53] A. Anselm and Z. Berezhiani, Nucl. Phys. B **484** (1997) 97-123

[54] Z. Berezhiani and A. Rossi, Nucl. Phys. B Proc. Suppl. **101** (2001) 410-420

[55] K.G. Chetyrkin, J.H. Kuhn and M. Steinhauser, Comput. Phys. Comm. **133** (2000) 43-65

[56] G. Colangelo, S. Lanz, H. Leutwyler and E. Passemar, Eur. Phys. J. C **78** (2018) no.11, 947

[57] B. Belfatto, R. Beradze and Z. Berezhiani, Eur. Phys. J. C **80** (2020) no.2, 149

[58] B. Belfatto and Z. Berezhiani, Eur. Phys. J. C **79** (2019) no.3, 202

[59] B. Belfatto and Z. Berezhiani, JHEP **10** (2021) 079

[60] G.C. Branco, J.T. Penedo, P.M.F. Pereira, M.N. Rebelo and J.I. Silva-Marcos, JHEP **07** (2021) 099

[61] A. Crivellin, M. Kirk, T. Kitahara and F. Mescia, JHEP **03** (2023) 234

[62] B. Belfatto and S. Trifinopoulos, Phys. Rev. D **108** (2023) no.3, 035022

[63] R. Dcruz and K.S. Babu, Phys. Rev. D **108** (2023) no.9, 095011

[64] O. Fischer, B. Mellado, S. Antusch *et al.* Eur. Phys. J. C **82** (2022) no.8, 665

[65] Z. Berezhiani, Phys. Lett. B **355** (1995) 178-186

[66] Z. Berezhiani and F. Nesti, JHEP **03** (2006) 041

[67] Z.. Berezhiani and R. Rattazzi, Phys. Lett. B **279** (1992) 124-130

[68] Z. Berezhiani and R. Rattazzi, Nucl. Phys. B **407** (1993) 249-270

[69] Z. Berezhiani and R. Rattazzi, JETP Lett. **56** (1992) 429-434

[70] Z.. Berezhiani and G. Dvali, Bull. Lebedev Phys. Inst. **5** (1989) 55-59

[71] Z. Berezhiani, C. Csaki and L. Randall, Nucl. Phys. B **444** (1995) 61-91

[72] G. Dvali and S. Pokorski, Phys. Rev. Lett. **78** (1997) 807-810

[73] R. Barbieri, G. Dvali, A. Strumia, Z. Berezhiani and L.J. Hall, Nucl. Phys. B **432** (1994) 49-67

[74] Z. Berezhiani, Phys. Lett. B **355** (1995) 481-491