PROCEEDINGS OF SCIENCE

Preparing nonleptonic NNLO B meson decays: revisiting semileptonic decays

Manuel Egner[a]

[a] *Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), Wolfgang-Gaede Straße 1, 76128 Karlsruhe, Germany*

E-mail: manuel.egner@kit.edu

The decay of B mesons can be described in the Heavy Quark Expansion as the decay of a free bottom quark plus corrections which are suppressed by powers of $1/m_b$. The focus of this contribution will be on the calculation of the NNLO corrections to semileptonic decays of a free bottom quark including a non-vanishing charm quark mass. For the semileptonic decays we obtain an analytic result by solving differential equations and fixing boundary conditions in the limit of a heavy charm quark. Our analytic expression can be compared to previous known results obtained via expansions in the mass ratio m_c/m_b. In a second approach we calculate precise numerical expansions of all master integrals with the help of differential equations which can describe the whole parameter space from $m_c/m_b = 1$ to $m_c/m_b = 0$. This calculation serves as a preparation for the nonleptonic B-meson decays, where the same techniques as in the semileptonic decays can be used.

16th International Symposium on Radiative Corrections: Applications of Quantum Field Theory to Phenomenology (RADCOR2023)
28th May - 2nd June, 2023
Crieff, Scotland, UK

https://pos.sissa.it/

PoS(RADCOR2023)041

1. Introduction

Lifetimes of B mesons can be calculated in Heavy Quark Effective expansion (HQE). In this effective theory, the decay width of the B meson, $\Gamma(B)$, is decomposed into the decay of a free b quark and additional contributions which are suppressed by powers of the heavy quark mass, m_b:

$$\Gamma(B) = \Gamma_3 + \Gamma_5 \frac{\langle O_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle O_6 \rangle}{m_b^3} + \cdots + 16\pi^2 \left(\Gamma_6 \frac{\langle \tilde{O}_6 \rangle}{m_b^3} + \Gamma_7 \frac{\langle \tilde{O}_7 \rangle}{m_b^4} + \ldots \right). \tag{1}$$

Since the bottom mass m_b is relatively large compared to the energy scale of the decay, the main contributions to the decay width is Γ_3, the decay width of the free b quark. In our work, we calculate QCD corrections to this quantity for weak decays of B mesons with a massive charm quark in the final state. These decays can be divided into two different decay channels, the semileptonic and the nonleptonic one. For the semileptonic decay channel, $b \to cl\bar{\nu}$, QCD corections are known up to N^{3}LO [1–5]. The higher order results for the semileptonic case are obtained with expansions around $m_c/m_b = 0$ [2] and $m_c/m_b = 1$ [3–5]. Both approaches show very good convergence in the region of physical charm mass. However, there is no exact analytic solution known in the literature starting from NNLO. In this contribution we summarize the results of ref.[6] where an exact analytic solution at NNLO for the dominant contribution to the semileptonic decay width has been obtained. The nonleptonic decays include the two decay channels $b \to c\bar{u}d$ and $b \to c\bar{c}s$. The calculation of these processes is more involved than the semileptonic case. The NLO corrections for both decays are known [7, 8]. At NNLO, first steps were made in ref.[9], however only one effective operator has been considered and massless quarks in the final state have been assumed.

The uncertainty contributions on B-meson lifetimes is dominated by the uncertainty induced by renormalization scale μ. This uncertainty will be reduced once higher order corrections are known. The methods described below [6] can be used to compute the NNLO corrections to the nonleptonic decay channels. To prepare for this calculation, we revisit the calculation of the semileptonic decays.

2. Calculation setup

The calculation is done by using the optical theorem. This leads to two loop diagrams at LO and therefore four loop diagrams at NNLO. However, only the imaginary part of these diagrams has to be calculated. The diagrams contributing to this process are generated with `QGRAF` [10]. We find 70 diagrams which are then mapped to scalar integral families using `tapir` [11] and `exp` [12, 13]. Using `Kira` [14, 15] we find 129 master integrals we have to calculate. Their calculation is described in the next section.

3. Calculation of master integrals

In the following, the calculation of the master integrals is outlined. First, we have a closer look at the 129 master integrals. They can be split into two different classes. The first class contains all the integrals with cuts through one charm quark, the second class also includes cuts through three charm quarks. Sample diagrams for these two classes are shown in figure 1. In the following

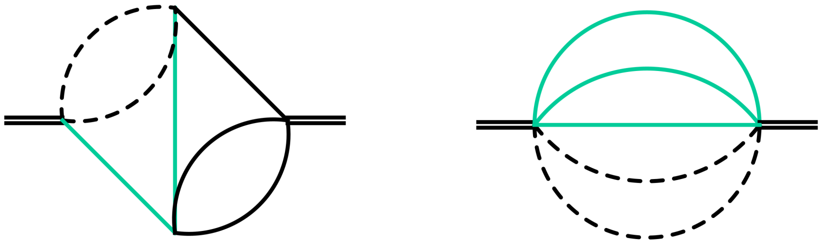

Figure 1: sample integrals with cuts through one (left) and three (right) charm quarks. Black lines have bottom quark mass, green lines have charm mass, dahsed lines are massless. The double external lines denote onshell bottom quarks with momentum q where $q^2 = m_b^2$. All possible cuts have to go through both dashed lines.

we show two different approaches to calculate these integrals. In the first approach, only the one-charm-cut integrals are considered. These are the main contributions to the decay width for the physical masses of the charm and bottom quarks. In the second approach we perform expansions around different points with precise numerical coefficients of the integrals. These expansions cover both, the one-charm and three-charm contributions.

3.1 Analytic calculation

In a first approach we calculate the master integrals analytically by solving the differential equation obtained with IBP relations. After the variable transformation

$$\frac{m_c}{m_b} = \frac{1-t^2}{1+t^2}, \tag{2}$$

we can perform a transformation of the differential equation in t into canonical form using `CANONICA` [16] and `Libra` [17]. The solution consists of iterated integrals over the alphabet

$$\frac{1}{t}, \quad \frac{1}{1+t}, \quad \frac{1}{1-t}, \quad \frac{t}{1+t^2}, \quad \frac{t^3}{1+t^4}, \tag{3}$$

In order to fix the integration constants, boundaries for some of the master integrals are needed. Here we calculate asymptotic expansions of the master integrals in the limit of a heavy c-quark, $m_c/m_b \approx 1$, using similar techniques as it was done in refs.[3, 4]. Since we are only interested in the imaginary part of the master integrals, we only calculate the imaginary part of the asymptotic expansions and therefore only cover the one-charm contribution. The three charm-contribution has no imaginary part for $m_c \to m_b$ and is therefore set to zero in this calculation. It would be possible to include this contribution by also calculating the real part of the integrals. However the real part is much more involved and therefore not considered here.

3.2 Numerical calculation

In our second approach, we perform a calculation of the master integrals which covers both, one-charm and three-charm contributions. We do this by using the method developed in refs.[18, 19]. We construct expansions of the master integrals around different kinematic points using differential equations. To do this, we make an expansion ansatz for the integrals with undetermined

coefficients. This ansatz is inserted in the differential equations which yields linear equations between the expansion coefficients. The linear equations can be solved for a small set of master coefficients using `Kira` and `FireFly` [20]. After finding the master coefficients, the expansions are matched to very precise numerical values of the integrals obtained with `AMFlow` [21], which allows us to determine the master coefficients and therefore all of the expansion coefficients numerically.
For different expansion points, we have to use different ansätze. Except for the singular points $0, 1/3, 1$, we can use simple Taylor expansions:

$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+4} \sum_{n=0}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\rho_0 - \rho)^n, \tag{4}$$

where the coefficients $c[i,j,m,n]$ have both real and imaginary parts. For the expansion around the three-charm threshhold, $\rho = m_c/m_b = 1/3$, we use the ansatz

$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+4} \sum_{n=0}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\rho - \rho_0)^n \log^m(\rho - \rho_0), \tag{5}$$

with $\rho_0 = 1/3$. When crossing the three-charm threshhold at $\rho = 1/3$ from $\rho > 1/3$ to $\rho < 1/3$, the argument of the logarithm gets negative and produces an additional imaginary part, which corresponds to the three-charm contribution. In our calculation we also allow for square roots of $(\rho - \rho_0)$ which appear in similar calculations for two-particle threshholds. However, they are not found for three-particle threshold, see ref.[6] for details.
For the expansions around $\rho = 0$, we use the same ansatz as given in equation (5) with $\rho_0 = 0$, for the expansion around $\rho = 1$, the ansatz has to be modified by replacing $(\rho - \rho_0)$ with $(\rho_0 - \rho)$, where $\rho_0 = 1$.

4. Results

Our analytic result can be compared to the known expansions around $m_c/m_b \approx 0$ [2] and $m_c/m_b \approx 1$ [3] by expanding the iterated integrals in the corresponding limits. In the limit $m_c/m_b \to 1$, we find perfect agreement with the literature. In the limit $m_c/m_b \to 0$ the expansion of our analytic result can not reproduce the result in result in ref.[2] since we do not include the three-charm contribution. However, we can separate the one-charm and three-charm contributions by taking the difference. This is shown in figure 2. The quantities shown in this figure are defined by

$$\Gamma(B \to X_c l\bar{\nu}) = \frac{A_{\mathrm{ew}} G_F^2 |V_{cb}|^2 m_b^5}{192\pi^3}\left[X_0 + \sum_{n>0}\left(\frac{\alpha_s}{\pi}\right)^n X_n\right] + O\left(\frac{\Lambda_{\mathrm{QCD}}^2}{m_b^2}\right),$$

where the NNLO correction is divided into two parts

$$X_2 = X_2^{1c} + X_2^{3c},$$

which correspond to the contributions with one and three charm quarks in the final state. One can see that both contributions are divergent in the limit $m_c/m_b \to 0$ but the total decay width is finite.

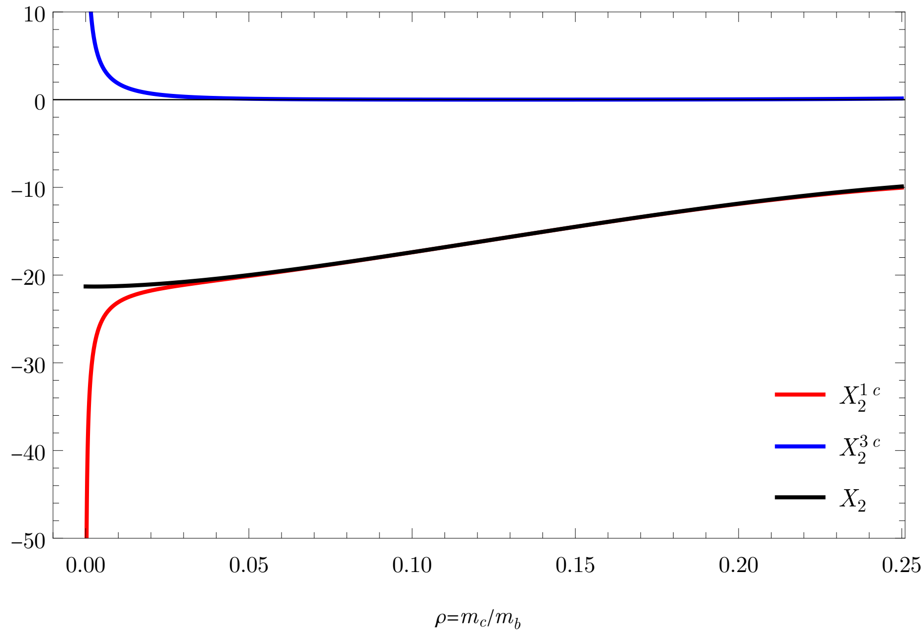

Figure 2: One-charm and three-charm contributions to the decay width in the limit $m_c/m_b \to 0$.

It is also important to note, that the three-charm contribution becomes very small in the region of physical charm mass, $m_c/m_b \in [0.2, 0.3]$ and is therefore not relevant for phenomenological analysis. For example, at $m_c/m_b = 0.2$, the branching ratio is $\Gamma\,(b \to c\bar{c}cl\bar{\nu}) = 4 \cdot 10^{-8}$.
With the numerical approach, we can reproduce the analytic expansion coefficients of the result in ref.[2] with 8 significant digits up to the order ρ^5. This accuracy is obtained by starting the "expand and match" method at $m_c/m_b = 1/2$, crossing the three particle threshold before matching to the expansion around zero. The accuracy can be improved significantly by matching directly to `AMFlow` results close to $m_c/m_b = 0$, for example $m_c/m_b = 1/100$. In this case we can reproduce the coefficients of the expansion with 50 digits.
The calculation shown here serves as preparation for the computation of nonleptonic decay rates of B-mesons at NNLO. In these cases the techniques used in the literature for the semileptonic decays to obtain expansions are either not applicable or very challenging. However, it should be straightforward to apply our ansatz with the "expand and match" method to this problem.

Acknowledgements

This work was done in collaboration with Matteo Fael, Kay Schönwald and Matthias Steinhauser. We thank Fabian Lange for the help with Kira, Vladyslav Shtabovenko for the help with Canonica and Kirill Melnikov for useful discussions. This research was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under grant 396021762 — TRR 257 “Particle Physics Phenomenology after the Higgs Discovery”.

PoS(RADCOR2023)041

References

[1] Yosef Nir. “The Mass Ratio m(c) / m(b) in Semileptonic B Decays”. In: *Phys. Lett. B* 221 (1989), pp. 184–190. DOI: 10.1016/0370-2693(89)91495-0.

[2] Alexey Pak and Andrzej Czarnecki. “Mass Effects in Muon and Semileptonic $b \to c$ Decays”. In: *Physical Review Letters* 100.24 (June 2008). DOI: 10.1103/physrevlett.100.241807. URL: https://doi.org/10.1103%2Fphysrevlett.100.241807.

[3] Matthew Dowling, Jan H. Piclum, and Andrzej Czarnecki. “Semileptonic decays in the limit of a heavy daughter quark”. In: *Physical Review D* 78.7 (Oct. 2008). DOI: 10.1103/physrevd.78.074024. URL: https://doi.org/10.1103%2Fphysrevd.78.074024.

[4] Matteo Fael, Kay Schönwald, and Matthias Steinhauser. “Third order corrections to the semileptonic $b \to c$ and the muon decays”. In: *Phys. Rev. D* 104 (1 July 2021), p. 016003. DOI: 10.1103/PhysRevD.104.016003. URL: https://link.aps.org/doi/10.1103/PhysRevD.104.016003.

[5] Michał Czakon, Andrzej Czarnecki, and Matthew Dowling. “Three-loop corrections to the muon and heavy quark decay rates”. In: *Physical Review D* 103.11 (June 2021). DOI: 10.1103/physrevd.103.l111301. URL: https://doi.org/10.1103%2Fphysrevd.103.l111301.

[6] Manuel Egner, Matteo Fael, Kay Schönwald, and Matthias Steinhauser. “Revisiting semileptonic B meson decays at next-to-next-to-leading order”. In: (Aug. 2023). accepted for publication in Journal of High Energy Physics. arXiv: 2308.01346 [hep-ph].

[7] E. Bagan, Patricia Ball, V.M. Braun, and P. Gosdzinsky. “Charm quark mass dependence of QCD corrections to nonleptonic inclusive B decays”. In: *Nuclear Physics B* 432.1-2 (Dec. 1994), pp. 3–38. DOI: 10.1016/0550-3213(94)90591-6. URL: https://doi.org/10.1016%2F0550-3213%2894%2990591-6.

[8] E. Bagan, Patricia Ball, B. Fiol, and P. Gosdzinsky. “Next-to-leading order radiative corrections to the decay b → ccs”. In: *Physics Letters B* 351.4 (June 1995), pp. 546–554. DOI: 10.1016/0370-2693(95)00437-p. URL: https://doi.org/10.1016%2F0370-2693%2895%2900437-p.

[9] Andrzej Czarnecki, Maciej Ślusarczyk, and Fyodor Tkachov. “Enhancement of hadronic b quark Decays”. In: *Physical Review Letters* 96.17 (May 2006). DOI: 10.1103/physrevlett.96.171803. URL: https://doi.org/10.1103%2Fphysrevlett.96.171803.

[10] P. Nogueira. “Automatic Feynman Graph Generation”. In: *Journal of Computational Physics* 105.2 (1993), pp. 279–289. ISSN: 0021-9991. DOI: https://doi.org/10.1006/jcph.1993.1074. URL: https://www.sciencedirect.com/science/article/pii/S0021999183710740.

[11] Marvin Gerlach, Florian Herren, and Martin Lang. “tapir: A tool for topologies, amplitudes, partial fraction decomposition and input for reductions”. In: *Computer Physics Communications* 282 (Jan. 2023), p. 108544. DOI: 10.1016/j.cpc.2022.108544. URL: https://doi.org/10.1016%2Fj.cpc.2022.108544.

[12] T. Seidensticker. “Automatic application of successive asymptotic expansions of Feynman diagrams”. In: *6th International Workshop on New Computing Techniques in Physics Research: Software Engineering, Artificial Intelligence Neural Nets, Genetic Algorithms, Symbolic Algebra, Automatic Calculation*. May 1999. arXiv: hep-ph/9905298.

[13] R. Harlander, T. Seidensticker, and M. Steinhauser. “Complete corrections of Order alpha alpha-s to the decay of the Z boson into bottom quarks”. In: *Phys. Lett. B* 426 (1998), pp. 125–132. DOI: 10.1016/S0370-2693(98)00220-2. arXiv: hep-ph/9712228.

[14] P. Maierhöfer, J. Usovitsch, and P. Uwer. “Kira—A Feynman integral reduction program”. In: *Computer Physics Communications* 230 (Sept. 2018), pp. 99–112. ISSN: 0010-4655. DOI: 10.1016/j.cpc.2018.04.012. URL: http://dx.doi.org/10.1016/j.cpc.2018.04.012.

[15] J. Klappert, F. Lange, P. Maierhöfer, and J. Usovitsch. “Integral reduction with Kira 2.0 and finite field methods”. In: *Computer Physics Communications* 266 (Sept. 2021), p. 108024. ISSN: 0010-4655. DOI: 10.1016/j.cpc.2021.108024. URL: http://dx.doi.org/10.1016/j.cpc.2021.108024.

[16] C. Meyer. “Algorithmic transformation of multi-loop master integrals to a canonical basis with CANONICA”. In: *Computer Physics Communications* 222 (Jan. 2018), pp. 295–312. ISSN: 0010-4655. DOI: 10.1016/j.cpc.2017.09.014. URL: http://dx.doi.org/10.1016/j.cpc.2017.09.014.

[17] R. N. Lee. “Libra: A package for transformation of differential systems for multiloop integrals”. In: *Comput. Phys. Commun.* 267 (2021), p. 108058. DOI: 10.1016/j.cpc.2021.108058. arXiv: 2012.00279 [hep-ph].

[18] Matteo Fael, Fabian Lange, Kay Schönwald, and Matthias Steinhauser. “A semi-analytic method to compute Feynman integrals applied to four-loop corrections to the $\overline{\text{MS}}$-pole quark mass relation”. In: *Journal of High Energy Physics* 2021.9 (Sept. 2021). DOI: 10.1007/jhep09(2021)152. URL: https://doi.org/10.1007%2Fjhep09%282021%29152.

[19] Matteo Fael, Fabian Lange, Kay Schönwald, and Matthias Steinhauser. “Singlet and non-singlet three-loop massive form factors”. In: *Phys. Rev. D* 106.3 (2022), p. 034029. DOI: 10.1103/PhysRevD.106.034029. arXiv: 2207.00027 [hep-ph].

[20] Jonas Klappert, Sven Yannick Klein, and Fabian Lange. “Interpolation of dense and sparse rational functions and other improvements in FireFly”. In: *Computer Physics Communications* 264 (July 2021), p. 107968. DOI: 10.1016/j.cpc.2021.107968. URL: https://doi.org/10.1016%2Fj.cpc.2021.107968.

[21] Xiao Liu and Yan-Qing Ma. “AMFlow: A Mathematica package for Feynman integrals computation via auxiliary mass flow”. In: *Computer Physics Communications* 283 (Feb. 2023), p. 108565. DOI: 10.1016/j.cpc.2022.108565. URL: https://doi.org/10.1016%2Fj.cpc.2022.108565.