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NNLO subtraction for any massless final state: a complete analytic expression

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Abstract: We use the Local Analytic Sector Subtraction scheme to construct a completely analytic set of expressions implementing a fully local infrared subtraction at NNLO for generic coloured massless final states. The cancellation of all explicit infrared poles appearing in the double-virtual contribution, in the real-virtual correction and in the integrated local infrared counterterms is explicitly verified, and all finite contributions arising from integrated local counterterms are analytically evaluated in terms of ordinary polylogarithms up to weight three. The resulting *subtraction formula* can readily be implemented in any numerical framework containing the relevant matrix elements up to NNLO.

Contents

1 Introduction

The coming decades will see a vast increase in the experimental precision of collider data, as the LHC experiments move into the high-luminosity era. At the same time, the complexity of the observables being probed in hadronic collisions is likely to increase as well, as more detailed information becomes available about multi-particle final states. This future evolution on the experimental side poses a significant challenge for the theory community, which is called upon to provide increasingly precise predictions for ever more intricate observables. As a result, a number of innovative theoretical tools for perturbative calculations have been developed over the last two decades, and continue to be refined and extended (for a recent review, see Ref. [1]). Predictions at the next-to-next-to-leading order (NNLO) in the strong coupling are rapidly becoming standard, even for relatively complex final states (see, for example, Ref. [2–5]), while the frontier has moved to the third perturbative order in the strong coupling ($\mathrm{N^3LO}$) for relatively simple processes [6, 7].

A necessary ingredient for the calculation of differential distributions to the required accuracy is an efficient and automatic treatment of infrared singularities, which must cancel between virtual corrections and the phase-space integrals of unresolved final-state radiation, or must be factorised in a universal manner in the case of collisions involving hadrons in the initial state. The theoretical foundations of this treatment are well understood (for a recent review, see [8]): the cancellation (or factorisation) is guaranteed by general theorems valid to all orders in perturbation theory [9–13], and hinges upon the factorisation properties of virtual corrections to scattering amplitudes [14–25] and of real-radiation matrix elements [26–28]. The anomalous dimensions required for the infrared factorisation of virtual corrections are fully known up to three loops [29, 30], while the real-radiation splitting kernels have been computed at order α_s^2 [26–28, 31–33], with near-complete information available also at α_s^3 [34–47].

Notwithstanding this extensive body of knowledge, the construction of general and efficient algorithms for infrared subtraction beyond NLO has proved to be a very difficult task. At NLO, the task of handling infrared singularities was first approached with *phase-space slicing* methods [48, 49], by isolating the phase-space regions where real radiation is singular, introducing for those regions approximate expressions of the relevant matrix elements, and integrating analytically up to the slicing parameter. To avoid residual dependence on the slicing parameter, *subtraction* methods [50–53], see also [54], were later introduced, which work by defining local counterterms in all regions of phase space affected by singularities, subtracting them from the full real-radiation matrix elements, and then adding back their exact integrals. Some of these methods have been developed in full generality, and versions of the corresponding algorithms are implemented in a number of multi-purpose NLO event generators [55–63], providing a solution of the problem at this accuracy.

Beyond NLO, the handling of infrared singularities becomes significantly more difficult, both conceptually and practically, due to the rapid increase in the number of overlapping singular regions, to the need for considering strongly-ordered infrared limits, and to the mixing between virtual poles and phase-space singularities. As a consequence, efforts to reach the same degree of universality and efficiency as was achieved at NLO already span almost two decades. Many different approaches have been proposed and pursued [64–85], as recently reviewed in Ref. [86]. Some of the methods proposed belong to the *slicing* family, or define non-local subtractions, as is the case for Ref. [75], while others adopt the *local-subtraction* viewpoint (for example [67, 70]); they also range from predominantly numerical methods, as in [85], to predominantly analytical ones, as for example [73]; finally, they have reached varying degrees of practical implementation, culminating with the first differential NNLO calculations for $2 \to 3$ collider processes with at least two QCD particles in the final state at Born level, in Refs. [2–5, 87, 88].

All approaches to infrared subtraction beyond NLO are affected by considerable computational complexity, either at the level of the analytic integration of counterterms, or at the level of numerical implementation. Even if the underlying physical mechanism for the cancellation is essentially simple and well understood, concrete technical implementations are intricate, and it is clear that there is room for improvement in the universality, versatility and efficiency of existing algorithms. With these goals in mind, we have developed an approach to infrared *subtraction*, which we call Local Analytic Sector Subtraction [83, 89]. We attempt to optimise the structure of the calculation at all stages, while maintaining full locality of the counterterms and complete universality for all hadronic final states, as well as providing completely analytic expressions for all required counterterms and their phase-space integrals, including finite contributions. We believe that the completion of this programme will provide an extremely versatile tool: once fully analytic expressions are available, the method can in principle be implemented within any existing numerical framework, and applications to multi-particle final states will be limited only by the available computing power and multi-loop matrix elements (see for instance Ref. [90]). In parallel, we are studying more formal aspects of subtraction, from the point of view of factorisation [84], with the hope of further optimising the structure of local counterterms, taking full advantage of the highly non-trivial structure of infrared factorisation and exponentiation. In that context, we provided a set of definitions for soft and collinear local counterterms which apply to all orders in perturbation theory, and we are currently studying the necessary organisation of strongly-ordered infrared configurations [91].

In the present paper, we complete our subtraction programme for the case of generic massless coloured final states. All relevant integrals were computed analytically in [92], requiring only standard techniques. In order to achieve this simplicity, we exploited as much as possible the existing freedom in the definition of local infrared counterterms. Specifically, a crucial element of our approach is the smooth partition of phase space in sectors, each of which contains only a minimal set of soft and collinear singularities, along the lines of Ref. [50]. The next important ingredient is a flexible family of phase-space parametrisations, which can be applied sector by sector, and in fact can be varied for each contribution to the local counterterms. This ultimately leads to a minimal and simple set of phase-space integrals to be performed. Our final result is a completely analytic subtraction formula, which gives the NNLO contribution to the differential distribution for any infrared-safe observable built out of massless coloured final states (as well as with an arbitrary set of massive or massless colourless final-state particles), and requires as input only the relevant matrix elements: the double-virtual correction to the Born-level process, the one-loop correction to the single-radiation process, and the tree-level expression for the double-real-emission contribution.

We present the architecture of our method in Section 2, beginning with a quick review of our approach at NLO, for massless final states, to introduce the relevant notations in a simple context[1]. The following sections give the details for the construction of all the ingredients entering the subtracted formula. Section 3 discusses the subtracted double-real contribution, which is integrable over the entire radiative phase space. Explicit expressions for all required counterterms are included, as well as a detailed analysis of phase-space mappings. Section 4 organises the integration procedure for all counterterms associated with double-real radiation, expressing the required integrals in terms of a small set of basic integrals, which were discussed in Ref. [92] and are collected here in Appendix E. Section 5 presents the subtracted real-virtual correction, providing an explicit expression for the real-virtual counterterm. By combining together the real-virtual correction with its local counterterm, and the integrals of the single-unresolved and the strongly-ordered counterterms, we build an expression that is both free of infrared poles and integrable in the radiative phase space. The integration of the real-virtual counterterm is discussed in Section 6, and again can be organised in terms of simple integrals. Finally, Section 7 gives the subtracted double-virtual contribution, which is free of infrared poles, and Section 8 summarises our results, putting them in

[1] Our method provides a complete subtraction formalism at NLO, including the case of initial state hadrons, as discussed in details in Ref. [89].

perspective. Several appendices collect useful formulas and some technical details of the formalism.

2 Subtraction for massless final states: a framework

We consider a generic process where an electroweak initial state with total momentum q, $q^2 \equiv s$, produces n massless final-state coloured particles at Born level, and we denote with $\mathcal{A}_n(k_i)$, $i = 1, \ldots, n$, the relevant scattering amplitude[2]. The perturbative expansion of the amplitude reads

$$\mathcal{A}_n(k_i) \;=\; \mathcal{A}_n^{(0)}(k_i) \,+\, \mathcal{A}_n^{(1)}(k_i) \,+\, \mathcal{A}_n^{(2)}(k_i) \,+\, \ldots\,, \tag{2.1}$$

where $\mathcal{A}_n^{(k)}$ is the k-loop correction, and includes the appropriate power of the strong coupling constant. For such a process, we consider a generic infrared-safe observable X, and we write the corresponding differential distribution as

$$\frac{d\sigma}{dX} \;=\; \frac{d\sigma_{\rm LO}}{dX} \,+\, \frac{d\sigma_{\rm NLO}}{dX} \,+\, \frac{d\sigma_{\rm NNLO}}{dX} \,+\, \ldots\,. \tag{2.2}$$

Our task is to express such differential distributions in a manifestly finite form, which is free of infrared poles, and integrable over the appropriate phase spaces. In order to introduce our method and notations, we begin with a brief review of the NLO calculation for massless final states.

2.1 Local Analytic Sector Subtraction at NLO

The standard expression for the NLO term in the distribution in Eq. (2.2) requires combining virtual corrections to the Born term, which contain IR poles in $\epsilon \equiv (4-d)/2$, where d is the number of space-time dimensions, and the phase-space integral of unresolved radiation, which is also singular in $d = 4$. One must then compute the combination

$$\frac{d\sigma_{\rm NLO}}{dX} \;=\; \lim_{d\to 4}\left[\int d\Phi_n\, V\, \delta_n(X) + \int d\Phi_{n+1}\, R\, \delta_{n+1}(X)\right]. \tag{2.3}$$

Here $\delta_m(X) \equiv \delta(X - X_m)$ fixes X_m, the expression for the observable X, computed for an m-particle configuration, to its prescribed value, $d\Phi_m$ denotes the Lorentz-invariant phase-space measure for m massless final-state particles, and

$$R \;=\; \left|\mathcal{A}_{n+1}^{(0)}\right|^2\,, \qquad V \;=\; 2\,\mathbf{Re}\left[\mathcal{A}_n^{(0)\dagger}\,\mathcal{A}_n^{(1)}\right]\,, \tag{2.4}$$

are the real and the ($\overline{\rm MS}$-renormalised) virtual contributions, respectively. To rewrite Eq. (2.3) in terms of finite quantities we need a sequence of steps. First, we must define a *local counterterm*, denoted here by K, which is required to reproduce the singular IR behaviour of the real-radiation matrix element R *locally* in phase space. At the same time, it is expected to be simple enough to be analytically integrated in the phase space of the unresolved radiation. In order to perform this integration, we need to introduce a parametrisation of the radiative phase space $d\Phi_{n+1}$, which must factorise as

$$d\Phi_{n+1} \;=\; \frac{\varsigma_{n+1}}{\varsigma_n}\, d\Phi_n\, d\Phi_{\rm rad}\,, \tag{2.5}$$

where, as before, $d\Phi_n$ is the phase space for n massless particles, while $d\Phi_{\rm rad}$ is the measure of integration for the degrees of freedom of the unresolved radiation, and we explicitly extracted the

[2]The subtraction presented in the following applies with no modifications to the case of an arbitrary number of colourless particles accompanying the n coloured ones in the final state, so that in general $\sum_i k_i \neq q$. Just for the sake of notational simplicity, we will assume n to coincide with the total number of final-state particles and the total momentum to be q.

ratio of the relevant symmetry factors ς_{n+1} and ς_n. The factorisation theorems for real radiation guarantee that the function K will be a combination of products of Born-level squared amplitudes (to be integrated over $d\Phi_n$) and infrared kernels, to be integrated in $d\Phi_{\rm rad}$. Once a parametrisation yielding Eq. (2.5) is in place, one can compute the *integrated counterterm*

$$I \, = \, \int d\Phi_{\rm rad} \, K \, . \tag{2.6}$$

Eq. (2.6) will reproduce, by construction, the infrared poles arising from the integration of the real-radiation squared matrix element. It is now possible to rewrite Eq. (2.3) identically as a combination of virtual corrections and real contributions that are separately finite, and therefore phase-space integrals can be performed numerically when needed. Using $\int d\Phi_{n+1} K = \int d\Phi_n \, I$, we obtain

$$\frac{d\sigma_{\rm NLO}}{dX} \, = \, \int d\Phi_n V_{\rm sub}(X) \, + \int d\Phi_{n+1} \, R_{\rm sub}(X) \, , \tag{2.7}$$

with

$$V_{\rm sub}(X) \, = \, \big(V + I\big) \, \delta_n(X) \, , \qquad\qquad R_{\rm sub}(X) \, = \, R \, \delta_{n+1}(X) - K \, \delta_n(X) \, . \tag{2.8}$$

The subtracted real matrix element $R_{\rm sub}(X)$ is free of phase-space singularities by construction, while $V_{\rm sub}(X)$ is finite as $\epsilon \to 0$ as a consequence of the KLN theorem, and both contributions are now suitable for a numerical implementation in four space-time dimensions. Notice that the IR safety of the observable X is necessary for the cancellation, which requires that $\delta_{n+1}(X)$ turns smoothly into $\delta_n(X)$ in all unresolved limits.

Eqs. (2.3)-(2.7) provide just an outline of NLO subtraction task: the actual definition of the required local counterterm is in fact not unique, and characterises the subtraction scheme. Furthermore, it is necessary to include a prescription to perform the phase-space mapping implied by Eq. (2.5). Within the context of Local Analytic Sector Subtraction at NLO, we proceed as follows.

- We define *projection operators* $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ that extract from the real-radiation squared matrix element R its singular behaviour in soft and collinear limits. In practice, one must pick specific phase-space variables in order to perform the projection: one could for example choose a Lorentz frame and define the soft limit in terms of the energy of particle i in that frame, and the collinear limit in terms of the angle between i and j, as was done in Ref. [50]. We prefer to use Lorentz-invariant quantities, as discussed in detail in Refs. [83] and [89]. Concretely, we introduce the variables

$$e_i \, \equiv \, \frac{s_{qi}}{s} \, , \qquad\qquad w_{ij} \, \equiv \, \frac{s \, s_{ij}}{s_{qi} s_{qj}} \, , \tag{2.9}$$

  where $s_{q\ell} = 2q \cdot k_\ell$. We then define $\mathbf{S}_i$ as extracting the leading power in e_i, and $\mathbf{C}_{ij} = \mathbf{C}_{ji}$ as extracting the leading power in w_{ij}. It is not difficult to verify that, with this definition, the two operators commute when acting on the squared matrix element, $\mathbf{S}_i \, \mathbf{C}_{ij} \, R = \mathbf{C}_{ij} \, \mathbf{S}_i \, R$.

- We then partition the radiative phase space into *sectors*, defined by introducing a set of *sector functions*, $\mathcal{W}_{ij}$, along the lines of Ref. [50], which constitute a partition of unity, namely a set of kinematical weights smoothly dampening all radiative singularities but those due to particle i becoming soft, or becoming collinear to a second particle j. Our sector functions are constructed in terms of Lorentz invariants. We indeed define

$$\sigma_{ij} \, \equiv \, \frac{1}{e_i w_{ij}} \, , \qquad\qquad \mathcal{W}_{ij} \, \equiv \, \frac{\sigma_{ij}}{\sum_{k \neq l} \sigma_{kl}} \, , \tag{2.10}$$

satisfying $\sum_{i\neq j}\mathcal{W}_{ij}=1$. These sector functions have the further defining property that their soft and collinear limits still form a partition of unity. Indeed, one easily verifies that

$$\mathbf{S}_i \sum_{k\neq i}\mathcal{W}_{ik} \;=\; 1\,, \qquad \mathbf{C}_{ij}\Big[\mathcal{W}_{ij}+\mathcal{W}_{ji}\Big] \;=\; 1\,. \tag{2.11}$$

Eq. (2.11) guarantees that, upon summing over sectors, the full soft and collinear singularities will be recovered, and sector functions will not explicitly appear in counterterms to be integrated.

- The purpose of introducing sector functions is to minimise the number of singular limits of $R\,\mathcal{W}_{ij}$, so that we can easily identify a combination which is by construction integrable in the radiative phase space. Indeed, in sector (ij)

$$(1-\mathbf{S}_i)\,(1-\mathbf{C}_{ij})\;R\,\mathcal{W}_{ij} \;=\; R\,\mathcal{W}_{ij}-\mathbf{L}^{(\mathbf{1})}_{ij}\;R\,\mathcal{W}_{ij} \;\;\to\;\; \text{integrable}\,, \tag{2.12}$$

where we introduced $\mathbf{L}^{(\mathbf{1})}_{ij}\equiv\mathbf{S}_i+\mathbf{C}_{ij}-\mathbf{S}_i\,\mathbf{C}_{ij}$. We stress here that the operators $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ are defined to act on all elements that lie to their right: therefore, if $\mathbf{L}$ denotes a generic singular limit, the relation $\mathbf{L}\,R\,\mathcal{W}_{ij}\equiv(\mathbf{L}\,R)\,(\mathbf{L}\,\mathcal{W}_{ij})$ is understood. Summing over sectors we get the expression

$$\sum_i\sum_{j\neq i}\mathbf{L}^{(\mathbf{1})}_{ij}R\,\mathcal{W}_{ij} \;\equiv\; \sum_i\sum_{j\neq i}\Big[\mathbf{S}_i+\mathbf{C}_{ij}\big(1-\mathbf{S}_i\big)\Big]R\,\mathcal{W}_{ij}\,, \tag{2.13}$$

which satisfies the requirement of reproducing the singular behaviour of R in all soft and collinear regions. Eq. (2.13), however, cannot yet be used directly in Eq. (2.7), since it does not properly factorise a Born-level squared matrix element involving n on-shell particles.

- For this purpose, we must introduce a set of *mappings* of the $(n+1)$-particle momenta $\{k\}$ onto the n-particle momenta $\{\bar{k}\}$, which must not affect soft and collinear limits at leading power. We adopt the Catani-Seymour mappings [51]

$$\bar{k}^{(abc)}_i \;=\; k_i\,,\;\; i\neq a,b,c\,; \qquad \bar{k}^{(abc)}_b \;=\; k_a+k_b-\frac{s_{ab}}{s_{ac}+s_{bc}}k_c\,; \qquad \bar{k}^{(abc)}_c \;=\; \frac{s_{abc}}{s_{ac}+s_{bc}}k_c\,, \tag{2.14}$$

where i runs from 1 to $n+1$. The mappings above satisfy the on-shell and momentum-conservation conditions

$$\big(\bar{k}^{(abc)}_j\big)^2 \;=\; 0\,,\;\; j \;=\; 1,\ldots,n\,; \qquad \sum_{j=1}^{n}\bar{k}^{(abc)}_j \;=\; \sum_{i=1}^{n+1}k_i\,. \tag{2.15}$$

One easily verifies the two sets of momenta coincide when k_a becomes soft, and when k_a becomes collinear to k_b.

- Finally, we can turn Eq. (2.13) into a *local counterterm*, by using the factorised expressions for soft and collinear limits of R, and evaluating the Born-level squared matrix elements with the mapped momenta defined Eq. (2.14), sector by sector in the radiative phase space. We do this by introducing *improved projection operators* $\overline{\mathbf{S}}_i$ and $\overline{\mathbf{C}}_{ij}$, which are defined at NLO to project on leading-power soft and collinear limits, and at the same time apply the selected phase-space mappings. For NLO massless final states their action is defined by

$$\begin{aligned}
\overline{\mathbf{S}}_i\,R &\equiv -\mathcal{N}_1\sum_{c\neq i}\sum_{d\neq i,c}\mathcal{E}^{(i)}_{cd}\,\bar{B}^{(icd)}_{cd}\,, \\
\overline{\mathbf{C}}_{ij}\,R &\equiv \mathcal{N}_1\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\bar{B}^{(ijr)}_{\mu\nu}\,, \\
\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R &\equiv 2\mathcal{N}_1\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\,\bar{B}^{(ijr)}\,, \qquad r=r_{ij}\,.
\end{aligned} \tag{2.16}$$

The quantities entering Eq. (2.17) are defined as follows. We denote by B the Born matrix element squared, $B = |\mathcal{A}_n^{(0)}|^2$, while $B_{\mu\nu}$ is its spin-correlated counterpart, defined by removing the gluon polarisation vectors from the matrix element and from its complex conjugate[3]. Similarly, B_{cd} is the colour-correlated Born, defined in Eq. (A.5). These three objects are evaluated in Eq. (2.17) with mapped momenta, and are therefore denoted with a bar and with a label identifying the specific mapping to be employed. Thus, for example, $\bar{B}^{(ilm)} \equiv B(\{\bar{k}\}^{(ilm)})$ is the Born squared matrix element with mapped momenta $\{\bar{k}\}^{(ilm)}$. Furthermore, C_{f_j} is the Casimir eigenvalue of the colour representation of parton j, while the eikonal kernel $\mathcal{E}_{cd}^{(i)}$ and the DGLAP kernels $P_{ij(r)}^{\mu\nu}$ are presented in Eq. (B.3) and Eq. (B.7) respectively[4]. The overall normalisation is given by

$$\mathcal{N}_1 \; = \; 8\pi\alpha_s \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^{\epsilon} . \tag{2.17}$$

Importantly, the improved operators must preserve the correct soft and collinear limits of R to ensure the locality of the subtraction procedure: in this case, one must verify that

$$\mathbf{S}_i \, \overline{\mathbf{S}}_i \, R \; = \; \mathbf{S}_i \, R \, , \quad \mathbf{C}_{ij} \, \overline{\mathbf{C}}_{ij} \, R \; = \; \mathbf{C}_{ij} \, R \, , \tag{2.18}$$

as well as

$$\mathbf{S}_i \, \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, R \; = \; \mathbf{S}_i \, \overline{\mathbf{C}}_{ij} \, R \, , \quad \mathbf{C}_{ij} \, \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, R \; = \; \mathbf{C}_{ij} \, \overline{\mathbf{S}}_i \, R \, . \tag{2.19}$$

These *consistency conditions* are indeed verified by Eq. (2.17). We also stress that $r = r_{ij}$ is any particle different from i, j, chosen according to the rule defined in Eq. (A.14) (in this case it means that the same r must be chosen for the pair ij and for the pair ji). In what follows, we will describe the action of the improved operators as realising *improved limits*. Notice that, at this stage, we have a residual freedom in the definition of improved limits of sector functions, subject to the preservation of the constraints in Eq. (2.11) and in Eq. (2.12).

- The definition of the improved operators given above contains a subtlety [89], which must be analysed with care. The DGLAP kernels $P_{ij(r)}^{\mu\nu}$ reported in Appendix B are written in terms of the invariants

$$x_i \; = \; \frac{s_{ir}}{s_{ir} + s_{jr}} \, , \qquad x_j \; = \; \frac{s_{jr}}{s_{ir} + s_{jr}} \, , \tag{2.20}$$

as opposed to the energy fractions $e_i/(e_i + e_j)$, $e_j/(e_i + e_j)$. This is a useful choice in view of analytical integration, and a legitimate one since x_i and x_j reduce to e_i and e_j in the collinear limit $\mathbf{C}_{ij}$. This choice, however, introduces *spurious singularities* in the collinear limits $\mathbf{C}_{ir}$ and $\mathbf{C}_{jr}$ in the sectors $\mathcal{W}_{ij}$ and $\mathcal{W}_{ji}$, so that the combination $(1 - \overline{\mathbf{S}}_i)(1 - \overline{\mathbf{C}}_{ij}) R \, \mathcal{W}_{ij}$ is not integrable in the limits $\mathbf{C}_{ir}$ and $\mathbf{C}_{jr}$. This problem can be solved by using our freedom to define the action of the improved operators $\overline{\mathbf{S}}_i$ and $\overline{\mathbf{C}}_{ij}$ on sector functions $\mathcal{W}_{ij}$ ($r = r_{ij}$):

$$\overline{\mathbf{S}}_i \, \mathcal{W}_{ij} \; \equiv \; \mathbf{S}_i \, \mathcal{W}_{ij} \; = \; \frac{\frac{1}{w_{ij}}}{\sum\limits_{l \neq i} \frac{1}{w_{il}}} \, , \qquad \overline{\mathbf{C}}_{ij} \, \mathcal{W}_{ij} \; \equiv \; \frac{e_j w_{jr}}{e_i w_{ir} + e_j w_{jr}} \, , \qquad \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, \mathcal{W}_{ij} \; \equiv \; 1 \, . \tag{2.21}$$

The presence of the angular factors w_{ir} and w_{jr}, vanishing in the $\mathbf{C}_{ir}$ and $\mathbf{C}_{jr}$ limits respectively, allows to verify the following *auxiliary consistency conditions*

$$\begin{aligned} \mathbf{C}_{ir} \left\{ 1 \, , \; \overline{\mathbf{S}}_i \, , \; \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i \right) \right\} R \, \mathcal{W}_{ij} \; &\to \; \text{integrable} \, , \\ \mathbf{C}_{jr} \left\{ 1 \, , \; \overline{\mathbf{S}}_i \, , \; \overline{\mathbf{C}}_{ij} \, , \; \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \right\} R \, \mathcal{W}_{ij} \; &\to \; \text{integrable} \, , \end{aligned} \tag{2.22}$$

[3] If the parent parton in the collinear ij splitting is not a gluon, the corresponding kernel is diagonal in spin space by helicity conservation, and $B_{\mu\nu}$ reduces to B.

[4] We note that, as seen in Appendix B, these and all other kernels are written in terms of Lorentz invariants and in a manifestly flavour-symmetric notation.

on top of the standard ones, corresponding to Eqs. (2.18) and (2.19), which now need to be written explicitly including the sector functions, as

$$\begin{aligned}
\mathbf{S}_i \Big\{ \big(1-\overline{\mathbf{S}}_i\big) \,,\; \overline{\mathbf{C}}_{ij} \, \big(1-\overline{\mathbf{S}}_i\big) \Big\} \, R\, \mathcal{W}_{ij} \;\; &\to \;\; \text{integrable} \,, \\
\mathbf{C}_{ij} \Big\{ \big(1-\overline{\mathbf{C}}_{ij}\big) \,,\; \overline{\mathbf{S}}_i \, \big(1-\overline{\mathbf{C}}_{ij}\big) \Big\} \, R\, \mathcal{W}_{ij} \;\; &\to \;\; \text{integrable} \,.
\end{aligned} \tag{2.23}$$

Recall that in Eq. (2.22) the index r labels the reference vector used to define the collinear kernel $P^{\mu\nu}_{ij(r)}$: in fact, all collinear projection operators should properly be labelled with the index r, which in general we omit for brevity. Notice also that our definition of improved limits of sector functions, Eq. (2.21), is not symmetric under $i \leftrightarrow j$. As a consequence, the two lines of Eq. (2.22) are not identical: in the first line, only the combination $\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i)$ gives an integrable result in the ir collinear limit, when acting on $R\,\mathcal{W}_{ij}$ (which is sufficient for our purposes), while in the second line $\overline{\mathbf{C}}_{ij}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}$ give separately integrable contributions in the same limit.

- With these definitions, our first expressions for the NLO local counterterm is

$$K \; = \sum_{i,j\neq i} K_{ij} \,, \qquad K_{ij} \; = \; \Big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\Big) R\,\mathcal{W}_{ij} \,. \tag{2.24}$$

so that the subtracted squared matrix element is given by

$$R_{\rm sub}(X) \; = \sum_{i,j\neq i} R^{\rm sub}_{ij}(X) \,, \quad R^{\rm sub}_{ij}(X) \; = \; R\,\mathcal{W}_{ij}\,\delta_{n+1}(X) - K_{ij}\,\delta_n(X) \,. \tag{2.25}$$

The counterterm defined in Eq. (2.24) is sufficient to construct a fully functional subtraction algorithm at NLO. There is however some room for optimisation: for example, we note that the sector functions $\mathcal{W}_{ij}$ are useful to identify the improved limits to be defined, and the consistency relations they must satisfy, but the stability of numerical integrations will improve when sectors involving the same parametrisations are combined. To pursue this idea, we introduce *symmetrised sector functions* defined by

$$\mathcal{Z}_{ij} \; = \; \mathcal{W}_{ij} + \mathcal{W}_{ji} \,. \tag{2.26}$$

The corresponding improved limits read

$$\overline{\mathbf{S}}_i \mathcal{Z}_{ij} \; = \; \overline{\mathbf{S}}_i \mathcal{W}_{ij} \; = \; \frac{\frac{1}{w_{ij}}}{\sum\limits_{l\neq i}\frac{1}{w_{il}}} \,, \qquad \overline{\mathbf{C}}_{ij}\mathcal{Z}_{ij} \; = \; 1 \,, \qquad \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\mathcal{Z}_{ij} \; = \; \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\mathcal{W}_{ij} \; = \; 1 \,. \tag{2.27}$$

This symmetrisation of the sector functions reduces the number of sectors and, to some extent, simplifies the scheme in view of an efficient numerical performance. In fact, the counterterm K, with symmetrised sector functions, can be written as

$$K \; = \sum_{i,j>i} K_{\{ij\}} \,, \qquad K_{\{ij\}} \; = \; \big(\overline{\mathbf{S}}_i + \overline{\mathbf{S}}_j + \overline{\mathbf{HC}}_{ij}\big) R\,\mathcal{Z}_{ij} \,, \tag{2.28}$$

where we have introduced the hard-collinear improved limit

$$\overline{\mathbf{HC}}_{ij}\, R \equiv \overline{\mathbf{C}}_{ij}\, \big(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\big)\, R \; = \; \mathcal{N}_1 \, \frac{P^{{\rm hc},\mu\nu}_{ij(r)}}{s_{ij}} \, \bar{B}^{(ijr)}_{\mu\nu} , \tag{2.29}$$

with the hard-collinear splitting kernel $P^{{\rm hc},\mu\nu}_{ij(r)}$ defined in Appendix B. The subtracted squared matrix element is now given by

$$R_{\rm sub}(X) \; = \sum_{i,j>i} R^{\rm sub}_{\{ij\}}(X) \,, \qquad R^{\rm sub}_{\{ij\}}(X) \; = \; R\,\mathcal{Z}_{ij}\,\delta_{n+1}(X) - K_{\{ij\}}\,\delta_n(X) \,. \tag{2.30}$$

A third expression for the NLO counterterm, important for analytic integration, is obtained by summing over all sectors. Using Eq. (2.11), one can then write

$$R_{\rm sub}(X) \;=\; R\,\delta_{n+1}(X) - K\,\delta_n(X)\,, \qquad K \;=\; \sum_i \overline{\mathbf{S}}_i\,R + \sum_{i,j>i} \overline{\mathbf{HC}}_{ij}\,R\,. \tag{2.31}$$

This expression for $R_{\rm sub}(X)$, though very compact, is not the most suited for numerical implementation: the expression in Eq. (2.30), with symmetrised sector functions, is to be preferred, since it allows to parallelise the contribution of different sectors, and to independently optimise their numerical evaluation.

As discussed in detail in [83, 89, 92], these definitions enable a straightforward integration of local counterterms, and yield an implementation of NLO subtraction that can be extended to initial-state radiation as well. We now turn to the case of NNLO massless final states.

2.2 Local Analytic Sector Subtraction at NNLO

The NNLO contribution to the differential cross section in Eq. (2.2) can be written as

$$\frac{d\sigma_{\rm NNLO}}{dX} = \lim_{d\to4}\bigg[\int d\Phi_n\,VV\,\delta_n(X) + \int d\Phi_{n+1}\,RV\,\delta_{n+1}(X) + \int d\Phi_{n+2}\,RR\,\delta_{n+2}(X)\bigg]\,, \tag{2.32}$$

where

$$RR \;=\; \Big|\mathcal{A}^{(0)}_{n+2}\Big|^2\,, \quad RV \;=\; 2\,\mathbf{Re}\Big[\mathcal{A}^{(0)\dagger}_{n+1}\,\mathcal{A}^{(1)}_{n+1}\Big]\,, \quad VV \;=\; \Big|\mathcal{A}^{(1)}_{n}\Big|^2 + 2\,\mathbf{Re}\Big[\mathcal{A}^{(0)\dagger}_{n+1}\,\mathcal{A}^{(2)}_{n+1}\Big]\,. \tag{2.33}$$

In this case, the $\overline{\rm MS}$-renormalised double-virtual contribution VV displays IR poles up to ϵ^{-4}, the double-real RR contains up to four phase-space singularities, and the $\overline{\rm MS}$-renormalised real-virtual term RV has poles up to ϵ^{-2} and up to two phase-space singularities. In order to rewrite Eq. (2.32) as a sum of finite contributions, we will define *four* local counterterms, which we label $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$, $K^{(\mathbf{12})}$ and $K^{(\mathbf{RV})}$. The counterterm $K^{(\mathbf{1})}$ is designed to reproduce all phase-space singularities of RR due to a single particle becoming unresolved, while $K^{(\mathbf{2})}$ takes care of situations where two particles become unresolved at the same rate. The two sets of singularities overlap, and $K^{(\mathbf{12})}$ is responsible for subtracting the double-counted overlap region. Finally, $K^{(\mathbf{RV})}$ will subtract the phase-space singularities arising from the single-real radiation in RV.

In order to integrate these counterterms, we will need to introduce phase-space parametrisations factorising single and double radiation, in analogy with Eq. (2.5). In this case we will need the factorisations

$$d\Phi_{n+2} = \frac{\varsigma_{n+2}}{\varsigma_{n+1}}\,d\Phi_{n+1}\,d\Phi_{\rm rad}\,, \qquad d\Phi_{n+2} = \frac{\varsigma_{n+2}}{\varsigma_{n}}\,d\Phi_{n}\,d\Phi_{\rm rad,2}\,, \qquad d\Phi_{n+1} = \frac{\varsigma_{n+1}}{\varsigma_{n}}\,d\Phi_{n}\,d\Phi_{\rm rad}\,. \tag{2.34}$$

Once a parametrisation yielding Eq. (2.34) is in place, one can define *integrated counterterms* as

$$\begin{aligned} I^{(\mathbf{1})} &\equiv \int d\Phi_{\rm rad}\,K^{(\mathbf{1})}\,, \qquad & I^{(\mathbf{2})} &\equiv \int d\Phi_{\rm rad,2}\,K^{(\mathbf{2})}\,, \\ I^{(\mathbf{12})} &\equiv \int d\Phi_{\rm rad}\,K^{(\mathbf{12})}\,, \qquad & I^{(\mathbf{RV})} &\equiv \int d\Phi_{\rm rad}\,K^{(\mathbf{RV})}\,. \end{aligned} \tag{2.35}$$

We are now ready to write down the master formula for our subtraction at NNLO: in the rest of the paper we will precisely define and construct all the necessary ingredients, generalising the discussion summarised in Section 2.1. We aim to construct an expression of the form

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n\,VV_{\rm sub}(X) + \int d\Phi_{n+1}\,RV_{\rm sub}(X) + \int d\Phi_{n+2}\,RR_{\rm sub}(X)\,, \tag{2.36}$$

where each one of the three contributions is finite in ϵ and is free of phase-space singularities.

Using the local counterterms introduced above, and their integrals over the radiative degrees of freedom, the subtracted matrix elements $VV_{\rm sub}$, $RV_{\rm sub}$ and $RR_{\rm sub}$ are given by

$$VV_{\rm sub}(X) \equiv \Big(VV + I^{\,(\mathbf{2})} + I^{(\mathbf{RV})}\Big)\, \delta_n(X)\,, \tag{2.37}$$

$$RV_{\rm sub}(X) \equiv \Big(RV + I^{\,(\mathbf{1})}\Big)\, \delta_{n+1}(X) \,-\, \Big(K^{(\mathbf{RV})} + I^{\,(\mathbf{12})}\Big)\, \delta_n(X)\,, \tag{2.38}$$

$$RR_{\rm sub}(X) \equiv RR\, \delta_{n+2}(X) - K^{\,(\mathbf{1})}\, \delta_{n+1}(X) - \Big(K^{\,(\mathbf{2})} - K^{\,(\mathbf{12})}\Big)\, \delta_n(X)\,. \tag{2.39}$$

Once again, Eqs. (2.36) and (2.37)-(2.39) provide an identical rewriting of Eq. (2.32): their logic is as follows:

- in Eq. (2.39), $RR_{\rm sub}(X)$ must be integrated in the full phase space Φ_{n+2}, and it is built out of tree-level quantities[5], therefore has no explicit IR poles. It has no phase-space singularities either, since single-unresolved contributions are subtracted by $K^{\,(\mathbf{1})}$, double-unresolved contributions are subtracted by $K^{\,(\mathbf{2})}$, and their double-counted overlap is reinstated by adding back $K^{\,(\mathbf{12})}$.

- in Eq. (2.38), RV must be integrated in Φ_{n+1}, and is affected by both explicit IR poles and phase-space singularities. The IR poles arising from the loop integration in RV are cancelled by the integral $I^{\,(\mathbf{1})}$, by virtue of general cancellation theorems; the first parenthesis is thus finite, but both terms are singular in the phase space of the radiated particle. By construction, the phase-space singularities of $I^{\,(\mathbf{1})}$ are cancelled by $I^{\,(\mathbf{12})}$, and $K^{(\mathbf{RV})}$ is designed to cancel the phase-space singularities of RV. This however does not guarantee that explicit IR poles will cancel in the second parenthesis. Anyway, one can fine-tune the definition of $K^{(\mathbf{RV})}$, by including explicit IR poles not associated with the phase-space singularities of RV, in order to make the second parenthesis finite as well. At this point, Eq. (2.38) is both finite and integrable.

- The complete cancellation of real and virtual singularities in Eq. (2.38) and Eq. (2.39) guarantees then, as a consequence of the KLN theorem, that Eq. (2.37), to be integrated in the Born-level phase space Φ_n, will be free of IR poles.

In the next sections we will construct explicit expressions for all counterterms, compute their integrals analytically, and finally obtain $RR_{\rm sub}(X)$, $RV_{\rm sub}(X)$ and $VV_{\rm sub}(X)$. As was the case at NLO, this will require identifying the relevant single- and double-unresolved limits, introducing an appropriate set of NNLO sector functions, and defining flexible and consistent phase-space mappings. Needless to say, the multiplicity of singular configurations and of their overlaps will lead to long and intricate expressions: therefore, detailed formulas for NNLO soft and collinear kernels, for the relevant mapped limits, and for the required integrals, as well as a number of notational shortcuts, will be presented in the Appendices.

3 The subtracted double-real contribution $RR_{\rm sub}$

In this section we provide a detailed construction of the subtracted squared matrix element for double-real radiation, $RR_{\rm sub}$. As noted in Eq. (2.39), this will require the definition of three separate local counterterms. From a combinatorial viewpoint, this task represents the most intricate part of the NNLO-subtraction programme, due to the large number of overlapping singular limits affecting

[5] We have implicitly understood the underlying Born reaction to be associated with tree-level diagrams; however, in case of loop-induced processes, all arguments and techniques presented in this article carry over.

double-real radiation. In analogy to Section 2.1, we will proceed as follows: first, in Section 3.1, we will list and briefly discuss the relevant singular limits, which can be single- or double-unresolved; next, in Section 3.2, we will introduce a set of sector functions, smoothly partitioning the $(n+2)$-particle phase space so as to minimise the number of singular configurations to be considered in any given sector. These sectors will naturally be grouped into three different *topologies*, corresponding to the structure of the limits relevant to each sector. Next, in Section 3.3, we will identify specific combinations of limits that yield integrable contributions in each topology, in the spirit of Eq. (2.12); we will then construct, in Section 3.4, a family of phase-space mappings in order to properly factorise the double-radiative phase space in all relevant configurations. Finally, in Section 3.5, we will introduce improved limits appropriate for each topology, discuss the required consistency conditions, and then use the improved limits to compose an expression for the subtracted double-real contribution $RR_{\rm sub}$. As was the case at NLO for single-real radiation, it is possible to improve upon the resulting expression for $RR_{\rm sub}$ by introducing symmetrised sector functions in order to optimise the subsequent numerical integration. This construction is discussed in Section 3.6. We note that the construction presented in this paper differs slightly in some technical choices from the one given in Ref. [83]: we will note the differences as we go along.

3.1 Singular limits for double-real radiation

Double-real radiation matrix elements are characterised by a variety of overlapping singular limits. It is important, from the outset, to pick a complete set of limits, in order to then study (and subtract) their overlaps, to avoid double counting. Clearly, single-unresolved soft and collinear limits are relevant also for double radiation, so our list must include the limits $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ introduced in Section 2.1. Next, we need to collect all possible double-unresolved limits. Importantly, when two particles become unresolved, one needs to distinguish *uniform* limits, where the rate at which the two particles become unresolved is the same, and *strongly-ordered* limits, where one particle becomes unresolved at a higher rate with respect to the second one. Obviously, this distinction becomes relevant starting at NNLO. Our set of fundamental uniform limits consists of four independent configurations. First, two particles i and j can become soft at the same rate, a limit which we denote by $\mathbf{S}_{ij}$; second, a single hard particle can branch into three collinear ones, i, j and k, a limit which we denote by $\mathbf{C}_{ijk}$; third, two hard partons can independently branch into two collinear pairs, which we denote by $\mathbf{C}_{ijkl}$, with (i,j) and (k,l) labelling the two independent pairs; finally, a particle i can become soft while another pair of particles, j and k, become collinear at the same rate[6], which we denote by $\mathbf{SC}_{ijk}$. In these four limits, the double-real-radiation squared matrix element factorises, with the relevant kernels derived and presented in Ref. [27]. Given these uniform limits, the strongly-ordered ones can be reached by acting iteratively: for example, the strongly-ordered double-soft limit, with particle i becoming soft faster than particle j, can be reached by computing $\mathbf{S}_i\,\mathbf{S}_{ij}$,while the strongly-ordered double-collinear limit, with particles i and j becoming collinear faster than the third particle k, will be given by the combination $\mathbf{C}_{ij}\,\mathbf{C}_{ijk}$. All singular configurations can be reached in this way.

In order to proceed, we need to characterise the limits more precisely, in terms of phase-space variables. As was the case at NLO, we choose to define the limits in terms of Mandelstam invariants, and we pay attention to the fact that all limits must commute when acting on the double-real radiation squared matrix element. Using the variables e_i and w_{ij} given in Eq. (2.9), the definitions of the independent limits, both single-and double-unresolved, are specified in Table 1. Importantly, our choice of independent limits is related to our choice of sector functions, which will be tuned so that only a minimal pre-defined set of the chosen limits will contribute in each sector.

[6] In Ref. [83], two strongly-ordered soft-collinear limits were considered, instead of the uniform one chosen here.

$\mathbf{S}_i$	$e_i \to 0$ (soft configuration of parton i)
$\mathbf{C}_{ij}$	$w_{ij} \to 0$ (collinear configuration of partons (i,j))
$\mathbf{S}_{ij}$	$e_i, e_j \to 0$ and $e_i/e_j \to$ constant (uniform double-soft configuration of partons (i,j))
$\mathbf{C}_{ijk}$	$w_{ij}, w_{ik}, w_{jk} \to 0$ and $w_{ij}/w_{ik}, w_{ij}/w_{jk}, w_{ik}/w_{jk} \to$ constant (uniform double-collinear configuration of partons (i,j,k))
$\mathbf{C}_{ijkl}$	$w_{ij}, w_{kl} \to 0$ and $w_{ij}/w_{kl} \to$ constant (uniform double-collinear configuration of partons (i,j) and (k,l))
$\mathbf{SC}_{ijk}$	$e_i, w_{jk} \to 0$ and $e_i/w_{jk} \to$ constant (uniform soft and collinear configuration for partons i and (j,k))

Table 1. Definitions of the single-unresolved singular limits $\mathbf{S}_i$, $\mathbf{C}_{ij}$ and of our set of basic independent double-unresolved singular limits $\mathbf{S}_{ij}$, $\mathbf{C}_{ijk}$, $\mathbf{C}_{ijkl}$, $\mathbf{SC}_{ijk}$.

3.2 Sector functions and topologies for double-real radiation

We now introduce a smooth unitary partition of the double-real-radiation phase space, in the spirit of Ref. [50]. Since at most four particles can be involved in singular infrared limits at NNLO, we label the sector functions with four indices, and denote them by $\mathcal{W}_{ijkl}$. We pick the first two indices to label the single-unresolved configurations assigned to the chosen sector. In particular, we will design the sector $(ijkl)$ to contain the limits $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ (thus we take $j \neq i$). We then need to distinguish sectors involving only three distinct particles from sectors involving four distinct particles. In sectors where only three particles are involved, the double-unresolved limit $\mathbf{C}_{ijk}$ will be relevant; furthermore, a second particle (besides i) may become soft, and it can be particle j *or* particle k. Correspondingly, we will have distinct sector functions $\mathcal{W}_{ijjk}$ and $\mathcal{W}_{ijkj}$, where we take the third index to indicate the second particle that can become soft. Similarly, if all four indices are distinct, we take $\mathcal{W}_{ijkl}$ to select the sector where particles i and k can become soft, while the possible collinear pairs are (i,j) and (k,l). Notice that in all cases the last three indices j, k and l are distinct from i, and $k \neq l$. We will refer to the three allowed combinations of sector indices, $(ijjk)$, $(ijkj)$ and $(ijkl)$ as *topologies*, and we will denote them collectively by $\tau \equiv abcd \in \{ijjk, ijkj, ijkl\}$.

We now need to introduce a precise definition of NNLO sector functions, which will enable us to list all the fundamental limits contributing to each topology. As was done at NLO (see Eq. (2.10)), we will define NNLO sector functions as ratios of the type

$$\mathcal{W}_{abcd} = \frac{\sigma_{abcd}}{\sigma}, \qquad \sigma = \sum_{a,b\neq a} \sum_{\substack{c\neq a \\ d\neq a,c}} \sigma_{abcd}, \tag{3.1}$$

so that

$$\sum_{a,b\neq a} \sum_{\substack{c\neq a \\ d\neq a,c}} \mathcal{W}_{abcd} = 1. \tag{3.2}$$

Such a partition allows us to rewrite the double-real squared matrix element RR as

$$RR = \sum_{i,\,j\neq i} \sum_{k\neq i} \sum_{l\neq i,k} RR\,\mathcal{W}_{ijkl} = \sum_{i,\,j\neq i} \sum_{k\neq i,j} \left[RR\,\mathcal{W}_{ijjk} + RR\,\mathcal{W}_{ijkj} + \sum_{l\neq i,j,k} RR\,\mathcal{W}_{ijkl} \right]. \tag{3.3}$$

Our choice for the functions σ_{abcd}[7] is given by

$$\sigma_{abcd} = \frac{1}{(e_a\, w_{ab})^\alpha} \frac{1}{(e_c + \delta_{bc}\, e_a)\, w_{cd}}, \qquad \alpha > 1. \tag{3.4}$$

[7] This choice corresponds to setting $\alpha = \beta$ in the NNLO sector functions introduced in Ref. [83].

Given Eq. (3.4), we can list which of the fundamental limits discussed in Section 3.1 will affect each topology. One finds that the combination $RR\,\mathcal{W}_\tau$ will be singular in the limits listed below.

$$
\begin{aligned}
&RR\,\mathcal{W}_{ijjk}\ : \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ij}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}\,; \\
&RR\,\mathcal{W}_{ijkj}\ : \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}\,, \quad \mathbf{SC}_{kij}\,; \\
&RR\,\mathcal{W}_{ijkl}\ : \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijkl}\,, \quad \mathbf{SC}_{ikl}\,, \quad \mathbf{SC}_{kij}\,.
\end{aligned}
\tag{3.5}
$$

In analogy with the NLO sum-rule requirements in Eq. (2.11), also NNLO sector functions which share a given singular configuration must form a unitary partition. This is a crucial feature in order to minimise the complexity of the counterterm structure in view of analytic integration. The choice of the functions σ_{abcd} in Eq. (3.4) guarantees that the required partial sums reduce to unity. For example, we report the sum rules for the double-unresolved limits in Table 1, which read

$$
\mathbf{S}_{ik}\left(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd}+\sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\right) \;=\; 1\,, \tag{3.6}
$$

$$
\mathbf{C}_{ijk}\sum_{abc\,\in\,\pi(ijk)}\big(\mathcal{W}_{abbc}+\mathcal{W}_{abcb}\big) \;=\; 1\,, \qquad \mathbf{C}_{ijkl}\sum_{\substack{ab\,\in\,\pi(ij)\\ cd\,\in\,\pi(kl)}}\big(\mathcal{W}_{abcd}+\mathcal{W}_{cdab}\big) \;=\; 1\,, \tag{3.7}
$$

$$
\mathbf{SC}_{ijk}\left(\sum_{\substack{d\neq i\\ ab\,\in\,\pi(jk)}}\mathcal{W}_{idab}+\sum_{\substack{d\neq i,a\\ ab\,\in\,\pi(jk)}}\mathcal{W}_{abid}\right) \;=\; 1\,, \tag{3.8}
$$

where by $\pi(ij)$ and $\pi(ijk)$ we denote respectively the sets $\{ij, ji\}$ and $\{ijk, ikj, jik, jki, kij, kji\}$.

In order for the double-real contribution to properly combine with the real-virtual correction, we require NNLO sector functions to factorise into NLO-like sector functions under the action of single-unresolved limits. As discussed in Ref. [83], and below in Section 5, this ensures the local cancellation of integrated phase-space singularities with the poles of the real-virtual correction, sector by sector in the single-radiative phase space: indeed RV needs to be partitioned with NLO-like sector functions, since it involves a single-real radiation. As an example, one may verify that the sector functions for the topology $(ijjk)$ satisfy

$$
\mathbf{S}_i\,\mathcal{W}_{ijjk}=\mathcal{W}_{jk}\,\mathbf{S}_i\,\mathcal{W}^{(\alpha)}_{ij}\,,\quad \mathbf{C}_{ij}\,\mathcal{W}_{ijjk}=\mathcal{W}_{[ij]k}\,\mathbf{C}_{ij}\,\mathcal{W}^{(\alpha)}_{ij}\,,\quad \mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ijjk}=\mathcal{W}_{jk}\,\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}^{(\alpha)}_{ij}\,, \tag{3.9}
$$

where $\mathcal{W}_{[ij]k}$ is the NLO sector function defined in the $(n+1)$-particle phase space including the parent parton $[ij]$ of the collinear pair (i, j), and we introduced the NLO-like, α-dependent sector functions

$$
\mathcal{W}^{(\alpha)}_{ij} \;\equiv\; \frac{\sigma^{(\alpha)}_{ij}}{\sum_{k\neq l}\sigma^{(\alpha)}_{kl}}\,, \qquad \sigma^{(\alpha)}_{ij} \;\equiv\; \frac{1}{(e_i\,w_{ij})^{\alpha}}\,, \qquad \alpha>1\,, \tag{3.10}
$$

so that ordinary NLO sector functions are given by $\mathcal{W}_{ij} = \mathcal{W}^{(1)}_{ij}$. Similar relations hold for the other two topologies.

3.3 Combining singular limits of topologies

As listed in Eq. (3.5), a limited number of products of IR projectors is sufficient to collect all singular configurations of the double-real squared matrix element in each topology. Since the action of the relevant limits on both RR and on the sector functions does not depend on the order they are applied, the following combinations are by construction integrable in the whole phase space

$$
\begin{aligned}
(1-\mathbf{S}_i)\,(1-\mathbf{C}_{ij})\,(1-\mathbf{S}_{ij})\,(1-\mathbf{C}_{ijk})\,(1-\mathbf{SC}_{ijk})\;RR\,\mathcal{W}_{ijjk} \;&\to\; \text{integrable}\,, \\
(1-\mathbf{S}_i)\,(1-\mathbf{C}_{ij})\,(1-\mathbf{S}_{ik})\,(1-\mathbf{C}_{ijk})\,(1-\mathbf{SC}_{ijk})\,(1-\mathbf{SC}_{kij})\;RR\,\mathcal{W}_{ijkj} \;&\to\; \text{integrable}\,, \\
(1-\mathbf{S}_i)\,(1-\mathbf{C}_{ij})\,(1-\mathbf{S}_{ik})\,(1-\mathbf{C}_{ijkl})\,(1-\mathbf{SC}_{ikl})\,(1-\mathbf{SC}_{kij})\;RR\,\mathcal{W}_{ijkl} \;&\to\; \text{integrable}\,.
\end{aligned}
\tag{3.11}
$$

Note that, in analogy to the definition used for NLO projection operators, if we take $\mathbf{L}$ to be any one of the singular limits in Table 1, the action $\mathbf{L}\, RR\, \mathcal{W}_{abcd} \equiv (\mathbf{L}\, RR)\,(\mathbf{L}\, \mathcal{W}_{abcd})$ is understood for all topologies.

Applying directly Eq. (3.11) would be quite cumbersome, as the three lines generate a total of 160 terms. Fortunately, the resulting combinations of limits are not all independent, and several non-trivial relations can be obtained exploiting the symmetries of the limits under exchanges of indices, as well as the definitions of the various limits involved as projection operators on singular terms of RR. Consider for example, in four-particle sector $\mathcal{W}_{ijkl}$, the projection $(1-\mathbf{S}_{ik})\, RR\, \mathcal{W}_{ijkl}$. This will contain only terms in RR that are not singular in sector $(ijkl)$ when the uniform soft limit is taken for particles i and k. As a consequence, if further projections involving *both* the i and k soft limits are taken, the result will be integrable. We conclude, for example, that

$$\mathbf{SC}_{ikl}\, \mathbf{SC}_{kij}(1-\mathbf{S}_{ik})\, RR\, \mathcal{W}_{ijkl} \;\rightarrow\; \text{integrable}\,. \tag{3.12}$$

Working in this way, topology by topology, we can write a set of finite relations, which help us remove redundant configurations contributing to Eq. (3.11). They read

$$\begin{aligned}
\mathbf{C}_{ij}\, \mathbf{SC}_{ijk}(1-\mathbf{S}_i)(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})\, RR\, \mathcal{W}_{ijjk} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{S}_i\, \mathbf{SC}_{kij}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\, RR\, \mathcal{W}_{ijkj} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{C}_{ij}\, \mathbf{SC}_{ijk}(1-\mathbf{S}_i)(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\, RR\, \mathcal{W}_{ijkj} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{C}_{ij}\, \mathbf{S}_{ik}(1-\mathbf{S}_i)(1-\mathbf{SC}_{kij})(1-\mathbf{C}_{ijk})\, RR\, \mathcal{W}_{ijkj} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{SC}_{ijk}\, \mathbf{SC}_{kij}(1-\mathbf{S}_{ik})\, RR\, \mathcal{W}_{ijkj} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{S}_i\, \mathbf{SC}_{kij}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})\, RR\, \mathcal{W}_{ijkl} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{S}_i\, \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik})(1-\mathbf{SC}_{ikl})\, RR\, \mathcal{W}_{ijkl} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{C}_{ij}\, \mathbf{SC}_{ikl}(1-\mathbf{S}_i)(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})\, RR\, \mathcal{W}_{ijkl} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{C}_{ij}\, \mathbf{S}_{ik}(1-\mathbf{S}_i)(1-\mathbf{SC}_{kij})(1-\mathbf{C}_{ijkl})\, RR\, \mathcal{W}_{ijkl} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{SC}_{ikl}\, \mathbf{SC}_{kij}(1-\mathbf{S}_{ik})\, RR\, \mathcal{W}_{ijkl} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{C}_{ijkl}\, \mathbf{S}_{ik}(1-\mathbf{SC}_{ikl})\, RR\, \mathcal{W}_{ijkl} \;&\rightarrow\; \text{integrable}\,,\\
\mathbf{C}_{ijkl}\, \mathbf{S}_{ik}(1-\mathbf{SC}_{kij})\, RR\, \mathcal{W}_{ijkl} \;&\rightarrow\; \text{integrable}\,.
\end{aligned} \tag{3.13}$$

These finite relations allow us to simplify considerably Eq. (3.11), leading to the integrable expression

$$RR\, \mathcal{W}_\tau - \left(\mathbf{L}^{(\mathbf{1})}_{ij} + \mathbf{L}^{(\mathbf{2})}_{\tau} - \mathbf{L}^{(\mathbf{12})}_{\tau}\right) RR\, \mathcal{W}_\tau \;\;\rightarrow\;\; \text{integrable}\,, \tag{3.14}$$

which is the NNLO equivalent of Eq. (2.12) for double-real radiation[8]. In Eq. (3.14) we distinguished, for each topology τ, the single-unresolved limit $\mathbf{L}^{(\mathbf{1})}_{ij}$, the uniform double-unresolved limit $\mathbf{L}^{(\mathbf{2})}_{\tau}$, and the strongly-ordered double-unresolved limit $\mathbf{L}^{(\mathbf{12})}_{\tau}$. Their explicit expressions for each

[8]Note that there is no ambiguity in the notation: we denote by (ij) the first two indices of the sector, which are common to all three topologies.

topology, in terms of the projectors discussed in Section 3.1, are

$$
\begin{aligned}
\mathbf{L}^{(\mathbf{1})}_{ij} &= \mathbf{S}_i + \mathbf{C}_{ij} \, (1 - \mathbf{S}_i) \, , \\
\mathbf{L}^{(\mathbf{2})}_{ijjk} &= \mathbf{S}_{ij} + \mathbf{SC}_{ijk} \, (1 - \mathbf{S}_{ij}) + \mathbf{C}_{ijk} \, (1 - \mathbf{S}_{ij}) \, (1 - \mathbf{SC}_{ijk}) \, , \\
\mathbf{L}^{(\mathbf{2})}_{ijkj} &= \mathbf{S}_{ik} + (\mathbf{SC}_{ijk} + \mathbf{SC}_{kij}) \, (1 - \mathbf{S}_{ik}) + \mathbf{C}_{ijk} \, (1 - \mathbf{S}_{ik}) \, (1 - \mathbf{SC}_{ijk} - \mathbf{SC}_{kij}) \, , \\
\mathbf{L}^{(\mathbf{2})}_{ijkl} &= \mathbf{S}_{ik} + (\mathbf{SC}_{ikl} + \mathbf{SC}_{kij}) \, (1 - \mathbf{S}_{ik}) + \mathbf{C}_{ijkl} \, (1 + \mathbf{S}_{ik} - \mathbf{SC}_{ikl} - \mathbf{SC}_{kij}) \, , \\
\mathbf{L}^{(\mathbf{12})}_{ijjk} &= \mathbf{S}_i \Big[\mathbf{S}_{ij} + \mathbf{SC}_{ijk} \, (1 - \mathbf{S}_{ij}) + \mathbf{C}_{ijk} \, (1 - \mathbf{S}_{ij}) \, (1 - \mathbf{SC}_{ijk}) \Big] \\
&\qquad\qquad - \mathbf{C}_{ij} \, (1 - \mathbf{S}_i) \Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk} \, (1 - \mathbf{S}_{ij}) \Big] , \\
\mathbf{L}^{(\mathbf{12})}_{ijkj} &= \mathbf{S}_i \Big[\mathbf{S}_{ik} + \mathbf{SC}_{ijk} \, (1 - \mathbf{S}_{ik}) + \mathbf{C}_{ijk} \, (1 - \mathbf{S}_{ik}) \, (1 - \mathbf{SC}_{ijk}) \Big] \\
&\qquad\qquad - \mathbf{C}_{ij} \, (1 - \mathbf{S}_i) \Big[\mathbf{SC}_{kij} + \mathbf{C}_{ijk} \, (1 - \mathbf{SC}_{kij}) \Big] , \\
\mathbf{L}^{(\mathbf{12})}_{ijkl} &= \mathbf{S}_i \Big[\mathbf{S}_{ik} + \mathbf{SC}_{ikl} \, (1 - \mathbf{S}_{ik}) \Big] - \mathbf{C}_{ij} \, (1 - \mathbf{S}_i) \Big[\mathbf{SC}_{kij} + \mathbf{C}_{ijkl} \, (1 - \mathbf{SC}_{kij}) \Big] .
\end{aligned}
\tag{3.15}
$$

The projection operators appearing in Eq. (3.15) are organised so as to display, in order, the soft ($\mathbf{S}$), the uniform soft and collinear ($\mathbf{SC}$) and the collinear ($\mathbf{C}$) singular contributions. Upon summing over sectors, Eq. (3.14) and Eq. (3.15) build up the equivalent at NNLO of Eq. (2.12) and Eq. (2.13), for double-real radiation: indeed, applying the limits defined in Eq. (3.15) on RR and on the sector functions gives the starting point to determine the form of the counterterms for each sector, since the limits contain all phase-space singularities of RR in a given sector, without double counting. In order to promote them to actual counterterms, it is now necessary to introduce phase-space mappings, allowing to properly factorise the $(n+2)$-body phase space into an $(n+1)$-body phase space times a single-radiation phase space for $\mathbf{L}^{(\mathbf{1})}$ and $\mathbf{L}^{(\mathbf{12})}$, and into an n-body phase space times a double-radiation phase space for $\mathbf{L}^{(\mathbf{2})}$, as shown in Eq. (2.34). We now turn to the discussion of these mappings.

3.4 Phase-space mappings for double-real radiation

There is considerable freedom to define phase-space mappings for double-real radiation (see for example [93]). We have chosen to use nested Catani-Seymour final-state mappings, which involve a minimal set of the $(n+2)$ momenta, and are built in terms of Mandelstam invariants, simplifying both the factorised expression for the $(n+2)$-body phase space and the dependence of the counterterms on the integration variables of the radiative phase spaces. In this framework, the mappings to factorise the $(n+2)$-body phase space into an $(n+1)$-body phase space times a single-radiation phase space, necessary for $\mathbf{L}^{(\mathbf{1})}$ and $\mathbf{L}^{(\mathbf{12})}$, can be constructed with the same procedure followed at NLO, and one is lead to Eq. (2.14) and Eq. (2.15), with i running from 1 to $n+2$, and j running from 1 to $n+1$.

For the construction of an on-shell, momentum conserving n-tuple of massless momenta in the $(n+2)$-particle phase space, necessary for $\mathbf{L}^{(\mathbf{2})}$, we distinguish the following three possibilities.

- We choose six final-state massless momenta k_a, k_b, k_c, k_d, k_e, k_f (all different) and construct the n-tuple (without k_a and k_b)

$$
\{\bar{k}\}^{(acd,bef)} = \Big\{ \{k\}_{\not a \not b \not c \not d \not e \not f} , \bar{k}_c^{(acd,bef)} , \bar{k}_d^{(acd,bef)} , \bar{k}_e^{(acd,bef)} , \bar{k}_f^{(acd,bef)} \Big\} \, , \tag{3.16}
$$

with

$$
\begin{aligned}
\bar{k}_c^{(acd,bef)} &= k_a + k_c - \frac{s_{ac}}{s_{[ac]d}} \, k_d \, , \qquad & \bar{k}_d^{(acd,bef)} &= \frac{s_{acd}}{s_{[ac]d}} \, k_d \, , \\
\bar{k}_e^{(acd,bef)} &= k_b + k_e - \frac{s_{be}}{s_{[be]f}} \, k_f \, , \qquad & \bar{k}_f^{(acd,bef)} &= \frac{s_{bef}}{s_{[be]f}} \, k_f \, ,
\end{aligned}
\tag{3.17}
$$

while all other momenta are left unchanged ($\bar{k}_n^{(acd,bef)} = k_n\,,\ n \neq a,b,c,d,e,f$). Here and in the following $s_{[ab]c} = s_{ac} + s_{bc}$.

- We choose five final-state massless momenta k_a, k_b, k_c, k_d, k_e (all different) and construct the n-tuple (without k_a and k_b)

$$\{\bar{k}\}^{(acd,bed)} \;=\; \Big\{\{k\}_{\cancel{a}\cancel{b}\cancel{c}\cancel{d}\cancel{e}},\, \bar{k}_c^{(acd,bed)}, \bar{k}_d^{(acd,bed)}, \bar{k}_e^{(acd,bed)}\Big\}\,, \tag{3.18}$$

with

$$\begin{aligned}
\bar{k}_c^{(acd,bed)} \;&=\; k_a + k_c - \frac{s_{ac}}{s_{[ac]d}}\,k_d\,, \qquad\qquad \bar{k}_d^{(acd,bed)} \;=\; \left(1 + \frac{s_{ac}}{s_{[ac]d}} + \frac{s_{be}}{s_{[be]d}}\right)k_d\,,\\
\bar{k}_e^{(acd,bed)} \;&=\; k_b + k_e - \frac{s_{be}}{s_{[be]d}}\,k_d\,,
\end{aligned} \tag{3.19}$$

while all other momenta are left unchanged ($\bar{k}_n^{(acd,bed)} = k_n\,,\ n \neq a,b,c,d,e$).

- We choose four final-state massless momenta k_a, k_b, k_c, k_d (all different) and construct the n-tuple (without k_a and k_b)

$$\{\bar{k}\}^{(acd,bcd)} \;=\; \{\bar{k}\}^{(abc,bcd)} \;=\; \{\bar{k}\}^{(abcd)} \;=\; \Big\{\{k\}_{\cancel{a}\cancel{b}\cancel{c}\cancel{d}},\, \bar{k}_c^{(abcd)}, \bar{k}_d^{(abcd)}\Big\}\,, \tag{3.20}$$

with

$$\bar{k}_c^{(abcd)} \;=\; k_a + k_b + k_c - \frac{s_{abc}}{s_{ad} + s_{bd} + s_{cd}}\,k_d\,, \qquad\qquad \bar{k}_d^{(abcd)} \;=\; \frac{s_{abcd}}{s_{ad} + s_{bd} + s_{cd}}\,k_d\,, \tag{3.21}$$

while all other momenta are left unchanged ($\bar{k}_n^{(abcd)} = k_n\,,\ n \neq a,b,c,d$).

With these tools, we are now ready to construct improved infrared projectors, with a proper factorised structure, and we can use them to define our counterterms.

3.5 Building $RR_{\rm sub}$ with improved singular limits

To write explicitly the counterterms we introduce *improved* versions of the limits in Table 1

$$\overline{\mathbf{S}}_a, \quad \overline{\mathbf{C}}_{ab}, \quad \overline{\mathbf{S}}_{ab}, \quad \overline{\mathbf{C}}_{abc}, \quad \overline{\mathbf{C}}_{abcd}, \quad \overline{\mathbf{SC}}_{abc} \quad .$$

They are to be interpreted as operators which, on top of extracting the corresponding singular limit on the objects they act on, convey a specific mapping of momenta, to be defined case by case, and may be further refined (for example by tuning their action on sector functions) in order to ensure the local cancellation of singularities after the implementation of phase-space mappings.

Given the definitions of the improved limits (to be discussed below) we can construct the expression for $RR_{\rm sub}$ in the following way. First, we define the improved version of the various $\mathbf{L}$ operators corresponding to the limits in Eq. (3.15), for each topology, denoting the improved operators by $\overline{\mathbf{L}}$. Next, we define our local counterterms, for each topology $\tau = ijjk, ijkj, ijkl$, as

$$K_\tau^{(\mathbf{1})} \;=\; \overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\,RR\,\mathcal{W}_\tau\,, \qquad\quad K_\tau^{(\mathbf{2})} \;=\; \overline{\mathbf{L}}_\tau^{(\mathbf{2})}\,RR\,\mathcal{W}_\tau\,, \qquad\quad K_\tau^{(\mathbf{12})} \;=\; \overline{\mathbf{L}}_\tau^{(\mathbf{12})}\,RR\,\mathcal{W}_\tau\,, \tag{3.22}$$

The subtracted double-real squared matrix element for topology τ is then given by

$$RR_\tau^{\rm sub}(X) \;=\; RR\,\mathcal{W}_\tau\,\delta_{n+2}(X) - K_\tau^{(\mathbf{1})}\,\delta_{n+1}(X) - \Big(K_\tau^{(\mathbf{2})} - K_\tau^{(\mathbf{12})}\Big)\delta_n(X)\,. \tag{3.23}$$

This allows to build the complete $RR_{\rm sub}(X)$ of Eq. (2.36) by summing the contributions from all sectors according to

$$RR_{\rm sub}(X) = \sum_{i,\, j\neq i} \sum_{k\neq i,j} \bigg[RR^{\rm sub}_{ijjk}(X) + RR^{\rm sub}_{ijkj}(X) + \sum_{l\neq i,j,k} RR^{\rm sub}_{ijkl}(X) \bigg] . \tag{3.24}$$

The structure of Eq. (2.39) is then recovered by using Eq. (3.3), and by defining

$$\begin{aligned} K^{(\mathbf{1})} &= \sum_{i,\, j\neq i} \sum_{k\neq i,j} \bigg[K^{(\mathbf{1})}_{ijjk} + K^{(\mathbf{1})}_{ijkj} + \sum_{l\neq i,j,k} K^{(\mathbf{1})}_{ijkl} \bigg] , \\ K^{(\mathbf{2})} &= \sum_{i,\, j\neq i} \sum_{k\neq i,j} \bigg[K^{(\mathbf{2})}_{ijjk} + K^{(\mathbf{2})}_{ijkj} + \sum_{l\neq i,j,k} K^{(\mathbf{2})}_{ijkl} \bigg] , \\ K^{(\mathbf{12})} &= \sum_{i,\, j\neq i} \sum_{k\neq i,j} \bigg[K^{(\mathbf{12})}_{ijjk} + K^{(\mathbf{12})}_{ijkj} + \sum_{l\neq i,j,k} K^{(\mathbf{12})}_{ijkl} \bigg] . \end{aligned} \tag{3.25}$$

We emphasise that the definitions of the counterterms are actually complete only after specifying both the action $\overline{\mathbf{L}}\, RR$ of improved limits on the double-real matrix element, as well as the action $\overline{\mathbf{L}}\, \mathcal{W}_\tau$ on sector functions. All the improved limits are reported in Appendix C, and are written in terms of the soft and collinear kernels listed in Appendix B, multiplying appropriate versions of the Born-level probabilities, expressed in terms of mapped momenta.

In order to give the reader a feeling for the kind of expressions that emerge from this procedure, we reproduce here two representative examples. First, the uniform double-unresolved double-soft improved limit $\overline{\mathbf{S}}_{ik}$ $(i \neq k)$ can be written as

$$\begin{aligned} \overline{\mathbf{S}}_{ik}\, RR \equiv \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,k \\ d\neq i,k,c}} \bigg\{ \mathcal{E}^{(i)}_{cd} \sum_{e\neq i,k,c,d} \bigg[\sum_{f\neq i,k,c,d,e} \mathcal{E}^{(k)}_{ef}\, \bar{B}^{(icd,kef)}_{cdef} + 4\, \mathcal{E}^{(k)}_{ed}\, \bar{B}^{(icd,ked)}_{cded} \bigg] \\ + 2\, \mathcal{E}^{(i)}_{cd}\, \mathcal{E}^{(k)}_{cd}\, \bar{B}^{(icd,kcd)}_{cdcd} + \mathcal{E}^{(ik)}_{cd}\, \bar{B}^{(ikcd)}_{cd} \bigg\} , \end{aligned} \tag{3.26}$$

where the NLO eikonal kernel $\mathcal{E}^{(i)}_{cd}$ and the NNLO eikonal kernel $\mathcal{E}^{(ik)}_{cd}$ are presented in Eqs. (B.3) and (B.4), and we employed six-, five- and four-particle mappings for the colour-correlated Born terms, according to the numbers of particles involved. Note in particular that all eikonal dipoles are mapped differently, which is essential for the analytic integration, as discussed in Ref. [92] and in Section 4 below.

For the strongly-ordered double-unresolved double-soft improved limit $\overline{\mathbf{S}}_i\, \overline{\mathbf{S}}_{ik}$ $(i \neq k)$, on the other hand, we write

$$\begin{aligned} \overline{\mathbf{S}}_i\, \overline{\mathbf{S}}_{ik}\, RR \equiv \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,k \\ d\neq i,k,c}} \bigg\{ \mathcal{E}^{(i)}_{cd} \bigg[\sum_{e\neq i,k,c,d} \bigg(\sum_{f\neq i,k,c,d,e} \bar{\mathcal{E}}^{(k)(icd)}_{ef}\, \bar{B}^{(icd,kef)}_{cdef} + 2\, \bar{\mathcal{E}}^{(k)(icd)}_{ed}\, \bar{B}^{(icd,ked)}_{cded} \bigg) \\ + 2 \sum_{e\neq i,k,c,d} \bar{\mathcal{E}}^{(k)(idc)}_{ed}\, \bar{B}^{(idc,ked)}_{cded} + 2\, \bar{\mathcal{E}}^{(k)(icd)}_{cd} \Big(\bar{B}^{(icd,kcd)}_{cdcd} + C_A\, \bar{B}^{(icd,kcd)}_{cd} \Big) \bigg] \\ - 2\, C_A \Big[\mathcal{E}^{(i)}_{kc}\, \bar{\mathcal{E}}^{(k)(ick)}_{cd}\, \bar{B}^{(ick,kcd)}_{cd} + \mathcal{E}^{(i)}_{kd}\, \bar{\mathcal{E}}^{(k)(ikd)}_{cd}\, \bar{B}^{(ikd,kcd)}_{cd} \Big] \bigg\} . \end{aligned} \tag{3.27}$$

As might be expected, the complexity of the kernels has diminished with respect to Eq. (3.26) (indeed the expression solely features NLO eikonal factors), but the combinatorics has become more intricate. Notice that we used mapped momenta also in the eikonal kernels corresponding to the least-unresolved particle k.

It is important to stress that, while there appears to be considerable freedom in the choice of the improved limits, there are also stringent constraints that must be satisfied. In particular, the improved limits $\overline{\mathbf{L}}\, RR$ must carry the same symmetries under index exchange as the respective unimproved countertparts, so that the improved collections $\overline{\mathbf{L}}^{(\mathbf{1})}, \overline{\mathbf{L}}^{(\mathbf{2})}, \overline{\mathbf{L}}^{(\mathbf{12})}$ are still consistent with Eq. (3.15), whose content is based on the validity of the integrable relations listed in Eq. (3.13). Within the limitations of this requirement, there is still a residual freedom to modify the action of the improved limits acting on both RR and sector functions, with respect to the bare result extracted by their unimproved version. However, we must make sure that this procedure does preserve the locality of the cancellation of singularities, or, analogously, the finiteness of $RR^{\rm sub}_{ijjk}$, $RR^{\rm sub}_{ijkj}$ and $RR^{\rm sub}_{ijkl}$, defined in Eq. (3.23). To this end, we checked the consistency of the improved limits listed in Appendix C by analytically verifying that, for any topology τ, the corresponding $RR^{\rm sub}_{\tau}$ is in fact integrable in all singular limits of that tolopogy. Specifically, we verified analytically that

$$\begin{aligned}
\big\{\mathbf{S}_i,\ \mathbf{C}_{ij},\ \mathbf{S}_{ij},\ \mathbf{C}_{ijk},\ \mathbf{SC}_{ijk}\big\}\, RR^{\rm sub}_{ijjk} &\to \text{integrable}\,, \\
\big\{\mathbf{S}_i,\ \mathbf{C}_{ij},\ \mathbf{S}_{ik},\ \mathbf{C}_{ijk},\ \mathbf{SC}_{ijk},\ \mathbf{SC}_{kij}\big\}\, RR^{\rm sub}_{ijkj} &\to \text{integrable}\,, \\
\big\{\mathbf{S}_i,\ \mathbf{C}_{ij},\ \mathbf{S}_{ik},\ \mathbf{C}_{ijkl},\ \mathbf{SC}_{ikl},\ \mathbf{SC}_{kij}\big\}\, RR^{\rm sub}_{ijkl} &\to \text{integrable}\,.
\end{aligned} \tag{3.28}$$

Furthermore, since the collinear kernels of Appendix B display spurious collinear singularities involving the reference momentum k_r, which are not always screened by the sector functions, we verified explicitly that also the following relations hold

$$\begin{aligned}
\big\{\mathbf{C}_{ir},\ \mathbf{C}_{jr},\ \mathbf{C}_{ijr}\big\}\, RR^{\rm sub}_{ijjk} &\to \text{integrable}\,, \\
\big\{\mathbf{C}_{ir},\ \mathbf{C}_{kr},\ \mathbf{C}_{ikr}\big\}\, RR^{\rm sub}_{ijkj} &\to \text{integrable}\,, \\
\big\{\mathbf{C}_{ir},\ \mathbf{C}_{kr}\big\}\, RR^{\rm sub}_{ijkl} &\to \text{integrable}\,.
\end{aligned} \tag{3.29}$$

Having passed these tests, the improved limits listed in Appendix C, when assembled according to Eqs. (3.22)-(3.24), provide a fully local subtraction of phase-space singularities for the double-real-emission contribution to the cross section, and Eq. (3.24) is indeed integrable in the $(n+2)$-particle phase space. We now go on to illustrate a different construction for $RR_{\rm sub}$ based on symmetrised sector functions, similarly to what was done in Section 2.1 at NLO.

3.6 $RR_{\rm sub}$ with symmetrised sector functions

The partition of the $(n+2)$-particle phase space by means of the sector functions $\mathcal{W}_{abcd}$ that we introduced in Section 3.2 is not the only possible way forward. Analogously to what we did at NLO (see Eqs. (2.29) and (2.31)), this sector structure can be adapted to meet certain symmetry conditions that reduce the actual number of sectors: in particular, sectors sharing the same double-collinear singularities would naturally be parametrised in the same way in a numerical implementation, whence grouping such sectors in a single contribution is expected to improve numerical stability. Exploiting the symmetries of the improved limit $\overline{\mathbf{C}}_{ijk}$, we thus sum up the 6 permutations of i, j, k in sectors $\mathcal{W}_{ijjk}, \mathcal{W}_{ijkj}$ introducing the *symmetrised sector functions*

$$\begin{aligned}
\mathcal{Z}_{ijk} \;=\; & \mathcal{W}_{ijjk} + \mathcal{W}_{ikkj} + \mathcal{W}_{jiik} + \mathcal{W}_{jkki} + \mathcal{W}_{kiij} + \mathcal{W}_{kjji} \\
& + \mathcal{W}_{ijkj} + \mathcal{W}_{ikjk} + \mathcal{W}_{jiki} + \mathcal{W}_{jkik} + \mathcal{W}_{kiji} + \mathcal{W}_{kjij}\,.
\end{aligned} \tag{3.30}$$

Similarly, in the four-particle sectors $\mathcal{W}_{ijkl}$, we can exploit the symmetries of the improved limit $\overline{\mathbf{C}}_{ijkl}$ to sum up the 8 permutations $ijkl$, $ijlk$, $jikl$, $jilk$, $klij$, $klji$, $lkij$, $lkji$, and define

$$\mathcal{Z}_{ijkl} \;=\; \mathcal{W}_{ijkl} + \mathcal{W}_{ijlk} + \mathcal{W}_{jikl} + \mathcal{W}_{jilk} + \mathcal{W}_{klij} + \mathcal{W}_{klji} + \mathcal{W}_{lkij} + \mathcal{W}_{lkji}\,. \tag{3.31}$$

We also introduce the NLO-type symmetric sector functions

$$\mathcal{Z}_{ij}^{(\alpha)} \;\equiv\; \mathcal{W}_{ij}^{(\alpha)} + \mathcal{W}_{ji}^{(\alpha)}\,, \qquad \mathcal{Z}_{ij} \;\equiv\; \mathcal{Z}_{ij}^{(1)}\,, \tag{3.32}$$

where $\mathcal{W}_{ij}^{(\alpha)}$ was defined in Eq. (3.10). We will also find it useful to introduce a notation for the soft limit of the symmetric sector functions

$$\mathcal{Z}_{\mathrm{s},ij}^{(\alpha)} \;\equiv\; \mathbf{S}_i\,\mathcal{Z}_{ij}^{(\alpha)} \;=\; \mathbf{S}_i\,\mathcal{W}_{ij}^{(\alpha)} \;=\; \frac{\frac{1}{w_{ij}^{\alpha}}}{\sum\limits_{l\neq i}\frac{1}{w_{il}^{\alpha}}}\,, \qquad \mathcal{Z}_{\mathrm{s},ij} \;\equiv\; \mathcal{Z}_{\mathrm{s},ij}^{(1)}\,. \tag{3.33}$$

The use of $\mathcal{Z}_{ijk}$ and $\mathcal{Z}_{ijkl}$, upon reducing the number of sectors, simplifies the expression of the counterterms. In fact, deriving the action of the generic improved limit $\overline{\mathbf{L}}$ on the new sector functions (which can be directly obtained from the $\overline{\mathbf{L}}\,\mathcal{W}_{abcd}$ definitions in Appendix C), we verify that, thanks to their symmetries, any improved limit involving either the operator $\overline{\mathbf{C}}_{ijk}$, or the operator $\overline{\mathbf{C}}_{ijkl}$, when acting on $\mathcal{Z}_{ijk}$ and $\mathcal{Z}_{ijkl}$ respectively, reduces them to unity, according to

$$\overline{\mathbf{C}}_{ijk}\big(\dots\big)\,RR\,\mathcal{Z}_{ijk} \;=\; \overline{\mathbf{C}}_{ijk}\big(\dots\big)\,RR\,, \qquad \overline{\mathbf{C}}_{ijkl}\big(\dots\big)\,RR\,\mathcal{Z}_{ijkl} \;=\; \overline{\mathbf{C}}_{ijkl}\big(\dots\big)\,RR\,, \tag{3.34}$$

where the ellipsis denotes a generic sequence of improved limits.

In analogy with Eq. (3.22), we now define our local counterterms with symmetrised sector functions by

$$K_{\{\sigma\}}^{(\mathbf{1})} \;=\; \overline{\mathbf{L}}_{\{\sigma\}}^{(\mathbf{1})}\,RR\,\mathcal{Z}_{\sigma}\,, \qquad K_{\{\sigma\}}^{(\mathbf{2})} \;=\; \overline{\mathbf{L}}_{\{\sigma\}}^{(\mathbf{2})}\,RR\,\mathcal{Z}_{\sigma}\,, \qquad K_{\{\sigma\}}^{(\mathbf{12})} \;=\; \overline{\mathbf{L}}_{\{\sigma\}}^{(\mathbf{12})}\,RR\,\mathcal{Z}_{\sigma}\,, \tag{3.35}$$

where we denote the symmetrised topologies by $\sigma \in \{ijk, ijkl\}$, and the limits $\overline{\mathbf{L}}_{\{\sigma\}}$ are symmetrised versions of the limits in Eq. (3.15), to be presented below. The subtracted double-real contribution for a given symmetrised sector, in analogy with Eq. (3.23), is then given by

$$RR_{\{\sigma\}}^{\mathrm{sub}}(X) \;\equiv\; RR\;\mathcal{Z}_{\sigma}\,\delta_{n+2}(X) - K_{\{\sigma\}}^{(\mathbf{1})}\,\delta_{n+1}(X) - \Big(K_{\{\sigma\}}^{(\mathbf{2})} - K_{\{\sigma\}}^{(\mathbf{12})}\Big)\,\delta_n(X)\,, \tag{3.36}$$

and finally the full expression for $RR_{\mathrm{sub}}(X)$ of Eq. (2.36) is obtained by summing the contributions from the symmetrised sectors $\mathcal{Z}_{ijk}$, $\mathcal{Z}_{ijkl}$. It reads

$$RR_{\mathrm{sub}}(X) \;=\; \sum_{i,\,j>i}\Bigg[\sum_{k>j} RR_{\{ijk\}}^{\mathrm{sub}}(X) + \sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq i,j\\l>k}} RR_{\{ijkl\}}^{\mathrm{sub}}(X)\Bigg]. \tag{3.37}$$

This expression can be written in the form of Eq. (2.39) by building the complete counterterms $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$ and $K^{(\mathbf{12})}$ in terms of symmetrised sector functions, as

$$\begin{aligned}
K^{(\mathbf{1})} \;&=\; \sum_{i,\,j>i}\Bigg[\sum_{k>j} K_{\{ijk\}}^{(\mathbf{1})} + \sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq i,j\\l>k}} K_{\{ijkl\}}^{(\mathbf{1})}\Bigg],\\
K^{(\mathbf{2})} \;&=\; \sum_{i,\,j>i}\Bigg[\sum_{k>j} K_{\{ijk\}}^{(\mathbf{2})} + \sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq i,j\\l>k}} K_{\{ijkl\}}^{(\mathbf{2})}\Bigg],\\
K^{(\mathbf{12})} \;&=\; \sum_{i,\,j>i}\Bigg[\sum_{k>j} K_{\{ijk\}}^{(\mathbf{12})} + \sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq i,j\\l>k}} K_{\{ijkl\}}^{(\mathbf{12})}\Bigg].
\end{aligned} \tag{3.38}$$

The symmetrised improved limits required to compute the symmetrised counterterms defined in Eq. (3.35) can be derived from the limits designed for the $\mathcal{W}_{abcd}$ sector functions, which were presented in Eq. (3.15) before improvement. The symmetrisation must be done carefully, in order not to overcount singular configurations. We adopt the following procedure. First, we expand all products in Eq. (3.15), and we express the corresponding *improved* limits as flat sums running over the respective sets of relevant singular limits. For example, we write

$$
\begin{aligned}
\overline{\mathbf{L}}^{(\mathbf{1})}_{ab} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{1})}_{ab}} \bar\ell\,, \qquad \text{where} \quad \mathcal{L}^{(\mathbf{1})}_{ab} = \Big\{ \overline{\mathbf{S}}_a,\, \overline{\mathbf{C}}_{ab},\, -\overline{\mathbf{S}}_a\,\overline{\mathbf{C}}_{ab} \Big\}\,, \\
\overline{\mathbf{L}}^{(\mathbf{2})}_{abbc} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{2})}_{abbc}} \bar\ell\,, \qquad \text{where} \quad \mathcal{L}^{(\mathbf{2})}_{abbc} = \Big\{ \overline{\mathbf{S}}_{ab},\, \overline{\mathbf{SC}}_{abc},\, -\overline{\mathbf{SC}}_{abc}\,\overline{\mathbf{S}}_{ab},\, \overline{\mathbf{C}}_{abc},\, -\overline{\mathbf{S}}_{ab}\,\overline{\mathbf{C}}_{abc}, \\
&\qquad\qquad\qquad\qquad\qquad -\overline{\mathbf{SC}}_{abc}\,\overline{\mathbf{C}}_{abc},\, \overline{\mathbf{SC}}_{abc}\,\overline{\mathbf{S}}_{ab}\,\overline{\mathbf{C}}_{abc} \Big\}\,,
\end{aligned}
\tag{3.39}
$$

and similarly for the remaining limits given in Eq. (3.15). Next, we introduce the index sets

$$
\begin{aligned}
\alpha &= \{ij, ji, ik, ki, jk, kj\}\,, & \beta &= \{ij, ji, kl, lk\}\,, \\
\gamma_1 &= \{ijjk, ikkj, jkki, jiik, kiij, kjji\}\,, & \gamma_2 &= \{ijkj, ikjk, jkik, jiki, kiji, kjij\}\,, \\
\delta &= \{ijkl, ijlk, jikl, jilk, klij, klji, lkij, lkji\}\,,
\end{aligned}
\tag{3.40}
$$

which enumerate the permutations that will need to be summed in order to perform the required symmetrisations. The limits $\overline{\mathbf{L}}^{(\mathbf{1})}_{\{\sigma\}}$, $\overline{\mathbf{L}}^{(\mathbf{2})}_{\{\sigma\}}$ and $\overline{\mathbf{L}}^{(\mathbf{12})}_{\{\sigma\}}$ can now be defined by sums running over unions of the sets $\mathcal{L}$. Specifically, we define

$$
\begin{aligned}
\overline{\mathbf{L}}^{(\mathbf{1})}_{\{ijk\}} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{1})}_{\alpha}} \bar\ell\,, \qquad \text{where} \quad \mathcal{L}^{(\mathbf{1})}_{\alpha} = \bigcup_{ab\,\in\,\alpha} \mathcal{L}^{(\mathbf{1})}_{ab}\,, \\
\overline{\mathbf{L}}^{(\mathbf{1})}_{\{ijkl\}} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{1})}_{\beta}} \bar\ell\,, \qquad \text{where} \quad \mathcal{L}^{(\mathbf{1})}_{\beta} = \bigcup_{ab\,\in\,\beta} \mathcal{L}^{(\mathbf{1})}_{ab}\,, \\
\overline{\mathbf{L}}^{(\mathbf{2})}_{\{ijk\}} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{2})}_{\gamma}} \bar\ell\,, \qquad \text{where} \quad \mathcal{L}^{(\mathbf{2})}_{\gamma} = \Bigg[\bigcup_{abbc\,\in\,\gamma_1} \mathcal{L}^{(\mathbf{2})}_{abbc} \Bigg] \cup \Bigg[\bigcup_{abcb\,\in\,\gamma_2} \mathcal{L}^{(\mathbf{2})}_{abcb} \Bigg]\,, \\
\overline{\mathbf{L}}^{(\mathbf{2})}_{\{ijkl\}} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{2})}_{\delta}} \bar\ell\,, \qquad \text{where} \quad \mathcal{L}^{(\mathbf{2})}_{\delta} = \bigcup_{abcd\,\in\,\delta} \mathcal{L}^{(\mathbf{2})}_{abcd}\,.
\end{aligned}
\tag{3.41}
$$

Similarly, the strongly-ordered double-unresolved limits $\overline{\mathbf{L}}^{(\mathbf{12})}_{\{\sigma\}}$ are given by analogous sums, where for $\sigma = ijk$ the sum runs over the collection $\mathcal{L}^{(\mathbf{12})}_{\gamma}$, and, for $\sigma = ijkl$, the sum runs over the collection $\mathcal{L}^{(\mathbf{12})}_{\delta}$, defined as in the last two lines of Eq. (3.41), with the replacement $(\mathbf{2}) \to (\mathbf{12})$. While assembling the set unions introduced in Eq. (3.41), one must take care to count only once all limits that coincide by symmetry: thus, for example, one should use the fact that $\overline{\mathbf{C}}_{ij} = \overline{\mathbf{C}}_{ji}$, and $\overline{\mathbf{SC}}_{ijk} = \overline{\mathbf{SC}}_{ikj}$. To further illustrate the procedure, we note that the first line of Eq. (3.41) becomes

$$
\begin{aligned}
\overline{\mathbf{L}}^{(\mathbf{1})}_{\{ijk\}} &= \overline{\mathbf{S}}_i + \overline{\mathbf{S}}_j + \overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{ij} + \overline{\mathbf{C}}_{ik} + \overline{\mathbf{C}}_{jk} \\
&\quad - \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_j\,\overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ik} - \overline{\mathbf{S}}_k\,\overline{\mathbf{C}}_{ik} - \overline{\mathbf{S}}_j\,\overline{\mathbf{C}}_{jk} - \overline{\mathbf{S}}_k\,\overline{\mathbf{C}}_{jk} \\
&= \overline{\mathbf{S}}_i + \overline{\mathbf{S}}_j + \overline{\mathbf{S}}_k + \overline{\mathbf{HC}}_{ij} + \overline{\mathbf{HC}}_{ik} + \overline{\mathbf{HC}}_{jk}\,,
\end{aligned}
\tag{3.42}
$$

properly including all relevant singular regions without double counting.

The explicit results for the sums in Eq. (3.41) appear rather cumbersome at first sight, but in fact they result in relatively compact expressions when the limits are evaluated. Indeed, thanks to the symmetry properties of $\mathcal{Z}_{ijk}$ and $\mathcal{Z}_{ijkl}$, it is possible to merge subsets of singular limits which factor identical combinations of symmetrised sector functions. One finds then that only certain combinations of singular limits survive in the result. In detail, all single-unresolved limits can be written explicitly as sums of single-soft limits $\overline{\mathbf{S}}_a$ plus hard-collinear combinations $\overline{\mathbf{HC}}_{ab}$, defined in Eq. (2.29). Furthermore, it is useful to introduce a soft-subtracted version of the uniform double-unresolved limit $\overline{\mathbf{SC}}_{abc}$, which is given by

$$\overline{\mathbf{SHC}}_{abc} \equiv \overline{\mathbf{SC}}_{abc} \left(1 - \overline{\mathbf{S}}_{ab} - \overline{\mathbf{S}}_{ac}\right) . \tag{3.43}$$

This combination can appear only when attached to either the $\overline{\mathbf{S}}_a$ or $\overline{\mathbf{C}}_{abc}$ limits: indeed, in any other case, the operators $\overline{\mathbf{SC}}_{abc}$ and $\overline{\mathbf{S}}_{ab}\,\overline{\mathbf{SC}}_{abc}$ do not share the same sector functions in the limit. Similarly, considering the double-unresolved improved collinear limit $\overline{\mathbf{C}}_{abc}$, we can distinguish three useful combinations, defined by

$$\begin{aligned}
\overline{\mathbf{HC}}_{abc} &\equiv \overline{\mathbf{C}}_{abc} \left(1 - \overline{\mathbf{S}}_{ab} - \overline{\mathbf{S}}_{bc} - \overline{\mathbf{S}}_{ac}\right) , \\
\overline{\mathbf{HC}}_{abc}^{(\mathbf{s})} &\equiv \overline{\mathbf{C}}_{abc} \left(1 - \overline{\mathbf{S}}_{ab} - \overline{\mathbf{S}}_{ac}\right) \left(1 - \overline{\mathbf{SC}}_{abc}\right) , \\
\overline{\mathbf{HC}}_{abc}^{(\mathbf{c})} &\equiv \overline{\mathbf{C}}_{abc} \left(1 - \overline{\mathbf{S}}_{ab} - \overline{\mathbf{SC}}_{cab}\right) ,
\end{aligned} \tag{3.44}$$

which reflect three different possible strategies for removing soft singularities from the collinear kernel. The superscripts $(\mathbf{s})$ and $(\mathbf{c})$ in the second and third line of Eq. (3.44) refer to the fact that the $(\mathbf{s})$ combination can appear only in association with a single-soft limit $\overline{\mathbf{S}}_d$ (with $d \in \{a, b, c\}$), while the $(\mathbf{c})$ combination can appear only in association with single hard-collinear limits $\overline{\mathbf{HC}}_{de}$, with $de \in \{ab, ac, bc\}$. Finally, for the four-particle double-collinear improved limit $\overline{\mathbf{C}}_{ijkl}$ we introduce

$$\begin{aligned}
\overline{\mathbf{HC}}_{abcd} &\equiv \overline{\mathbf{C}}_{abcd} \left(1 + \overline{\mathbf{S}}_{ac} + \overline{\mathbf{S}}_{bc} + \overline{\mathbf{S}}_{ad} + \overline{\mathbf{S}}_{bd} - \overline{\mathbf{SC}}_{acd} - \overline{\mathbf{SC}}_{bcd} - \overline{\mathbf{SC}}_{cab} - \overline{\mathbf{SC}}_{dab}\right) , \\
\overline{\mathbf{HC}}_{abcd}^{(\mathbf{c})} &\equiv \overline{\mathbf{C}}_{abcd} \left(1 - \overline{\mathbf{SC}}_{cab} - \overline{\mathbf{SC}}_{dab}\right) ,
\end{aligned} \tag{3.45}$$

where again the notation $(\mathbf{c})$ refers to the fact that the combined limit in the second line of Eq. (3.45) can only appear in association with the hard-collinear single-unresolved limits $\overline{\mathbf{HC}}_{ab}$ and $\overline{\mathbf{HC}}_{cd}$.

Using these preliminary definitions, we can write down explicit expressions for the symmetrised improved limits defined in Eq. (3.41). They are

$$\begin{aligned}
\overline{\mathbf{L}}^{(\mathbf{1})}_{\{ijk\}} &= \overline{\mathbf{S}}_i + \overline{\mathbf{S}}_j + \overline{\mathbf{S}}_k + \overline{\mathbf{HC}}_{ij} + \overline{\mathbf{HC}}_{jk} + \overline{\mathbf{HC}}_{ik} \,, \\
\overline{\mathbf{L}}^{(\mathbf{1})}_{\{ijkl\}} &= \overline{\mathbf{S}}_i + \overline{\mathbf{S}}_j + \overline{\mathbf{S}}_k + \overline{\mathbf{S}}_l + \overline{\mathbf{HC}}_{ij} + \overline{\mathbf{HC}}_{kl} \,, \\
\overline{\mathbf{L}}^{(\mathbf{2})}_{\{ijk\}} &= \overline{\mathbf{S}}_{ij} + \overline{\mathbf{S}}_{jk} + \overline{\mathbf{S}}_{ik} + \overline{\mathbf{SC}}_{ijk}(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}) + \overline{\mathbf{SC}}_{jik}(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{jk}) + \overline{\mathbf{SC}}_{kij}(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}) \\
&\quad + \overline{\mathbf{HC}}_{ijk} - \overline{\mathbf{C}}_{ijk}\big(\,\overline{\mathbf{SHC}}_{ijk} + \overline{\mathbf{SHC}}_{jik} + \overline{\mathbf{SHC}}_{kij}\,\big) \,, \\
\overline{\mathbf{L}}^{(\mathbf{2})}_{\{ijkl\}} &= \overline{\mathbf{S}}_{ik} + \overline{\mathbf{S}}_{jk} + \overline{\mathbf{S}}_{il} + \overline{\mathbf{S}}_{jl} + \overline{\mathbf{SC}}_{ikl}\left(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{il}\right) + \overline{\mathbf{SC}}_{jkl}\left(1-\overline{\mathbf{S}}_{jk}-\overline{\mathbf{S}}_{jl}\right) \\
&\quad + \overline{\mathbf{SC}}_{kij}\left(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}\right) + \overline{\mathbf{SC}}_{lij}\left(1-\overline{\mathbf{S}}_{il}-\overline{\mathbf{S}}_{jl}\right) + \overline{\mathbf{HC}}_{ijkl} \,, \\
\overline{\mathbf{L}}^{(\mathbf{12})}_{\{ijk\}} &= \overline{\mathbf{S}}_i\left(\overline{\mathbf{S}}_{ij} + \overline{\mathbf{S}}_{ik} + \overline{\mathbf{SHC}}_{ijk}\right) + \overline{\mathbf{S}}_j\left(\overline{\mathbf{S}}_{ij} + \overline{\mathbf{S}}_{jk} + \overline{\mathbf{SHC}}_{jik}\right) + \overline{\mathbf{S}}_k\left(\overline{\mathbf{S}}_{ik} + \overline{\mathbf{S}}_{jk} + \overline{\mathbf{SHC}}_{kij}\right) \\
&\quad + \left(\overline{\mathbf{S}}_i + \overline{\mathbf{S}}_j + \overline{\mathbf{S}}_k\right)\overline{\mathbf{HC}}^{(\mathbf{s})}_{ijk} + \overline{\mathbf{HC}}_{ij}\left(\overline{\mathbf{S}}_{ij} + \overline{\mathbf{SC}}_{kij} + \overline{\mathbf{HC}}^{(\mathbf{c})}_{ijk}\right) \\
&\quad + \overline{\mathbf{HC}}_{jk}\left(\overline{\mathbf{S}}_{jk} + \overline{\mathbf{SC}}_{ijk} + \overline{\mathbf{HC}}^{(\mathbf{c})}_{ijk}\right) + \overline{\mathbf{HC}}_{ik}\left(\overline{\mathbf{S}}_{ik} + \overline{\mathbf{SC}}_{jik} + \overline{\mathbf{HC}}^{(\mathbf{c})}_{ijk}\right) \,; \\
\overline{\mathbf{L}}^{(\mathbf{12})}_{\{ijkl\}} &= \overline{\mathbf{S}}_i\left(\overline{\mathbf{S}}_{ik} + \overline{\mathbf{S}}_{il}\right) + \overline{\mathbf{S}}_j\left(\overline{\mathbf{S}}_{jk} + \overline{\mathbf{S}}_{jl}\right) + \overline{\mathbf{S}}_k\left(\overline{\mathbf{S}}_{ik} + \overline{\mathbf{S}}_{jk}\right) + \overline{\mathbf{S}}_l\left(\overline{\mathbf{S}}_{il} + \overline{\mathbf{S}}_{jl}\right) \\
&\quad + \overline{\mathbf{S}}_i\,\overline{\mathbf{SHC}}_{ikl} + \overline{\mathbf{S}}_j\,\overline{\mathbf{SHC}}_{jkl} + \overline{\mathbf{S}}_k\,\overline{\mathbf{SHC}}_{kij} + \overline{\mathbf{S}}_l\,\overline{\mathbf{SHC}}_{lij} \\
&\quad + \overline{\mathbf{HC}}_{ij}\left(\overline{\mathbf{SC}}_{kij} + \overline{\mathbf{SC}}_{lij}\right) + \overline{\mathbf{HC}}_{kl}\left(\overline{\mathbf{SC}}_{ikl} + \overline{\mathbf{SC}}_{jkl}\right) + \left(\overline{\mathbf{HC}}_{ij} + \overline{\mathbf{HC}}_{kl}\right)\overline{\mathbf{HC}}^{(\mathbf{c})}_{ijkl} \,.
\end{aligned} \tag{3.46}$$

The actions of all these improved limits on RR and on the symmetrised sector functions $\mathcal{Z}_{ijk}$, $\mathcal{Z}_{ijkl}$ are reported in Appendix C.

Comparing the collections of singular projectors relevant to $\mathcal{W}_{abcd}$ sector functions in Eq. (3.15) with the ones reported in Eq. (3.46) for the symmetrised case, it is immediate to notice that the number of different non-trivial singular limits contributing to a given sector changes, depending on the type of partition we introduce. In particular, this number increases for our choice of $\mathcal{Z}_{ijk}$ and $\mathcal{Z}_{ijkl}$. Despite this, though, the ordered sums in Eq. (3.37), building up the relevant integrable contributions, lead to a significantly more compact final expression (in terms of the number of different objects one needs to define and evaluate). This is a feature that will translate into a gain in computational time and resources in the final numerical implementation.

4 Integration of the double-real-radiation counterterms

In the previous section we constructed $RR_{\rm sub}$ of Eq. (2.39), a combination which is integrable everywhere in the double-radiative phase space, by subtracting the local counterterms $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$ and $K^{(\mathbf{12})}$ (given in Eq. (3.25), or equivalently in Eq. (3.38)) from the double-real squared matrix element RR. These counterterms must now be added back, after integrating out one or two emissions, yielding the integrated counterterms $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$, $I^{(\mathbf{12})}$. The integration procedure in the presence of sectors involves rather intricate combinatorics, and generates lengthy expressions in the intermediate stages. However, all integrals that need to be computed are remarkably simple, and in almost all cases have trivial (logarithmic) dependence on the Mandelstam invariants [92].

We will begin, in Section 4.1, by introducing the relevant phase-space factorisations and parameterisations, derived from the nested Catani-Seymour mappings introduced in Section 3.4. Then, in Section 4.2, we will report the integration of the counterterms $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$, $K^{(\mathbf{12})}$, specifying how each singular contribution is treated. The resulting expressions can be simplified, by relabelling the momenta and rewriting the flavour sums of the original $(n+2)$-body phase space, as explained in Section 4.3. It is then possible to recombine the contributions carrying different mappings, resulting in relatively compact collections of integrals for $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$, $I^{(\mathbf{12})}$, presented in Section 4.4. At this stage, the results can be directly employed in the subtraction formula, Eq. (2.36).

It is natural to define $I^{(\mathbf{1})}$ as the integral of $K^{(\mathbf{1})}$ in the single-unresolved radiation, and $I^{(\mathbf{2})}$ as the integral of $K^{(\mathbf{2})}$ in both unresolved emissions. For the strongly-ordered counterterm $K^{(\mathbf{12})}$ both possibilities are in principle viable. In our framework, we define $I^{(\mathbf{12})}$ as the integral of $K^{(\mathbf{12})}$ in a single radiation[9], corresponding to the 'most unresolved' radiated particle, as explicitly noted in Eq. (2.35). As a consequence, before performing the integrations, we rewrite both $K^{(\mathbf{1})}$ and $K^{(\mathbf{12})}$ by summing up the sector functions related to the most unresolved radiation (the ones carrying the suffix α), while keeping the sector functions for the second (least-unresolved) radiation untouched. Note however that these remaining sector functions carry mapped kinematics. In this way, it will be possible to combine directly the integrated counterterms $I^{(\mathbf{1})}$ and $I^{(\mathbf{12})}$ with the real-virtual contribution RV, and with the real-virtual counterterm $K^{(\mathbf{RV})}$, in Eq. (2.38), sector by sector in Φ_{n+1}. For the sake of simplicity, in the following all integrations are described using the expressions for $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$ and $K^{(\mathbf{12})}$ in terms of symmetrised sector functions, as given in Eq. (3.38), but the resulting expressions for $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$ and $I^{(\mathbf{12})}$ will be given also in terms of the $\mathcal{W}$ sector functions.

4.1 Phase-space parametrisations

We start by giving precise definitions for the measures of integration in the radiative phase spaces $d\Phi_{\rm rad}$ and $d\Phi_{\rm rad,2}$, according to Eq. (2.5), but now highlighting the dependence on the chosen mappings (discussed in Section 3.4), and making specific choices of integration variables.

The single-unresolved counterterm $K^{(\mathbf{1})}$ contains just single mappings of the type $\{\bar{k}\}^{(acd)}$ (a, c, d all different) and is going to be integrated in the corresponding single-radiation phase space.

[9] We note that in the context of the `CoLoRFul` approach to subtraction [94, 95], the strongly-ordered counterterm is integrated directly in both unresolved radiations.

On the contrary, $K^{(\mathbf{12})}$ and $K^{(\mathbf{2})}$ are built by means of iterated mappings of the type $\{\bar{k}\}^{(acd,bef)}$ (a,c,d all different and b,e,f all different). However, while $K^{(\mathbf{12})}$ needs to be integrated just in the phase space of the single radiation corresponding to the first mapping, $K^{(\mathbf{2})}$ must be integrated in the whole double-radiation phase space.

We start specifying Eq. (2.34), needed for the integration of $K^{(\mathbf{1})}$ and $K^{(\mathbf{12})}$. We write

$$\int d\Phi_{n+2}(\{k\}) \;=\; \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \int d\Phi_{n+1}^{(acd)} \int d\Phi_{\mathrm{rad}}^{(acd)} \,, \tag{4.1}$$

where we defined

$$d\Phi_{n+1}^{(acd)} \;\equiv\; d\Phi_{n+1}(\{\bar{k}\}^{(acd)})\,. \tag{4.2}$$

The explicit expression for the radiative measure is

$$\int d\Phi_{\mathrm{rad}}^{(acd)} \;=\; N(\epsilon) \left(\bar{s}_{cd}^{(acd)}\right)^{1-\epsilon} \int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \left[y(1-y)^2 z(1-z)\right]^{-\epsilon} (1-y)\,, \tag{4.3}$$

where

$$N(\epsilon) \;\equiv\; \frac{(4\pi)^{\epsilon-2}}{\sqrt{\pi}\,\Gamma\!\left(\frac{1}{2}-\epsilon\right)}\,. \tag{4.4}$$

The invariants composed by the momenta k_a, k_c, k_d are related to the integration variables y and z by

$$s_{ac} \;=\; y\,\bar{s}_{cd}^{(acd)}\,, \qquad s_{ad} \;=\; z(1-y)\,\bar{s}_{cd}^{(acd)}\,, \qquad s_{cd} \;=\; (1-z)(1-y)\,\bar{s}_{cd}^{(acd)}\,, \tag{4.5}$$

so that $s_{acd} \;=\; s_{ac}+s_{ad}+s_{cd} = \bar{s}_{cd}^{(acd)}$.

To parametrise the double-radiative phase space, needed for $K^{(\mathbf{2})}$, we employ double mappings of three different types, as discussed in Section 3.4. We examine them in turn.

The six-particle mapping $\{\bar{k}\}^{(acd,bef)}$ (a,b,c,d,e,f all different) presented in Eqs. (3.16) and (3.17) induces the factorization

$$\int d\Phi_{n+2}(\{k\}) \;=\; \frac{\varsigma_{n+2}}{\varsigma_n} \int d\Phi_n^{(acd,bef)} \int d\Phi_{\mathrm{rad},2}^{(acd,bef)}\,, \qquad d\Phi_n^{(acd,bef)} \;\equiv\; d\Phi_n(\{\bar{k}\}^{(acd,bef)})\,, \tag{4.6}$$

and the radiative measure of integration is

$$\begin{aligned}\int d\Phi_{\mathrm{rad},2}^{(acd,bef)} \;=&\; N^2(\epsilon) \left(\bar{s}_{cd}^{(acd,bef)}\,\bar{s}_{ef}^{(acd,bef)}\right)^{1-\epsilon} \int_0^\pi d\phi'\,(\sin\phi')^{-2\epsilon} \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon} \\ &\times \int_0^1 dy \int_0^1 dz \left[y'(1-y')^2\,z'(1-z')\,y(1-y)^2\,z(1-z)\right]^{-\epsilon} (1-y')(1-y)\,,\end{aligned} \tag{4.7}$$

where the expressions for relevant invariants in terms of the integration variables are

$$\begin{aligned} s_{ac} \;&=\; y'\,\bar{s}_{cd}^{(acd,bef)}\,, \qquad s_{ad} \;=\; z'\,(1-y')\,\bar{s}_{cd}^{(acd,bef)}\,, \qquad s_{cd} \;=\; (1-z')(1-y')\,\bar{s}_{cd}^{(acd,bef)}\,, \\ s_{be} \;&=\; y\,\bar{s}_{ef}^{(acd,bef)}\,, \qquad s_{bf} \;=\; z\,(1-y)\,\bar{s}_{ef}^{(acd,bef)}\,, \qquad s_{ef} \;=\; (1-z)(1-y)\,\bar{s}_{ef}^{(acd,bef)}\,, \end{aligned} \tag{4.8}$$

so that $s_{acd} \;=\; s_{ac}+s_{ad}+s_{cd} = \bar{s}_{cd}^{(acd,bef)} \;=\; \bar{s}_{cd}^{(acd)}$, and $s_{bef} \;=\; s_{be}+s_{bf}+s_{ef} = \bar{s}_{ef}^{(acd,bef)} \;=\; \bar{s}_{ef}^{(bef)}$.

The five-particle mapping $\{\bar{k}\}^{(acd,bed)}$ (a,b,c,d,e all different) presented in Eqs. (3.18) and (3.19) induces the factorization

$$\int d\Phi_{n+2}(\{k\}) \;=\; \frac{\varsigma_{n+2}}{\varsigma_n} \int d\Phi_n^{(acd,bed)} \int d\Phi_{\mathrm{rad},2}^{(acd,bed)}\,, \qquad d\Phi_n^{(acd,bed)} \;\equiv\; d\Phi_n(\{\bar{k}\}^{(acd,bed)})\,, \tag{4.9}$$

and we write

$$\int d\Phi^{(acd,bed)}_{\mathrm{rad},2} = N^2(\epsilon)\,\Big(\bar s^{(acd,bed)}_{cd}\,\bar s^{(acd,bed)}_{ed}\Big)^{1-\epsilon}\int_0^\pi d\phi'\,(\sin\phi')^{-2\epsilon}\int_0^1 dy'\int_0^1 dz'\int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon}\int_0^1 dy$$
$$\times\int_0^1 dz\,\Big[y'(1-y')^2\,z'(1-z')\,y(1-y)^3\,z(1-z)\Big]^{-\epsilon}(1-y')(1-y)^2\,, \qquad (4.10)$$

with

$$\begin{aligned}
s_{ac} &= y'\,(1-y)\,\bar s^{(acd,bed)}_{cd}\,, & s_{ad} &= z'\,(1-y')(1-y)\,\bar s^{(acd,bed)}_{cd}\,,\\
s_{be} &= y\,\bar s^{(acd,bed)}_{ed}\,, & s_{bd} &= (1-y')\,z\,(1-y)\,\bar s^{(acd,bed)}_{ed}\,,\\
s_{cd} &= (1-y')(1-z')(1-y)\,\bar s^{(acd,bed)}_{cd}\,, & s_{ed} &= (1-y')(1-z)(1-y)\,\bar s^{(acd,bed)}_{ed}\,, \qquad (4.11)
\end{aligned}$$

so that the five-parton invariant $s_{abcde} = s_{ab}+s_{ac}+s_{ad}+s_{ae}+s_{bc}+s_{bd}+s_{be}+s_{cd}+s_{ce}+s_{de}$ is equal to $\bar s^{(acd,bed)}_{cde} = \bar s^{(acd,bed)}_{cd}+\bar s^{(acd,bed)}_{ce}+\bar s^{(acd,bed)}_{de}$.

Finally, we have the four-particle mapping, $\{\bar k\}^{(acd,bcd)} = \{\bar k\}^{(abcd)}$, ($a,b,c,d$ all different), presented in Eqs. (3.20) and (3.21). This is the most intricate mapping, inducing the factorization

$$\int d\Phi_{n+2}(\{k\}) = \frac{\varsigma_{n+2}}{\varsigma_n}\int d\Phi^{(abcd)}_n\int d\Phi^{(abcd)}_{\mathrm{rad},2}\,, \qquad d\Phi^{(abcd)}_n \equiv d\Phi_n(\{\bar k\}^{(abcd)})\,, \qquad (4.12)$$

where we write

$$\int d\Phi^{(abcd)}_{\mathrm{rad},2} = 2^{-2\epsilon}\,N^2(\epsilon)\,\Big(\bar s^{(abcd)}_{cd}\Big)^{2-2\epsilon}\int_0^1 dw'\int_0^1 dy'\int_0^1 dz'\int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon}\int_0^1 dy\int_0^1 dz$$
$$\times\Big[w'(1-w')\Big]^{-1/2-\epsilon}\Big[y'(1-y')^2\,z'(1-z')\,y^2(1-y)^2\,z(1-z)\Big]^{-\epsilon}(1-y')\,y\,(1-y)\,,$$

with

$$\begin{aligned}
s_{ab} &= y'\,y\,\bar s^{(abcd)}_{cd}\,, \qquad s_{ac} = z'(1-y')\,y\,\bar s^{(abcd)}_{cd}\,, \qquad s_{bc} = (1-y')(1-z')\,y\,\bar s^{(abcd)}_{cd}\,,\\
s_{cd} &= (1-y')(1-y)(1-z)\,\bar s^{(abcd)}_{cd}\,,\\
s_{ad} &= (1-y)\Big[y'(1-z')(1-z)+z'z-2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,\bar s^{(abcd)}_{cd}\,,\\
s_{bd} &= (1-y)\Big[y'z'(1-z)+(1-z')z+2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,\bar s^{(abcd)}_{cd}\,, \qquad (4.13)
\end{aligned}$$

so that $s_{abcd} = s_{ab}+s_{ac}+s_{ad}+s_{bc}+s_{bd}+s_{cd} = \bar s^{(abcd)}_{cd}$.

4.2 Integration of $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$ and $K^{(\mathbf{12})}$

We now have all the ingredients to actually perform the required integrations. Our task is simplified by the fact that the integrals of the azimuthal parts of the collinear kernels (see (B.7)) vanish, as shown in Appendix D. All remaining integrals are then computed following the techniques explained in [92]. We will later recombine the components that were differently mapped by relabelling momenta, in order to compose the complete results, which will be considerably simpler.

For the single-unresolved counterterm $K^{(\mathbf{1})}$ the required integral is

$$\int d\Phi_{n+2}\,K^{(\mathbf{1})} = \int d\Phi_{n+2}\Bigg\{\sum_{i,j\neq i}\sum_{\substack{k\neq i\\k>j}}\overline{\mathbf{S}}_i\,RR\,\bar{\mathcal{Z}}_{jk}+\sum_{i,j>i}\sum_{k\neq i}\sum_{\substack{l\neq i\\l>k}}\overline{\mathbf{HC}}_{ij}\,RR\,\bar{\mathcal{Z}}_{kl}\Bigg\}\,. \qquad (4.14)$$

The integrand on the right-hand side has been obtained from $K^{(\mathbf{1})}$ of Eq. (3.38) by summing up the NLO sector functions with label α of Eqs. (C.92)-(C.93). As explained in Appendix C, the mapped sector functions $\bar{\mathcal{Z}}_{ij}$ are understood to carry the same mapping as the matrix elements they multiply. Since Eq. (4.14) will have to be combined with the real-virtual contribution RV, as part of Eq. (2.38), we need to express the integral in Eq. (4.14) as a sum of terms in which the integration over the single-particle radiative phase space has been performed, a specific parametrisation for the $(n+1)$-particle phase space has been identified, and the full single-real-radiation squared matrix element R has been factored, and computed in the chosen variables. The results for the summands of the two terms in Eq. (4.14) take the form

$$\int d\Phi_{n+2}\, \overline{\mathbf{S}}_i\, RR\, \bar{\mathcal{Z}}_{jk} \;=\; -\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}} \sum_{c\neq i} \sum_{d\neq i,c} \int d\Phi_{n+1}^{(icd)}\; J_s^{icd}\; \bar{R}_{cd}^{(icd)}\; \bar{\mathcal{Z}}_{jk}^{(icd)}\,, \tag{4.15}$$

$$\int d\Phi_{n+2}\, \overline{\mathbf{HC}}_{ij}\, RR\, \bar{\mathcal{Z}}_{kl} \;=\; \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \int d\Phi_{n+1}^{(ijr)}\; J_{\rm hc}^{ijr}\; \bar{R}^{(ijr)}\; \bar{\mathcal{Z}}_{kl}^{(ijr)}\,, \qquad r = r_{ijkl}\,, \tag{4.16}$$

where the measure of integration $d\Phi_{n+1}^{(acd)}$ was defined in Eq. (4.2). The integration over the appropriate $d\Phi_{\rm rad}$ has been performed, yielding the integrals J_s^{icd} and $J_{\rm hc}^{ijr}$, whose explicit expressions are given in Eq. (E.1) and in Eq. (E.7), respectively. The choice of $r = r_{ijkl} \neq i, j, k, l$, according to the rule of Eq. (A.14), which reflects the choice made for $\overline{\mathbf{HC}}_{ij}\, RR$ in Eq. (C.11), causes a dependence of the integrated kernel $J_{\rm hc}^{ijr}$ on the indices k and l of the sector function $\bar{\mathcal{Z}}_{kl}^{(ijr)}$. Notice that the choice $r = r_{ijkl}$ implies the need for at least five massless partons in Φ_{n+2}, namely three massless final-state partons at Born level. A solution for the case of two massless final-state partons in the Born phase space requires minor technical modifications, which have been developed, and will be presented elsewhere.

We now turn to the integration of $K^{\,(\mathbf{2})}$, which is the most involved part of the calculation. In this case, since $I^{\,(\mathbf{2})}$ enters Eq. (2.37), which lives in Φ_n, we start from $K^{\,(\mathbf{2})}$ in Eq. (3.38) and perform the complete sum over sector functions, exploiting their sum rules (see for example Eqs. (3.6)-(3.8)). This gives

$$\begin{aligned}\int d\Phi_{n+2}\, K^{\,(\mathbf{2})} \;=\; \int d\Phi_{n+2} \bigg[\sum_{i,j>i} \overline{\mathbf{S}}_{ij}\, RR + \sum_{i,j\neq i} \sum_{\substack{k\neq i\\ k>j}} \overline{\mathbf{SHC}}_{ijk} \left(1 - \overline{\mathbf{C}}_{ijk}\right) RR \\ + \sum_{i,\,j>i} \sum_{k>j} \overline{\mathbf{HC}}_{ijk} RR + \sum_{i,\,j>i} \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq j\\ l>k}} \overline{\mathbf{HC}}_{ijkl}\, RR \bigg]. \end{aligned} \tag{4.17}$$

Each of the four terms in Eq. (4.17) must be written as a sum of contributions, where the double-radiation kernels have been integrated over the parametrised radiative phase space, and one is left with a Born-level factor, expressed in terms of mapped momenta. To guide the eye of the reader through the following rather intricate expressions, we note that, for each one of the limits involved, the results are of the form

$$\int d\Phi_{n+2}\, \overline{\mathbf{L}}^{\,(\mathbf{2})}_{\ldots}\, RR \;=\; \text{constant} \sum_{\{\mu\}} \int d\Phi_n^{(\mu)}\; J^{\mu}_{\rm limit}\; \bar{B}^{(\mu)}_{\rm colour}\,, \tag{4.18}$$

where the overall constant is related to multiplicities, the sum is over the set $\{\mu\}$ of mappings that have been employed, the Born factor may have different colour correlations, and J will always denote the results of the integration of the kernels appropriate to the limit being taken[10]. The relevant J's will be listed in Appendix E. Beginning with the integrated double-soft limit in Eq. (4.17), we find

[10]Note that, since the limit $\overline{\mathbf{L}}$ is expressed as a sum of terms that can be mapped differently, several J's will contribute to each $\overline{\mathbf{L}}$.

the explicit expression

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_{ij}\,RR = \frac{1}{2}\,\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg\{\sum_{e\neq i,j,c,d}\Bigg[\sum_{f\neq i,j,c,d,e}\int d\Phi_n^{(icd,jef)}\,J_{\mathrm{s\otimes s}}^{ijcdef}\,\bar{B}_{cdef}^{(icd,jef)} + 4\int d\Phi_n^{(icd,jed)}\,J_{\mathrm{s\otimes s}}^{ijcde}\,\bar{B}_{cded}^{(icd,jed)}\Bigg] + \int d\Phi_n^{(ijcd)}\Big[2\,J_{\mathrm{s\otimes s}}^{ijcd}\,\bar{B}_{cdcd}^{(ijcd)} + J_{\mathrm{ss}}^{ijcd}\,\bar{B}_{cd}^{(ijcd)}\Big]\Bigg\}\,, \tag{4.19}$$

where we collected colour correlations involving four, three and two partons, and each term has been mapped differently, to simplify the corresponding integration. The integrals relevant for double-soft radiation are presented in Eq. (E.3). We now turn to the second term in Eq. (4.17), and we find (with $r = r_{ijk}$)

$$\begin{aligned}\int d\Phi_{n+2}\,\overline{\mathbf{SHC}}_{ijk}\,\big(1-\overline{\mathbf{C}}_{ijk}\big)\,RR = \\ -\frac{\varsigma_{n+2}}{\varsigma_n}\Bigg\{&\sum_{\substack{c\neq i,j,k,r\\ d\neq i,j,k,r,c}}\int d\Phi_n^{(jkr,icd)}\,J_{\mathrm{s\otimes hc}}^{jkricd}\,\bar{B}_{cd}^{(jkr,icd)} + 2\sum_{c\neq i,j,k,r}\int d\Phi_n^{(jkr,icr)}\,J_{\mathrm{s\otimes hc}}^{jkricr}\,\bar{B}_{cr}^{(jkr,icr)}\\ &+\Bigg[\sum_{c\neq i,j,k}\int d\Phi_n^{(krj,icj)}\,J_{\mathrm{s\otimes hc}}^{krjic}\Big(\rho_{jk}^{(C)}\,\bar{B}_{[jk]c}^{(krj,icj)} + \tilde{f}_{jk}^{\,q\bar{q}}\,\bar{\mathcal{B}}_{[jk]c}^{(krj,icj)}\Big)\\ &+C_{f_{[jk]}}\,\rho_{jk}^{(C)}\int d\Phi_n^{(krj,ijr)}\,J_{\mathrm{s\otimes hc}}^{krjir}\,\bar{B}^{(krj,ijr)} + (j\leftrightarrow k)\Bigg]\Bigg\}\,,\end{aligned} \tag{4.20}$$

where $[jk]$ represents the parent particle of the pair (j,k), the factors $\rho_{jk}^{(C)}$, involving combinations of Casimir eigenvalues, are defined in Eq. (A.8), the flavour factors such as $\tilde{f}_{jk}^{\,q\bar{q}}$ are presented in Eq. (A.3), and $\mathcal{B}_{cd}$ is a colour projection of the Born contribution involving the symmetric tensor d_{ABC}, defined in Eq. (A.6); furthermore, the phase-space integrals $J_{\mathrm{s\otimes hc}}$ are presented in Eq. (E.14). The remaining contributions to Eq. (4.17) are purely hard-collinear. For the integral of the emission of a cluster of three hard-collinear particles we find

$$\int d\Phi_{n+2}\,\overline{\mathbf{HC}}_{ijk}RR = \frac{\varsigma_{n+2}}{\varsigma_n}\int d\Phi_n^{(ijkr)}\,J_{\mathrm{hcc}}^{ijkr}\,\bar{B}^{(ijkr)}\,, \qquad r = r_{ijk}\,, \tag{4.21}$$

while for the emission of two distinct pairs of hard-collinear particles the integral reads

$$\int d\Phi_{n+2}\,\overline{\mathbf{HC}}_{ijkl}\,RR = \frac{\varsigma_{n+2}}{\varsigma_n}\int d\Phi_n^{(ijr,klr)}\,J_{\mathrm{hc\otimes hc}}^{ijklr}\,\bar{B}^{(ijr,klr)}\,, \qquad r = r_{ijkl}\,, \tag{4.22}$$

where the integrals J_{hcc} and $J_{\mathrm{hc\otimes hc}}$ are reported in Eq. (E.10) and in Eq. (E.12), respectively.

We finally turn to the integration of the strongly-ordered counterterm $K^{(\mathbf{12})}$. As announced, we integrate $K^{(\mathbf{12})}$ only in the phase space of the most unresolved radiation, so the integrals involved will be the same that appeared in the case of $K^{(\mathbf{1})}$. Starting from the expression for $K^{(\mathbf{12})}$ in Eq. (3.38), we then sum up the NLO sector functions with label α of Eqs. (C.96)-(C.97), and we get

$$\begin{aligned}\int d\Phi_{n+2}\,K^{(\mathbf{12})} = \int d\Phi_{n+2}\Bigg\{&\sum_{i,j\neq i}\overline{\mathbf{S}}_i\Bigg[\sum_{k\neq i,j}\overline{\mathbf{S}}_{ij}\,RR\,\bar{\mathcal{Z}}_{\mathrm{s},jk} + \sum_{\substack{k\neq i\\ k>j}}\big(\overline{\mathbf{SHC}}_{ijk} + \overline{\mathbf{HC}}_{ijk}^{\,(\mathbf{s})}\big)\,RR\Bigg]\\ &+\sum_{i,\,j>i}\sum_{k\neq i,j}\overline{\mathbf{HC}}_{ij}\Bigg[\overline{\mathbf{S}}_{ij}\,RR\,\bar{\mathcal{Z}}_{\mathrm{s},jk} + \sum_{l\neq i,k}\overline{\mathbf{SC}}_{kij}\,RR\,\bar{\mathcal{Z}}_{\mathrm{s},kl}\\ &+\overline{\mathbf{HC}}_{ijk}^{\,(\mathbf{c})}\,RR + \sum_{\substack{l\neq i,j\\ l>k}}\overline{\mathbf{HC}}_{ijkl}^{\,(\mathbf{c})}\,RR\Bigg]\Bigg\}\,,\end{aligned} \tag{4.23}$$

where again the mapped sector functions $\bar{\mathcal{Z}}_{\mathrm{s},ab}$ carry the same mapping as the matrix elements they multiply. No other sector functions appear in Eq. (4.23), since the use of symmetrised sector functions has allowed to perform the corresponding sector sums, thus replacing sector functions by unity. Once again, to highlight the general structure of the expressions listed below, we note that they are all of the form

$$\int d\Phi_{n+2}\,\overline{\mathbf{L}}^{\,(\mathbf{12})}_{\dots}\,RR \;=\; \text{constant}\sum_{\{\mu_1,\mu_2\}}\int d\Phi^{(\mu_1)}_{n+1}\;J^{\mu_1}_{\mathrm{limit}}\;\bar{\mathcal{K}}^{(\mu_1)}_{\mu_2}\;\bar{B}^{(\mu_1,\mu_2)}_{\mathrm{colour}}\,. \tag{4.24}$$

In this case, the only integrals required for the most unresolved radiation will again be J_s^{ilm} and J_{hc}^{ijr}, given in Eq. (E.1) and in Eq. (E.7) respectively, and we denoted by $\bar{\mathcal{K}}$ a contribution to either a soft or a collinear kernel, associated with the second radiation, which carries mapping (μ_1), i.e. the first one in the nested mapping (μ_1,μ_2) of the Born matrix elements. Proceeding in the order of Eq. (4.23), the integrated strongly-ordered double-soft limit is given by

$$\begin{aligned}
&\int d\Phi_{n+2}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ij}\,RR\,\bar{\mathcal{Z}}_{\mathrm{s},jk} \;= \qquad\qquad (4.25)\\
&\mathcal{N}_1\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg\{\int d\Phi^{(icd)}_{n+1}\,J_s^{icd}\Bigg[\sum_{e\neq i,j,c,d}\Bigg(\frac{1}{2}\sum_{f\neq i,j,c,d,e}\bar{\mathcal{E}}^{(j)(icd)}_{ef}\,\bar{B}^{(icd,jef)}_{cdef}+\bar{\mathcal{E}}^{(j)(icd)}_{ed}\,\bar{B}^{(icd,jed)}_{cded}\Bigg)\\
&\qquad\qquad+\bar{\mathcal{E}}^{(j)(icd)}_{cd}\Big(\bar{B}^{(icd,jcd)}_{cdcd}+C_A\bar{B}^{(icd,jcd)}_{cd}\Big)\Bigg]\bar{\mathcal{Z}}^{(icd)}_{\mathrm{s},\,jk}\\
&\qquad+\int d\Phi^{(idc)}_{n+1}\,J_s^{idc}\sum_{e\neq i,j,c,d}\bar{\mathcal{E}}^{(j)(idc)}_{ed}\,\bar{B}^{(idc,jed)}_{cded}\;\bar{\mathcal{Z}}^{(idc)}_{\mathrm{s},\,jk}\\
&\qquad-C_A\int d\Phi^{(icj)}_{n+1}\,J_s^{icj}\,\bar{\mathcal{E}}^{(j)(icj)}_{cd}\,\bar{B}^{(icj,jcd)}_{cd}\;\bar{\mathcal{Z}}^{(icj)}_{\mathrm{s},\,jk}\\
&\qquad-C_A\int d\Phi^{(ijd)}_{n+1}\,J_s^{ijd}\,\bar{\mathcal{E}}^{(j)(ijd)}_{cd}\,\bar{B}^{(ijd,jcd)}_{cd}\;\bar{\mathcal{Z}}^{(ijd)}_{\mathrm{s},\,jk}\Bigg\}\,,
\end{aligned}$$

and it is entirely expressed in terms of the simple one-loop eikonal kernels given in Eq. (B.3). Next, we need the integral (with $r=r_{ijk}$)

$$\begin{aligned}
&\int d\Phi_{n+2}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{SHC}}_{ijk}\,RR \;= \qquad\qquad (4.26)\\
&-\mathcal{N}_1\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_{c\neq i,j,k}\Bigg\{\sum_{d\neq i,j,k,c}\int d\Phi^{(icd)}_{n+1}\,J_s^{icd}\,\frac{\bar{P}^{(icd)\mathrm{hc},\mu\nu}_{jk}}{\bar{s}^{(icd)}_{jk}}\,\bar{B}^{(icd,jkr)}_{\mu\nu,cd}\\
&\qquad+\Bigg[\int d\Phi^{(ijc)}_{n+1}\,J_s^{ijc}\,\frac{\bar{P}^{(ijc)\mathrm{hc},\mu\nu}_{jk(r)}}{2\bar{s}^{(ijc)}_{jk}}\Big(\rho^{(C)}_{jk}\,\bar{B}^{(ijc,krj)}_{\mu\nu,[jk]c}+\tilde{f}^{\,q\bar{q}}_{jk}\,\bar{\mathcal{B}}^{(ijc,krj)}_{\mu\nu,[jk]c}\Big)+(j\leftrightarrow k)\Bigg]\\
&\qquad+\Bigg[\int d\Phi^{(icj)}_{n+1}\,J_s^{icj}\,\frac{\bar{P}^{(icj)\mathrm{hc},\mu\nu}_{jk(r)}}{2\bar{s}^{(icj)}_{jk}}\Big(\rho^{(C)}_{jk}\,\bar{B}^{(icj,krj)}_{\mu\nu,[jk]c}+\tilde{f}^{\,q\bar{q}}_{jk}\,\bar{\mathcal{B}}^{(icj,krj)}_{\mu\nu,[jk]c}\Big)+(j\leftrightarrow k)\Bigg]\Bigg\}\,,
\end{aligned}$$

where the hard-collinear kernels are given in Eq. (B.10). We now turn to limits involving triple-collinear configurations. First we need

$$
\begin{aligned}
&\int d\Phi_{n+2}\, \overline{\mathbf{S}}_i\, \overline{\mathbf{HC}}^{\,(\mathbf{s})}_{ijk}\, RR \;= \qquad\qquad (4.27)\\
&\mathcal{N}_1\, \frac{\varsigma_{n+2}}{\varsigma_{n+1}}\, \frac{C_{f_{[jk]}}}{2} \Bigg\{ \Bigg[\rho^{(C)}_{jk} \int d\Phi^{(ijr)}_{n+1}\, J^{ijr}_{s}\, \frac{\bar{P}^{(ijr)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}} \Big(\bar{B}^{(ijr,jkr)}_{\mu\nu} - \bar{B}^{(ijr,krj)}_{\mu\nu} \Big) + (j \leftrightarrow k) \Bigg] \\
&\qquad + \Bigg[\rho^{(C)}_{jk} \int d\Phi^{(irj)}_{n+1}\, J^{irj}_{s}\, \frac{\bar{P}^{(irj)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(irj)}_{jk}} \Big(\bar{B}^{(irj,jkr)}_{\mu\nu} - \bar{B}^{(irj,krj)}_{\mu\nu} \Big) + (j \leftrightarrow k) \Bigg] \\
&\qquad - \rho^{(C)}_{[jk]} \Bigg[\int d\Phi^{(ijk)}_{n+1}\, J^{ijk}_{s}\, \frac{\bar{P}^{(ijk)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ijk)}_{jk}}\, \bar{B}^{(ijk,jkr)}_{\mu\nu} + (j \leftrightarrow k) \Bigg] \Bigg\} , \qquad r = r_{ijk} .
\end{aligned}
$$

Next we consider (again with $r = r_{ijk}$)

$$
\int d\Phi_{n+2}\, \overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{S}}_{ij}\, RR\, \bar{\mathcal{Z}}_{\mathrm{s},\,jk} \;=\; -\mathcal{N}_1\, \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \int d\Phi^{(ijr)}_{n+1}\, J^{ijr}_{\mathrm{hc}} \sum_{\substack{c\neq i,j\\ d\neq i,j,c}} \bar{\mathcal{E}}^{(j)(ijr)}_{cd}\, \bar{B}^{(ijr,jcd)}_{cd}\, \bar{\mathcal{Z}}^{(ijr)}_{\mathrm{s},\,jk} , \tag{4.28}
$$

where the choice of r different from i, j, k, analogously to the integral of $\overline{\mathbf{HC}}_{ij}\, RR$ in Eq. (4.16), causes a dependence of the integrated kernel J^{ijr}_{hc} on the index k of the sector function $\bar{\mathcal{Z}}^{(ijr)}_{\mathrm{s},\,jk}$. Next we have ($r = r_{ijk}$)

$$
\begin{aligned}
\int d\Phi_{n+2}\, \overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{SC}}_{kij}\, RR\, \bar{\mathcal{Z}}_{\mathrm{s},\,kl} \;=\; -\mathcal{N}_1\, \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \int d\Phi^{(ijr)}_{n+1}\, J^{ijr}_{\mathrm{hc}} \Bigg[\sum_{\substack{c\neq i,j,k,r\\ d\neq i,j,k,r,c}} \bar{\mathcal{E}}^{(k)(ijr)}_{cd}\, \bar{B}^{(ijr,kcd)}_{cd} \qquad (4.29)\\
+ 2 \sum_{c\neq i,j,k,r} \bar{\mathcal{E}}^{(k)(ijr)}_{cr}\, \bar{B}^{(ijr,kcr)}_{cr} + 2 \sum_{c\neq i,j,k} \bar{\mathcal{E}}^{(k)(ijr)}_{jc}\, \bar{B}^{(ijr,kcj)}_{jc} \Bigg] \bar{\mathcal{Z}}^{(ijr)}_{\mathrm{s},\,kl} .
\end{aligned}
$$

Finally we need to handle strongly-ordered hard-collinear limits. First, with a collinear cluster of three particles we find

$$
\int d\Phi_{n+2}\, \overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{HC}}^{\,(\mathbf{c})}_{ijk}\, RR \;=\; \mathcal{N}_1\, \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \int d\Phi^{(ijr)}_{n+1}\, J^{ijr}_{\mathrm{hc}}\, \frac{\bar{P}^{(ijr)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\, \bar{B}^{(ijr,jkr)}_{\mu\nu} , \qquad r = r_{ijk} . \tag{4.30}
$$

Then, with two independent pairs of collinear particles, we find

$$
\int d\Phi_{n+2}\, \overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{HC}}^{\,(\mathbf{c})}_{ijkl}\, RR \;=\; \mathcal{N}_1\, \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \int d\Phi^{(ijr)}_{n+1}\, J^{ijr}_{\mathrm{hc}}\, \frac{\bar{P}^{(ijr)\mathrm{hc},\mu\nu}_{kl(r)}}{\bar{s}^{(ijr)}_{kl}}\, \bar{B}^{(ijr,klr)}_{\mu\nu} , \qquad r = r_{ijkl} . \tag{4.31}
$$

This concludes the list of all required integrals for double-real radiation.

4.3 Relabelling of momenta and flavour sums

Our next step will be to collect the results of the different sectors and combine them by renaming the mapped momenta in each sector. More precisely, in all $(n+1)$-body phase spaces $d\Phi^{(abc)}_{n+1}$ appearing in the integrals of $K^{\,(\mathbf{1})}$ and $K^{\,(\mathbf{12})}$, we rename the sets of mapped momenta $\{\bar{k}^{(abc)}\}_{n+1}$ as a unique set of $(n+1)$ momenta $\{k\}_{n+1}$. With this new labelling, the indices of the mapped momenta refer directly to the particles of the unique $(n+1)$-body phase space, and the reference to the first mapping can be simply removed. The relabelling thus leads to

$$
\begin{aligned}
&d\Phi^{(abc)}_{n+1} \to d\Phi_{n+1} , \qquad \bar{\mathcal{Z}}^{(abc)}_{\dots} \to \mathcal{Z}_{\dots} , \qquad \bar{R}^{(abc)}_{\dots} \to R_{\dots} , \qquad \bar{B}^{(abc,def)}_{\dots} \to \bar{B}^{(def)}_{\dots} , \\
&\bar{s}^{(abc)}_{ij} \to s_{ij} , \qquad \bar{P}^{(abc)\mathrm{hc},\mu\nu}_{ij(r)} \to P^{\mathrm{hc},\mu\nu}_{ij(r)} , \qquad \bar{\mathcal{E}}^{(i)(abc)}_{lm} \to \mathcal{E}^{(i)}_{lm} .
\end{aligned} \tag{4.32}
$$

Similarly, in the n-body phase spaces $d\Phi_n^{(abc,def)}$ appearing in the integral of $K^{(\mathbf{2})}$, the sets of mapped momenta $\{\bar{k}^{(abc,def)}\}_n$ are renamed as a unique set of n momenta $\{k\}_n$, which in practice means performing the substitutions

$$d\Phi_n^{(abc,def)} \;\to\; d\Phi_n\,, \qquad \bar{B}_{\dots}^{(abc,def)} \;\to\; B_{\dots} \qquad \bar{s}_{ij}^{(abc,def)} \;\to\; s_{ij}\,. \tag{4.33}$$

In particular, in the integral of $\overline{\mathbf{SHC}}_{ijk}(1-\overline{\mathbf{C}}_{ijk})RR$, which involves a collinear splitting of partons j and k, the momenta $\bar{k}_k^{(jkr,icd)}$, $\bar{k}_k^{(jkr,icr)}$, $\bar{k}_j^{(krj,icj)}$ and $\bar{k}_j^{(krj,ijr)}$ are all renamed as k_p, where p is the parent particle of j and k.

At this stage, in all integrated counterterms, the only recollection of the particles of the original $(n+2)$-body phase space is confined to the flavour factors f_i^q, $f_i^{\bar{q}}$, f_i^g. These can be summed up, and, if needed, translated into flavour factors for the particles of the $(n+1)$-body and n-body phase spaces. We now give the rules to perform these sums.

Let us begin with the simple case in which only one particle is integrated out, which is what happens for $K^{(\mathbf{1})}$ and $K^{(\mathbf{12})}$. In this case the following rules apply.

- When going from an $(n+2)$-body phase space to an $(n+1)$-body phase space by discarding particle i, which happens when particle i is a soft gluon, the sum over flavour factors satisfies

$$\frac{\varsigma_{n+2}}{\varsigma_{n+1}} \sum_i f_i^g = 1\,. \tag{4.34}$$

  For example, if all $n+2$ particles are gluons, one has $\varsigma_{n+2} = 1/(n+2)!$ and $\varsigma_{n+1} = 1/(n+1)!$, and the sum yields the missing factor of $n+2$.

- When going from an $(n+2)$-body phase space to an $(n+1)$-body phase space by replacing two particles i, j with their parent particle p, which happens when i and j form a collinear pair, the sum over the flavour factors of particles i, j can be written as a sum over flavour factors for particle p according to the rules

$$\begin{aligned}
\frac{\varsigma_{n+2}}{\varsigma_{n+1}} \sum_{i,j>i} f_{ij}^{q\bar{q}} &= N_f \sum_p f_p^g\,,\\
\frac{\varsigma_{n+2}}{\varsigma_{n+1}} \sum_{i,j>i} \left(f_{ij}^{gq} + f_{ij}^{g\bar{q}}\right) &= \sum_p \left(f_p^q + f_p^{\bar{q}}\right),\\
\frac{\varsigma_{n+2}}{\varsigma_{n+1}} \sum_{i,j>i} f_{ij}^{gg} &= \frac{1}{2} \sum_p f_p^g\,.
\end{aligned} \tag{4.35}$$

  As an example, consider the production of n gluons and a collinear $q\bar{q}$ pair. In this case the first line of Eq. (4.35) applies, and one must take into account the fact that quark flavours must be summed, since the quark pair is integrated out. One then has $\varsigma_{n+2} = N_f/n!$ and $\varsigma_{n+1} = 1/(n+1)!$, since the new final state involves $(n+1)$ gluons. For the same reason, the *r.h.s.* yields $N_f(n+1)$.

Not surprisingly, the flavour sum rules for the integrated $K^{(\mathbf{2})}$ are both more varied and more intricate, since one is integrating out two particles, either by removing them (when they are soft), or by replacing them with their (grand)parent particles when they form collinear sets. We consider the various cases in turn.

- When going from an $(n+2)$-body phase space to an n-body phase space by discarding two particles i, j, the sum over particles i, j satisfies

$$\frac{\varsigma_{n+2}}{\varsigma_n} \sum_i \sum_{j>i} f_{ij}^{gg} = \frac{1}{2}\,, \qquad \frac{\varsigma_{n+2}}{\varsigma_n} \sum_i \sum_{j>i} f_{ij}^{q\bar{q}} = N_f\,. \tag{4.36}$$

As before, the first equality is easily verified when all $n+2$ particles are gluons, as is the second one when the final state consists of n gluons and a quark-antiquark pair.

- When going from an $(n+2)$-body phase space to an n-body phase space by replacing two particles j, k with their parent particle p, and by discarding particle i, the sum over particles i, j, k can be written as a sum over p according to the following rules.

$$\begin{aligned}
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,j\neq i} \sum_{\substack{k\neq i\\ k>j}} f_i^g f_{jk}^{q\bar{q}} &= N_f \sum_p f_p^g \,, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,j\neq i} \sum_{\substack{k\neq i\\ k>j}} f_i^g (f_{jk}^{gq} + f_{jk}^{g\bar{q}}) &= \sum_p (f_p^q + f_p^{\bar{q}}) \,, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,j\neq i} \sum_{\substack{k\neq i\\ k>j}} f_i^g f_{jk}^{gg} &= \frac{1}{2} \sum_p f_p^g \,,
\end{aligned} \tag{4.37}$$

where it is important to pay attention to the range of the various sums.

- When going from an $(n+2)$-body phase space to an n-body phase space by replacing three particles i, j, k with their grandparent particle p, the sum over particles i, j, k can be replaced by a sum over p according to the following rules.

$$\begin{aligned}
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\, j>i} \sum_{k>j} (f_{ijk}^{q\bar{q}q'} + f_{ijk}^{q\bar{q}\bar{q}'}) &= N_f \sum_p (f_p^q + f_p^{\bar{q}}) \,, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\, j>i} \sum_{k>j} (f_{ijk}^{q\bar{q}q} + f_{ijk}^{q\bar{q}\bar{q}}) &= \frac{1}{2} \sum_p (f_p^q + f_p^{\bar{q}}) \,, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\, j>i} \sum_{k>j} f_{ijk}^{q\bar{q}g} &= N_f \sum_p f_p^g \,, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\, j>i} \sum_{k>j} (f_{ijk}^{ggq} + f_{ijk}^{gg\bar{q}}) &= \frac{1}{2} \sum_p (f_p^q + f_p^{\bar{q}}) \,, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\, j>i} \sum_{k>j} f_{ijk}^{ggg} &= \frac{1}{6} \sum_p f_p^g \,,
\end{aligned} \tag{4.38}$$

where one easily recognises in the five lines the five possible partonic channels involving the production of a cluster of three collinear particles: in the first line, the final quark-antiquark pair can have any flavour (including that of the grandparent (anti)quark, which is the same as that of the final (anti)quark q'), while in the second line all three (anti)quarks have the same flavour.

- The most intricate channel for flavour sums arises when going from an $(n+2)$-body phase space to an n-body phase space by replacing two pairs of particles i, j and k, l with their parent particles, p and t respectively. In this case, the sum over particles i, j, k, l can be replaced by a sum over p and t according to the following rules.

$$
\begin{aligned}
\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{i,\,j>i}\sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq j\\l>k}} f_{ij}^{q\bar q} f_{kl}^{q'\bar q'} &= \frac{N_f^2}{2}\sum_{p,t\neq p} f_{pt}^{gg}\,,\\
\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{i,\,j>i}\sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq j\\l>k}} \Big[f_{ij}^{q\bar q}(f_{kl}^{gq'}+f_{kl}^{g\bar q'})+(f_{ij}^{gq'}+f_{ij}^{g\bar q'})f_{kl}^{q\bar q}\Big] &= \frac{N_f}{2}\sum_{p,t\neq p}(f_{pt}^{gq}+f_{pt}^{g\bar q})\,,\\
\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{i,\,j>i}\sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq j\\l>k}} (f_{ij}^{q\bar q} f_{kl}^{gg}+f_{ij}^{gg} f_{kl}^{q\bar q}) &= \frac{N_f}{2}\sum_{p,t\neq p} f_{pt}^{gg}\,,\\
\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{i,\,j>i}\sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq j\\l>k}} (f_{ij}^{gq}+f_{ij}^{g\bar q})(f_{kl}^{gq'}+f_{kl}^{g\bar q'}) &= \frac{1}{2}\sum_{p,t\neq p}(f_{p}^{q}+f_{p}^{\bar q})(f_{t}^{q'}+f_{t}^{\bar q'})\,,\\
\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{i,\,j>i}\sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq j\\l>k}} \Big[(f_{ij}^{gq}+f_{ij}^{g\bar q})f_{kl}^{gg}+f_{ij}^{gg}(f_{kl}^{gq}+f_{kl}^{g\bar q})\Big] &= \frac{1}{4}\sum_{p,t\neq p}(f_{pt}^{gq}+f_{pt}^{g\bar q})\,,\\
\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{i,\,j>i}\sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq j\\l>k}} f_{ij}^{gg} f_{kl}^{gg} &= \frac{1}{8}\sum_{p,t\neq p} f_{pt}^{gg}\,.
\end{aligned}
\tag{4.39}
$$

We emphasise that the flavour sum rules listed in this section apply for any final-state multiplicity and flavour structure. We now have all the tools to assemble the complete integrated counterterms, which will be naturally organised according to the flavour structures of the $(n+1)$-particle and of the n-particle phase spaces, as needed.

4.4 Assembling the complete integrated counterterms

After summing all contributions that were differently mapped, relabelling momenta, and making use of the flavour rules listed in Section 4.3, the resulting integrated counterterms do not bear any remaining trace of the original $(n+2)$-body phase space, and we can actually get full results for $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$, $I^{(\mathbf{12})}$, as defined in Eq. (2.35). The simplest case is the integral of the single-unresolved counterterm $I^{(\mathbf{1})}$, which reads

$$
\begin{aligned}
I^{(\mathbf{1})} &= \sum_{i,j\neq i} I_{ij}^{(\mathbf{1})}\,\mathcal{W}_{ij} = \sum_{i,j>i} I_{ij}^{(\mathbf{1})}\,\mathcal{Z}_{ij}\,,\\
I_{ij}^{(\mathbf{1})} &= -\sum_{c,d\neq c} J_s(s_{cd})\,R_{cd} + \sum_k J_{\mathrm{hc}}^k(s_{kr})\,R\,, \qquad r = r_{ijk}\,.
\end{aligned}
\tag{4.40}
$$

Here R is the full squared matrix element for single real radiation, defined in Eq. (2.4), and R_{cd} is its colour-correlated counterpart, defined in Eq. (A.7). The single-soft integral J_s is given in Eq. (E.2), and the collinear integral J_{hc}^k is given in Eq. (E.9). Because of the rule $r = r_{ijk}$, a dependence of $J_{\mathrm{hc}}^k(s_{kr})$ on i and j is left, excluding the possibility to sum over sectors in the hard-collinear part of $I^{(\mathbf{1})}$.

The integral of the double-unresolved counterterm, $I^{(\mathbf{2})}$, is more intricate, and we assemble it according to

$$
I^{(\mathbf{2})} = I_{\mathrm{SS}}^{(\mathbf{2})} + I_{\mathrm{SHC}}^{(\mathbf{2})} + I_{\mathrm{HCC}}^{(\mathbf{2})} + I_{\mathrm{HCHC}}^{(\mathbf{2})}\,,
\tag{4.41}
$$

distinguishing double-soft, soft-times-hard-collinear and double-hard-collinear contributions, the last of which may involve three or four Born-level particles. For $I_{\mathrm{SS}}^{(\mathbf{2})}$ we get contributions containing

Born-level colour correlations involving four, three and two particles, and we write

$$
\begin{aligned}
I^{(\mathbf{2})}_{\rm SS} \;=\; \frac{1}{4}\sum_{c,d\neq c}\bigg\{ & \sum_{e\neq c,d}\bigg[\sum_{f\neq c,d,e} J^{(4)}_{\rm s\otimes s}(s_{cd},s_{ef})B_{cdef} + 4\,J^{(3)}_{\rm s\otimes s}(s_{cd},s_{ed})B_{cded}\bigg] \\
& + 2\,J^{(2)}_{\rm s\otimes s}(s_{cd})B_{cdcd} + 2\Big[2\,N_f\,T_R\,J^{(q\bar{q})}_{\rm ss}(s_{cd}) - C_A\,J^{(gg)}_{\rm ss}(s_{cd})\Big]B_{cd}\bigg\}\,,
\end{aligned}
\tag{4.42}
$$

where the constituent integrals are given in Eq. (E.4). The soft-times-hard-collinear contribution yields

$$
\begin{aligned}
I^{(\mathbf{2})}_{\rm SHC} \;=\; -\sum_k\bigg\{ & J^k_{\rm hc}(s_{kr})\sum_{c,d\neq c}J_{\rm s}(s_{cd})B_{cd} + J^k_{\rm shc}(s_{kr})B + J^{k,{\rm A}}_{\rm shc}(s_{kr})B_{kr} \\
& + \sum_{c\neq k,r}\Big[J^{k,{\rm B}}_{\rm shc}(s_{kr},s_{kc})B_{kc} + J^{k,{\rm B}}_{\rm shc}(s_{kr},s_{cr})B_{cr}\Big]\bigg\}\,, \qquad r=r_k\,,
\end{aligned}
\tag{4.43}
$$

where the rule $r = r_k$, as defined in Eq. (A.14), prevents r from being equal to k. In Eq. (4.43) we have introduced the following soft-times-hard-collinear integrals

$$
\begin{aligned}
J^k_{\rm shc}(s) \;=\;& (f^q_k+f^{\bar{q}}_k)\Big\{2\,C_F\,J^{gqq}_{\rm s\otimes hc}(s) + C_A\Big[J^{gqg}_{\rm s\otimes hc}(s) - J^{gqg}_{\rm s\otimes hc}(s)\Big]\Big\} \\
& + f^g_k\,C_A\Big[2\,N_f\,J^{gqq}_{\rm s\otimes hc}(s) + J^{ggg}_{\rm s\otimes hc}(s)\Big]\,, \\
J^{k,{\rm A}}_{\rm shc}(s) \;=\;& (f^q_k+f^{\bar{q}}_k)\Big\{2\,J^{gqq}_{\rm s\otimes hc}(s) + \frac{C_A}{C_F}\Big[J^{gqg}_{\rm s\otimes hc}(s) - J^{gqg}_{\rm s\otimes hc}(s)\Big] - 2\,J^{4(2g)}_{\rm s\otimes hc}(s,s)\Big\} \\
& + f^g_k\Big\{2\,N_f\Big[J^{gqq}_{\rm s\otimes hc}(s) - J^{4(1g)}_{\rm s\otimes hc}(s,s)\Big] + J^{ggg}_{\rm s\otimes hc}(s) - J^{4(3g)}_{\rm s\otimes hc}(s,s)\Big\}\,, \\
J^{k,{\rm B}}_{\rm shc}(s,s') \;=\;& (f^q_k+f^{\bar{q}}_k)\Big[2\,J^{3(2g)}_{\rm s\otimes hc}(s,s') - 2\,J^{4(2g)}_{\rm s\otimes hc}(s,s')\Big] \\
& + f^g_k\Big\{2\,N_f\Big[J^{3(1g)}_{\rm s\otimes hc}(s,s') - J^{4(1g)}_{\rm s\otimes hc}(s,s')\Big] + J^{3(3g)}_{\rm s\otimes hc}(s,s') - J^{4(3g)}_{\rm s\otimes hc}(s,s')\Big\}\,,
\end{aligned}
\tag{4.44}
$$

whose constituent integrals can be found in Eq. (E.15). Next, we turn to the double-hard-collinear integral involving three Born-level particles, which reads

$$
\begin{aligned}
I^{(\mathbf{2})}_{\rm HCC} \;=\; \sum_k\bigg\{ & (f^q_k+f^{\bar{q}}_k)\Big[N_f\,J^{(0g)}_{\rm hcc}(s_{kr}) + \frac{1}{2}\,J^{(0g,{\rm id})}_{\rm hcc}(s_{kr}) + \frac{1}{2}\,J^{(2g)}_{\rm hcc}(s_{kr})\Big] \\
& + f^g_k\Big[N_f\,J^{(1g)}_{\rm hcc}(s_{kr}) + \frac{1}{6}\,J^{(3g)}_{\rm hcc}(s_{kr})\Big]\bigg\}B\,, \qquad r=r_k\,,
\end{aligned}
$$

where the relevant constituent integrals are given in Eq. (E.11). Finally, we come to the double-hard-collinear integral involving four Born-level particles, which reads

$$
\begin{aligned}
I^{(\mathbf{2})}_{\rm HCHC} \;=\; \frac{1}{2}\sum_{j,l\neq j}\bigg\{ & (f^q_j+f^{\bar{q}}_j)(f^{q'}_l+f^{\bar{q}'}_l)J^{\rm qgqg}_{\rm hc\otimes hc}(s_{jr}s_{lr}) \\
& + (f^{gq}_{jl}+f^{g\bar{q}}_{jl})\Big[N_f\,J^{\rm qqqg}_{\rm hc\otimes hc}(s_{jr}s_{lr}) + \frac{1}{2}\,J^{\rm qggg}_{\rm hc\otimes hc}(s_{jr}s_{lr})\Big] \\
& + f^{gg}_{jl}\Big[N_f^2 J^{\rm qqqq}_{\rm hc\otimes hc}(s_{jr}s_{lr}) + N_f J^{\rm qqgg}_{\rm hc\otimes hc}(s_{jr}s_{lr}) + \frac{1}{4}J^{\rm gggg}_{\rm hc\otimes hc}(s_{jr}s_{lr})\Big]\bigg\}B\,, \qquad r=r_{jl}\,,
\end{aligned}
\tag{4.45}
$$

where the constituent integrals are given in Eq. (E.13). Similarly to $I^{(\mathbf{1})}$, for the integral of the strongly-ordered counterterm, $I^{(\mathbf{12})}$, we provide expressions with both unsymmetrised and symmetrised sector functions, so as to make it straightforward to prove that $I^{(\mathbf{12})}$ compensates sector by sector the phase-space singularities of $I^{(\mathbf{1})}$. Beginning with the expression involving the original sector functions $\mathcal{W}_{ij}$, we write

$$I^{(\mathbf{12})} \;=\; \sum_{i,j\neq i} I^{(\mathbf{12})}_{ij}\,, \qquad\qquad I^{(\mathbf{12})}_{ij} \;=\; I^{(\mathbf{12})}_{\mathrm{S},ij}\,\mathcal{W}_{\mathrm{s},ij} + I^{(\mathbf{12})}_{\mathrm{C},ij} - I^{(\mathbf{12})}_{\mathrm{SC},ij}\,, \tag{4.46}$$

where the soft limit of sector functions $\mathcal{W}_{\mathrm{s},ij}$ is given in Eq. (C.4). The soft integral $I^{(\mathbf{12})}_{\mathrm{S},ij}$ can again be organised in terms of quadruple, triple and simple Born-level colour correlations, which in this case will be multiplied times eikonal kernels for the second radiation, and NLO-type soft and hard-collinear integrals. We find ($r = r_{ik}$, $r' = r_{ij}$)

$$\begin{aligned} I^{(\mathbf{12})}_{\mathrm{S},ij} \;=\; \mathcal{N}_1 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd} \bigg\{ \frac{1}{2} \sum_{\substack{e\neq i\\ f\neq i,e}} J_s(s_{ef})\, \bar{B}^{(icd)}_{cdef} + \sum_{e\neq i,d} J_s(s_{de}) \left(\bar{B}^{(icd)}_{cded} - \bar{B}^{(idc)}_{cded} \right) \\ - C_A \Big[J_s(s_{ic}) + J_s(s_{id}) - J_s(s_{cd}) \Big] \bar{B}^{(icd)}_{cd} - J^{i}_{\mathrm{hc}}(s_{ir'}) \bar{B}^{(icd)}_{cd} \bigg\} \\ - \mathcal{N}_1 \sum_{k\neq i} J^{k}_{\mathrm{hc}}(s_{kr}) \bigg[\sum_{\substack{c\neq i,k,r\\ d\neq i,c,k,r}} \mathcal{E}^{(i)}_{cd}\, \bar{B}^{(icd)}_{cd} + 2 \sum_{c\neq i,k,r} \mathcal{E}^{(i)}_{cr}\, \bar{B}^{(icr)}_{cr} + 2 \sum_{c\neq i,k} \mathcal{E}^{(i)}_{kc}\, \bar{B}^{(ick)}_{kc} \bigg]\,, \end{aligned} \tag{4.47}$$

where the component integrals are given in Eq. (E.2) and in Eq. (E.9). We notice that the expression contains two different reference particles r and r', both built according to the rule in Eq. (A.14). In particular $r' = r_{ij}$ introduces a dependence in $I^{(\mathbf{12})}_{\mathrm{S},ij}$ on the particle j of the soft sector function $\mathcal{W}_{\mathrm{s},ij}$. The collinear integral $I^{(\mathbf{12})}_{\mathrm{C},ij}$ is expressed in terms of spin-correlated Born-level squared matrix elements, which in this case are multiplied times LO collinear kernels for the least-unresolved collinear splitting, and times suitable combinations of the same constituent integrals as in Eq. (4.47). We find (with $r = r_{ij}$, $r' = r_{ijk}$)

$$\begin{aligned} I^{(\mathbf{12})}_{\mathrm{C},ij} \;=\; -\mathcal{N}_1 \frac{P^{\mu\nu}_{ij(r)}}{s_{ij}} \bigg\{ \sum_{c\neq i,j} \sum_{d\neq i,j,c} J_s(s_{cd})\, \bar{B}^{(ijr)}_{\mu\nu,cd} + C_{f_{[ij]}}\, \rho^{(C)}_{[ij]}\, J_s(s_{ij})\, \bar{B}^{(ijr)}_{\mu\nu} \\ + \bigg[\sum_{c\neq i,j} J_s(s_{ic}) \left(\rho^{(C)}_{ij}\, \bar{B}^{(jri)}_{\mu\nu,[ij]c} + \tilde{f}^{\,q\bar{q}}_{ij}\, \bar{\mathcal{B}}^{(jri)}_{\mu\nu,[ij]c} \right) \\ + C_{f_{[ij]}}\, \rho^{(C)}_{ij}\, J_s(s_{ir}) \left(\bar{B}^{(jri)}_{\mu\nu} - \bar{B}^{(ijr)}_{\mu\nu} \right) + (i \leftrightarrow j) \bigg] \bigg\}\, \mathcal{W}_{\mathrm{c},ij(r)} \\ + \mathcal{N}_1 \frac{P^{\mu\nu}_{ij(r)}}{s_{ij}} \Big[J^{i}_{\mathrm{hc}}(s_{ir}) + J^{j}_{\mathrm{hc}}(s_{jr}) \Big] \bar{B}^{(ijr)}_{\mu\nu}\, \mathcal{W}_{\mathrm{c},ij(r)} + \mathcal{N}_1 \sum_{k\neq i,j} \frac{P^{\mu\nu}_{ij(r')}}{s_{ij}}\, J^{k}_{\mathrm{hc}}(s_{kr'})\, \bar{B}^{(ijr')}_{\mu\nu}\, \mathcal{W}_{\mathrm{c},ij(r')}\,, \end{aligned} \tag{4.48}$$

where the collinear limit of sector functions $\mathcal{W}_{\mathrm{c},ij}$ is given in Eq. (C.5), and again two reference particles have to be introduced. Finally, the soft-collinear integral has a similar structure and reads (with $r = r_{ij}$, $r' = r_{ijk}$)

$$\begin{aligned} I^{(\mathbf{12})}_{\mathrm{SC},ij} \;=\; -\,2\,\mathcal{N}_1\, \mathcal{E}^{(i)}_{jr} \bigg\{ C_{f_j} \sum_{c\neq i,j} \sum_{d\neq i,j,c} J_s(s_{cd})\, \bar{B}^{(ijr)}_{cd} - C_{f_j} C_A\, J_s(s_{ij})\, \bar{B}^{(ijr)} \\ + C_A \bigg[\sum_{c\neq i,j} J_s(s_{ic})\, \bar{B}^{(jri)}_{[ij]c} + C_{f_j}\, J_s(s_{ir}) \Big(\bar{B}^{(jri)} - \bar{B}^{(ijr)} \Big) \bigg] \\ + (2C_{f_j} - C_A) \bigg[\sum_{c\neq i,j} J_s(s_{jc})\, \bar{B}^{(irj)}_{[ij]c} + C_{f_j}\, J_s(s_{jr}) \Big(\bar{B}^{(irj)} - \bar{B}^{(ijr)} \Big) \bigg] \bigg\} \\ + 2\mathcal{N}_1 C_{f_j} \mathcal{E}^{(i)}_{jr} \Big[J^{i}_{\mathrm{hc}}(s_{ir}) + J^{j}_{\mathrm{hc}}(s_{jr}) \Big] \bar{B}^{(ijr)} + 2\mathcal{N}_1 C_{f_j} \mathcal{E}^{(i)}_{jr'} \sum_{k\neq i,j} J^{k}_{\mathrm{hc}}(s_{kr'})\, \bar{B}^{(ijr')}. \end{aligned} \tag{4.49}$$

As already noted, a more compact expression can be obtained using symmetrised sector functions. We can write

$$I^{(\mathbf{12})} = \sum_{i,j>i} I^{(\mathbf{12})}_{\{ij\}} \,, \qquad\qquad I^{(\mathbf{12})}_{\{ij\}} = I^{(\mathbf{12})}_{\mathrm{S},ij}\, \mathcal{Z}_{\mathrm{s},ij} + I^{(\mathbf{12})}_{\mathrm{S},ji}\, \mathcal{Z}_{\mathrm{s},ji} + I^{(\mathbf{12})}_{\mathrm{HC},ij} \,, \tag{4.50}$$

where the soft contributions are given by Eq. (4.47) and the hard-collinear contribution $I^{(\mathbf{12})}_{\mathrm{HC},ij}$ reads ($r = r_{ij}$, $r' = r_{ijk}$)

$$\begin{aligned}
I^{(\mathbf{12})}_{\mathrm{HC},ij} &= I^{(\mathbf{12})}_{\mathrm{C},ij} + I^{(\mathbf{12})}_{\mathrm{C},ji} - I^{(\mathbf{12})}_{\mathrm{SC},ij} - I^{(\mathbf{12})}_{\mathrm{SC},ji} \\
&= -\mathcal{N}_1 \, \frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}} \bigg\{ \sum_{c\neq i,j} \sum_{d\neq i,j,c} J_s(s_{cd})\, \bar{B}^{(ijr)}_{\mu\nu,cd} + C_{f_{[ij]}}\, \rho^{(C)}_{[ij]}\, J_s(s_{ij})\, \bar{B}^{(ijr)}_{\mu\nu} \\
&\qquad + \bigg[\sum_{c\neq i,j} J_s(s_{ic}) \left(\rho^{(C)}_{ij}\, \bar{B}^{(jri)}_{\mu\nu,[ij]c} + \tilde{f}^{\,q\bar{q}}_{ij}\, \bar{\mathcal{B}}^{(jri)}_{\mu\nu,[ij]c} \right) \\
&\qquad\quad + C_{f_{[ij]}}\, \rho^{(C)}_{ij}\, J_s(s_{ir}) \left(\bar{B}^{(jri)}_{\mu\nu} - \bar{B}^{(ijr)}_{\mu\nu} \right) + (i \leftrightarrow j) \bigg] \bigg\} \\
&\quad + \mathcal{N}_1 \, \frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}} \Big[J^{i}_{\mathrm{hc}}(s_{ir}) + J^{j}_{\mathrm{hc}}(s_{jr}) \Big] \bar{B}^{(ijr)}_{\mu\nu} + \mathcal{N}_1 \sum_{k\neq i,j} \frac{P^{\mathrm{hc},\mu\nu}_{ij(r')}}{s_{ij}}\, J^{k}_{\mathrm{hc}}(s_{kr'})\, \bar{B}^{(ijr')}_{\mu\nu} \,.
\end{aligned} \tag{4.51}$$

This concludes the list of integrated counterterms for double-real radiation. We now turn to the treatment of real-virtual contributions.

5 The subtracted real-virtual contribution RV_{sub}

Let us take stock of what we have achieved so far. After subtracting the appropriate combination of the local counterterms $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$ and $K^{(\mathbf{12})}$ from the double-real squared matrix element RR, and after adding back the corresponding integrated counterterms, $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$ and $I^{(\mathbf{12})}$, we can write a partially subtracted expression for the differential distribution in Eq. (2.32). It reads

$$\begin{aligned}
\frac{d\sigma_{\mathrm{NNLO}}}{dX} &= \int d\Phi_n \Big[VV + I^{(\mathbf{2})} \Big] \delta_n(X) \\
&\quad + \int d\Phi_{n+1} \Big[\Big(RV + I^{(\mathbf{1})} \Big) \delta_{n+1}(X) - I^{(\mathbf{12})}\, \delta_n(X) \Big] \\
&\quad + \int d\Phi_{n+2}\, RR_{\mathrm{sub}}(X) \,.
\end{aligned} \tag{5.1}$$

Notice that no approximations have been made in reaching Eq. (5.1), since all local terms that were subtracted from Eq. (2.32) were added back exactly in integrated form. At this stage, RR_{sub}, given in Eq. (3.24) or in Eq. (3.37), is free of phase-space singularities in Φ_{n+2}, and (evidently) does not contain explicit poles in ϵ. Therefore it can be directly integrated in four dimensions, as desired. We now focus on the second line of Eq. (5.1). While the introduction of the integrated counterterm $I^{(\mathbf{1})}$ exactly cancels the ϵ poles of RV (in the same way as, at next-to-leading order, I cancels the poles of V), new poles in ϵ are introduced through $I^{(\mathbf{12})}$; on top of this, the combination in square brackets is still affected by phase-space singularities in Φ_{n+1}. To be more precise, the second line of Eq. (5.1) verifies now two crucial properties that follow from general cancellation theorems and from the definitions given so far. Specifically

$$\begin{aligned}
&(1) \qquad \Big(RV + I^{(\mathbf{1})} \Big)\, \delta_{n+1}(X) \;\to\; \text{finite} \,, \\
&(2) \qquad I^{(\mathbf{1})}\, \delta_{n+1}(X) - I^{(\mathbf{12})}\, \delta_n(X) \;\to\; \text{integrable} \,,
\end{aligned} \tag{5.2}$$

where the integrated couterterms are defined in Eq. (4.40) and Eq. (4.46). The first property is expected from the KLN theorem: indeed, $I^{(\mathbf{1})}$ is the integral over the most unresolved radiation of RR, and its IR poles must compensate the virtual poles arising when one of the two unresolved particles becomes virtual, while the other one is unaffected. These are precisely the poles of RV. To check this, which provides a test of the results obtained so far, it is sufficient to perform the ϵ expansion of $I^{(\mathbf{1})}$, as given in Eq. (4.40), writing

$$I^{(\mathbf{1})} = I^{(\mathbf{1})}_{\rm poles} + I^{(\mathbf{1})}_{\rm fin} + \mathcal{O}(\epsilon)\,. \tag{5.3}$$

Performing the sum over sectors in $I^{(\mathbf{1})}_{\rm poles}$, we get

$$I^{(\mathbf{1})}_{\rm poles} = \frac{\alpha_s}{2\pi}\left[\frac{1}{\epsilon^2}\,\Sigma_C\,R + \frac{1}{\epsilon}\left(\Sigma_\gamma\,R + \sum_{c,d\neq c} L_{cd}\,R_{cd}\right)\right] = -\,RV_{\rm poles}\,. \tag{5.4}$$

Keeping the complete dependence on sector functions in $I^{(\mathbf{1})}_{\rm fin}$, we have

$$\begin{aligned} I^{(\mathbf{1})}_{\rm fin} &= \sum_{i,j\neq i} I^{(\mathbf{1})}_{{\rm fin},ij}\,\mathcal{W}_{ij} = \sum_{i,j>i} I^{(\mathbf{1})}_{{\rm fin},ij}\,\mathcal{Z}_{ij}\,, \\ I^{(\mathbf{1})}_{{\rm fin},ij} &= \frac{\alpha_s}{2\pi}\left[\left(\Sigma_\phi - \sum_k \gamma_k^{\rm hc}\,L_{kr}\right)R + \sum_{c,d\neq c} L_{cd}\left(2-\frac{1}{2}\,L_{cd}\right)R_{cd}\right], \qquad r = r_{ijk}\,. \end{aligned} \tag{5.5}$$

In Eqs. (5.4)-(5.5), $L_{ab} = \log(s_{ab}/\mu^2)$, and the numerical coefficients are given in Eqs. (A.8)-(A.11). One easily verifies that $I^{(\mathbf{1})}_{\rm poles}$ matches the explicit poles of the real-virtual matrix element $RV_{\rm poles}$, which have the well-known universal NLO structure (see for example [18, 48]), upon replacing the n-point amplitude with the $(n+1)$-point amplitude.

In order to prove the second property in Eq. (5.2), we start from the decompositions of Eqs. (4.40)-(4.46) in terms of the sector fuctions $\mathcal{W}_{ij}$ and write

$$I^{(\mathbf{1})}\delta_{n+1}(X) - I^{(\mathbf{12})}\delta_n(X) = \sum_{i,j\neq i}\left\{I^{(\mathbf{1})}_{ij}\mathcal{W}_{ij}\,\delta_{n+1}(X) - \left[I^{(\mathbf{12})}_{{\rm S},ij}\,\mathcal{W}_{{\rm s},ij} + I^{(\mathbf{12})}_{{\rm C},ij} - I^{(\mathbf{12})}_{{\rm SC},ij}\right]\delta_n(X)\right\}, \tag{5.6}$$

where the NLO sector functions $\mathcal{W}_{ij}$ and $\mathcal{W}_{{\rm s},ij}$ are defined in Eq. (2.10) and Eq. (C.4) respectively. The second property in Eq. (5.2) is thus satisfied at the level of single sectors $\mathcal{W}_{ij}$ owing to the relations

$$\begin{aligned} &\mathbf{S}_i\left[I^{(\mathbf{1})}_{ij}\,\mathcal{W}_{ij} - I^{(\mathbf{12})}_{{\rm S},ij}\,\mathcal{W}_{{\rm s},ij}\right] \to \text{integrable}\,, \qquad \mathbf{S}_i\left[I^{(\mathbf{12})}_{{\rm C},ij} - I^{(\mathbf{12})}_{{\rm SC},ij}\right] \to \text{integrable}\,, \\ &\mathbf{C}_{ij}\left[I^{(\mathbf{1})}_{ij}\,\mathcal{W}_{ij} - I^{(\mathbf{12})}_{{\rm C},ij}\right] \to \text{integrable}\,, \qquad \mathbf{C}_{ij}\left[I^{(\mathbf{12})}_{{\rm S},ij}\,\mathcal{W}_{{\rm s},ij} - I^{(\mathbf{12})}_{{\rm SC},ij}\right] \to \text{integrable}\,. \end{aligned} \tag{5.7}$$

For concreteness, consider the first relation. Under soft limit, the $(n+1)$-particle matrix element in $I^{(\mathbf{1})}_{ij}$ returns a sum of products of eikonal factors and Born-level, colour-correlated matrix elements, and its sector function $\mathcal{W}_{ij}$ becomes equal to $\mathcal{W}_{{\rm s},ij}$. At the same time, when the operator $\mathbf{S}_i$ acts on $I^{(\mathbf{12})}_{{\rm S},ij}$, it effectively removes the phase-space mappings, so that Eq. (4.47) tends to the $\mathbf{S}_i$ limit of the square parenthesis in Eq. (4.40), up to the overall sign. Similar steps show the validity of the other relations in Eq. (5.7).

At this point, on the one hand we have shown that the combination $\left(RV + I^{(\mathbf{1})}\right)\delta_{n+1}(X)$ is free of explicit poles, but it still contains phase-space singularities. On the other hand, we have proven that $I^{(\mathbf{1})}\,\delta_{n+1}(X) - I^{(\mathbf{12})}\,\delta_n(X)$ is integrable in Φ_{n+1}, but may still contain poles in ϵ. In order to build a fully subtracted real-virtual matrix element $RV_{\rm sub}$, free of poles in ϵ and integrable in the whole $(n+1)$-body phase space, we need to define, in each sector ij, a real-virtual counterterm

$K_{ij}^{(\mathbf{RV})}$ satisfying the two further properties

$$
\begin{aligned}
&(3) \qquad K_{ij}^{(\mathbf{RV})} + I_{ij}^{(\mathbf{12})} \;\to\; \text{finite}\,, \\
&(4) \qquad RV\,\mathcal{W}_{ij}\,\delta_{n+1}(X) - K_{ij}^{(\mathbf{RV})}\,\delta_n(X) \;\to\; \text{integrable}\,.
\end{aligned}
\tag{5.8}
$$

With a real-virtual counterterm satisfying the two properties in Eq. (5.8), the subtracted real-virtual contribution to the cross section, defined in Eq. (2.38), is manifestly finite and integrable in Φ_{n+1}. To construct $RV^{\,\rm sub}$ explicitly, we rewrite it here as a sum over sectors:

$$
RV_{\rm sub}(X) \;=\; \sum_{i,j\neq i}\left[\left(RV + I_{ij}^{(\mathbf{1})}\right)\mathcal{W}_{ij}\,\delta_{n+1}(X) - \left(K_{ij}^{(\mathbf{RV})} + I_{ij}^{(\mathbf{12})}\right)\delta_n(X)\right]. \tag{5.9}
$$

Thanks to the presence of sector functions, the second condition of Eq. (5.8) actually simplifies to

$$
RV\,\mathcal{W}_{ij}\,\delta_{n+1}(X) - K_{ij}^{(\mathbf{RV})}\,\delta_n(X) \;\to\; \text{integrable in the limits } \mathbf{S}_i,\,\mathbf{C}_{ij}\,. \tag{5.10}
$$

In order to find a suitable definition for $K_{ij}^{(\mathbf{RV})}$, satisfying the required properties, we start by introducing soft and collinear improved limits, $\overline{\mathbf{S}}_i$ and $\overline{\mathbf{C}}_{ij}$, for the real-virtual squared matrix element. On the one hand, these limits must reproduce the singular behaviour of RV, so that

$$
\begin{aligned}
\mathbf{S}_i\left[\left(1-\overline{\mathbf{S}}_i\right)RV\,\mathcal{W}_{ij}\right] &\to \text{integrable}\,, \qquad \mathbf{S}_i\left[\overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i\right)RV\,\mathcal{W}_{ij}\right] \;\to\; \text{integrable}\,, \\
\mathbf{C}_{ij}\left[\left(1-\overline{\mathbf{C}}_{ij}\right)RV\,\mathcal{W}_{ij}\right] &\to \text{integrable}\,, \qquad \mathbf{C}_{ij}\left[\overline{\mathbf{S}}_i\left(1-\overline{\mathbf{C}}_{ij}\right)RV\,\mathcal{W}_{ij}\right] \;\to\; \text{integrable}\,.
\end{aligned}
\tag{5.11}
$$

On the other hand, the improved limits must feature appropriate mappings, such that they fulfil momentum conservation and on-shell conditions for the Born-level particles, and, at the same time, they simplify as much as possible the analytic integration over the radiation phase space. Following the discussion presented at NLO, and the choices made in Ref. [92], we introduce

$$
\begin{aligned}
\overline{\mathbf{S}}_i\,RV\,\mathcal{W}_{ij} &\equiv -\mathcal{N}_1\sum_{\substack{c\neq i\\ d\neq i,c}}\left[\mathcal{E}_{cd}^{(i)}\,\bar{V}_{cd}^{(icd)} - \frac{\alpha_s}{2\pi}\left(\tilde{\mathcal{E}}_{cd}^{(i)} + \mathcal{E}_{cd}^{(i)}\,\frac{\beta_0}{2\epsilon}\right)\bar{B}_{cd}^{(icd)} + \alpha_s\sum_{e\neq i,c,d}\tilde{\mathcal{E}}_{cde}^{(i)}\,\bar{B}_{cde}^{(icd)}\right]\mathcal{W}_{{\rm s},ij}\,, \\
\overline{\mathbf{C}}_{ij}\,RV\,\mathcal{W}_{ij} &\equiv \frac{\mathcal{N}_1}{s_{ij}}\left[P_{ij(r)}^{\mu\nu}\,\bar{V}_{\mu\nu}^{(ijr)} + \frac{\alpha_s}{2\pi}\left(\tilde{P}_{ij(r)}^{\mu\nu} - P_{ij(r)}^{\mu\nu}\,\frac{\beta_0}{2\epsilon}\right)\bar{B}_{\mu\nu}^{(ijr)}\right]\mathcal{W}_{{\rm c},ij}\,, \\
\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,RV\,\mathcal{W}_{ij} &\equiv 2\mathcal{N}_1\,C_{f_j}\left[\mathcal{E}_{jr}^{(i)}\bar{V}^{(ijr)} - \frac{\alpha_s}{2\pi}\left(\tilde{\mathcal{E}}_{jr}^{(i)} + \mathcal{E}_{jr}^{(i)}\,\frac{\beta_0}{2\epsilon}\right)\bar{B}^{(ijr)}\right], \qquad\qquad r = r_{ij}\,.
\end{aligned}
\tag{5.12}
$$

The kernels $\mathcal{E}_{cd}^{(i)}$ and $P_{ij(r)}^{\mu\nu}$ are the eikonal and collinear kernels from tree-level factorisation, introduced already at NLO, and given in Eq. (B.3) and in Eq. (B.7), respectively. In addition, $\tilde{\mathcal{E}}_{cd}^{(i)}$, $\tilde{\mathcal{E}}_{cde}^{(i)}$ and $\tilde{P}_{ij(r)}^{\mu\nu}$ are the genuine real-virtual soft and collinear kernels [32, 33], presented here in Eq. (B.5) and in Eq. (B.24), respectively.

Since the combination $(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})\,RV\,\mathcal{W}_{ij}$ is integrable everywhere in Φ_{n+1}, one would expect to define the counterterm $K_{ij}^{(\mathbf{RV})}$ simply as an NLO-like combination of improved limits, namely

$$
K_{ij,\,{\rm expected}}^{(\mathbf{RV})} \;=\; \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i\right)\right]RV\,\mathcal{W}_{ij}\,. \tag{5.13}
$$

Although such a choice preserves the minimal structure of the real-virtual counterterm, and automatically fulfils the condition (4) of Eq. (5.8), explicit computations show that it spoils the condition (3) of Eq. (5.8). In principle, it would have been natural to expect that the poles of Eq. (5.13) would cancel those of $I_{ij}^{(\mathbf{12})}$. Indeed, the poles of Eq. (5.13) are designed to match the poles of RV that are accompanied by phase-space singularities. At the same time, $I^{(\mathbf{12})}_{ij}$ is the result of integrating the strongly-ordered counterterm over the phase space of the most unresolved radiation: thus, it collects precisely terms that have phase-space singularities in the remaining radiation, as well as poles that should match their virtual counterpart, given by RV. On the other hand, there are subtleties that prevent the poles of $I^{(\mathbf{12})}_{ij}$ from matching exactly those of $K^{(\mathbf{RV})}_{ij,\text{expected}}$. The first subtlety stems from the specific phase-space mappings one has to adopt in order to define the improved limits in Eq. (5.12). Since such contributions are affected by both double poles in ϵ and by phase-space singularities, they feature single poles in ϵ with coefficients depending on kinematic invariants. This generates a mismatch: in fact, we notice that in Eqs. (4.47)-(4.49) the residues of the poles in $I^{(\mathbf{12})}_{ij}$ that depend on kinematics are proportional to logarithms of Lorentz invariants constructed with *unmapped* momenta, *i.e.* with $(n+1)$-body kinematics. In contrast, the residues of the poles in the real-virtual improved limits of Eq. (5.12) can also depend on logarithms of *mapped* invariants, obtained via momentum mappings from the $(n+1)$- to the n-particle phase space. This is the case, for instance, for the virtual component of the soft limit: the pole content of $\bar{V}^{(icd)}_{cd}$ includes terms of the type $\log\big(\bar{s}^{(icd)}_{ef}/\mu^2\big)$, which cannot appear in $I^{(\mathbf{12})}_{ij}$. More involved mismatches occur in the collinear sector, where the kinematics of the poles of $I^{(\mathbf{12})}_{ij}$ fails to match that of $K^{(\mathbf{RV})}_{ij,\text{expected}}$ out of the collinear region, irrespectively of mappings.

The fact that all discrepancies in the single pole in ϵ disappear in the singular regions of phase space, as they must, gives us the possibility to refine the definition of $K^{(\mathbf{RV})}_{ij,\text{expected}}$, by adding back precisely the mismatched terms, thus obtaining the desired cancellation of the $I^{(\mathbf{12})}_{ij}$ poles, without introducing new phase-space singularities. Schematically, we define

$$K^{(\mathbf{RV})}_{ij} \;\equiv\; K^{(\mathbf{RV})}_{ij,\,\text{expected}} + \Delta_{ij} \;=\; \Big[\,\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}\,\big(1-\overline{\mathbf{S}}_i\big)\Big] RV\,\mathcal{W}_{ij} + \Delta_{ij}\,. \tag{5.14}$$

The extra term Δ_{ij} appearing in Eq. (5.14) is required not to spoil condition (4) of Eq. (5.8), and therefore cannot have any phase-space singularity in the limits $\mathbf{S}_i$ and $\mathbf{C}_{ij}$. Thus we impose that

$$\mathbf{S}_i\,\Delta_{ij} \;\to\; \text{integrable}\,, \qquad \mathbf{C}_{ij}\,\Delta_{ij} \;\to\; \text{integrable}\,. \tag{5.15}$$

At the same time, Δ_{ij} has the crucial role of matching the explicit ϵ poles of $I^{(\mathbf{12})}_{ij}$, implying the finiteness of the combination $K^{(\mathbf{RV})}_{ij} + I^{(\mathbf{12})}_{ij}$, in agreement with condition (3) of Eq. (5.8). In practice, we introduce for Δ_{ij} a decomposition into soft, collinear and soft-collinear components, along the lines discussed for $I^{(\mathbf{12})}_{ij}$ in Eq.(4.46), and we write

$$\Delta_{ij} \;\equiv\; \Delta_{\mathrm{S},i}\,\mathcal{W}_{\mathrm{s},ij} + \Delta_{\mathrm{C},ij} - \Delta_{\mathrm{SC},ij}\,. \tag{5.16}$$

Using this decomposition, the properties Eq. (5.15) can be better detailed, and read

$$\begin{aligned} \mathbf{S}_i\,\Delta_{\mathrm{S},i}\,\mathcal{W}_{\mathrm{s},ij} \;&\to\; \text{integrable}\,, & \mathbf{S}_i\big(\Delta_{\mathrm{C},ij} - \Delta_{\mathrm{SC},ij}\big) \;&\to\; \text{integrable}\,, \\ \mathbf{C}_{ij}\,\Delta_{\mathrm{C},ij} \;&\to\; \text{integrable}\,, & \mathbf{C}_{ij}\big(\Delta_{\mathrm{S},i}\,\mathcal{W}_{\mathrm{s},ij} - \Delta_{\mathrm{SC},ij}\big) \;&\to\; \text{integrable}\,. \end{aligned} \tag{5.17}$$

Furthermore, we can enforce the desired cancellation between $K^{(\mathbf{RV})}_{ij}$ and $I^{(\mathbf{12})}_{ij}$ for each component. Specifically, we require that

$$\begin{aligned} \Big[\overline{\mathbf{S}}_i\,RV\,\mathcal{W}_{ij} + \Big(\Delta_{\mathrm{S},i} + I^{(\mathbf{12})}_{\mathrm{S},ij}\Big)\,\mathcal{W}_{\mathrm{s},ij}\Big]_{\text{poles}} \;&=\; 0\,, \\ \Big[\overline{\mathbf{C}}_{ij}\,RV\,\mathcal{W}_{ij} + \Big(\Delta_{\mathrm{C},ij} + I^{(\mathbf{12})}_{\mathrm{C},ij}\Big)\Big]_{\text{poles}} \;&=\; 0\,, \\ \Big[\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,RV\,\mathcal{W}_{ij} + \Big(\Delta_{\mathrm{SC},ij} + I^{(\mathbf{12})}_{\mathrm{SC},ij}\Big)\Big]_{\text{poles}} \;&=\; 0\,. \end{aligned} \tag{5.18}$$

Since the pole parts of both $I^{(\mathbf{12})}_{ij}$ and $K^{(\mathbf{RV})}_{ij,\,\mathrm{expected}}$ are explicitly known, the necessary compensating terms are easily determined. An expression for the three components of Δ_{ij} can be constructed in a minimal way by considering all and only the single poles of $I^{(\mathbf{12})}_{ij}$ with mismatching kinematics. Since they consist in differences of logarithms, or differences of Born matrix elements (which vanish in the soft or collinear limit), we decided to promote the differences of logarithms to ratios of scales, raised to a power vanishing with ϵ. This non-minimal structure simplifies subsequent integrations, and it only affects finite parts, without introducing new phase-space singularities. Beginning with the soft term $\Delta_{\mathrm{S},i}$, we define

$$\begin{aligned}\Delta_{\mathrm{S},i} \;=\; & -\frac{\alpha_s}{2\pi}\mathcal{N}_1\sum_{\substack{c\neq i\\ d\neq i,c}}\mathcal{E}^{(i)}_{cd}\Bigg\{\;\frac{1}{2\epsilon^2}\sum_{\substack{e\neq i,c\\ f\neq i,c,e}}\Bigg[\bigg(\frac{s_{ef}}{\bar{s}^{(icd)}_{ef}}\bigg)^{-\epsilon}-1\Bigg]\bar{B}^{(icd)}_{efcd}+\frac{1}{\epsilon^2}\sum_{e\neq i,d}\Bigg[\bigg(\frac{s_{ed}}{\bar{s}^{(icd)}_{ed}}\bigg)^{-\epsilon}-1\Bigg]\bar{B}^{(icd)}_{edcd}\\ & \qquad\qquad +\Bigg[\bigg(\frac{1}{\epsilon^2}+\frac{2}{\epsilon}\bigg)2\,C_{f_c}+\frac{\gamma^{\mathrm{hc}}_c}{\epsilon}\Bigg]\Big(\bar{B}^{(icd)}_{cd}-\bar{B}^{(idc)}_{cd}\Big)\Bigg\}\\ & -\frac{\alpha_s}{2\pi}\mathcal{N}_1\sum_{\substack{k\neq i\\ c\neq i,k,r}}\mathcal{E}^{(i)}_{cr}\,\frac{\gamma^{\mathrm{hc}}_k}{\epsilon}\Big(\bar{B}^{(irc)}_{cr}-\bar{B}^{(icr)}_{cr}\Big)\,, \qquad\qquad r=r_{ik}\,. \end{aligned} \tag{5.19}$$

Thanks to the fact that in the soft limit the mapped momenta coincide with the unmapped ones, the first Eq. (5.17) is fulfilled in a straightforward way. The first relation in Eq. (5.18) is less evident, but can be proven by simply performing the ϵ expansion of $\overline{\mathbf{S}}_i\,RV$, $\Delta_{\mathrm{S},i}$ and $I^{(\mathbf{12})}_{\mathrm{S},ij}$. For the collinear component, we define ($r=r_{ij}$, $r'=r_{ijk}$)

$$\begin{aligned}\Delta_{\mathrm{C},ij} \;=\; & \frac{\alpha_s}{2\pi}\mathcal{N}_1\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\frac{1}{\epsilon^2}\sum_{c\neq i,j}\Bigg\{\sum_{d\neq i,j,c}\Bigg[\bigg(\frac{s_{cd}}{\bar{s}^{(ijr)}_{cd}}\bigg)^{-\epsilon}-1\Bigg]\bar{B}^{(ijr)}_{\mu\nu,cd}+2\Bigg[1-\bigg(\frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}}\bigg)^{-\epsilon}\Bigg]\bar{B}^{(ijr)}_{\mu\nu,[ij]c}\\ & \qquad +\Bigg\{\rho^{(C)}_{ij}\Bigg[\bigg(\frac{\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}}\bigg)^{-\epsilon}-\bigg(\frac{s_{ir}\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}s_{ic}}\bigg)^{-\epsilon}\Bigg]\bar{B}^{(jri)}_{\mu\nu,[ij]c}\\ & \qquad +\tilde{f}^{\,q\bar{q}}_{ij}\Bigg[\bigg(\frac{\bar{s}^{(jri)}_{ic}}{\mu^2}\bigg)^{-\epsilon}-\bigg(\frac{\bar{s}^{(jri)}_{ic}}{s_{ic}}\bigg)^{-\epsilon}\Bigg]\bar{\mathcal{B}}^{(jri)}_{\mu\nu,[ij]c}+(i\leftrightarrow j)\Bigg\}\Bigg\}\,\mathcal{W}_{\mathrm{c},ij(r)}\\ & +\frac{\alpha_s}{2\pi}\mathcal{N}_1\sum_{k\neq i,j}\bigg(\frac{\gamma^{\mathrm{hc}}_k}{\epsilon}+\phi^{\mathrm{hc}}_k\bigg)\Bigg[\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\bar{B}^{(ijr)}_{\mu\nu}\,\mathcal{W}_{\mathrm{c},ij(r)}-\frac{P^{\mu\nu}_{ij(r')}}{s_{ij}}\,\bar{B}^{(ijr')}_{\mu\nu}\,\mathcal{W}_{\mathrm{c},ij(r')}\Bigg]\,,\end{aligned} \tag{5.20}$$

where $\rho^{(C)}_{ij}$, $\tilde{f}^{\,q\bar{q}}_{ij}$, γ^{hc}_k, ϕ^{hc}_k and $\bar{\mathcal{B}}$ are defined in Appendix A, and $\mathcal{W}_{\mathrm{c},ij(r)}$ is given in Eq. (C.5). The third Eq. (5.17) can be verified by considering that in the collinear limit $\mathbf{C}_{ij}$ we have

$$\bar{k}^{(ijr)}_j,\bar{k}^{(jri)}_i\xrightarrow{\mathbf{C}_{ij}}k_{[ij]},\qquad \bar{k}^{(ijr)}_r,\bar{k}^{(jri)}_r\xrightarrow{\mathbf{C}_{ij}}k_r,\qquad \bar{k}^{(ijr)}_c,\bar{k}^{(jri)}_c\xrightarrow{\mathbf{C}_{ij}}k_c. \tag{5.21}$$

Again the second Eq. (5.18) can be proven upon expansion in ϵ. Finally for the soft-collinear component we introduce (with $r=r_{ij}$, $r'=r_{ijk}$)

$$\begin{aligned}\Delta_{\mathrm{SC},ij} \;=\; & \frac{\alpha_s}{2\pi}2\mathcal{N}_1\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\,\frac{1}{\epsilon^2}\sum_{c\neq i,j}\Bigg\{\sum_{d\neq i,j,c}\Bigg[\bigg(\frac{s_{cd}}{\bar{s}^{(ijr)}_{cd}}\bigg)^{-\epsilon}-1\Bigg]\bar{B}^{(ijr)}_{cd}+2\Bigg[\bigg(\frac{s_{jr}}{s_{[ij]r}}\bigg)^{-\epsilon}-\bigg(\frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}}\bigg)^{-\epsilon}\Bigg]\bar{B}^{(ijr)}_{[ij]c}\\ & +\frac{C_A}{C_{f_j}}\Bigg[\bigg(\frac{\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}}\bigg)^{-\epsilon}-\bigg(\frac{s_{ir}\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}s_{ic}}\bigg)^{-\epsilon}\Bigg]\bar{B}^{(jri)}_{[ij]c}+\frac{2\,C_{f_j}-C_A}{C_{f_j}}\Bigg[\bigg(\frac{\bar{s}^{(irj)}_{jc}}{\bar{s}^{(irj)}_{jr}}\bigg)^{-\epsilon}-\bigg(\frac{s_{jr}\bar{s}^{(irj)}_{jc}}{\bar{s}^{(irj)}_{jr}s_{jc}}\bigg)^{-\epsilon}\Bigg]\bar{B}^{(irj)}_{[ij]c}\Bigg\}\\ & +\frac{\alpha_s}{2\pi}2\mathcal{N}_1\,C_{f_j}\sum_{k\neq i,j}\bigg(\frac{\gamma^{\mathrm{hc}}_k}{\epsilon}+\phi^{\mathrm{hc}}_k\bigg)\Big[\mathcal{E}^{(i)}_{jr}\,\bar{B}^{(ijr)}-\mathcal{E}^{(i)}_{jr'}\,\bar{B}^{(ijr')}\Big]\,.\end{aligned} \tag{5.22}$$

With the latter definition we are able to prove the second and the fourth Eq. (5.17), by exploiting the colour algebra of the colour-connected matrix elements, and the cancellation of the ϵ poles in the third Eq. (5.18). The explicit expression of the components of Δ_{ij} in Eq. (5.16) completes the list of definitions required to implement the subtracted real-virtual squared matrix element $RV_{\rm sub}$. Given its finiteness in $d = 4$, we can now rewrite Eq. (5.9) as

$$RV_{\rm sub}(X) \;=\; \sum_{i,j\neq i}\bigg[\Big(RV_{\rm fin} + I^{(\mathbf{1})}_{{\rm fin},ij}\Big)\mathcal{W}_{ij}\,\delta_{n+1}(X) - \Big(K^{(\mathbf{RV})}_{{\rm fin},ij} + I^{(\mathbf{12})}_{{\rm fin},ij}\Big)\,\delta_n(X)\bigg]\,, \tag{5.23}$$

where the subscript emphasises that, at this stage, all the explicit poles have already been cancelled. The finite component $I^{(\mathbf{1})}_{{\rm fin},ij}$ is given in Eq. (5.5), while $I^{(\mathbf{12})}_{{\rm fin},ij}$ can easily be derived from Eqs. (4.47)-(4.49). Finally, we obtain the finite contribution $K^{(\mathbf{RV})}_{{\rm fin},ij}$ by computing the expansion in powers of ϵ of the sum of Eqs. (5.12) and (5.19)-(5.22). We refrain from giving here the explicit expression for the quantities in Eq. (5.23), as we will derive a more compact result for $RV_{\rm sub}(X)$, in terms of symmetrised sector functions in the next section.

5.1 $RV_{\rm sub}$ with symmetrised sector functions

In the previous section we presented the construction of the subtracted real-virtual matrix element. We started by introducing the general properties of $RV_{\rm sub}$, and we discussed the main steps necessary to provide an explicit form for all the terms that contribute to its definition, according to Eq. (5.9). We then proved that $RV_{\rm sub}$ is free of both explicit poles and phase-space singularities in each $\mathcal{W}_{ij}$ sector separately. As was mentioned in Section 2.1 and in Section 3.6, however, one can improve the numerical performance of the scheme by appropriately symmetrising the sector functions. In this section we present explicit expressions for $RV_{\rm sub}$ in terms of symmetrised sector functions.

In analogy to the procedure applied at NLO in Eq. (2.28), and later generalised to $RR_{\rm sub}$ in Section 3.6, we rewrite the real-virtual counterterm $K^{(\mathbf{RV})}$ in terms the symmetrised sector counterterms $K^{(\mathbf{RV})}_{\{ij\}}$, defined as

$$K^{(\mathbf{RV})}_{\{ij\}} \;=\; K^{(\mathbf{RV})}_{ij} + K^{(\mathbf{RV})}_{ji}\,, \qquad\qquad K^{(\mathbf{RV})} \;=\; \sum_{i,j\neq i} K^{(\mathbf{RV})}_{ij} \;=\; \sum_{i,j>i} K^{(\mathbf{RV})}_{\{ij\}}\,. \tag{5.24}$$

Starting from Eq.(5.23), it is then straightforward to obtain

$$RV_{\rm sub}(X) \;=\; \sum_{i,j>i}\bigg\{\Big[RV_{\rm fin} + I^{(\mathbf{1})}_{{\rm fin},ij}\Big]\mathcal{Z}_{ij}\,\delta_{n+1}(X) + \Big[K^{(\mathbf{RV})}_{{\rm fin},\{ij\}} + I^{(\mathbf{12})}_{{\rm fin},\{ij\}}\Big]\,\delta_n(X)\bigg\}\,, \tag{5.25}$$

with $I^{(\mathbf{1})}_{{\rm fin},ij}$ given in Eq. (5.5). To present explicitly the other finite terms featuring in Eq. (5.25), we organise them in terms of soft and hard-collinear components, writing

$$K^{(\mathbf{RV})}_{{\rm fin},\{ij\}} + I^{(\mathbf{12})}_{{\rm fin},\{ij\}} \;=\; K^{(\mathbf{RV+12})}_{{\rm S},ij}\,\mathcal{Z}_{{\rm s},ij} + K^{(\mathbf{RV+12})}_{{\rm S},ji}\,\mathcal{Z}_{{\rm s},ji} + K^{(\mathbf{RV+12})}_{{\rm HC},ij}\,, \tag{5.26}$$

where the soft limit of the symmetrised sector functions, $\mathcal{Z}_{{\rm s},ij}$, is defined in Eq. (3.33). The finite soft counterterm $K^{(\mathbf{RV+12})}_{{\rm S},ij}$ is obtained by combining Eq. (4.47) with Eqs. (5.12) and (5.19), dropping the explicit poles. The result is extremely compact, and, except for the process-dependent finite part of the single-virtual squared matrix element, it displays only simple logarithmic dependence on the kinematics. We find ($r = r_{ik}$, $r' = r_{ij}$)

$$
\begin{aligned}
K^{(\mathbf{RV+12})}_{\mathrm{S},ij} \;=\; 4\,\alpha_s^2 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd} \Bigg\{ & \sum_{\substack{e\neq i\\ f\neq i,e}} \left(L_{ef} - \frac{1}{4} L_{ef}^2 \right) \bar{B}^{(icd)}_{cdef} + 2 \sum_{e\neq i,d} \left(L_{ed} - \frac{1}{4} L_{ed}^2 \right) \left(\bar{B}^{(icd)}_{cded} - \bar{B}^{(idc)}_{cded} \right) \\
& + \sum_{e\neq i,d} \ln^2 \frac{\bar{s}^{(icd)}_{de}}{s_{de}} \, \bar{B}^{(icd)}_{cded} - \frac{1}{2} \ln^2 \frac{\bar{s}^{(icd)}_{cd}}{s_{cd}} \, \bar{B}^{(icd)}_{cdcd} - 2\pi \sum_{e\neq i,c,d} \ln \frac{s_{id} s_{ie}}{\mu^2\, s_{de}} \, \bar{B}^{(icd)}_{cde} \\
& + \Bigg[\left(6 - \frac{7}{2}\,\zeta_2 \right) \left(\Sigma_C + 2 C_{f_d} - 2 C_{f_c} \right) + \sum_k \phi^{\rm hc}_k - \sum_{k\neq i} \gamma^{\rm hc}_k \, L_{kr} - \gamma^{\rm hc}_i \, L_{ir'} \\
& \quad + C_A \left(6 - \zeta_2 - \ln \frac{s_{ic}}{s_{cd}} \ln \frac{s_{id}}{s_{cd}} - 2 \ln \frac{s_{ic} s_{id}}{\mu^2 s_{cd}} \right) \Bigg] \bar{B}^{(icd)}_{cd} \Bigg\} \\
+ 4\,\alpha_s^2 \sum_{k\neq i} \left(\phi^{\rm hc}_k - \gamma^{\rm hc}_k L_{kr} \right) & \Bigg[\sum_{c\neq i,k} \mathcal{E}^{(i)}_{kc} \left(\bar{B}^{(ick)}_{kc} - \bar{B}^{(ikc)}_{kc} \right) + \sum_{c\neq i,k,r} \mathcal{E}^{(i)}_{cr} \left(\bar{B}^{(icr)}_{cr} - \bar{B}^{(irc)}_{cr} \right) \Bigg] \\
+ 8\pi\,\alpha_s \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd} \, \bar{V}^{(icd)}_{\mathrm{fin},cd} \,, &
\end{aligned}
\tag{5.27}
$$

where $\bar{V}^{(icd)}_{\mathrm{fin},cd}$ is the finite part of the colour-correlated, single-virtual squared matrix element, expressed in the mapped kinematics. We notice that, as happened for $I^{(\mathbf{12})}_{\mathrm{S},ij}$, the presence of the reference particle $r' = r_{ij}$ introduces a dependence on the particle j of the soft sector function $\mathcal{Z}_{\mathrm{s},ij}$ which multiplies $K^{(\mathbf{RV+12})}_{\mathrm{S},ij}$.

To conclude this section, we also report the finite hard-collinear counterterm $K^{(\mathbf{RV+12})}_{\mathrm{HC},ij}$, which is the result of summing Eqs. (4.48) and (4.49) with Eqs. (5.12), (5.20), and (5.22). We find (with $r = r_{ij}$, $r' = r_{ijk}$)

$$
\begin{aligned}
K^{(\mathbf{RV+12})}_{\mathrm{HC},ij} \;=\; 4\,\alpha_s^2 \, \frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}} \Bigg\{ & \sum_{c\neq i,j} \Bigg[\ln^2 \frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}} \, \bar{B}^{(ijr)}_{\mu\nu,[ij]c} - \frac{1}{2} \sum_{d\neq i,j,c} \left(4 L_{cd} - L_{cd}^2 \right) \bar{B}^{(ijr)}_{\mu\nu,cd} \Bigg] \\
& - \sum_{c\neq i,j,r} \Bigg[\ln^2 \frac{\bar{s}^{(ijr)}_{cr}}{s_{cr}} \, \bar{B}^{(ijr)}_{\mu\nu,cr} + \frac{\rho^{(C)}_{ij}}{2} \, \mathcal{L}_{ijcr} \, \bar{B}^{(jri)}_{\mu\nu,[ij]c} + \frac{\rho^{(C)}_{ji}}{2} \, \mathcal{L}_{jicr} \, \bar{B}^{(irj)}_{\mu\nu,[ij]c} \Bigg] \\
& - \frac{1}{2} \sum_{c\neq i,j} \tilde{f}^{\,q\bar{q}}_{ij} \left(\tilde{\mathcal{L}}_{ijcr} \, \bar{\mathcal{B}}^{(jri)}_{\mu\nu,[ij]c} - \tilde{\mathcal{L}}_{jicr} \, \bar{\mathcal{B}}^{(irj)}_{\mu\nu,[ij]c} \right) \\
& - \Bigg[\left(6 - \frac{7}{2}\,\zeta_2 \right) \left(\Sigma_C - C_{f_{[ij]}} \rho^{(C)}_{[ij]} \right) + C_{f_{[ij]}} \frac{\rho^{(C)}_{[ij]}}{2} \left(4 L_{ij} - L_{ij}^2 \right) \\
& \quad - C_{f_{[ij]}} \frac{\rho^{(C)}_{ij}}{2} \left(4 L_{ir} - L_{ir}^2 \right) - C_{f_{[ij]}} \frac{\rho^{(C)}_{ji}}{2} \left(4 L_{jr} - L_{jr}^2 \right) + \Sigma^{\rm hc}_{\phi} \Bigg] \bar{B}^{(ijr)}_{\mu\nu} \Bigg\} \\
- \; 4\,\alpha_s^2 & \left[2\, C_{f_j} \, \mathcal{E}^{(i)}_{jr} \, C_{f_{[ij]}} \ln^2 \frac{s_{jr}}{s_{[ij]r}} \, \bar{B}^{(ijr)} + (i \leftrightarrow j) \right] \\
+ \; 4\,\alpha_s^2 & \left[\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}} \left(\gamma^{\rm hc}_i L_{ir} + \gamma^{\rm hc}_j L_{jr} \right) \bar{B}^{(ijr)}_{\mu\nu} + \sum_{k\neq i,j} \frac{P^{\mathrm{hc},\mu\nu}_{ij(r')}}{s_{ij}} \, \gamma^{\rm hc}_k \, L_{kr'} \, \bar{B}^{(ijr')}_{\mu\nu} \right] \\
- \; 4\,\alpha_s^2 & \, \frac{\tilde{P}^{\mathrm{hc},\mu\nu}_{\mathrm{fin},ij(r)}}{s_{ij}} \, \bar{B}^{(ijr)}_{\mu\nu} - 8\pi\,\alpha_s \, \frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}} \, \bar{V}^{(ijr)}_{\mathrm{fin},\mu\nu} \,,
\end{aligned}
\tag{5.28}
$$

where we introduced the shorthand notation

$$
\mathcal{L}_{ijcr} \;=\; 2 \ln \frac{s_{ic}}{s_{ir}} \left[2 - L_{ic} + \ln \frac{\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}} \right] , \qquad \tilde{\mathcal{L}}_{ijcr} \;=\; 2 L_{ic} \left[2 - L_{ic} + \ln \frac{\bar{s}^{(jri)}_{ic}}{\mu^2} \right] . \tag{5.29}
$$

Notice that also in Eq. (5.28) the kinematic dependence is expressed only in terms of simple logarithms. Our next step is now to integrate the real-virtual counterterm, and add back the result to complete Eq. (2.37).

6 Integration of the real-virtual counterterm

In Eqs. (5.14), (5.24) we have defined the counterterm $K^{(\mathbf{RV})}$, that enabled us to build the subtracted real-virtual squared matrix element $RV_{\rm sub}$, integrable in the whole $(n+1)$-body phase space, and free of poles in ϵ. The $K^{(\mathbf{RV})}$ counterterm needs to be integrated in $d = 4-2\epsilon$ dimensions in the radiation phase space, and then the result must be added back, according to the subtraction structure given in Eqs. (2.36)-(2.38). In order to compute the integrated counterterm, $I^{(\mathbf{RV})}$, as defined in Eq. (2.35), we proceed by summing over all sectors $\mathcal{W}_{ij}$, so that sector functions drop out of the calculation, owing to the sum rules they satisfy (like for example those in (2.11)). We then perform the integration over the radiative phase space, with the measure $d\Phi_{\rm rad}^{(acd)}$, naturally induced by the mapping (acd), according to

$$\int d\Phi_{n+1}(\{k\}) \;=\; \frac{\varsigma_{n+1}}{\varsigma_n}\int d\Phi_n^{(acd)}\int d\Phi_{\rm rad}^{(acd)}\,, \qquad\qquad d\Phi_n^{(acd)} \;\equiv\; d\Phi_n(\{\bar k\}^{(acd)})\,, \tag{6.1}$$

where $d\Phi_{\rm rad}^{(acd)}$ is defined in Eq. (4.3). The integration of $K^{(\mathbf{RV})}$ is carried out following the methods described in Ref. [92], and using the fact that the spin-correlated contributions proportional to the kernels $Q^{\mu\nu}_{ij(r)}$ and $\tilde Q^{\mu\nu}_{ij(r)}$ vanish upon integration, as discussed in Appendix D. The formal expression for the integrated version of $K^{(\mathbf{RV})}$ can be written as

$$\int d\Phi_{n+1}\, K^{(\mathbf{RV})} \;=\; \int d\Phi_{n+1}\left[\sum_i \left(\overline{\mathbf{S}}_i\, RV + \Delta_{{\rm S},i}\right) + \sum_{i,\,j>i}\left(\overline{\mathbf{HC}}_{ij}\, RV + \Delta_{{\rm HC},ij}\right)\right], \tag{6.2}$$

where the integrands are defined in Eqs. (5.12) and (5.19)-(5.22) and we use the shorthand notations (see Eq. (2.29))

$$\overline{\mathbf{HC}}_{ij}\, RV \equiv \overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j)\, RV\,, \qquad\qquad \Delta_{{\rm HC},ij} \equiv \Delta_{{\rm C},ij}+\Delta_{{\rm C},ji}-\Delta_{{\rm SC},ij}-\Delta_{{\rm SC},ji}\,. \tag{6.3}$$

Before integrating, we can further simplify the expressions for $\Delta_{{\rm S},i}$ and $\Delta_{{\rm C},ij}$, given in (5.19)-(5.20). In fact, since $\bar s_{ef}^{(icd)} = s_{ef}$ for $e,f \neq i,c,d$, and $\bar s_{cd}^{(ijr)} = s_{cd}$ for $c,d\neq i,j,r$, one finds that

$$\begin{aligned}
\frac12\sum_{\substack{e\neq i,c\\ f\neq i,c,e}}\left[\left(\frac{s_{ef}}{\bar s_{ef}^{(icd)}}\right)^{-\epsilon}-1\right]\bar B_{efcd}^{(icd)} + \sum_{e\neq i,d}\left[\left(\frac{s_{ed}}{\bar s_{ed}^{(icd)}}\right)^{-\epsilon}-1\right]\bar B_{edcd}^{(icd)} \;=\; \\
=\; 2\sum_{e\neq i,c,d}\left[\left(\frac{s_{ed}}{\bar s_{ed}^{(icd)}}\right)^{-\epsilon}-1\right]\bar B_{edcd}^{(icd)} + \left[\left(\frac{s_{cd}}{\bar s_{cd}^{(icd)}}\right)^{-\epsilon}-1\right]\bar B_{cdcd}^{(icd)}\,,
\end{aligned} \tag{6.4}$$

as well as

$$\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\left[\left(\frac{s_{cd}}{\bar s_{cd}^{(ijr)}}\right)^{-\epsilon}-1\right]\bar B_{\mu\nu,cd}^{(ijr)} \;=\; 2\sum_{c\neq i,j,r}\left[\left(\frac{s_{cr}}{\bar s_{cr}^{(ijr)}}\right)^{-\epsilon}-1\right]\bar B_{\mu\nu,cr}^{(ijr)}\,. \tag{6.5}$$

After integration, the soft contributions to Eq. (6.2) read

$$\begin{aligned}
\int d\Phi_{n+1}\,\overline{\mathbf{S}}_i\, RV \;=\; -\,\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{\substack{c\neq i\\ d\neq i,c}}\int d\Phi_n^{(icd)}\Bigg[\, & J_{\rm s}^{icd}\,\bar V_{cd}^{(icd)} - \frac{\alpha_s}{2\pi}\left(\tilde J_{\rm s}^{icd} + J_{\rm s}^{icd}\,\frac{\beta_0}{2\epsilon}\right)\bar B_{cd}^{(icd)} \\
& + \alpha_s\sum_{e\neq i,c,d}\tilde J_{\rm s}^{icde}\,\bar B_{cde}^{(icd)}\Bigg]\,,
\end{aligned} \tag{6.6}$$

while $(r = r_{ik})$

$$\int d\Phi_{n+1}\,\Delta_{\mathrm{S},i} \;=\; -\frac{\alpha_s}{2\pi}\,\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{\substack{c\neq i\\ d\neq i,c}}\Bigg\{\int d\Phi_n^{(icd)}\Bigg[\sum_{e\neq i,c,d} J_{\Delta_{\mathrm{s}}}^{icd(e)}\,\bar B^{(icd)}_{edcd} + J_{\Delta_{\mathrm{s}}}^{icd}\,\bar B^{(icd)}_{cdcd}\Bigg] \\ +\Bigg[2\,C_{f_c}\left(\frac{1}{\epsilon^2}+\frac{2}{\epsilon}\right)+\frac{\gamma_c^{\mathrm{hc}}}{\epsilon}\Bigg]\Bigg[\int d\Phi_n^{(icd)}J_{\mathrm{s}}^{icd}\,\bar B_{cd}^{(icd)} - \int d\Phi_n^{(idc)}J_{\mathrm{s}}^{idc}\,\bar B_{cd}^{(idc)}\Bigg]\Bigg\} \\ -\frac{\alpha_s}{2\pi}\,\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{\substack{k\neq i\\ c\neq i,k,r}}\frac{\gamma_k^{\mathrm{hc}}}{\epsilon}\Bigg[\int d\Phi_n^{(irc)}J_{\mathrm{s}}^{irc}\,\bar B_{cr}^{(irc)} - \int d\Phi_n^{(icr)}J_{\mathrm{s}}^{icr}\,\bar B_{cr}^{(icr)}\Bigg]. \tag{6.7}$$

Explicit expressions for the constituent integrals $\tilde J_s^{icd}$, $\tilde J_{\mathrm{s}}^{icde}$, $J_{\Delta_{\mathrm{s}}}^{icd(e)}$ and $J_{\Delta_{\mathrm{s}}}^{icd}$ are given in Eq. (E.5), while the NLO integral J_{s}^{icd} is given in Eq. (E.1). We notice that the soft integrated real-virtual counterterm in Eq. (6.6) receives contributions from the triple-colour-correlated squared matrix element $\bar B_{cde}$. However, the pole content of such term vanishes upon performing the appropriate colour sums (see Ref. [92] for further details). This cancellation represents a strong test for the method: it is protected by the fact that no singular contributions proportional to colour tripoles can arise from double-virtual nor from double-real corrections. On the other hand, integrating the tripole contribution to the soft real-virtual kernel requires the non-trivial procedure described in Ref. [92], which is necessary in order to verify the pole cancellation, and to compute the finite remainder. To complete the discussion we also report the integrated hard-collinear component,

$$\int d\Phi_{n+1}\,\overline{\mathbf{HC}}_{ij}\,RV \;=\; \frac{\varsigma_{n+1}}{\varsigma_n}\int d\Phi_n^{(ijr)}\left[J_{\mathrm{hc}}^{ijr}\,\bar V^{(ijr)} + \frac{\alpha_s}{2\pi}\left(\tilde J_{\mathrm{hc}}^{ijr} - J_{\mathrm{hc}}^{ijr}\frac{\beta_0}{2\epsilon}\right)\bar B^{(ijr)}\right], \qquad r = r_{ij}\,, \tag{6.8}$$

while the compensating hard-collinear term integrates to $(r = r_{ij},\ r' = r_{ijk})$

$$\int d\Phi_{n+1}\,\Delta_{\mathrm{HC},ij} \;=\; \frac{\alpha_s}{2\pi}\,\frac{\varsigma_{n+1}}{\varsigma_n}\Bigg\{\int d\Phi_n^{(ijr)}\Bigg[\sum_{c\neq i,j,r} J_{\Delta\mathrm{hc}}^{ijr}\,\bar B_{cr}^{(ijr)} + \sum_{c\neq i,j} J_{\Delta\mathrm{hc}}^{ijrc}\,\bar B_{[ij]c}^{(ijr)}\Bigg] \\ +\sum_{c\neq i,j,r}\Bigg[\int d\Phi_n^{(jri)}\,J_{\Delta\mathrm{hc}}^{jri,c}\,\bar B_{[ij]c}^{(jri)} + \int d\Phi_n^{(irj)}\,J_{\Delta\mathrm{hc}}^{irj,c}\,\bar B_{[ij]c}^{(irj)}\Bigg] \\ +\sum_{k\neq i,j}\left(\frac{\gamma_k^{\mathrm{hc}}}{\epsilon}+\phi_k^{\mathrm{hc}}\right)\Bigg[\int d\Phi_n^{(ijr)}J_{\mathrm{hc}}^{ijr}\,\bar B^{(ijr)} - \int d\Phi_n^{(ijr')}J_{\mathrm{hc}}^{ijr'}\,\bar B^{(ijr')}\Bigg] \\ +\tilde f_{ij}^{q\bar q}\sum_{c\neq i,j}\Bigg[\int d\Phi_n^{(jri)}\,\tilde J_{\Delta\mathrm{hc}}^{jri,c}\,\bar{\mathcal B}_{[ij]c}^{(jri)} - \int d\Phi_n^{(irj)}\,\tilde J_{\Delta\mathrm{hc}}^{irj,c}\,\bar{\mathcal B}_{[ij]c}^{(irj)}\Bigg]\Bigg\}. \tag{6.9}$$

Explicit expressions for the hard-collinear constituent integrals $\tilde J_{\mathrm{hc}}^{ijr}$, $J_{\Delta\mathrm{hc}}^{ijr}$, $J_{\Delta\mathrm{hc}}^{ijrc}$, $J_{\Delta\mathrm{hc}}^{jri,c}$, and $\tilde J_{\Delta\mathrm{hc}}^{jri,c}$ are given in Eq. (E.16), while the NLO hard-collinear integral J_{hc}^{ijr} is given in Eq. (E.7).

Having computed all relevant integrals, we now recombine them, following a procedure analogous to the one described at the end of Section 4.2. We rename the sets of mapped momenta $\{\bar k^{(abc)}\}_n$ to the same set of Born-level momenta $\{k\}_n$ by means of the replacements

$$d\Phi_n^{(abc)} \;\to\; d\Phi_n\,, \qquad \bar B_{\dots}^{(abc)} \;\to\; B_{\dots}\,, \qquad \bar{\mathcal B}_{\dots}^{(abc)} \;\to\; \bar{\mathcal B}_{\dots}\,, \qquad \bar s_{lm}^{(abc)} \;\to\; s_{lm}\,, \tag{6.10}$$

where the ellipsis in the Born-level matrix element stands for a generic colour correlation. In particular, in the integral of $\Delta_{\mathrm{HC},ij}$, all momenta $\bar k_j^{(ijr)}$, $\bar k_i^{(jri)}$, $\bar k_j^{(irj)}$, and $\bar k_j^{(ijr')}$ are renamed as k_p, where p is the label of the parent particle splitting into i and j. As a consequence of this renaming, the integrals involving $\bar{\mathcal B}_{[ij]c}$ can be recombined, and do not contribute to the integrated counterterm. Indeed

$$
\begin{aligned}
&\int d\Phi_n^{(jri)}\, \tilde{J}^{\,jri,c}_{\Delta\mathrm{hc}}\, \bar{\mathcal{B}}^{(jri)}_{[ij]c} - \int d\Phi_n^{(irj)}\, \tilde{J}^{\,irj,c}_{\Delta\mathrm{hc}}\, \bar{\mathcal{B}}^{(irj)}_{[ij]c} \;= \\
&\qquad = \int d\Phi_n^{(jri)}\, \tilde{J}^{\,c}_{\Delta\mathrm{hc}}\Big(\bar{s}^{(jri)}_{ir}, \bar{s}^{(jri)}_{ic}\Big)\, \bar{\mathcal{B}}^{(jri)}_{[ij]c} - \int d\Phi_n^{(irj)}\, \tilde{J}^{\,c}_{\Delta\mathrm{hc}}\Big(\bar{s}^{(irj)}_{jr}, \bar{s}^{(irj)}_{jc}\Big)\, \bar{\mathcal{B}}^{(irj)}_{[ij]c} \\
&\qquad \to \int d\Phi_n\, \tilde{J}^{\,c}_{\Delta\mathrm{hc}}\Big(s_{pr}, s_{pc}\Big)\, \mathcal{B}_{pc} - \int d\Phi_n\, \tilde{J}^{\,c}_{\Delta\mathrm{hc}}\Big(s_{pr}, s_{pc}\Big)\, \mathcal{B}_{pc} \;=\; 0\,.
\end{aligned}
\tag{6.11}
$$

The dependence on the $(n+1)$-body phase-space particles is now limited to the flavour factors f_i^q, $f_i^{\bar{q}}$ and f_i^g, which can be translated into flavour factors for the n-body-phase-space particles, as was done in Section 4.3 for the double-real contribution. In particular, when going from an $(n+1)$-body phase space to an n-body phase space the relations in Eq. (4.34) and Eq. (4.35) apply, with the formal replacement $n \to n-1$. After performing the flavour sums, no dependence on the original $(n+1)$-body phase space remains. Simplifying the colour correlations where possible, we finally get

$$
\begin{aligned}
I^{(\mathbf{RV})} \;=\; & - \sum_{c,d\neq c} \Bigg[J_{\mathrm{s}}(s_{cd})\, V_{cd} + J_{\mathrm{sRV}}(s_{cd})\, B_{cd} + J^{(2)}_{\mathrm{sRV}}(s_{cd})\, B_{cdcd} + \sum_{e\neq c,d} J^{cde}_{\mathrm{sRV}}\, B_{cde} \Bigg] \\
& + \sum_j \Bigg\{ J^j_{\mathrm{hc}}(s_{jr})\, V + J^j_{\mathrm{hcRV}}(s_{jr})\, B + J^{j,\mathrm{A}}_{\mathrm{hcRV}}(s_{jr})\, B_{jr} + \sum_{c\neq j,r} \Bigg[J^{j,\mathrm{B}}_{\mathrm{hcRV}}(s_{jc})\, B_{jc} + J^{j,\mathrm{C}}_{\mathrm{hcRV}}(s_{jr})\, B_{cr} \Bigg] \\
& \qquad + \frac{\alpha_s}{2\pi} \sum_{k\neq j} \left(\frac{\gamma^{\mathrm{hc}}_k}{\epsilon} + \phi^{\mathrm{hc}}_k \right) \Big[J^j_{\mathrm{hc}}(s_{jr}) - J^j_{\mathrm{hc}}(s_{jr'}) \Big] \Bigg\}\,,
\end{aligned}
\tag{6.12}
$$

where we introduced the following combinations of constituent integrals:

$$
J_{\mathrm{sRV}}(s) \;=\; -\frac{\alpha_s}{2\pi} \Bigg[C_A\, \tilde{J}_{\mathrm{s}}(s) + \frac{\beta_0}{2\epsilon}\, J_{\mathrm{s}}(s) + 2\, C_{f_c}\, J^{(3)}_{\Delta\mathrm{s}}(s) \Bigg]\,,
\tag{6.13}
$$

$$
J^{(2)}_{\mathrm{sRV}}(s) \;=\; \frac{\alpha_s}{2\pi} \Big[J^{(2)}_{\Delta\mathrm{s}}(s) - J^{(3)}_{\Delta\mathrm{s}}(s) \Big]\,,
\tag{6.14}
$$

$$
J^{cde}_{\mathrm{sRV}} \;=\; -\frac{\alpha_s^2}{2\pi} \Bigg[\frac{1}{2} \ln\frac{s_{ce}}{s_{de}} \ln^2\frac{s_{cd}}{\mu^2} + \frac{1}{6} \ln^3\frac{s_{ce}}{s_{de}} + \mathrm{Li}_3\Big(-\frac{s_{ce}}{s_{de}}\Big) + \mathcal{O}(\epsilon) \Bigg]\,,
\tag{6.15}
$$

$$
\begin{aligned}
J^j_{\mathrm{hcRV}}(s) \;=\; \frac{\alpha_s}{2\pi} \Bigg\{ & \big(f^q_j + f^{\bar{q}}_j\big) \Bigg[\tilde{J}^{(1\mathrm{g})}_{\mathrm{hc}}(s) - \frac{\beta_0}{2\epsilon}\, J^{(1\mathrm{g})}_{\mathrm{hc}}(s) - C_F\, J^{(1\mathrm{g})}_{\Delta\mathrm{hc,A}}(s) - C_F\, J^{\mathrm{qg}}_{\Delta\mathrm{hc,A}}(s) - C_F\, J^{\mathrm{gq}}_{\Delta\mathrm{hc,A}}(s) \Bigg] \\
& + f^g_j \Bigg[\frac{1}{2} \left(\tilde{J}^{(2\mathrm{g})}_{\mathrm{hc}}(s) - \frac{\beta_0}{2\epsilon}\, J^{(2\mathrm{g})}_{\mathrm{hc}}(s) - C_A\, J^{(2\mathrm{g})}_{\Delta\mathrm{hc,A}}(s) - 2\, C_A\, J^{\mathrm{gg}}_{\Delta\mathrm{hc,A}}(s) \right) \\
& \qquad + N_f \left(\tilde{J}^{(0\mathrm{g})}_{\mathrm{hc}}(s) - \frac{\beta_0}{2\epsilon}\, J^{(0\mathrm{g})}_{\mathrm{hc}}(s) - C_A\, J^{(0\mathrm{g})}_{\Delta\mathrm{hc,A}}(s) - 2\, C_A\, J^{\mathrm{qq}}_{\Delta\mathrm{hc,A}}(s) \right) \Bigg] \Bigg\}\,,
\end{aligned}
\tag{6.16}
$$

$$
\begin{aligned}
J^{j,\mathrm{A}}_{\mathrm{hcRV}}(s) \;=\; \frac{\alpha_s}{2\pi} \Bigg\{ & \big(f^q_j + f^{\bar{q}}_j\big) \Big(J^{(1\mathrm{g})}_{\Delta\mathrm{hc,B}}(s) - J^{\mathrm{qg}}_{\Delta\mathrm{hc,A}}(s) - J^{\mathrm{gq}}_{\Delta\mathrm{hc,A}}(s) \Big) \\
& + f^g_j \Bigg[\frac{1}{2} \Big(J^{(2\mathrm{g})}_{\Delta\mathrm{hc,B}}(s) - 2\, J^{\mathrm{gg}}_{\Delta\mathrm{hc,A}}(s) \Big) + N_f \Big(J^{(0\mathrm{g})}_{\Delta\mathrm{hc,B}}(s) - 2\, J^{\mathrm{qq}}_{\Delta\mathrm{hc,A}}(s) \Big) \Bigg] \Bigg\}\,,
\end{aligned}
\tag{6.17}
$$

$$J^{j,\mathrm{B}}_{\mathrm{hcRV}}(s) \; = \; \frac{\alpha_s}{2\pi} \Bigg\{ \left(f^q_j + f^{\bar{q}}_j\right) \left(J^{(1\mathrm{g})}_{\Delta\mathrm{hc,B}}(s) + J^{\mathrm{qg}}_{\Delta\mathrm{hc,B}}(s) + J^{\mathrm{gq}}_{\Delta\mathrm{hc,B}}(s) \right) \\ + f^g_j \left[\frac{1}{2} \left(J^{(2\mathrm{g})}_{\Delta\mathrm{hc,B}}(s) + 2\, J^{\mathrm{gg}}_{\Delta\mathrm{hc,B}}(s) \right) + N_f \left(J^{(0\mathrm{g})}_{\Delta\mathrm{hc,B}}(s) + 2\, J^{\mathrm{qq}}_{\Delta\mathrm{hc,B}}(s) \right) \right] \Bigg\} , \tag{6.18}$$

$$J^{j,\mathrm{C}}_{\mathrm{hcRV}}(s) \; = \; \frac{\alpha_s}{2\pi} \left\{ \left(f^q_j + f^{\bar{q}}_j\right) J^{(1\mathrm{g})}_{\Delta\mathrm{hc}}(s) + f^g_j \left[\frac{1}{2} J^{(2\mathrm{g})}_{\Delta\mathrm{hc}}(s) + N_f \, J^{(0\mathrm{g})}_{\Delta\mathrm{hc}}(s) \right] \right\} . \tag{6.19}$$

All new constituent integrals appearing in the above results are listed in Appendix E: the soft integrals are presented in Eq. (E.6), the hard-collinear integrals in Eq. (E.17), and the integrals arising from the compensating Δ_{ij} terms in Eqs. (E.18)-(E.20). We note once again that all integrals involved are single-scale, and thus involve only simple logarithms. Interestingly, the only exception is Eq. (6.15), a uniform-weight-three function featuring three scales and a single trilogarithm: this integral arises as a finite remainder of the non-trivial integration of the tripole term.

The integrated counterterm $I^{(\mathbf{RV})}$ given in Eq. (6.12), which features Born-level kinematics, contains explicit poles in ϵ, that must be combined with those of the integrated counterterm $I^{(\mathbf{2})}$, and must, together, cancel the singularities of the double-virtual squared matrix element. In the next section we turn to the proof of this statement, which provides a highly non-trivial test of all our calculations, and completes the subtraction programme for generic massless final states.

7 The subtracted double-virtual contribution VV_{sub}

Finally, we turn our attention to the first line in Eq. (2.37), which we rewrite here as

$$VV_{\mathrm{sub}}(X) \; = \; \Big[VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \Big] \, \delta_n(X) \, . \tag{7.1}$$

It is our task to show that the equation above is free of ϵ poles. To verify this, we first explicitly derive the ϵ poles of VV, and then we provide the complete ϵ expansion of $I^{(\mathbf{2})} + I^{(\mathbf{RV})}$, including $\mathcal{O}(\epsilon^0)$ terms, obtained by combining Eq. (4.41) and Eq. (6.12).

7.1 The pole part of the double-virtual matrix element VV

All infrared poles of gauge-theory scattering amplitudes can be expressed in a factorised form through the formula [18, 19, 21, 24, 25]

$$\mathcal{A}\left(\frac{k_i}{\mu}, \alpha_s(\mu), \epsilon\right) \; = \; \mathbf{Z}\left(\frac{k_i}{\mu}, \alpha_s(\mu), \epsilon\right) \mathcal{H}\left(\frac{k_i}{\mu}, \alpha_s(\mu), \epsilon\right) , \tag{7.2}$$

where $\mathcal{H}$ is finite as $\epsilon \to 0$, and $\mathbf{Z}$ is a colour operator with a universal form, to be discussed below. The infrared operator $\mathbf{Z}$ obeys a (matrix) renormalisation-group equation, which can be solved in exponential form, with a trivial initial condition, in terms of an anomalous-dimension matrix $\mathbf{\Gamma}$. One may write

$$\mathbf{Z}\left(\frac{k_i}{\mu}, \alpha_s(\mu), \epsilon\right) \; = \; \mathcal{P} \exp\left[\int_0^\mu \frac{d\lambda}{\lambda} \, \mathbf{\Gamma}\left(\frac{k_i}{\lambda}, \alpha_s(\lambda), \epsilon\right) \right] , \tag{7.3}$$

where the integral converges at $\lambda = 0$ in dimensional regularisation thanks to the behaviour of the β function in $d = 4 - 2\epsilon$, for $\epsilon < 0$ ($d > 4$). Indeed, in dimensional regularisation one has

$$\mu \frac{d\alpha_s}{d\mu} \; \equiv \; \beta\left(\epsilon, \alpha_s\right) \; = \; -\, 2\epsilon \, \alpha_s \; - \; \frac{\alpha_s^2}{2\pi} \, \beta_0 \; + \; \mathcal{O}\left(\alpha_s^3\right) , \tag{7.4}$$

whose solution implies [16] that the d-dimensional running coupling $\alpha_s(\mu, \epsilon)$ vanishes at $\mu = 0$ for $\epsilon < 0$, so that the corresponding initial condition is $\mathbf{Z}(\mu = 0) = \mathbf{1}$, leading to Eq. (7.3). For the purposes of NNLO subtraction (and thus at two loops for virtual amplitudes), $\mathbf{\Gamma}$ is given by the dipole formula [21, 24]

$$\mathbf{\Gamma}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda), \epsilon\right) = \frac{1}{2}\,\widehat{\gamma}_K\left(\alpha_s(\lambda, \epsilon)\right) \sum_{i,j>i} \ln\left(\frac{s_{ij}\, e^{\mathrm{i}\pi\sigma_{ij}}}{\lambda^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j + \sum_i \gamma_i\left(\alpha_s(\lambda, \epsilon)\right) . \tag{7.5}$$

In Eq. (7.5), the phases σ_{ij} are given by $\sigma_{ij} = +1$ if partons i and j are either both in the initial state or both in the final state, while $\sigma_{ij} = 0$ otherwise. For our present final-state application, we can thus henceforth replace all phase factors using $e^{\mathrm{i}\pi\sigma_{ij}} = -1$, with the understanding that the logarithm is taken above the cut.

The anomalous dimensions appearing in Eq. (7.5) are the cusp anomalous dimension $\widehat{\gamma}_K\left(\alpha_s\right)$ and the collinear anomalous dimensions $\gamma_i\left(\alpha_s\right)$. More precisely, in the derivation of Eq. (7.5) it has been assumed that the (light-like) cusp anomalous dimension $\gamma_K^{(r)}(\alpha_s)$, in colour representation r, obeys 'Casimir scaling', *i.e.* it can be written as

$$\gamma_K^{(r)}(\alpha_s) = C_r\, \widehat{\gamma}_K(\alpha_s)\,, \tag{7.6}$$

where C_r is the quadratic Casimir eigenvalue for colour representation r, while $\widehat{\gamma}_K(\alpha_s)$ is a universal (representation-independent) function. This assumption is known to fail at four loops [96, 97]. The collinear anomalous dimensions $\gamma_i(\alpha_s)$ are related to the anomalous dimensions of quark and gluon fields, and can be derived from essentially colour-singlet calculations such as those of form factors.

One important consequence of the dipole formula is that the scale integration in Eq. (7.3) can be performed without affecting the colour structure (which is scale-independent): one may therefore omit the path-ordering in Eq. (7.3), simplifying considerably the necessary calculations. Expanding the various ingredients perturbatively as

$$\widehat{\gamma}_K(\alpha_s) = \sum_{n=1}^{\infty} \widehat{\gamma}_K^{(n)} \left(\frac{\alpha_s}{2\pi}\right)^n , \quad \gamma_i(\alpha_s) = \sum_{n=1}^{\infty} \gamma_i^{(n)} \left(\frac{\alpha_s}{2\pi}\right)^n , \quad \mathbf{\Gamma}(\alpha_s) = \sum_{n=1}^{\infty} \mathbf{\Gamma}^{(n)} \left(\frac{\alpha_s}{2\pi}\right)^n , \tag{7.7}$$

one gets at NLO

$$\mathbf{\Gamma}^{(1)} = \frac{1}{4}\,\widehat{\gamma}_K^{(1)} \sum_{i,j\neq i} \ln\left(\frac{-s_{ij} + \mathrm{i}\eta}{\mu^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j + \sum_i \gamma_i^{(1)} - \frac{1}{4}\,\widehat{\gamma}_K^{(1)} \ln\left(\frac{\mu^2}{\lambda^2}\right) \sum_i C_{f_i}\,, \tag{7.8}$$

and consequently

$$\mathbf{Z}^{(1)}\left(\frac{p_i}{\mu}, \epsilon\right) = -\frac{1}{\epsilon^2}\frac{\widehat{\gamma}_K^{(1)}}{8}\Sigma_{\mathcal{C}} - \frac{1}{\epsilon}\left(\frac{\widehat{\gamma}_K^{(1)}}{8} \sum_{i,j\neq i} L_{ij}\, \mathbf{T}_i \cdot \mathbf{T}_j + \frac{1}{2}\Sigma_\gamma\right) + \mathrm{i}\pi\, \frac{\gamma_K^{(1)}}{8\epsilon}\, \Sigma_{\mathcal{C}}\,, \tag{7.9}$$

where $L_{ij} = \ln(s_{ij}/\mu^2)$. Eq. (7.9) is in agreement with [18, 24], with the one-loop anomalous-dimension coefficients given by

$$\widehat{\gamma}_K^{(1)} = 4\,, \quad \gamma_i^{(1)} \equiv \gamma_i = \frac{3}{2}\, C_F\left(f_i^q + f_i^{\bar{q}}\right) + \frac{1}{2}\,\beta_0\, f_i^g\,, \quad \Sigma_{\mathcal{C}} = \sum_i C_{f_i}\,, \quad \Sigma_\gamma = \sum_i \gamma_i\,, \tag{7.10}$$

where we noted that in the text we have sometimes used the notation γ_i for the one-loop coefficient denoted here by $\gamma_i^{(1)}$. Expanding the anomalous dimensions to two loops and performing the relevant integrals, the NNLO result for the $\mathbf{Z}$ factor is

$$
\begin{aligned}
\mathbf{Z}^{(2)} =\; & \frac{1}{\epsilon^4}\frac{\left(\widehat{\gamma}_K^{(1)}\right)^2}{128}\Sigma_C^2 \\
& + \frac{1}{\epsilon^3}\frac{\widehat{\gamma}_K^{(1)}}{64}\Sigma_C\left[3\beta_0 + 4\Sigma_\gamma + \widehat{\gamma}_K^{(1)}\sum_{i,j\neq i}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j\right] \\
& + \frac{1}{\epsilon^2}\frac{1}{8}\left[\frac{\beta_0\,\widehat{\gamma}_K^{(1)}}{4}\sum_{i,j\neq i}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \beta_0\Sigma_\gamma - \frac{\widehat{\gamma}_K^{(2)}}{4}\Sigma_C\right. \\
& \qquad + \Sigma_\gamma^2 + \frac{\widehat{\gamma}_K^{(1)}}{2}\Sigma_\gamma\sum_{i,j\neq i}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j \\
& \qquad \left. + \frac{\left(\widehat{\gamma}_K^{(1)}\right)^2}{16}\sum_{\substack{i,j\neq i\\k,l\neq k}}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j\,\mathbf{T}_k\cdot\mathbf{T}_l\right] \\
& - \frac{1}{\epsilon}\frac{1}{4}\left[\frac{\widehat{\gamma}_K^{(2)}}{4}\sum_{i,j\neq i}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \Sigma_\gamma^{(2)}\right], \qquad (7.11)
\end{aligned}
$$

which agrees with [24], with the anomalous dimension coefficients given in Eq. (A.13), and where we defined $\Sigma_\gamma^{(2)} = \sum_i \gamma_i^{(2)}$. Having deduced the $\mathbf{Z}$ elements up to the needed order, we can now interfere the double-virtual amplitude with the Born, and extract the poles. The perturbative expansion of (7.2) yields

$$
\begin{aligned}
\mathcal{A}^{(0)} &= \mathcal{H}^{(0)}, \\
\mathcal{A}^{(1)} &= \frac{\alpha_s}{2\pi}\left[\mathcal{H}^{(1)} + \mathbf{Z}^{(1)}\mathcal{H}^{(0)}\right] \equiv \frac{\alpha_s}{2\pi}A^{(1)}, \\
\mathcal{A}^{(2)} &= \left(\frac{\alpha_s}{2\pi}\right)^2\left[\mathcal{H}^{(2)} + \mathbf{Z}^{(1)}\mathcal{H}^{(1)} + \mathbf{Z}^{(2)}\mathcal{H}^{(0)}\right] \equiv \left(\frac{\alpha_s}{2\pi}\right)^2 A^{(2)}, \qquad (7.12)
\end{aligned}
$$

implying

$$
\begin{aligned}
|\mathcal{A}|^2 =\; & \left|\mathcal{H}^{(0)}\right|^2 + \frac{\alpha_s}{2\pi}\,2\,\mathbf{Re}\left[\left(\mathcal{H}^{(0)}\right)^\dagger\mathcal{H}^{(1)} + \left(\mathcal{H}^{(0)}\right)^\dagger\mathbf{Z}^{(1)}\mathcal{H}^{(0)}\right] \qquad (7.13) \\
& + \left(\frac{\alpha_s}{2\pi}\right)^2\left[2\,\mathbf{Re}\left(\left(\mathcal{H}^{(0)}\right)^\dagger\mathcal{H}^{(2)} + \left(\mathcal{H}^{(0)}\right)^\dagger\mathbf{Z}^{(1)}\mathcal{H}^{(1)} + \left(\mathcal{H}^{(0)}\right)^\dagger\mathbf{Z}^{(2)}\mathcal{H}^{(0)}\right)\right. \\
& \qquad \left. + \left|\mathcal{H}^{(1)}\right|^2 + \left(\mathcal{H}^{(0)}\right)^\dagger\left(\mathbf{Z}^{(1)}\right)^\dagger\mathbf{Z}^{(1)}\mathcal{H}^{(0)} + 2\,\mathbf{Re}\left(\left(\mathcal{H}^{(1)}\right)^\dagger\mathbf{Z}^{(1)}\mathcal{H}^{(0)}\right)\right] + \mathcal{O}(\alpha_s^3).
\end{aligned}
$$

We are interested in the divergent contributions to Eq. (7.13) at $\mathcal{O}(\alpha_s^2)$: we extract them in turn.

First, the direct contribution of the two-loop $\mathbf{Z}$ matrix is given by

$$\begin{aligned}
2\,\mathbf{Re}\Big(\left(\mathcal{H}^{(0)}\right)^{\dagger}\mathbf{Z}^{(2)}\mathcal{H}^{(0)}\Big) \;&=\; \mathcal{H}^{(0)\dagger}(\mathbf{Z}^{(2)}+\mathbf{Z}^{(2)\dagger})\mathcal{H}^{(0)} \\
&= \frac{1}{\epsilon^4}\,\frac{1}{4}\Sigma_C^2\,B+\frac{1}{\epsilon^3}\,\frac{1}{2}\Sigma_C\left[\left(\frac{3}{4}\,\beta_0\;+\;\Sigma_\gamma\right)B+\sum_{i,j\neq i}L_{ij}\,B_{ij}\right] \\
&+\;\frac{1}{\epsilon^2}\,\frac{1}{4}\left[\left(\beta_0\Sigma_\gamma\;-\;\frac{\widehat{\gamma}_K^{(2)}}{4}\Sigma_C+\Sigma_\gamma^2\right)B\;+\;(\beta_0+2\,\Sigma_\gamma)\sum_{i,j\neq i}L_{ij}\,B_{ij}\right. \\
&\qquad\left.+\,\frac{1}{2}\sum_{\substack{i,j\neq i\\k,l\neq k}}\left(L_{ij}\,L_{kl}-\pi^2\right)B_{ijkl}\right] \\
&-\;\frac{1}{\epsilon}\,\frac{1}{8}\left[4\,\Sigma_\gamma^{(2)}\,B\;+\;\widehat{\gamma}_K^{(2)}\sum_{i,j\neq i}L_{ij}\,B_{ij}\right], \qquad (7.14)
\end{aligned}$$

where again $L_{ij}=\ln(s_{ij}/\mu^2)$, and the colour-correlated Born amplitudes B_{ij} and B_{ijkl} are defined in Eq. (A.5). The square of the one-loop $\mathbf{Z}$ matrix contributes

$$\begin{aligned}
\mathcal{H}^{(0)\dagger}\mathbf{Z}^{(1)\dagger}\mathbf{Z}^{(1)}\mathcal{H}^{(0)} \;&=\; \frac{1}{\epsilon^4}\,\frac{1}{4}\,\Sigma_C^2 B+\frac{1}{\epsilon^3}\,\frac{1}{2}\,\Sigma_C\left[\Sigma_\gamma\,B+\sum_{i,j\neq i}L_{ij}\,B_{ij}\right] \\
&+\;\frac{1}{\epsilon^2}\,\frac{1}{4}\left[\Sigma_\gamma^2\,B+2\,\Sigma_\gamma\sum_{i,j\neq i}L_{ij}\,B_{ij}+\frac{1}{2}\sum_{\substack{i,j\neq i\\k,l\neq k}}\left(L_{ij}\,L_{kl}+\pi^2\right)B_{ijkl}\right]. \qquad (7.15)
\end{aligned}$$

Note that in Eq. (7.14) and in Eq. (7.15), for simplicity, we already substituted $\widehat{\gamma}_K^{(1)}=4$. Finally, terms involving the product of the one-loop hard part and the one-loop $\mathbf{Z}$ matrix give

$$\begin{aligned}
2\,\mathbf{Re}\left(\mathcal{H}^{(0)\dagger}\mathbf{Z}^{(1)}\mathcal{H}^{(1)}+\mathcal{H}^{(1)\dagger}\mathbf{Z}^{(1)}\mathcal{H}^{(0)}\right) \;&=\; \mathcal{H}^{(0)\dagger}\left(\mathbf{Z}^{(1)}+\mathbf{Z}^{(1)\dagger}\right)\mathcal{H}^{(1)} \\
&+\mathcal{H}^{(1)\dagger}\left(\mathbf{Z}^{(1)}+\mathbf{Z}^{(1)\dagger}\right)\mathcal{H}^{(0)}\,. \qquad (7.16)
\end{aligned}$$

In order to make use in practice of Eq. (7.16), it is useful to rewrite $\mathcal{H}^{(1)}$ in terms of the full virtual amplitude $A^{(1)}$, using

$$\mathcal{H}^{(1)} \;=\; A^{(1)}-\mathbf{Z}^{(1)}\mathcal{H}^{(0)}\,. \qquad (7.17)$$

Eq. (7.16) then becomes

$$\begin{aligned}
2\,\mathbf{Re}\left(\mathcal{H}^{(0)\dagger}\mathbf{Z}^{(1)}\mathcal{H}^{(1)}+\mathcal{H}^{(1)\dagger}\mathbf{Z}^{(1)}\mathcal{H}^{(0)}\right) \;&=\; \mathcal{H}^{(0)\dagger}\left(\mathbf{Z}^{(1)}+\mathbf{Z}^{(1)\dagger}\right)A^{(1)}+A^{(1)\dagger}\left(\mathbf{Z}^{(1)}+\mathbf{Z}^{(1)\dagger}\right)\mathcal{H}^{(0)} \\
&-\mathcal{H}^{(0)\dagger}\left(\mathbf{Z}^{(1)2}+2\,\mathbf{Z}^{(1)\dagger}\mathbf{Z}^{(1)}+\mathbf{Z}^{(1)\dagger 2}\right)\mathcal{H}^{(0)}\,. \qquad (7.18)
\end{aligned}$$

The term on the second line of Eq. (7.18) is easily computed using Eq. (7.9) and yields

$$\begin{aligned}
-\,\mathcal{H}^{(0)\dagger}\left(\mathbf{Z}^{(1)2}+2\,\mathbf{Z}^{(1)\dagger}\mathbf{Z}^{(1)}+\mathbf{Z}^{(1)\dagger 2}\right)\mathcal{H}^{(0)} \;&=\; -\,\frac{1}{\epsilon^4}\,\Sigma_C^2 B-\frac{1}{\epsilon^3}2\,\Sigma_C\left[\Sigma_\gamma\,B+\sum_{i,j\neq i}L_{ij}\,B_{ij}\right] \\
&-\,\frac{1}{\epsilon^2}\left[\Sigma_\gamma^2 B+2\,\Sigma_\gamma\sum_{i,j\neq i}L_{ij}\,B_{ij}+\frac{1}{2}\sum_{\substack{i,j\neq i\\k,l\neq k}}L_{ij}\,L_{kl}\,B_{ijkl}\right]. \qquad (7.19)
\end{aligned}$$

The first two terms on the *r.h.s.* of Eq. (7.18) can be expressed in terms of the one-loop virtual correction to the cross section. One finds

$$\begin{aligned}
&\frac{\alpha_s}{2\pi}\Big[\mathcal{H}^{(0)\dagger}\Big(\mathbf{Z}^{(1)}+\mathbf{Z}^{(1)\dagger}\Big)A^{(1)}+A^{(1)\dagger}\Big(\mathbf{Z}^{(1)}+\mathbf{Z}^{(1)\dagger}\Big)\mathcal{H}^{(0)}\Big] \\
&\qquad = \mathcal{H}^{(0)\dagger}\Bigg[-\frac{1}{\epsilon^2}\frac{\widehat{\gamma}_K^{(1)}}{4}\Sigma_{\mathcal{C}}-\frac{1}{\epsilon}\Bigg(\frac{\widehat{\gamma}_K^{(1)}}{4}\sum_{i,j\neq i}L_{ij}\,\mathbf{T}_i\cdot\mathbf{T}_j+\Sigma_\gamma\Bigg)\Bigg]\mathcal{A}^{(1)}\;+\;\mathrm{h.c.} \\
&\qquad = -\frac{1}{\epsilon^2}\,\Sigma_{\mathcal{C}}\,V-\frac{1}{\epsilon}\Sigma_\gamma\,V-\frac{1}{\epsilon}\sum_{i,j\neq i}L_{ij}\,V_{ij}\,,
\end{aligned} \tag{7.20}$$

where the colour-correlated virtual correction V_{ij} is defined in Eq. (A.7). Combining Eq. (7.14) with Eq. (7.15) and Eq. (7.20), we get a complete and explicit expression for the pole part of the double-virtual contribution to the cross section,

$$\begin{aligned}
VV\Big|_{\mathrm{poles}} = \left(\frac{\alpha_s}{2\pi}\right)^2\Bigg\{&-\frac{1}{\epsilon^4}\frac{1}{2}\Sigma_{\mathcal{C}}^2\,B+\frac{1}{\epsilon^3}\Sigma_{\mathcal{C}}\Bigg[\left(\frac{3}{8}\beta_0-\Sigma_\gamma\right)B-\sum_{i,j\neq i}L_{ij}\,B_{ij}\Bigg] \\
&+\frac{1}{\epsilon^2}\frac{1}{4}\Bigg[\left(\beta_0\Sigma_\gamma-\frac{\widehat{\gamma}_K^{(2)}}{4}\Sigma_{\mathcal{C}}-2\,\Sigma_\gamma^2\right)B \\
&\qquad+\Big(\beta_0-4\,\Sigma_\gamma\Big)\sum_{i,j\neq i}L_{ij}\,B_{ij}-\sum_{\substack{i,j\neq i\\ k,l\neq k}}L_{ij}\,L_{kl}\,B_{ijkl}\Bigg] \\
&-\frac{1}{\epsilon}\frac{1}{8}\Bigg[4\,\Sigma_\gamma^{(2)}\,B+\widehat{\gamma}_K^{(2)}\sum_{i,j\neq i}L_{ij}\,B_{ij}\Bigg]\Bigg\} \\
&-\frac{\alpha_s}{2\pi}\Bigg[\frac{1}{\epsilon^2}\Sigma_{\mathcal{C}}\,V+\frac{1}{\epsilon}\Sigma_\gamma\,V+\frac{1}{\epsilon}\sum_{i,j\neq i}L_{ij}\,V_{ij}\Bigg].
\end{aligned} \tag{7.21}$$

Eq. (7.21) can now be combined with the integrals of the double-radiative and the real-virtual counterterms to form the subtracted double-virtual contribution to the cross section, VV_{sub}, given in Eq. (7.1).

7.2 Integrated counterterms for double-virtual poles

The expressions for the relevant integrated counterterms, $I^{\,(\mathbf{2})}$ and $I^{(\mathbf{RV})}$, were given in Eq. (4.41) and in Eq. (6.12), respectively. We only need to expand these expressions in powers of ϵ, including terms up to $\mathcal{O}(\epsilon^0)$. We define

$$I^{\,(\mathbf{2})}+I^{(\mathbf{RV})}\;\equiv\;I^{(\mathbf{2+RV})}_{\mathrm{poles}}+I^{(\mathbf{2+RV})}_{\mathrm{fin}}+\mathcal{O}(\epsilon)\,. \tag{7.22}$$

As expected, the pole part $I^{(\mathbf{2+RV})}_{\mathrm{poles}}$ exactly cancels Eq. (7.21):

$$I^{(\mathbf{2+RV})}_{\mathrm{poles}}\;=\;-\,VV\Big|_{\mathrm{poles}}\,. \tag{7.23}$$

We note in particular that *it is not necessary* to compute NLO virtual corrections up to $\mathcal{O}(\epsilon^2)$, since the last term in Eq. (7.21), containing virtual corrections multiplied times explicit poles up to ϵ^{-2}, is exactly reproduced by $I^{(\mathbf{2+RV})}_{\mathrm{poles}}$, so that $\mathcal{O}(\epsilon)$ contributions to NLO corrections never appear in our subtraction formula[11]. This was anticipated in Ref. [98] and emerges clearly in our approach

[11] This understands the technical capability by a two-loop provider to turn off the $\mathcal{O}(\epsilon)$ NLO virtual contribution in the computation of VV. Were this is not the case, the evaluation of $I^{\,(\mathbf{2})}$ as well would have to be performed with such a contribution turned on.

thanks to the factorisation properties of the one-loop amplitude, and the minimal scheme we adopt for the factorisation of virtual corrections. The finite part of the integrated counterterms can be written as ($r = r_j$, $r' = r_{jl}$)

$$
\begin{aligned}
I^{(\mathbf{2+RV})}_{\rm fin} \;=\; \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \bigg[\; I^{(0)} + \sum_j I^{(1)}_j \, L_{jr} + \sum_j I^{(2)}_j \, L^2_{jr} + \frac{1}{2} \sum_{j,l\neq j} \gamma^{\rm hc}_j \, \gamma^{\rm hc}_l \, L_{jr'} L_{lr'} \bigg] B \\
& + \sum_j \Big[I^{(0)}_{jr} + I^{(1)}_{jr} \, L_{jr} \Big] B_{jr} - 2 \left(1 - \zeta_2\right) \sum_{j,c\neq j,r} \gamma^{\rm hc}_j \left(2 - L_{cr}\right) B_{cr} \\
& + \sum_{c,d\neq c} L_{cd} \bigg[I^{(0)}_{cd} + I^{(1)}_{cd} \, L_{cd} + \frac{\beta_0}{12} \, L^2_{cd} - \frac{1}{2} \big(4 - L_{cd}\big) \sum_j \gamma^{\rm hc}_j \, L_{jr} \bigg] B_{cd} \\
& + \sum_{c,d\neq c} \bigg[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2\big(1 - \zeta_3\big) \, L_{cd} \bigg] B_{cdcd} \\
& + \big(1 - \zeta_2\big) \sum_{\substack{c,d\neq c \\ e\neq d}} L_{cd} \, L_{ed} \, B_{cded} + \sum_{\substack{c,d\neq c \\ e,f\neq e}} L_{cd} \, L_{ef} \bigg[1 - \frac{1}{2} \, L_{cd} \bigg(1 - \frac{1}{8} L_{ef}\bigg) \bigg] B_{cdef} \\
& + \pi \sum_{\substack{c,d\neq c \\ e\neq c,d}} \bigg[\ln\frac{s_{ce}}{s_{de}} \, L^2_{cd} + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2\,{\rm Li}_3\bigg(-\frac{s_{ce}}{s_{de}}\bigg) \bigg] B_{cde} \Bigg\} \\
+ \frac{\alpha_s}{2\pi} \bigg[& \bigg(\Sigma_\phi - \sum_j \gamma^{\rm hc}_j \, L_{jr}\bigg) V^{\rm fin} + \sum_{c,d\neq c} L_{cd} \left(2 - \frac{1}{2} L_{cd}\right) V^{\rm fin}_{cd} \bigg] ,
\end{aligned}
\tag{7.24}
$$

where $V^{\rm fin}$ and $V^{\rm fin}_{cd}$ are the $\mathcal{O}(\epsilon^0)$ terms in the virtual and colour-correlated virtual contributions, which are obtained from the full virtual contributions V and V_{cd} by subtracting the IR poles given explicitly by Eq. (7.9). We emphasise that the kinematic dependence of Eq. (7.24) is only through simple logarithms of kinematic invariants, with the single exception of the trilogarithm multiplying the tripole Born-level colour correlation B_{cde} on the one-but-last line of Eq. (7.9). All the integral coefficients appearing in Eq. (7.9) are pure numbers, and they are given by

$$
\begin{aligned}
I^{(0)} \;=\; & N_q^2 C_F^2 \left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4 \right] + N_g N_q C_F \left[C_A \left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4 \right) + \beta_0 \left(\frac{77}{12} - \frac{53}{12}\zeta_2 \right) \right] \\
& + N_g^2 \left[C_A^2 \left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4 \right) + \beta_0^2 \left(\frac{73}{72} - \frac{1}{8}\zeta_2 \right) + C_A \beta_0 \left(-\frac{1}{9} - \frac{11}{3}\zeta_2 \right) \right] \\
& + N_q C_F \left[C_F \left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4 \right) + C_A \left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4 \right) \right. \\
& \qquad \left. + \beta_0 \left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3 \right) \right] \\
& + N_g \left[C_F C_A \left(-\frac{737}{48} + 11\zeta_3 \right) + C_F \beta_0 \left(\frac{67}{16} - 3\zeta_3 \right) + \beta_0^2 \left(\frac{73}{72} - \frac{3}{8}\zeta_2 \right) \right. \\
& \qquad \left. + C_A^2 \left(-\frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4 \right) + C_A \beta_0 \left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3 \right) \right] ,
\end{aligned}
$$

$$
\begin{aligned}
I^{(1)}_j \;=\;& (f^q_j + f^{\bar q}_j)\, C_{\scriptscriptstyle F} \bigg[N_q C_{\scriptscriptstyle F} \bigg(\frac{5}{2} - \frac{7}{4}\zeta_2\bigg) + N_g C_{\scriptscriptstyle A} \bigg(\frac{1}{3} - \frac{7}{4}\zeta_2\bigg) + \frac{2}{3} N_g \beta_0 \\
& \qquad + C_{\scriptscriptstyle F} \bigg(- \frac{3}{8} - 4\zeta_2 + 2\zeta_3\bigg) + C_{\scriptscriptstyle A} \bigg(\frac{25}{12} - 3\zeta_2 + 3\zeta_3\bigg) + \beta_0 \bigg(\frac{1}{24} + \zeta_2\bigg)\bigg] \\
& + f^g_j \bigg[N_q C_{\scriptscriptstyle F} C_{\scriptscriptstyle A} (10 - 7\zeta_2) - N_q C_{\scriptscriptstyle F} \beta_0 \bigg(\frac{5}{2} - \frac{7}{4}\zeta_2\bigg) + N_g C^2_{\scriptscriptstyle A} \bigg(\frac{4}{3} - 7\zeta_2\bigg) + N_g C_{\scriptscriptstyle A} \beta_0 \bigg(\frac{7}{3} + \frac{7}{4}\zeta_2\bigg) \\
& \qquad - \frac{2}{3}(N_g + 1)\beta_0^2 + \frac{11}{4} C_{\scriptscriptstyle F} C_{\scriptscriptstyle A} - \frac{3}{4} C_{\scriptscriptstyle F} \beta_0 + C^2_{\scriptscriptstyle A} \bigg(\frac{28}{3} - \frac{23}{2}\zeta_2 + 5\zeta_3\bigg) - C_{\scriptscriptstyle A} \beta_0 \bigg(\frac{2}{3} - \frac{5}{2}\zeta_2\bigg)\bigg] , \\
I^{(2)}_j \;=\;& \frac{1}{8}\big(15\, C_A - 7\,\beta_0\big) C_{f_j} - \frac{1}{4}\big(5\, C_A - 2\,\beta_0\big)\gamma_j + \frac{1}{8}\big(16\,\zeta_2 - 15\big) C^2_{f_j} , \\
I^{(0)}_{jr} \;=\;& \big(- 1 + 3\zeta_2 - 2\zeta_3\big) C_A - \frac{1}{2}\big(13 + 10\zeta_2 + 2\zeta_3\big) C_{f_j} + \big(5 + 2\zeta_3\big)\gamma_j , \\
I^{(1)}_{jr} \;=\;& \big(1 - \zeta_2\big) C_A + \frac{1}{2}\big(4 + 7\zeta_2\big) C_{f_j} - \big(2 + \zeta_2\big)\gamma_j , \\
I^{(0)}_{cd} \;=\;& \bigg(\frac{20}{9} - 2\zeta_2 - \frac{7}{2}\zeta_3\bigg) C_A + \frac{31}{9}\,\beta_0 + 2\,\Sigma_\phi + 8\,\big(1 - \zeta_2\big)\, C_{f_d} , \\
I^{(1)}_{cd} \;=\;& - \bigg(\frac{1}{3} - \frac{1}{2}\zeta_2\bigg) C_A - \frac{11}{12}\,\beta_0 - \frac{1}{2}\,\Sigma_\phi \, .
\end{aligned}
\tag{7.25}
$$

We stress that, as expected, the pole part $I^{(\mathbf{2+RV})}_{\rm poles}$ does not depend on reference momenta r, r'; conversely, the dependence on r, r' arising in the finite part $I^{(\mathbf{2+RV})}_{\rm fin}$ is necessary to cancel the one explicit in the counterterms $K^{\,(\mathbf{2})}$ and $K^{(\mathbf{RV})}$.

8 Status and perspective

We have presented a complete analytic solution to the NNLO subtraction problem for general massless coloured final states, within the framework of Local Analytic Sector Subtraction, which can be implemented in conjunction with any numerical code providing the appropriate one- and two-loop matrix elements, and an efficient phase-space integrator.

The main ingredients for our construction are the following. Beginning with the double-radiative contribution, we introduce a smooth partition of the radiative phase space into *sectors*, each containing a minimal number of soft and collinear singularities, following the basic logic of Ref. [50]. Next, we list all *uniform* soft and collinear limits, with up to two particles becoming unresolved, that contribute to each sector. Denoting these limits by $\ell_{{\rm sect},\,i}$, we then follow the strategy of Ref. [64], and construct combinations of the form $\prod_i (1 - \ell_{{\rm sect},\,i})$, which are guaranteed to be integrable in the relevant phase spaces, and for which double-counted nested limits have been properly subtracted. Crucially, in all cases we define *commuting* limits, which significantly simplify subsequent manipulations. Exploiting the soft- and collinear-factorisation properties of matrix elements, all relevant limits can be expressed as products of known splitting kernels times lower-multiplicity matrix elements. In order to properly exploit this factorised structure for matrix elements, we then introduce a flexible set of phase-space *mappings*, which lead to the complete factorisation of the phase-space integration, separating the on-shell Born-level configuration from radiative factors. Using these mappings, we construct *improved limits* $\bar\ell_{{\rm sect},\,i}$, which are highly optimised, with different choices of mappings for different limits, different sectors and different terms in each sector; furthermore, the action of the improved limits on sector functions is tuned, when needed. Importantly, these optimisations must pass stringent *consistency conditions*, ensuring that nested improved limits with different mappings remain aligned with the underlying physical soft and collinear limits. An analogous procedure is followed for the real-virtual contribution, where the radiative phase-space structure is much simpler, but splitting kernels (and thus improved limits) are more intricate.

This lengthy optimisation work pays off when the local counterterms, thus obtained, are integrated over the radiative degrees of freedom. All counterterms can be analytically integrated, and all singular contributions to the integrated counterterms are given by single-scale integrals, with trivial logarithmic dependence on Born-level kinematic invariants. When integrated counterterms are properly combined with the singular part of the double-virtual contribution to the cross section, all poles are analytically shown to cancel. All finite contributions can also be obtained analytically, and they are of similar simplicity, with just a single contribution (proportional to a colour tripole) displaying a weight-three polylogarithm depending on two physical scales. In a sense, the existing tension between the remarkable simplicity of double-virtual singularities and the increasing intricacy of real-virtual and double-real radiative contributions is resolved by a judicious choice of sectors and mappings. Indeed, the simplicity of the integrals associated with both double-real and real-virtual counterterms kindles hopes that a generalisation to N^{3}LO subtraction with the same degree of generality might be achievable. On the other hand, our approach is undeniably costly from the combinatorial viewpoint, and requires a fast-growing number of consistency checks, which would be challenging to tackle at higher orders.

The future developments of our work are clearly outlined. First of all, the formalism must be numerically implemented and tested for efficiency. This work is under way, and was completed at NLO in Ref. [89]. In that paper, the NLO formalism was also extended to initial-state coloured particles, without significantly raising the technical difficulties. Obviously, the inclusion of non-trivial initial states is a high-priority goal at NNLO as well, in view of LHC applications. Also in this case, this generalisation is not expected to involve new major technical obstacles: as observed at NLO, new classes of mappings are needed, and collinear factorisation must be consistently implemented, but all of these developments are expected to be comparatively straightforward. Importantly, new phase-space integrals are expected to be of the same level of complexity as those presented here, so that a completely analytic result is expected to be within reach. Work is in progress also on this front. In the longer run, an important further ingredient to achieve complete generality for NNLO subtraction is the inclusion of massive particles in the final state. This task is going to be simplified by the fact that the number and type of singular limits associated with massive coloured particles are limited, since collinear limits for real radiation are non-singular in this case. Since our approach is combinatorially intensive, this is expected to be a significant advantage. On the other hand, massive particles will require adjustments in phase-space mappings, and will likely involve new classes of integrals, with a more intricate scale dependence. We are, nonetheless, confident that a complete analytic expression can be derived also in that case.

Finally, we believe that, notwithstanding the simplicity of our analytic results, there is further room for optimisation, which would be very important in view of a future generalisation of our approach to N^{3}LO. We note for example that the minimal interference between soft and collinear singularities which is suggested by factorisation at amplitude level still emerges in our formalism as an output rather than being introduced from the outset. We hope that a more detailed understanding of the factorisation structure for real radiation, in particular for strongly-ordered limits, along the lines of Refs. [84, 91] will provide further insights in this direction. Simplifications in the structure of nested infrared limits would likely improve significantly the combinatorial challenges of our approach, and open the way to higher orders.

In summary, we believe that our results bring the goal of establishing a completely general, local, analytic and efficient NNLO-subtraction formalism one step closer.

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A General notation

We denote by s the squared centre-of-mass energy and by $q^\mu = (\sqrt{s}, \vec{0}\,)$ the centre-of-mass four-momentum. Given two final-state momenta k_i^μ and k_j^μ, we define

$$\begin{aligned} s_{qi} &= 2\,q\cdot k_i\,, \qquad s_{ij} \;=\; 2\,k_i\cdot k_j\,, \qquad L_{ij} \;=\; \ln\frac{s_{ij}}{\mu^2}\,, \\ e_i &= \frac{s_{qi}}{s}\,, \qquad\quad w_{ij} \;=\; \frac{s\,s_{ij}}{s_{qi}\,s_{qj}}\,. \end{aligned} \tag{A.1}$$

In addition, given four final-state momenta k_a^μ, k_b^μ, k_c^μ and k_d^μ, we define

$$\begin{aligned} s_{abc} &= s_{ab}+s_{bc}+s_{ac}\,, \qquad s_{[ab]c} \;=\; s_{ac}+s_{bc}\,, \qquad k_{[ab]}^\mu \;=\; k_a^\mu+k_b^\mu\,, \\ s_{abcd} &= s_{ab}+s_{ac}+s_{ad}+s_{bc}+s_{bd}+s_{cd}\,, \qquad s_{[abc]d} \;=\; s_{ad}+s_{bd}+s_{cd}\,. \end{aligned} \tag{A.2}$$

For the sake of compactness, we define the following flavour structures:

$$f_i^q = \begin{cases} 1 & \text{if } i \text{ is a quark} \\ 0 & \text{if } i \text{ is not a quark} \end{cases} \qquad f_i^{\bar q} = \begin{cases} 1 & \text{if } i \text{ is an antiquark} \\ 0 & \text{if } i \text{ is not an antiquark} \end{cases} \qquad f_i^g = \begin{cases} 1 & \text{if } i \text{ is a gluon} \\ 0 & \text{if } i \text{ is not a gluon} \end{cases}$$

$$f_{ij}^{q\bar q} = f_i^q f_j^{\bar q} + f_i^{\bar q} f_j^q\,, \qquad f_{ij}^{gg} = f_i^g f_j^g\,, \qquad f_{ijk}^{ggg} = f_i^g f_j^g f_k^g\,, \qquad \tilde f_{ij}^{q\bar q} = f_i^q f_j^{\bar q} - f_i^{\bar q} f_j^q\,, \tag{A.3}$$

which are special cases of the general rule

$$f_{i_1\ldots i_n}^{f_1\ldots f_n} = \sum_{\substack{g_1,\ldots,g_n= \\ P(f_1,\ldots,f_n)}} f_{i_1}^{g_1}\cdots f_{i_n}^{g_n}\,, \qquad \tilde f_{i_1\ldots i_n}^{f_1\ldots f_n} = \sum_{\substack{g_1,\ldots,g_n= \\ P(f_1,\ldots,f_n)}} \mathrm{sign}(P)\, f_{i_1}^{g_1}\cdots f_{i_n}^{g_n}\,, \tag{A.4}$$

where $P(f_1,\ldots,f_n)$ is a generic permutation of indices $f_1,\ldots,f_n$.
We introduce a compact notation for Born-level colour correlations:

$$B_{cd} \equiv \mathcal{A}_n^{(0)\dagger}\,\mathbf{T}_c\cdot\mathbf{T}_d\,\mathcal{A}_n^{(0)}\,, \qquad B_{cdef} \equiv \mathcal{A}_n^{(0)\dagger}\,\{\mathbf{T}_c\cdot\mathbf{T}_d, \mathbf{T}_e\cdot\mathbf{T}_f\}\,\mathcal{A}_n^{(0)}\,, \tag{A.5}$$

$$\mathcal{B}_{cd} \equiv f_c^g\,\mathcal{A}_n^{(0)\dagger}\,\mathcal{T}_c\cdot\mathbf{T}_d\,\mathcal{A}_n^{(0)}\,, \qquad (\mathcal{T}_A)_{BC} = d_{ABC}\,. \tag{A.6}$$

Analogously, the colour-correlated real and virtual matrix elements are defined as

$$V_{cd} \equiv 2\,\mathbf{Re}\Big[\mathcal{A}_n^{(1)\dagger}\,\mathbf{T}_c\cdot\mathbf{T}_d\,\mathcal{A}_n^{(0)}\Big]\,, \qquad R_{cd} \equiv \mathcal{A}_{n+1}^{(0)\dagger}\,\mathbf{T}_c\cdot\mathbf{T}_d\,\mathcal{A}_{n+1}^{(0)}\,, \tag{A.7}$$

which are of relative order α_s with respect to the corresponding Born-level terms.
We define the following combinations of Casimir operators,

$$\rho_{ab}^{(C)} = \frac{C_{f_{[ab]}}+C_{f_a}-C_{f_b}}{C_{f_{[ab]}}}\,, \qquad \rho_{[ab]}^{(C)} = \frac{C_{f_{[ab]}}-C_{f_a}-C_{f_b}}{C_{f_{[ab]}}}\,, \qquad \Sigma_C = \sum_a C_{f_a}\,, \tag{A.8}$$

and

$$\gamma_a \;=\; \frac{3}{2}\,C_F\,(f_a^q+f_a^{\bar q})+\frac{1}{2}\,\beta_0\,f_a^g\,, \qquad \Sigma_\gamma \;=\; \sum_a \gamma_a\,, \qquad \gamma_a^{\rm hc} \;=\; \gamma_a - 2C_{f_a}\,, \tag{A.9}$$

$$\phi_a \;=\; \frac{13}{3}\,C_F\,(f_a^q+f_a^{\bar q})+\frac{4}{3}\,\beta_0\,f_a^g+\left(\frac{2}{3}-\frac{7}{2}\zeta_2\right)C_{f_a}\,, \qquad \Sigma_\phi \;=\; \sum_a \phi_a\,, \tag{A.10}$$

$$\phi_a^{\rm hc} \;=\; \frac{13}{3}\,C_F\,(f_a^q+f_a^{\bar q})+\frac{4}{3}\,\beta_0\,f_a^g-\frac{16}{3}\,C_{f_a}\,, \qquad \Sigma_\phi^{\rm hc} \;=\; \sum_a \phi_a^{\rm hc}\,, \tag{A.11}$$

where the sums run over all final-state QCD partons and

$$\beta_0 = \frac{11C_A - 4\,T_R\,N_f}{3}\,. \tag{A.12}$$

The two-loop anomalous dimensions are given by

$$\begin{aligned}
\widehat{\gamma}_K^{(2)} \;&=\; 4\left\{\left(\frac{67}{18}-\zeta_2\right)C_A - \frac{10}{9}T_R\,N_f\right\} \;=\; \left(\frac{8}{3}-4\zeta_2\right)C_A+\frac{10}{3}\,\beta_0\,,\\
\gamma_i^{(2)} \;&=\; (f_i^q+f_i^{\bar q})\,C_F\left[3\left(\frac{1}{8}-\zeta_2+2\zeta_3\right)C_F+\left(\frac{41}{36}-\frac{13}{2}\zeta_3\right)C_A+\left(\frac{65}{72}+\frac{3}{4}\zeta_2\right)\beta_0\right]\\
&\quad + f_i^g\left\{C_A\left[-\frac{11}{4}\,C_F+\left(-\frac{1}{9}-\frac{1}{2}\zeta_3\right)C_A\right]+\beta_0\left[\frac{3}{4}\,C_F+\left(\frac{16}{9}-\frac{1}{4}\zeta_2\right)C_A\right]\right\}.
\end{aligned} \tag{A.13}$$

As for the labelling of particles we introduce the notation

$$r_{i_1\ldots i_n} = R_n(i_1,\ldots,i_n) \neq i_1,\ldots,i_n\,, \tag{A.14}$$

to indicate a generic particle label different from $i_1,\ldots,i_n$, defined following a specific rule R_n. Such a rule is arbitrary to some extent, and could for instance assign $r_{i_1\ldots i_n}$ as the smallest label different from all $i_1,\ldots,i_n$, or the largest, and so on. A crucial feature, however, is that R_n must be symmetric under permutations of all indices $i_1,\ldots,i_n$, and must be the same for all $r_{i_1\ldots i_n}$ with the same n. As a consequence, the notation $r_{i_1\ldots i_n}$ always refers to the rule $R_n(i_1,\ldots,i_n)$, which is a symmetric function of its indices $i_1,\ldots,i_n$, and just depends on n.

B Infrared kernels

B.1 Soft kernels at tree level

We introduce the kernels associated with the real emission of one or two soft partons, as given in Ref. [92], relevant for both NLO (with the emission of just one parton) and NNLO corrections (with the emission of either one or two partons). We express all kernels in terms of Lorentz-invariant quantities, and using the flavour structures defined in Appendix A. The resulting expressions are

$$\mathcal{I}_{cd}^{(i)} \;=\; f_i^g\,\frac{s_{cd}}{s_{ic}\,s_{id}}\,, \qquad \mathcal{I}_{cd}^{(ij)} \;=\; f_{ij}^{q\bar q}\,2\,T_R\,\mathcal{I}_{cd}^{(q\bar q)(ij)} - f_{ij}^{gg}\,2\,C_A\,\mathcal{I}_{cd}^{(gg)(ij)}\,, \tag{B.1}$$

where

$$\mathcal{I}_{cd}^{(q\bar q)(ij)} \;=\; \frac{s_{ic}s_{jd}+s_{id}s_{jc}-s_{ij}s_{cd}}{s_{ij}^2\,s_{[ij]c}\,s_{[ij]d}}\,, \tag{B.2}$$

$$\mathcal{I}_{cd}^{(gg)(ij)} \;=\; \frac{(1-\epsilon)(s_{ic}s_{jd}+s_{id}s_{jc})-2s_{ij}s_{cd}}{s_{ij}^2\,s_{[ij]c}\,s_{[ij]d}} + s_{cd}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}-s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}}\left[1-\frac{1}{2}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}}{s_{[ij]c}\,s_{[ij]d}}\right].$$

We also define the combinations of eikonal kernels

$$\begin{aligned}
\mathcal{E}_{cd}^{(i)} &\equiv \mathcal{I}_{cd}^{(i)} \;=\; f_i^g \,\frac{s_{cd}}{s_{ic}\,s_{id}}\,, \\
\mathcal{E}_{cd}^{(ij)} &\equiv \mathcal{I}_{cd}^{(ij)} - \frac{1}{2}\mathcal{I}_{cc}^{(ij)} - \frac{1}{2}\mathcal{I}_{dd}^{(ij)} \;=\; f_{ij}^{q\bar{q}}\, 2\,T_R\,\mathcal{E}_{cd}^{(q\bar{q})(ij)} - f_{ij}^{gg}\, 2\,C_A\,\mathcal{E}_{cd}^{(gg)(ij)}\,,
\end{aligned} \tag{B.3}$$

with

$$\begin{aligned}
\mathcal{E}_{cd}^{(q\bar{q})(ij)} &= \frac{1}{s_{ij}^2}\left[\frac{s_{ic}s_{jd}+s_{id}s_{jc}}{s_{[ij]c}s_{[ij]d}} - \frac{s_{ic}s_{jc}}{s_{[ij]c}^2} - \frac{s_{id}s_{jd}}{s_{[ij]d}^2}\right] - \frac{s_{cd}}{s_{ij}s_{[ij]c}s_{[ij]d}}\,, \\
\mathcal{E}_{cd}^{(gg)(ij)} &= \frac{1-\epsilon}{s_{ij}^2}\left[\frac{s_{ic}s_{jd}+s_{id}s_{jc}}{s_{[ij]c}s_{[ij]d}} - \frac{s_{ic}s_{jc}}{s_{[ij]c}^2} - \frac{s_{id}s_{jd}}{s_{[ij]d}^2}\right] - 2\,\frac{s_{cd}}{s_{ij}s_{[ij]c}s_{[ij]d}} \\
&\quad + s_{cd}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}-s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}}\left[1-\frac{1}{2}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}}{s_{[ij]c}s_{[ij]d}}\right].
\end{aligned} \tag{B.4}$$

B.2 Soft kernels at one loop

We introduce kernels associated to the emission of a single-soft gluon at one-loop level, relevant for the soft part of the real-virtual counterterm at NNLO,

$$\begin{aligned}
\tilde{\mathcal{E}}_{cd}^{(i)} &\equiv f_i^g\,C_A\,\frac{\Gamma^3(1+\epsilon)\Gamma^4(1-\epsilon)}{\epsilon^2\,\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}\,\frac{s_{cd}}{s_{ic}s_{id}}\left(\frac{e^{\gamma_E}\mu^2 s_{cd}}{s_{ic}s_{id}}\right)^{\epsilon} \\
&= C_A\,\mathcal{E}_{cd}^{(i)}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{s_{ic}s_{id}}{\mu^2\,s_{cd}} - \frac{5}{2}\zeta_2 + \frac{1}{2}\ln^2\frac{s_{ic}s_{id}}{\mu^2\,s_{cd}} + \mathcal{O}(\epsilon)\right], \\
\tilde{\mathcal{E}}_{cde}^{(i)} &\equiv f_i^g\,\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)}\,\frac{s_{cd}}{s_{ic}s_{id}}\left(\frac{e^{\gamma_E}\mu^2\,s_{de}}{s_{id}s_{ie}}\right)^{\epsilon} \\
&= \mathcal{E}_{cd}^{(i)}\left[\frac{1}{\epsilon} - \ln\frac{s_{id}s_{ie}}{\mu^2\,s_{de}} + \mathcal{O}(\epsilon)\right],
\end{aligned} \tag{B.5}$$

where ϵ is the dimensional regulator ($d = 4 - 2\epsilon$).

B.3 Collinear and hard-collinear kernels at tree level

In order to define the kernel associated to the tree-level emission of two collinear final-state particles i and j (labelled *single-collinear*), we choose a reference momentum k_r, with $r \neq i, j$, and introduce the following kinematic structures:

$$x_i \;=\; \frac{s_{ir}}{s_{[ij]r}}\,, \qquad x_j \;=\; \frac{s_{jr}}{s_{[ij]r}}\,, \qquad \tilde{k}_i \;=\; x_i\,k_j - x_j\,k_i - (1-2x_j)\frac{s_{ij}}{s_{[ij]r}}\,k_r\,. \tag{B.6}$$

Then, the collinear (Altarelli-Parisi) kernels $P_{ij(r)}^{\mu\nu}$ are defined as

$$P_{ij(r)}^{\mu\nu} \;=\; -\,P_{ij(r)}\,g^{\mu\nu} + Q_{ij(r)}^{\mu\nu}\,, \qquad Q_{ij(r)}^{\mu\nu} \;=\; Q_{ij(r)}\,d_i^{\mu\nu}\,, \tag{B.7}$$

where the azimuthal tensor reads

$$d_i^{\mu\nu} \;=\; -g^{\mu\nu} + (d-2)\,\frac{\tilde{k}_i^\mu\tilde{k}_i^\nu}{\tilde{k}_i^2}\,, \tag{B.8}$$

and

$$\begin{aligned}
P_{ij(r)} &= P_{ij(r)}^{(0\mathrm{g})}\,f_{ij}^{q\bar{q}} + P_{ij(r)}^{(1\mathrm{g})}\,f_i^g(f_j^q + f_j^{\bar{q}}) + P_{ji(r)}^{(1\mathrm{g})}\,(f_i^q + f_i^{\bar{q}})f_j^g + P_{ij(r)}^{(2\mathrm{g})}\,f_{ij}^{gg}\,, \\
Q_{ij(r)} &= T_R\,\frac{2x_ix_j}{1-\epsilon}\,f_{ij}^{q\bar{q}} - 2\,C_A\,x_ix_j\,f_{ij}^{gg}\,, \\
P_{ij(r)}^{(0\mathrm{g})} &= T_R\left(1 - \frac{2x_ix_j}{1-\epsilon}\right), \quad P_{ij(r)}^{(1\mathrm{g})} = C_F\left[2\,\frac{x_j}{x_i} + (1-\epsilon)x_i\right], \quad P_{ij(r)}^{(2\mathrm{g})} = 2\,C_A\left(\frac{x_i}{x_j} + \frac{x_j}{x_i} + x_ix_j\right).
\end{aligned} \tag{B.9}$$

The hard-collinear kernels $P^{\rm hc,\mu\nu}_{ij(r)}$ are defined as

$$P^{\rm hc,\mu\nu}_{ij(r)} \;\equiv\; P^{\mu\nu}_{ij(r)} + s_{ij}\Big[2\,C_{f_j}\,\mathcal{E}^{(i)}_{jr} + 2\,C_{f_i}\,\mathcal{E}^{(j)}_{ir}\Big]g^{\mu\nu} \;\equiv\; -\,P^{\rm hc}_{ij(r)}\,g^{\mu\nu} + Q^{\mu\nu}_{ij(r)}\,, \tag{B.10}$$

where

$$\begin{aligned}
P^{\rm hc}_{ij(r)} &= P^{\rm hc,(0g)}_{ij(r)}\,f^{q\bar q}_{ij} + P^{\rm hc,(1g)}_{ij(r)}\,f^{g}_{i}(f^{q}_{j}+f^{\bar q}_{j}) + P^{\rm hc,(1g)}_{ji(r)}\,(f^{q}_{i}+f^{\bar q}_{i})f^{g}_{j} + P^{\rm hc,(2g)}_{ij(r)}\,f^{gg}_{ij}\,, \qquad (B.11)\\
P^{\rm hc,(0g)}_{ij(r)} &= P^{(0g)}_{ij(r)} = T_R\bigg(1-\frac{2x_ix_j}{1-\epsilon}\bigg), \quad P^{\rm hc,(1g)}_{ij(r)} = C_F(1-\epsilon)x_i\,, \quad P^{\rm hc,(2g)}_{ij(r)} = 2\,C_A\,x_ix_j\,.
\end{aligned}$$

The kernel associated to the emission of three collinear final-state partons i, j and k (labelled *double-collinear*) relies on the choice of a reference momentum k_r, with $r \neq i,j,k$, and on the following kinematic structures,

$$\begin{aligned}
z_a &= \frac{s_{ar}}{s_{[ijk]r}}\,, \qquad\qquad z_{ab} \;=\; z_a+z_b\,, \qquad\qquad a,b \;=\; i,j,k \qquad (B.12)\\
\tilde k^\mu_a &= k^\mu_a - z_a(k^\mu_i+k^\mu_j+k^\mu_k) - (s_{[ijk]a}-2\,z_a s^2_{ijk})\frac{k^\mu_r}{s_{[ijk]r}}\,, \qquad\qquad a,b,c=i,j,k\,,\\
\tilde k^2_a &= z_a(z_a s^2_{ijk} - s_{[ijk]a}) \;=\; z_a(s_{bc}-z_{bc}s_{ijk})\,.
\end{aligned}$$

The double-collinear kernels $P^{\mu\nu}_{ijk(r)}$ are defined as

$$P^{\mu\nu}_{ijk(r)} \equiv -P_{ijk(r)}\,g^{\mu\nu} + Q^{\mu\nu}_{ijk(r)}\,, \qquad Q^{\mu\nu}_{ijk(r)} \;=\; \sum_{a=i,j,k} Q^{a}_{ijk(r)}\,d^{\mu\nu}_a\,. \tag{B.13}$$

The $P_{ijk(r)}$ kernels, organised by flavour structures, are given by

$$\begin{aligned}
P_{ijk(r)} \;=\;& P^{(0g)}_{ijk(r)}\,f^{q\bar q}_{ij}\,(f^{q'}_{k}+f^{\bar q'}_{k}) + P^{(0g)}_{jki(r)}\,f^{q\bar q}_{jk}\,(f^{q'}_{i}+f^{\bar q'}_{i}) + P^{(0g)}_{kij(r)}\,f^{q\bar q}_{ik}\,(f^{q'}_{j}+f^{\bar q'}_{j})\\
+\;& P^{(0g,id)}_{ijk(r)}(f^{q}_{i}f^{q}_{j}f^{\bar q}_{k}+f^{\bar q}_{i}f^{\bar q}_{j}f^{q}_{k}) + P^{(0g,id)}_{jki(r)}(f^{q}_{j}f^{q}_{k}f^{\bar q}_{i}+f^{\bar q}_{j}f^{\bar q}_{k}f^{q}_{i}) + P^{(0g,id)}_{kij(r)}(f^{q}_{i}f^{q}_{k}f^{\bar q}_{j}+f^{\bar q}_{i}f^{\bar q}_{k}f^{q}_{j})\\
+\;& P^{(1g)}_{ijk(r)}\,f^{q\bar q}_{ij}\,f^{g}_{k} + P^{(1g)}_{jki(r)}\,f^{q\bar q}_{jk}\,f^{g}_{i} + P^{(1g)}_{kij(r)}\,f^{q\bar q}_{ik}\,f^{g}_{j}\\
+\;& P^{(2g)}_{ijk(r)}\,f^{gg}_{ij}\,(f^{q}_{k}+f^{\bar q}_{k}) + P^{(2g)}_{jki(r)}\,f^{gg}_{jk}\,(f^{q}_{i}+f^{\bar q}_{i}) + P^{(2g)}_{kij(r)}\,f^{gg}_{ik}\,(f^{q}_{j}+f^{\bar q}_{j})\\
+\;& P^{(3g)}_{ijk(r)}\,f^{ggg}_{ijk}\,,
\end{aligned} \tag{B.14}$$

where q' is a quark of flavour equal to or different from that of q; similarly, the azimuthal tensor kernel can be written as

$$Q^{a}_{ijk(r)} \;=\; Q^{(1g),a}_{ijk(r)}\,f^{q\bar q}_{ij}\,f^{g}_{k} + Q^{(1g),a}_{jki(r)}\,f^{q\bar q}_{jk}\,f^{g}_{i} + Q^{(1g),a}_{kij(r)}\,f^{q\bar q}_{ik}\,f^{g}_{j} + Q^{(3g),a}_{ijk(r)}\,f^{ggg}_{ijk}\,. \tag{B.15}$$

The expressions for $P^{(0g)}_{ijk(r)}$, $P^{(0g,id)}_{ijk(r)}$, $P^{(1g)}_{ijk(r)}$, $P^{(2g)}_{ijk(r)}$, and $P^{(3g)}_{ijk(r)}$ read:

$$P^{(0g)}_{ijk(r)} \;=\; C_FT_R\left\{-\frac{s^2_{ijk}}{2s^2_{ij}}\left(\frac{s_{jk}}{s_{ijk}}-\frac{s_{ik}}{s_{ijk}}+\frac{z_i-z_j}{z_{ij}}\right)^2+\frac{s_{ijk}}{s_{ij}}\left[2\,\frac{z_k-z_iz_j}{z_{ij}}+(1-\epsilon)z_{ij}\right]-\frac12+\epsilon\right\}, \tag{B.16}$$

$$\begin{aligned}
P^{(0g,id)}_{ijk(r)} \;=\;& C_F(2C_F-C_A)\Bigg\{-\frac{s^2_{ijk}\,z_k}{2s_{jk}s_{ik}}\left[\frac{1+z_k^2}{z_{jk}z_{ik}}-\epsilon\left(\frac{z_{ik}}{z_{jk}}+\frac{z_{jk}}{z_{ik}}+1+\epsilon\right)\right]+(1-\epsilon)\left[\frac{s_{ij}}{s_{jk}}+\frac{s_{ij}}{s_{ik}}-\epsilon\right]\\
&+\frac{s_{ijk}}{2s_{jk}}\left[\frac{1+z_k^2-\epsilon z_{jk}^2}{z_{ik}}-2(1-\epsilon)\frac{z_j}{z_{jk}}-\epsilon(1+z_k)-\epsilon^2\,z_{jk}\right]\\
&+\frac{s_{ijk}}{2s_{ik}}\left[\frac{1+z_k^2-\epsilon z_{ik}^2}{z_{jk}}-2(1-\epsilon)\frac{z_i}{z_{ik}}-\epsilon(1+z_k)-\epsilon^2\,z_{ik}\right]\Bigg\},
\end{aligned} \tag{B.17}$$

$$
\begin{aligned}
P^{(1\mathrm{g})}_{ijk(r)} \;=\; & C_F T_R \left\{ \frac{2s^2_{ijk}}{s_{ik}s_{jk}} \left[1 + z_k^2 - \frac{z_k + 2z_i z_j}{1-\epsilon} \right] - (1-\epsilon)\left[\frac{s_{ij}}{s_{jk}} + \frac{s_{ij}}{s_{ik}} \right] - 2 \right. \\
& \left. - \frac{s_{ijk}}{s_{jk}} \left[1 + 2z_k + \epsilon - \frac{2z_{jk}}{1-\epsilon} \right] - \frac{s_{ijk}}{s_{ik}} \left[1 + 2z_k + \epsilon - \frac{2z_{ik}}{1-\epsilon} \right] \right\} \\
& + C_A T_R \left\{ - \frac{s^2_{ijk}}{2s^2_{ij}} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 - \frac{s^2_{ijk}}{s_{ik}s_{jk}} \left[1 + z_k^2 - \frac{z_k + 2z_i z_j}{1-\epsilon} \right] \right. \\
& + \frac{s^2_{ijk}}{2s_{ij}s_{ik}} \, \frac{z_i}{z_k z_{ij}} \left[z_{ij}^3 - z_k^3 - \frac{2z_i(z_{jk} - 2z_j z_k)}{1-\epsilon} \right] \\
& + \frac{s^2_{ijk}}{2s_{ij}s_{jk}} \, \frac{z_j}{z_k z_{ij}} \left[z_{ij}^3 - z_k^3 - \frac{2z_j(z_{ik} - 2z_i z_k)}{1-\epsilon} \right] \\
& + \frac{s_{ijk}}{2s_{ik}} \, \frac{z_{ik}}{z_k z_{ij}} \left[1 + z_k z_{ij} - \frac{2z_j z_{ik}}{1-\epsilon} \right] + \frac{s_{ijk}}{2s_{jk}} \, \frac{z_{jk}}{z_k z_{ij}} \left[1 + z_k z_{ij} - \frac{2z_i z_{jk}}{1-\epsilon} \right] \\
& \left. + \frac{s_{ijk}}{s_{ij}} \, \frac{1}{z_k z_{ij}} \left[1 + z_k^3 + \frac{z_k(z_i - z_j)^2 - 2z_i z_j(1+z_k)}{1-\epsilon} \right] - \frac{1}{2} + \epsilon \right\} ,
\end{aligned}
\tag{B.18}
$$

$$
\begin{aligned}
P^{(2\mathrm{g})}_{ijk(r)} \;=\; & C_F^2 \left\{ \frac{s^2_{ijk}\, z_k}{2s_{ik}s_{jk}} \left[\frac{1 + z_k^2 - \epsilon z_{ij}^2}{z_i z_j} + \epsilon(1-\epsilon) \right] - (1-\epsilon)^2 \, \frac{s_{jk}}{s_{ik}} + \epsilon(1-\epsilon) \right. \\
& \left. + \frac{s_{ijk}}{s_{ik}} \left[\frac{z_k z_{jk} + z_{ik}^3 - \epsilon z_{ik} z_{ij}^2}{z_i z_j} + \epsilon\, z_{ik} + \epsilon^2\,(1 + z_k) \right] \right\} \\
& + C_F C_A \left\{ (1-\epsilon) \frac{s^2_{ijk}}{4s^2_{ij}} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 - \frac{s^2_{ijk}\, z_k}{4s_{ik}s_{jk}} \left[\frac{z_{ij}^2(1-\epsilon) + 2z_k}{z_i z_j} + \epsilon(1-\epsilon) \right] \right. \\
& + \frac{s^2_{ijk}}{2s_{ij}s_{ik}} \left[\frac{z_{ij}^2(1-\epsilon) + 2z_k}{z_j} + \frac{z_j^2(1-\epsilon) + 2z_{ik}}{z_{ij}} \right] + \frac{1}{4}(1-\epsilon)(1-2\epsilon) \\
& + \frac{s_{ijk}}{2s_{ik}} \left[(1-\epsilon) \frac{z_{ik}^3 + z_k^2 - z_j}{z_j z_{ij}} - 2\epsilon\, \frac{z_{ik}(z_j - z_k)}{z_j z_{ij}} \right. \\
& \qquad \left. - \frac{z_k z_{jk} + z_{ik}^3}{z_i z_j} + \epsilon\, z_{ik} \frac{z_{ij}^2}{z_i z_j} - \epsilon(1 + z_k) - \epsilon^2 z_{ik} \right] \\
& \left. + \frac{s_{ijk}}{2s_{ij}} \left[(1-\epsilon) \frac{z_i(2z_{jk} + z_i^2) - z_j(6z_{ik} + z_j^2)}{z_j z_{ij}} + 2\epsilon\, \frac{z_k(z_i - 2z_j) - z_j}{z_j z_{ij}} \right] \right\} \\
& + (i \leftrightarrow j) ,
\end{aligned}
\tag{B.19}
$$

$$
\begin{aligned}
P^{(3\mathrm{g})}_{ijk(r)} \;=\; & C_A^2 \left\{ (1-\epsilon) \frac{s^2_{ijk}}{4s^2_{ij}} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{3}{4}(1-\epsilon) \right. \\
& + \frac{s^2_{ijk}}{2s_{ij}s_{ik}} \left[\frac{2z_i z_j z_{ik}(1-2z_k)}{z_k z_{ij}} + \frac{1+2z_i+2z_i^2}{z_{ik} z_{ij}} + \frac{1-2z_i z_{jk}}{z_j z_k} + 2z_j z_k + z_i(1+2z_i) - 4 \right] \\
& \left. + \frac{s_{ijk}}{s_{ij}} \left[4\, \frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}\, z_k + \frac{3}{2} \right] \right\} \\
& + (\text{ 5 permutations }) .
\end{aligned}
\tag{B.20}
$$

The azimuthal kernels $Q^{(1g),a}_{ijk(r)}$ and $Q^{(3g),a}_{ijk(r)}$ are defined according to the following expressions:

$$Q^{(1g),i}_{ijk(r)} = T_R \frac{\tilde{k}_i^2}{1-\epsilon}\frac{s_{ijk}}{s_{ik}s_{jk}}\left\{C_A\left[1-\frac{2z_j}{z_k}\frac{s_{ij}+2s_{jk}}{s_{ij}^2}s_{ik}+\frac{z_i s_{jk}+z_j s_{ik}}{z_{ij}s_{ij}}+\left(\frac{z_i z_j}{z_k z_{ij}}-\frac{1-\epsilon}{2}\right)\frac{s_{ik}-s_{jk}}{s_{ij}}\right]-2C_F\right\},$$
$$Q^{(1g),j}_{ijk(r)} = T_R \frac{\tilde{k}_j^2}{1-\epsilon}\frac{s_{ijk}}{s_{ik}s_{jk}}\left\{C_A\left[1-\frac{2z_i}{z_k}\frac{s_{ij}+2s_{ik}}{s_{ij}^2}s_{jk}+\frac{z_i s_{jk}+z_j s_{ik}}{z_{ij}s_{ij}}+\left(\frac{z_i z_j}{z_k z_{ij}}-\frac{1-\epsilon}{2}\right)\frac{s_{jk}-s_{ik}}{s_{ij}}\right]-2C_F\right\},$$
$$Q^{(1g),k}_{ijk(r)} = T_R \frac{\tilde{k}_k^2}{1-\epsilon}\frac{s_{ijk}}{s_{ik}s_{jk}}\left\{C_A\left[\frac{z_i z_j}{z_k z_{ij}}\frac{4s_{ik}s_{jk}+s_{ij}s_{[ij]k}}{s_{ij}^2}+\frac{z_i-z_j}{2z_{ij}}\frac{s_{ik}-s_{jk}}{s_{ij}}-\epsilon\frac{s_{ijk}+s_{ij}}{2s_{ij}}\right]+2C_F\,\epsilon\right\},$$

$$\begin{aligned}\sum_{a=i,j,k} Q^{(3g),a}_{ijk(r)} d^{\mu\nu}_a = C_A^2\frac{s_{ijk}}{s_{ij}}\Bigg\{ &\left[\frac{2z_j}{z_k}\frac{1}{s_{ij}}+\left(\frac{z_j z_{ik}}{z_k z_{ij}}-\frac{3}{2}\right)\frac{1}{s_{ik}}\right]\tilde{k}_i^2\, d^{\mu\nu}_i \\ &+\left[\frac{2z_i}{z_k}\frac{1}{s_{ij}}-\left(\frac{z_j z_{ik}}{z_k z_{ij}}-\frac{3}{2}-\frac{z_i}{z_k}+\frac{z_i}{z_{ij}}\right)\frac{1}{s_{ik}}\right]\tilde{k}_j^2\, d^{\mu\nu}_j \\ &-\left[\frac{2z_i z_j}{z_{ij}z_k}\frac{1}{s_{ij}}+\left(\frac{z_j z_{ik}}{z_k z_{ij}}-\frac{3}{2}-\frac{z_i}{z_j}+\frac{z_i}{z_{ik}}\right)\frac{1}{s_{ik}}\right]\tilde{k}_k^2\, d^{\mu\nu}_k\Bigg\} + (5\text{ permutations})\,. \end{aligned} \tag{B.21}$$

The hard-double-collinear kernels $P^{\mathrm{hc},\mu\nu}_{ijk(r)}$ are defined as

$$P^{\mathrm{hc},\mu\nu}_{ijk(r)} \equiv -P^{\mathrm{hc}}_{ijk(r)}\, g^{\mu\nu} + Q^{\mu\nu}_{ijk(r)}\,, \tag{B.22}$$

where $Q^{\mu\nu}_{ijk(r)}$ is given in Eq. (B.13) and

$$P^{\mathrm{hc}}_{ijk(r)} \equiv P_{ijk(r)} - s_{ijk}^2\Big[C_{f_k}\Big(4\,C_{f_k}\mathcal{E}^{(i)}_{kr}\mathcal{E}^{(j)}_{kr} - \mathcal{E}^{(ij)}_{kr}\Big) + (i\leftrightarrow k) + (j\leftrightarrow k)\Big]. \tag{B.23}$$

B.4 Collinear and hard-collinear kernels at one loop

The collinear contribution to the real-virtual counterterm at NNLO depends on the one-loop, single-collinear kernel which reads ($r \neq i, j$):

$$\tilde{P}^{\mu\nu}_{ij(r)} \equiv \frac{\Gamma^2(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}\left(\frac{e^{\gamma_E}\mu^2}{s_{ij}}\right)^{\epsilon}\left\{\frac{C_{f_{[ij]}}}{\epsilon^2}\Big[\rho^{(C)}_{[ij]}+\rho^{(C)}_{ij}F(x_i)+\rho^{(C)}_{ji}F(x_j)\Big]P^{\mu\nu}_{ij(r)}+\hat{P}^{\mu\nu}_{ij(r)}\right\}, \tag{B.24}$$

where the function $F(x)$ is defined by

$$F(x) \equiv 1 - {}_2F_1\left(1,-\epsilon;1-\epsilon;\frac{x-1}{x}\right) = \epsilon\ln x + \sum_{n=2}^{+\infty}\epsilon^n\,\mathrm{Li}_n\left(\frac{x-1}{x}\right)\,, \tag{B.25}$$

and $\hat{P}^{\mu\nu}_{ij(r)}$ reads

$$\begin{aligned}\hat{P}^{\mu\nu}_{ij(r)} = &\left[-g_{\mu\nu}+4x_i x_j\frac{\tilde{k}_i^\mu\tilde{k}_i^\nu}{\tilde{k}_i^2}\right]\frac{T_R}{1-2\epsilon}\left[\frac{1}{\epsilon}\big(\beta_0-3\,C_F\big)+C_A-2C_F+\frac{C_A+4\,T_R\,N_f}{3(3-2\epsilon)}\right]f^{q\bar{q}}_{ij}\,, \\ &-g_{\mu\nu}\,C_F\,\frac{C_A-C_F}{1-2\epsilon}\Big[\big(1-\epsilon x_i\big)f^g_i\big(f^q_j+f^{\bar{q}}_j\big)+\big(1-\epsilon x_j\big)\big(f^q_i+f^{\bar{q}}_i\big)f^g_j\Big] \\ &+\frac{\tilde{k}_i^\mu\tilde{k}_i^\nu}{\tilde{k}_i^2}\,4\,C_A\,\frac{2\,T_R\,N_f-C_A(1-\epsilon)}{(1-2\epsilon)(2-2\epsilon)(3-2\epsilon)}\,\big(1-2\epsilon x_i x_j\big)f^{gg}_{ij}\,. \end{aligned} \tag{B.26}$$

The expansion of $\tilde{P}^{\mu\nu}_{ij(r)}$ in the dimensional regulator ϵ gives

$$\begin{aligned}
\tilde{P}^{\mu\nu}_{ij(r)} \; = \; & P^{\mu\nu}_{ij(r)} \, C_{f_{[ij]}} \bigg\{ \, \rho^{(C)}_{[ij]} \bigg[\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \, \ln \frac{s_{ij}}{\mu^2} - \frac{1}{2} \bigg(7 \, \zeta_2 - \ln^2 \frac{s_{ij}}{\mu^2} \bigg) \bigg] \\
& \qquad + \bigg[\frac{1}{\epsilon} - \ln \frac{s_{ij}}{\mu^2} \bigg] \big(\rho^{(C)}_{ij} \ln x_i + \rho^{(C)}_{ji} \ln x_j \big) + \rho^{(C)}_{ij} \mathrm{Li}_2 \bigg(\frac{-x_j}{x_i} \bigg) + \rho^{(C)}_{ji} \mathrm{Li}_2 \bigg(\frac{-x_i}{x_j} \bigg) \bigg\} \\
& + \bigg[- g_{\mu\nu} + 4 x_i x_j \, \frac{\tilde{k}^\mu_i \tilde{k}^\nu_i}{\tilde{k}^2_i} \bigg] \, f^{q\bar{q}}_{ij} \; T_R \, \bigg[\, \bigg(\frac{1}{\epsilon} - \ln \frac{s_{ij}}{\mu^2} \bigg) \big(\beta_0 - 3 \, C_F \big) + \frac{7}{3} C_A + \frac{5}{3} \beta_0 - 8 C_F \bigg] \\
& - \, g_{\mu\nu} \, (f^{gq}_{ij} + f^{g\bar{q}}_{ij}) \; C_F \, (C_A - C_F) + \frac{\tilde{k}^\mu_i \tilde{k}^\nu_i}{\tilde{k}^2_i} \; f^{gg}_{ij} \; C_A \, (3 C_A - \beta_0) + \mathcal{O}(\epsilon) \, .
\end{aligned} \tag{B.27}$$

The one-loop collinear kernel $\hat{P}^{\mu\nu}_{ij(r)}$ can be rewritten according to the same structure as in Eq. (B.7),

$$\hat{P}^{\mu\nu}_{ij(r)} \; = \; - \, \hat{P}_{ij(r)} \, g^{\mu\nu} + \hat{Q}^{\mu\nu}_{ij(r)} \, , \qquad \hat{Q}^{\mu\nu}_{ij(r)} \; = \; \hat{Q}_{ij(r)} \, d^{\mu\nu}_i \, , \tag{B.28}$$

where we have introduced

$$\begin{aligned}
\hat{P}_{ij(r)} \; = \; & \frac{T_R}{1 - 2\epsilon} \bigg[1 - \frac{2 x_i x_j}{1 - \epsilon} \bigg] \bigg[\frac{1}{\epsilon} \big(\beta_0 - 3 \, C_F \big) + C_A - 2 C_F + \frac{C_A + 4 \, T_R \, N_f}{3(3 - 2\epsilon)} \bigg] f^{q\bar{q}}_{ij} \\
& + C_F \, \frac{C_A - C_F}{1 - 2\epsilon} \Big[\big(1 - \epsilon x_i \big) f^g_i (f^q_j + f^{\bar{q}}_j) + \big(1 - \epsilon x_j \big) (f^q_i + f^{\bar{q}}_i) f^g_j \Big] \\
& + 4 \, C_A \, \frac{C_A (1 - \epsilon) - 2 \, T_R \, N_f}{(1 - 2\epsilon)(2 - 2\epsilon)^2 (3 - 2\epsilon)} \, \big(1 - 2 \epsilon x_i x_j \big) f^{gg}_{ij} \, , \\
\hat{Q}_{ij(r)} \; = \; & 2 x_i x_j \, \frac{T_R}{(1 - 2\epsilon)(1 - \epsilon)} \, \bigg[\frac{1}{\epsilon} \big(\beta_0 - 3 \, C_F \big) + C_A - 2 C_F + \frac{C_A + 4 \, T_R \, N_f}{3(3 - 2\epsilon)} \bigg] f^{q\bar{q}}_{ij} \\
& + 4 \, C_A \, \frac{2 \, T_R \, N_f - C_A (1 - \epsilon)}{(1 - 2\epsilon)(2 - 2\epsilon)^2 (3 - 2\epsilon)} \, \big(1 - 2 \epsilon x_i x_j \big) f^{gg}_{ij} \, .
\end{aligned} \tag{B.29}$$

Analogously, the ϵ expansion $\tilde{P}^{\mu\nu}_{ij(r)}$ can be recast in the same form, as

$$\tilde{P}^{\mu\nu}_{ij(r)} \; = \; - \, \tilde{P}_{ij(r)} \, g^{\mu\nu} + \tilde{Q}^{\mu\nu}_{ij(r)} \, , \qquad \tilde{Q}^{\mu\nu}_{ij(r)} \; = \; \tilde{Q}_{ij(r)} \, d^{\mu\nu}_i \, , \tag{B.30}$$

where $\tilde{P}_{ij(r)}$ and $\tilde{Q}_{ij(r)}$ are given by ($\mathcal{F} = P, Q$)

$$\tilde{\mathcal{F}}_{ij(r)} \; = \; \frac{\Gamma^2(1 + \epsilon) \Gamma^3(1 - \epsilon)}{\Gamma(1 + 2\epsilon) \Gamma^2(1 - 2\epsilon)} \bigg(\frac{e^{\gamma_E} \mu^2}{s_{ij}} \bigg)^{\epsilon} \bigg\{ \frac{C_{f_{[ij]}}}{\epsilon^2} \Big[\rho^{(C)}_{[ij]} + \rho^{(C)}_{ij} F(x_i) + \rho^{(C)}_{ji} F(x_j) \Big] \mathcal{F}_{ij(r)} + \hat{\mathcal{F}}_{ij(r)} \bigg\} \, . \tag{B.31}$$

The hard-collinear real-virtual kernel, expanded in the regulator ϵ, reads

$$\begin{aligned}
\tilde{P}^{\mathrm{hc},\mu\nu}_{ij(r)} \; \equiv \; & \tilde{P}^{\mu\nu}_{ij(r)} - s_{ij} \Big[2 \, C_{f_j} \, \tilde{\mathcal{E}}^{(i)}_{jr} + 2 \, C_{f_i} \, \tilde{\mathcal{E}}^{(j)}_{ir} \Big] g^{\mu\nu} \\
= \; & \tilde{P}^{\mathrm{hc},\mu\nu}_{\mathrm{fin},ij(r)} + C_{f_{[ij]}} \bigg[\rho^{(C)}_{[ij]} \bigg(\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \, \ln \frac{s_{ij}}{\mu^2} \bigg) + \frac{1}{\epsilon} \Big(\rho^{(C)}_{ij} \ln x_i + \rho^{(C)}_{ji} \ln x_j \Big) \bigg] \, P^{\mathrm{hc},\mu\nu}_{ij(r)} \\
& - \frac{4}{\epsilon} \, \bigg[f^g_i C^2_{f_j} \, \frac{x_j}{x_i} \ln x_j + f^g_j C^2_{f_i} \, \frac{x_i}{x_j} \ln x_i \bigg] \, g^{\mu\nu} - \frac{T_R}{\epsilon} \, \big(\beta_0 - 3 C_F \big) \, f^{q\bar{q}}_{ij} \bigg[g_{\mu\nu} - 4 x_i x_j \, \frac{\tilde{k}^\mu_i \tilde{k}^\nu_i}{\tilde{k}^2_i} \bigg] + \mathcal{O}(\epsilon) \, ,
\end{aligned} \tag{B.32}$$

where

$$
\begin{aligned}
\tilde{P}^{\mathrm{hc},\mu\nu}_{\mathrm{fin},ij(r)} \;=\; & P^{\mathrm{hc},\mu\nu}_{ij(r)}\, C_{f_{[ij]}} \Bigg\{ \rho^{(C)}_{[ij]} \Bigg[\bigg(\frac{1}{2} \ln^2 \frac{s_{ij}}{\mu^2} - \frac{7}{2}\,\zeta_2 \bigg) \Bigg] \\
& \qquad + \rho^{(C)}_{ij} \bigg[\mathrm{Li}_2\bigg(\frac{-x_j}{x_i}\bigg) - \ln \frac{s_{ij}}{\mu^2} \ln x_i \bigg] + \rho^{(C)}_{ji} \bigg[\mathrm{Li}_2\bigg(\frac{-x_i}{x_j}\bigg) - \ln \frac{s_{ij}}{\mu^2} \ln x_j \bigg] \Bigg\} \\
& - g^{\mu\nu}\, 2\, f^g_i\, C_{f_j}\, \frac{x_j}{x_i} \bigg\{ C_A \Big[\ln^2 x_j + 2\,\mathrm{Li}_2(x_i) \Big] + 2C_{f_j} \bigg[\mathrm{Li}_2\bigg(\frac{-x_i}{x_j}\bigg) - \ln \frac{s_{ij}}{\mu^2} \ln x_j \bigg] \bigg\} \\
& - g^{\mu\nu}\, 2\, f^g_j\, C_{f_i}\, \frac{x_i}{x_j} \bigg\{ C_A \Big[\ln^2 x_i + 2\,\mathrm{Li}_2(x_j) \Big] + 2C_{f_i} \bigg[\mathrm{Li}_2\bigg(\frac{-x_j}{x_i}\bigg) - \ln \frac{s_{ij}}{\mu^2} \ln x_i \bigg] \bigg\} \\
& - \bigg[g_{\mu\nu} - 4 x_i x_j \frac{\tilde{k}^\mu_i \tilde{k}^\nu_i}{\tilde{k}^2_i} \bigg] f^{q\bar{q}}_{ij}\, T_R \bigg[\ln \frac{s_{ij}}{\mu^2} (3\, C_F - \beta_0) + \frac{7}{3} C_A + \frac{5}{3} \beta_0 - 8 C_F \bigg] \\
& - g_{\mu\nu}\, (f^{gq}_{ij} + f^{g\bar{q}}_{ij})\, C_F\, (C_A - C_F) + \frac{\tilde{k}^\mu_i \tilde{k}^\nu_i}{\tilde{k}^2_i}\, f^{gg}_{ij}\, C_A\, (3 C_A - \beta_0)\,,
\end{aligned} \tag{B.33}
$$

and

$$
\begin{aligned}
\mathrm{Li}_2\bigg(-\frac{x_i}{x_j}\bigg) \;&=\; -\,\mathrm{Li}_2(x_i) - \frac{1}{2} \ln^2 x_j = \mathrm{Li}_2(x_j) + \ln x_i \ln x_j - \frac{1}{2} \ln^2 x_j - \zeta_2\,, \\
\mathrm{Li}_2\bigg(-\frac{x_j}{x_i}\bigg) \;&=\; -\,\mathrm{Li}_2(x_j) - \frac{1}{2} \ln^2 x_i = \mathrm{Li}_2(x_i) + \ln x_i \ln x_j - \frac{1}{2} \ln^2 x_i - \zeta_2\,.
\end{aligned} \tag{B.34}
$$

Equivalently we can write $\tilde{P}^{\mathrm{hc},\mu\nu}_{ij(r)}$ in the form

$$
\tilde{P}^{\mathrm{hc},\mu\nu}_{ij(r)} \;=\; -\,\tilde{P}^{\mathrm{hc}}_{ij(r)}\, g^{\mu\nu} + \tilde{Q}^{\mu\nu}_{ij(r)}\,, \tag{B.35}
$$

with

$$
\begin{aligned}
\tilde{P}^{\mathrm{hc}}_{ij(r)} \;\equiv\; & \tilde{P}_{ij(r)} + s_{ij} \Big[2\, C_{f_j}\, \tilde{\mathcal{E}}^{(i)}_{jr} + 2\, C_{f_i}\, \tilde{\mathcal{E}}^{(j)}_{ir} \Big] \qquad\qquad (\mathrm{B.36}) \\
=\; & \frac{\Gamma^2(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)} \bigg(\frac{e^{\gamma_E} \mu^2}{s_{ij}} \bigg)^{\epsilon} \bigg\{ \frac{C_{f_{[ij]}}}{\epsilon^2} \Big[\rho^{(C)}_{[ij]} + \rho^{(C)}_{ij} F(x_i) + \rho^{(C)}_{ji} F(x_j) \Big] P_{ij(r)} + \hat{P}_{ij(r)} \\
& \qquad + 2\, C_A \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\epsilon^2} \bigg[f^g_i C_{f_j} \bigg(\frac{x_j}{x_i} \bigg)^{1+\epsilon} + f^g_j C_{f_i} \bigg(\frac{x_i}{x_j} \bigg)^{1+\epsilon} \bigg] \bigg\}\,.
\end{aligned}
$$

C Improved limits

In this Appendix we provide three Sections collecting the building blocks for the construction of our local counterterms, namely we explicitly define the action of

- improved limits on the double-real matrix element RR (Section C.1);
- improved limits on sector functions $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$, $\mathcal{W}_{ijkl}$ (Section C.2);
- improved limits on symmetrised sector functions $\mathcal{Z}_{ijk}$, $\mathcal{Z}_{ijkl}$ (Section C.3).

The content of each section is organised according to the nature of the singular limits involved, which can be single-unresolved, uniform double-unresolved, and strongly-ordered double-unresolved. The action of improved limits $\overline{\mathbf{L}}$ on matrix elements times sector functions is specified by $\overline{\mathbf{L}}\, RR\, \mathcal{W}_{abcd} \equiv \big(\overline{\mathbf{L}}\, RR\big)\, \big(\overline{\mathbf{L}}\, \mathcal{W}_{abcd}\big)$, and similarly for $\mathcal{Z}$ functions. When acting on sector functions, single-unresolved and strongly-ordered improved limits imply the latter to be evaluated with mapped kinematics. Mapped sector functions are indicated generically as $\overline{\mathcal{W}}$ or $\bar{\mathcal{Z}}$ with no mapping labels in Sections

C.2, C.3, understanding that the actual mapping to be used must be adapted to the one of the matrix elements the sector function is associated to. To be more precise, for each term of an improved limit, the mapping of $\overline{\mathcal{W}}$ or $\bar{\mathcal{Z}}$ is always the same as the first mapping of matrix elements in that term.

To give an explicit example, let us apply this rule to the $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR\,\mathcal{W}_{ijkl}$ contribution to $K^{(\mathbf{12})}_{ijkl}$ counterterm. Starting with the definitions

$$
\begin{aligned}
&\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR \;\equiv\\
&\frac{\mathcal{N}_1^2}{2}\sum_{\substack{c\neq i,k\\ d\neq i,k,c}}\Bigg\{\mathcal{E}^{(i)}_{cd}\Bigg[\sum_{e\neq i,k,c,d}\Bigg(\sum_{f\neq i,k,c,d,e}\bar{\mathcal{E}}^{(k)(icd)}_{ef}\,\bar{B}^{(icd,kef)}_{cdef}+2\,\bar{\mathcal{E}}^{(k)(icd)}_{ed}\,\bar{B}^{(icd,ked)}_{cded}\Bigg)\\
&\qquad+2\sum_{e\neq i,k,c,d}\bar{\mathcal{E}}^{(k)(idc)}_{ed}\,\bar{B}^{(idc,ked)}_{cded}+2\,\bar{\mathcal{E}}^{(k)(icd)}_{cd}\Big(\bar{B}^{(icd,kcd)}_{cdcd}+C_A\,\bar{B}^{(icd,kcd)}_{cd}\Big)\Bigg]\\
&\qquad-2\,C_A\Big[\mathcal{E}^{(i)}_{kc}\,\bar{\mathcal{E}}^{(k)(ick)}_{cd}\,\bar{B}^{(ick,kcd)}_{cd}+\mathcal{E}^{(i)}_{kd}\,\bar{\mathcal{E}}^{(k)(ikd)}_{cd}\,\bar{B}^{(ikd,kcd)}_{cd}\Big]\Bigg\}\,,
\end{aligned}
\tag{C.1}
$$

and

$$
\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\mathcal{W}_{ijkl}\;\equiv\;\overline{\mathcal{W}}_{\mathrm{s},\,kl}\,\mathcal{W}^{(\alpha)}_{\mathrm{s},\,ij}\,,
\tag{C.2}
$$

according to the procedure detailed above, the explicit expression for $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR\,\mathcal{W}_{ijkl}$ results in

$$
\begin{aligned}
&\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR\,\mathcal{W}_{ijkl} \;\equiv\\
&\frac{\mathcal{N}_1^2}{2}\sum_{\substack{c\neq i,k\\ d\neq i,k,c}}\Bigg\{\mathcal{E}^{(i)}_{cd}\Bigg[\sum_{e\neq i,k,c,d}\Bigg(\sum_{f\neq i,k,c,d,e}\bar{\mathcal{E}}^{(k)(icd)}_{ef}\,\bar{B}^{(icd,kef)}_{cdef}+2\,\bar{\mathcal{E}}^{(k)(icd)}_{ed}\,\bar{B}^{(icd,ked)}_{cded}\Bigg)\overline{\mathcal{W}}^{(icd)}_{\mathrm{s},\,kl}\\
&\qquad+2\sum_{e\neq i,k,c,d}\bar{\mathcal{E}}^{(k)(idc)}_{ed}\,\bar{B}^{(idc,ked)}_{cded}\,\overline{\mathcal{W}}^{(idc)}_{\mathrm{s},\,kl}+2\,\bar{\mathcal{E}}^{(k)(icd)}_{cd}\Big(\bar{B}^{(icd,kcd)}_{cdcd}+C_A\,\bar{B}^{(icd,kcd)}_{cd}\Big)\overline{\mathcal{W}}^{(icd)}_{\mathrm{s},\,kl}\Bigg]\\
&\qquad-2\,C_A\Big[\mathcal{E}^{(i)}_{kc}\,\bar{\mathcal{E}}^{(k)(ick)}_{cd}\,\bar{B}^{(ick,kcd)}_{cd}\,\overline{\mathcal{W}}^{(ick)}_{\mathrm{s},\,kl}+\mathcal{E}^{(i)}_{kd}\,\bar{\mathcal{E}}^{(k)(ikd)}_{cd}\,\bar{B}^{(ikd,kcd)}_{cd}\,\overline{\mathcal{W}}^{(ikd)}_{\mathrm{s},\,kl}\Big]\Bigg\}\mathcal{W}^{(\alpha)}_{\mathrm{s},\,ij}\,,
\end{aligned}
\tag{C.3}
$$

where it is evident that each $\overline{\mathcal{W}}_{ab}$ contribution is mapped according to the first mapping of the Born matrix element it accompanies.

Finally, we introduce a shorthand notation to simplify the treatment in section C.2: we define single-unresolved improved limits on NLO sector functions as

$$
\mathcal{W}^{(\alpha)}_{\mathrm{s},\,ij}\;\equiv\;\overline{\mathbf{S}}_i\,\mathcal{W}^{(\alpha)}_{ij}\;\equiv\;\frac{\frac{1}{w^\alpha_{ij}}}{\sum\limits_{l\neq i}\frac{1}{w^\alpha_{il}}}\,,\qquad \mathcal{W}_{\mathrm{s},\,ij}\;\equiv\;\mathcal{W}^{(1)}_{\mathrm{s},\,ij}\,,
\tag{C.4}
$$

$$
\mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\;\equiv\;\overline{\mathbf{C}}_{ij}\,\mathcal{W}^{(\alpha)}_{ij}\;\equiv\;\frac{e^\alpha_j w^\alpha_{jr}}{e^\alpha_i w^\alpha_{ir}+e^\alpha_j w^\alpha_{jr}}\,,\qquad \mathcal{W}_{\mathrm{c},\,ij(r)}\;\equiv\;\mathcal{W}^{(1)}_{\mathrm{c},\,ij(r)}\,,
\tag{C.5}
$$

depending on a reference particle $r\neq i,j$, whose choice will be specified case by case; as for NNLO sector functions, we introduce

$$
\hat{\sigma}_{abcd(r)}\;=\;\frac{1}{(e_a\,w_{ab}\,w_{ar})^\alpha}\,\frac{1}{(e_c\,w_{cr}+\delta_{bc}\,e_a\,w_{ar})\,w_{cd}}\,,
\tag{C.6}
$$

and

$$
\begin{aligned}
\hat{\sigma}_{\{ijk\}(r)}\;=\;&\hat{\sigma}_{ijjk(r)}+\hat{\sigma}_{ikjk(r)}+\hat{\sigma}_{jiik(r)}+\hat{\sigma}_{jkik(r)}+\hat{\sigma}_{ijkj(r)}+\hat{\sigma}_{ikkj(r)}\\
+\;&\hat{\sigma}_{kiij(r)}+\hat{\sigma}_{kjij(r)}+\hat{\sigma}_{jiki(r)}+\hat{\sigma}_{jkki(r)}+\hat{\sigma}_{kiji(r)}+\hat{\sigma}_{kjji(r)}\,.
\end{aligned}
\tag{C.7}
$$

C.1 Improved limits of RR

Single-unresolved improved limits

For the single-unresolved improved limits we have ($j \neq i$)

$$\overline{\mathbf{S}}_i \, RR \equiv -\mathcal{N}_1 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd} \, \bar{R}^{(icd)}_{cd} \,, \tag{C.8}$$

$$\overline{\mathbf{C}}_{ij} \, RR \equiv \mathcal{N}_1 \frac{P^{\mu\nu}_{ij(r)}}{s_{ij}} \, \bar{R}^{(ijr)}_{\mu\nu} \,, \tag{C.9}$$

$$\overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, RR \equiv \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ji} \, RR \; \equiv \; \mathcal{N}_1 \, 2 \, C_{f_j} \, \mathcal{E}^{(i)}_{jr} \, \bar{R}^{(ijr)} \,; \tag{C.10}$$

$$\overline{\mathbf{HC}}_{ij} \, RR \equiv \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\right) RR \; = \; \mathcal{N}_1 \, \frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}} \, \bar{R}^{(ijr)}_{\mu\nu} \,. \tag{C.11}$$

In these equations r must be chosen according to the rule of Eq. (A.14) as $r = r_{ijkl} \neq i,j,k,l$, where i,j,k,l are the indices appearing in the NNLO sector functions multiplying the improved limits $\overline{\mathbf{C}}_{ij}$, $\overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij}$, $\overline{\mathbf{HC}}_{ij}$. This means that in the topologies $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$ the index $r = r_{ijk}$ is different from the three indices of the sector, while for the topology $\mathcal{W}_{ijkl}$ (i,j,k,l all different) the index $r = r_{ijkl}$ is different from the four indices of the sector. We stress that, having defined $r = r_{ijkl}$, one needs at least five massless partons in Φ_{n+2}, namely three massless final-state partons at Born level. We work under this assumption throughout the paper.

Uniform double-unresolved improved limits

The double-soft improved limit is given by ($k \neq i$)

$$\begin{aligned} \overline{\mathbf{S}}_{ik} \, RR \equiv \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,k\\ d\neq i,k,c}} \Bigg\{ \mathcal{E}^{(i)}_{cd} \sum_{e\neq i,k,c,d} \Bigg[\sum_{f\neq i,k,c,d,e} \mathcal{E}^{(k)}_{ef} \, \bar{B}^{(icd,kef)}_{cdef} + 4 \, \mathcal{E}^{(k)}_{ed} \, \bar{B}^{(icd,ked)}_{cded} \Bigg] \\ + 2 \, \mathcal{E}^{(i)}_{cd} \mathcal{E}^{(k)}_{cd} \, \bar{B}^{(icd,kcd)}_{cdcd} + \mathcal{E}^{(ik)}_{cd} \, \bar{B}^{(ikcd)}_{cd} \Bigg\} . \end{aligned} \tag{C.12}$$

The soft-collinear improved limits $\overline{\mathbf{SC}}_{ikl}$ and its double-soft version $\overline{\mathbf{S}}_{ik} \, \overline{\mathbf{SC}}_{ikl}$ read ($k \neq i$, $l \neq i,k$, and $r = r_{ikl} \neq i,k,l$ defined with the rule of Eq. (A.14))

$$\begin{aligned} \overline{\mathbf{SC}}_{ikl} \, RR \equiv -\mathcal{N}_1^2 \, \frac{P^{\mu\nu}_{kl(r)}}{s_{kl}} \Bigg\{ \sum_{c\neq i,k,l,r} \Bigg[\sum_{d\neq i,k,l,r,c} \mathcal{E}^{(i)}_{cd} \, \bar{B}^{(klr,icd)}_{\mu\nu,cd} + 2 \, \mathcal{E}^{(i)}_{cr} \, \bar{B}^{(klr,icr)}_{\mu\nu,cr} \Bigg] \\ + \sum_{c\neq i,k,l} \Bigg[\mathcal{E}^{(i)}_{kc} \left(\rho^{(C)}_{kl} \, \bar{B}^{(lrk,ick)}_{\mu\nu,[kl]c} + \tilde{\bar{B}}^{(lrk,ick)}_{\mu\nu,[kl]c} \, \tilde{f}^{\,q\bar{q}}_{kl} \right) + (k \leftrightarrow l) \Bigg] \Bigg\} , \end{aligned} \tag{C.13}$$

$$\begin{aligned} \overline{\mathbf{S}}_{ik} \, \overline{\mathbf{SC}}_{ikl} \, RR &\equiv \overline{\mathbf{S}}_{ki} \, \overline{\mathbf{SC}}_{ikl} \, RR \; \equiv \; \overline{\mathbf{S}}_{ik} \, \overline{\mathbf{SC}}_{ilk} \, RR \\ &\equiv -2 \mathcal{N}_1^2 \, \mathcal{E}^{(k)}_{lr} \Bigg\{ \; C_{f_l} \sum_{c\neq i,k,l,r} \Bigg[\sum_{d\neq i,k,l,r,c} \mathcal{E}^{(i)}_{cd} \, \bar{B}^{(klr,icd)}_{cd} + 2 \, \mathcal{E}^{(i)}_{cr} \, \bar{B}^{(klr,icr)}_{cr} \Bigg] \\ &\qquad + \sum_{c\neq i,k,l} \Bigg[C_A \, \mathcal{E}^{(i)}_{kc} \, \bar{B}^{(lrk,ick)}_{[kl]c} + (2C_{f_l} - C_A) \, \mathcal{E}^{(i)}_{lc} \, \bar{B}^{(krl,icl)}_{[kl]c} \Bigg] \Bigg\} . \end{aligned} \tag{C.14}$$

The improved limits $\overline{\mathbf{SC}}_{ijk}$, $\overline{\mathbf{SC}}_{kij}$, $\overline{\mathbf{S}}_{ij} \, \overline{\mathbf{SC}}_{ijk}$, $\overline{\mathbf{S}}_{ik} \, \overline{\mathbf{SC}}_{ijk}$, $\overline{\mathbf{S}}_{ik} \, \overline{\mathbf{SC}}_{kij}$ can be obtained from these limits with a renaming of indices. For the uniform double-unresolved limits involving $\overline{\mathbf{C}}_{ijk}$, we have ($j \neq i$, $k \neq i,j$ and $r = r_{ijk} \neq i,j,k$)

$$\overline{\mathbf{C}}_{ijk} \, RR \equiv \frac{\mathcal{N}_1^2}{s^2_{ijk}} \, P^{\mu\nu}_{ijk(r)} \, \bar{B}^{(ijkr)}_{\mu\nu} \,; \tag{C.15}$$

$$\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR \equiv \overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ikj}\,RR \;\equiv\; \overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{kij}\,RR \;\equiv\; \mathcal{N}_1^{\,2}\,C_{f_k}\bigg[4\,C_{f_k}\,\mathcal{E}^{(i)}_{kr}\,\mathcal{E}^{(j)}_{kr} - \mathcal{E}^{(ij)}_{kr}\bigg]\bar{B}^{(ijkr)}\,, \tag{C.16}$$

$$\overline{\mathbf{HC}}_{ijk}\,RR \equiv \overline{\mathbf{C}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}\right)RR \;=\; \frac{\mathcal{N}_1^{\,2}}{s_{ijk}^2}\,P^{\mathrm{hc},\mu\nu}_{ijk(r)}\,\bar{B}^{(ijkr)}_{\mu\nu}\,; \tag{C.17}$$

$$\begin{aligned}\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,RR &\equiv \overline{\mathbf{C}}_{jki}\,\overline{\mathbf{SC}}_{ijk}\,RR\\ &\equiv \mathcal{N}_1^{\,2}\,C_{f_{[jk]}}\,\frac{P^{\mu\nu}_{jk(r)}}{s_{jk}}\bigg[\rho^{(C)}_{jk}\,\mathcal{E}^{(i)}_{jr}\,\bar{B}^{(krj,ijr)}_{\mu\nu} + \rho^{(C)}_{kj}\,\mathcal{E}^{(i)}_{kr}\,\bar{B}^{(jrk,ikr)}_{\mu\nu}\bigg]\,,\end{aligned} \tag{C.18}$$

$$\begin{aligned}\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,RR &\equiv \overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ikj}\,\overline{\mathbf{SC}}_{ikj}\,RR = \overline{\mathbf{S}}_{ji}\,\overline{\mathbf{C}}_{jki}\,\overline{\mathbf{SC}}_{ijk}\,RR\\ &\equiv 2\,\mathcal{N}_1^{\,2}\,C_{f_k}\,\mathcal{E}^{(j)}_{kr}\Big[C_A\,\mathcal{E}^{(i)}_{jr}\,\bar{B}^{(krj,ijr)} + (2C_{f_k}-C_A)\,\mathcal{E}^{(i)}_{kr}\,\bar{B}^{(jrk,ikr)}\Big]\,,\end{aligned} \tag{C.19}$$

$$\begin{aligned}\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SHC}}_{ijk}\,RR &\equiv \overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}\right)RR\\ &\equiv \mathcal{N}_1^{\,2}\,C_{f_{[jk]}}\,\frac{P^{\mathrm{hc},\mu\nu}_{jk(r)}}{s_{jk}}\bigg[\rho^{(C)}_{jk}\,\mathcal{E}^{(i)}_{jr}\,\bar{B}^{(krj,ijr)}_{\mu\nu} + \rho^{(C)}_{kj}\,\mathcal{E}^{(i)}_{kr}\,\bar{B}^{(jrk,ikr)}_{\mu\nu}\bigg]\,,\end{aligned} \tag{C.20}$$

$$\begin{aligned}\left(1-\overline{\mathbf{C}}_{ijk}\right)\overline{\mathbf{SHC}}_{ijk}\,RR &\equiv \left(1-\overline{\mathbf{C}}_{ijk}\right)\overline{\mathbf{SC}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}\right)RR\\ &\equiv -\,\mathcal{N}_1^{\,2}\,\frac{P^{\mathrm{hc},\mu\nu}_{jk(r)}}{s_{jk}}\Bigg\{\sum_{c\neq i,j,k,r}\Bigg[\sum_{d\neq i,j,k,r,c}\mathcal{E}^{(i)}_{cd}\,\bar{B}^{(jkr,icd)}_{\mu\nu,cd} + 2\,\mathcal{E}^{(i)}_{cr}\,\bar{B}^{(jkr,icr)}_{\mu\nu,cr}\Bigg]\\ &\qquad + \sum_{c\neq i,j,k}\bigg[\mathcal{E}^{(i)}_{jc}\Big(\rho^{(C)}_{jk}\,\bar{B}^{(krj,icj)}_{\mu\nu,[jk]c} + \tilde{\bar{B}}^{(krj,icj)}_{\mu\nu,[jk]c}\,\tilde{f}^{\,q\bar{q}}_{jk}\Big) + (j\leftrightarrow k)\bigg]\\ &\qquad + C_{f_{[jk]}}\bigg[\rho^{(C)}_{jk}\,\mathcal{E}^{(i)}_{jr}\,\bar{B}^{(krj,ijr)}_{\mu\nu} + \rho^{(C)}_{kj}\,\mathcal{E}^{(i)}_{kr}\,\bar{B}^{(jrk,ikr)}_{\mu\nu}\bigg]\Bigg\}\,.\end{aligned} \tag{C.21}$$

Finally, the limits involving $\overline{\mathbf{C}}_{ijkl}$ are given by ($j \neq i$, $k \neq i, j$, $l \neq i, j, k$ and $r = r_{ijkl} \neq i, j, k, l$)

$$\overline{\mathbf{C}}_{ijkl}\,RR \equiv \mathcal{N}_1^{\,2}\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\frac{P^{\rho\sigma}_{kl(r)}}{s_{kl}}\,\bar{B}^{(ijr,klr)}_{\mu\nu\rho\sigma}\,, \tag{C.22}$$

$$\begin{aligned}\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijkl}\,RR &\equiv \overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{jikl}\,RR \;\equiv\; \overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijlk}\,RR \;\equiv\; \overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{jilk}\,RR\\ &\equiv 4\,\mathcal{N}_1^{\,2}\,C_{f_j}\,C_{f_l}\,\mathcal{E}^{(i)}_{jr}\,\mathcal{E}^{(k)}_{lr}\,\bar{B}^{(ijr,klr)}\,;\end{aligned} \tag{C.23}$$

$$\begin{aligned}\overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{C}}_{ijkl}\,RR &\equiv \overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{C}}_{klij}\,RR \;\equiv\; \overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{C}}_{jikl}\,RR \;\equiv\; \overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{C}}_{klji}\,RR\\ &\equiv 2\,\mathcal{N}_1^{\,2}\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\,\frac{P^{\mu\nu}_{kl(r)}}{s_{kl}}\,\bar{B}^{(ijr,klr)}_{\mu\nu}\,;\end{aligned} \tag{C.24}$$

$$\begin{aligned}\overline{\mathbf{HC}}_{ijkl}\,RR &\equiv \overline{\mathbf{C}}_{ijkl}\left(1+\overline{\mathbf{S}}_{ik}+\overline{\mathbf{S}}_{jk}+\overline{\mathbf{S}}_{il}+\overline{\mathbf{S}}_{jl}-\overline{\mathbf{SC}}_{ikl}-\overline{\mathbf{SC}}_{jkl}-\overline{\mathbf{SC}}_{kij}-\overline{\mathbf{SC}}_{lij}\right)RR\\ &= \mathcal{N}_1^{\,2}\,\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\,\frac{P^{\mathrm{hc},\rho\sigma}_{kl(r)}}{s_{kl}}\,\bar{B}^{(ijr,klr)}_{\mu\nu\rho\sigma}\,.\end{aligned} \tag{C.25}$$

Strongly-ordered double-unresolved improved limits

The improved limit $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}$ is given by ($k \neq i$)

$$\begin{aligned}\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR \equiv \frac{\mathcal{N}_1^{\,2}}{2}\sum_{\substack{c\neq i,k\\ d\neq i,k,c}}\Bigg\{\mathcal{E}^{(i)}_{cd}\Bigg[&\sum_{e\neq i,k,c,d}\Bigg(\sum_{f\neq i,k,c,d,e}\bar{\mathcal{E}}^{(k)(icd)}_{ef}\,\bar{B}^{(icd,kef)}_{cdef} + 2\,\bar{\mathcal{E}}^{(k)(icd)}_{ed}\,\bar{B}^{(icd,ked)}_{cded}\Bigg)\\ &+ 2\sum_{e\neq i,k,c,d}\bar{\mathcal{E}}^{(k)(idc)}_{ed}\,\bar{B}^{(idc,ked)}_{cded} + 2\,\bar{\mathcal{E}}^{(k)(icd)}_{cd}\Big(\bar{B}^{(icd,kcd)}_{cdcd} + C_A\,\bar{B}^{(icd,kcd)}_{cd}\Big)\Bigg]\\ &- 2\,C_A\Big[\mathcal{E}^{(i)}_{kc}\,\bar{\mathcal{E}}^{(k)(ick)}_{cd}\,\bar{B}^{(ick,kcd)}_{cd} + \mathcal{E}^{(i)}_{kd}\,\bar{\mathcal{E}}^{(k)(ikd)}_{cd}\,\bar{B}^{(ikd,kcd)}_{cd}\Big]\Bigg\}\,.\end{aligned} \tag{C.26}$$

For $\overline{\mathbf{S}}_i\,\overline{\mathbf{SC}}_{ikl}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}$ we have ($k \neq i$, $l \neq i,k$, and $r = r_{ikl} \neq i,k,l$)

$$\begin{aligned}
\overline{\mathbf{S}}_i\,\overline{\mathbf{SC}}_{ikl}\,RR \,\equiv\, -\mathcal{N}_1^2\!\sum_{c\neq i,k,l}\Bigg\{ & \sum_{d\neq i,k,l,c}\mathcal{E}^{(i)}_{cd}\,\frac{\bar{P}^{(icd)\mu\nu}_{kl(r)}}{\bar{s}^{(icd)}_{kl}}\,\bar{B}^{(icd,klr)}_{\mu\nu,cd} \\
&+\left[\mathcal{E}^{(i)}_{kc}\,\frac{\bar{P}^{(ikc)\mu\nu}_{kl(r)}}{2\,\bar{s}^{(ikc)}_{kl}}\Big(\rho^{(C)}_{kl}\,\bar{B}^{(ikc,lrk)}_{\mu\nu,[kl]c}+\tilde{\bar{B}}^{(ikc,lrk)}_{\mu\nu,[kl]c}\,\tilde{f}^{\,q\bar{q}}_{\,kl}\Big)+(k\leftrightarrow l)\right] \\
&+\left[\mathcal{E}^{(i)}_{kc}\,\frac{\bar{P}^{(ick)\mu\nu}_{kl(r)}}{2\,\bar{s}^{(ick)}_{kl}}\Big(\rho^{(C)}_{kl}\,\bar{B}^{(ick,lrk)}_{\mu\nu,[kl]c}+\tilde{\bar{B}}^{(ick,lrk)}_{\mu\nu,[kl]c}\,\tilde{f}^{\,q\bar{q}}_{\,kl}\Big)+(k\leftrightarrow l)\right]\Bigg\}\,,
\end{aligned} \tag{C.27}$$

$$\begin{aligned}
\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}\,RR \,\equiv\, & \overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ilk}\,RR \\
\equiv\, & -\mathcal{N}_1^2\sum_{c\neq i,k,l}\Bigg[\,2\,C_{f_l}\sum_{d\neq i,k,l,c}\mathcal{E}^{(i)}_{cd}\,\bar{\mathcal{E}}^{(k)(icd)}_{lr}\,\bar{B}^{(icd,klr)}_{cd} \\
&+C_A\,\mathcal{E}^{(i)}_{kc}\Big(\bar{\mathcal{E}}^{(k)(ikc)}_{lr}\,\bar{B}^{(ikc,lrk)}_{lc}+\bar{\mathcal{E}}^{(k)(ick)}_{lr}\,\bar{B}^{(ick,lrk)}_{lc}\Big) \\
&+(2C_{f_l}\!-\!C_A)\,\mathcal{E}^{(i)}_{lc}\Big(\bar{\mathcal{E}}^{(k)(ilc)}_{lr}\,\bar{B}^{(ilc,krl)}_{lc}+\bar{\mathcal{E}}^{(k)(icl)}_{lr}\,\bar{B}^{(icl,krl)}_{lc}\Big)\Bigg]\,.
\end{aligned} \tag{C.28}$$

Combining the previous definitions we have ($j \neq i$, $k \neq i,j$, and $r = r_{ijk} \neq i,j,k$)

$$\begin{aligned}
\overline{\mathbf{S}}_i\,\overline{\mathbf{SHC}}_{ijk}\,RR \,\equiv\, & \overline{\mathbf{S}}_i\,\overline{\mathbf{SC}}_{ijk}\big(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}\big)\,RR \\
\equiv\, & -\mathcal{N}_1^2\!\sum_{c\neq i,j,k}\Bigg\{\sum_{d\neq i,j,k,c}\mathcal{E}^{(i)}_{cd}\,\frac{\bar{P}^{(icd)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(icd)}_{jk}}\,\bar{B}^{(icd,jkr)}_{\mu\nu,cd} \\
&+\left[\mathcal{E}^{(i)}_{jc}\,\frac{\bar{P}^{(ijc)\mathrm{hc},\mu\nu}_{jk(r)}}{2\,\bar{s}^{(ijc)}_{jk}}\Big(\rho^{(C)}_{jk}\,\bar{B}^{(ijc,krj)}_{\mu\nu,[jk]c}+\tilde{\bar{B}}^{(ijc,krj)}_{\mu\nu,[jk]c}\,\tilde{f}^{\,q\bar{q}}_{\,jk}\Big)+(j\leftrightarrow k)\right] \\
&+\left[\mathcal{E}^{(i)}_{jc}\,\frac{\bar{P}^{(icj)\mathrm{hc},\mu\nu}_{jk(r)}}{2\,\bar{s}^{(icj)}_{jk}}\Big(\rho^{(C)}_{jk}\,\bar{B}^{(icj,krj)}_{\mu\nu,[jk]c}+\tilde{\bar{B}}^{(icj,krj)}_{\mu\nu,[jk]c}\,\tilde{f}^{\,q\bar{q}}_{\,jk}\Big)+(j\leftrightarrow k)\right]\Bigg\}\,.
\end{aligned} \tag{C.29}$$

For the strongly-ordered double-unresolved limits involving $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk}$, we have ($j \neq i$, $k \neq i,j$, $r = r_{ijk} \neq i,j,k$)

$$\begin{aligned}
&\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk}(1\!-\!\overline{\mathbf{SC}}_{ijk})RR \,\equiv \\
&\mathcal{N}_1^2\,\frac{C_{f_{[jk]}}}{2}\Bigg\{\rho^{(C)}_{jk}\,\mathcal{E}^{(i)}_{jr}\left[\frac{\bar{P}^{(ijr)\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\Big(\bar{B}^{(ijr,jkr)}_{\mu\nu}\!-\!\bar{B}^{(ijr,krj)}_{\mu\nu}\Big)+\frac{\bar{P}^{(irj)\mu\nu}_{jk(r)}}{\bar{s}^{(irj)}_{jk}}\Big(\bar{B}^{(irj,jkr)}_{\mu\nu}\!-\!\bar{B}^{(irj,krj)}_{\mu\nu}\Big)\right] \\
&\qquad+\rho^{(C)}_{kj}\,\mathcal{E}^{(i)}_{kr}\left[\frac{\bar{P}^{(ikr)\mu\nu}_{jk(r)}}{\bar{s}^{(ikr)}_{jk}}\Big(\bar{B}^{(ikr,jkr)}_{\mu\nu}\!-\!\bar{B}^{(ikr,jrk)}_{\mu\nu}\Big)+\frac{\bar{P}^{(irk)\mu\nu}_{jk(r)}}{\bar{s}^{(irk)}_{jk}}\Big(\bar{B}^{(irk,jkr)}_{\mu\nu}\!-\!\bar{B}^{(irk,jrk)}_{\mu\nu}\Big)\right] \\
&\qquad-\rho^{(C)}_{[jk]}\,\mathcal{E}^{(i)}_{jk}\left[\frac{\bar{P}^{(ijk)\mu\nu}_{jk(r)}}{\bar{s}^{(ijk)}_{jk}}\,\bar{B}^{(ijk,jkr)}_{\mu\nu}+\frac{\bar{P}^{(ikj)\mu\nu}_{jk(r)}}{\bar{s}^{(ikj)}_{jk}}\,\bar{B}^{(ikj,jkr)}_{\mu\nu}\right]\Bigg\}\,,
\end{aligned} \tag{C.30}$$

$$\begin{aligned}
&\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}(1\!-\!\overline{\mathbf{SC}}_{ijk})RR \,\equiv\, \overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ikj}(1\!-\!\overline{\mathbf{SC}}_{ikj})RR \\
&\equiv \mathcal{N}_1^2\,C_{f_k}\Bigg\{C_A\,\mathcal{E}^{(i)}_{jr}\Big[\bar{\mathcal{E}}^{(j)(ijr)}_{kr}\Big(\bar{B}^{(ijr,jkr)}\!-\!\bar{B}^{(ijr,krj)}\Big)+\bar{\mathcal{E}}^{(j)(irj)}_{kr}\Big(\bar{B}^{(irj,jkr)}\!-\!\bar{B}^{(irj,krj)}\Big)\Big] \\
&\qquad+(2C_{f_k}\!-\!C_A)\,\mathcal{E}^{(i)}_{kr}\Big[\bar{\mathcal{E}}^{(j)(ikr)}_{kr}\Big(\bar{B}^{(ikr,jkr)}\!-\!\bar{B}^{(ikr,jrk)}\Big)+\bar{\mathcal{E}}^{(j)(irk)}_{kr}\Big(\bar{B}^{(irk,jkr)}\!-\!\bar{B}^{(irk,jrk)}\Big)\Big] \\
&\qquad+C_A\,\mathcal{E}^{(i)}_{jk}\Big[\bar{\mathcal{E}}^{(j)(ijk)}_{kr}\,\bar{B}^{(ijk,jkr)}+\bar{\mathcal{E}}^{(j)(ikj)}_{kr}\,\bar{B}^{(ikj,jkr)}\Big]\Bigg\}\,,
\end{aligned} \tag{C.31}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{HC}}^{(\mathbf{s})}_{ijk}\,RR \;\equiv\; \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}\right)\left(1-\overline{\mathbf{SC}}_{ijk}\right)RR \tag{C.32}$$
$$= \mathcal{N}_1^{\,2}\,\frac{C_{f_{[jk]}}}{2}\Bigg\{\;\rho^{(C)}_{jk}\,\mathcal{E}^{(i)}_{jr}\left[\frac{\bar P^{(ijr)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar s^{(ijr)}_{jk}}\left(\bar B^{(ijr,jkr)}_{\mu\nu}-\bar B^{(ijr,krj)}_{\mu\nu}\right)+\frac{\bar P^{(irj)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar s^{(irj)}_{jk}}\left(\bar B^{(irj,jkr)}_{\mu\nu}-\bar B^{(irj,krj)}_{\mu\nu}\right)\right]$$
$$+\,\rho^{(C)}_{kj}\,\mathcal{E}^{(i)}_{kr}\left[\frac{\bar P^{(ikr)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar s^{(ikr)}_{jk}}\left(\bar B^{(ikr,jkr)}_{\mu\nu}-\bar B^{(ikr,jrk)}_{\mu\nu}\right)+\frac{\bar P^{(irk)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar s^{(irk)}_{jk}}\left(\bar B^{(irk,jkr)}_{\mu\nu}-\bar B^{(irk,jrk)}_{\mu\nu}\right)\right]$$
$$-\,\rho^{(C)}_{[jk]}\,\mathcal{E}^{(i)}_{jk}\left[\frac{\bar P^{(ijk)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar s^{(ijk)}_{jk}}\bar B^{(ijk,jkr)}_{\mu\nu}+\frac{\bar P^{(ikj)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar s^{(ikj)}_{jk}}\bar B^{(ikj,jkr)}_{\mu\nu}\right]\Bigg\}\,.$$

For $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}$ we have ($j\neq i$, $k\neq i,j$, $r=r_{ijk}\neq i,j,k$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,RR \equiv -\mathcal{N}_1^{\,2}\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\Bigg\{\sum_{c\neq i,j,k,r}\Bigg[\sum_{d\neq i,j,k,r,c}\bar{\mathcal{E}}^{(k)(ijr)}_{cd}\,\bar B^{(ijr,kcd)}_{\mu\nu,cd}+2\,\bar{\mathcal{E}}^{(k)(ijr)}_{cr}\,\bar B^{(ijr,kcr)}_{\mu\nu,cr}\Bigg]$$
$$+\,2\sum_{c\neq i,j,k}\bar{\mathcal{E}}^{(k)(ijr)}_{jc}\,\bar B^{(ijr,kcj)}_{\mu\nu,jc}\Bigg\}\,, \tag{C.33}$$
$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{SC}}_{kji}\,RR$$
$$\equiv -2\,\mathcal{N}_1^{\,2}\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\Bigg\{\sum_{c\neq i,j,k,r}\Bigg[\sum_{d\neq i,j,k,r,c}\bar{\mathcal{E}}^{(k)(ijr)}_{cd}\,\bar B^{(ijr,kcd)}_{cd}+2\,\bar{\mathcal{E}}^{(k)(ijr)}_{cr}\,\bar B^{(ijr,kcr)}_{cr}\Bigg]$$
$$+\,2\sum_{c\neq i,j,k}\bar{\mathcal{E}}^{(k)(ijr)}_{jc}\,\bar B^{(ijr,kcj)}_{jc}\Bigg]\,, \tag{C.34}$$
$$\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,RR \equiv \overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\right)\overline{\mathbf{SC}}_{kij}\,RR$$
$$= -\mathcal{N}_1^{\,2}\,\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\Bigg\{\sum_{c\neq i,j,k,r}\Bigg[\sum_{d\neq i,j,k,r,c}\bar{\mathcal{E}}^{(k)(ijr)}_{cd}\,\bar B^{(ijr,kcd)}_{\mu\nu,cd}+2\,\bar{\mathcal{E}}^{(k)(ijr)}_{cr}\,\bar B^{(ijr,kcr)}_{\mu\nu,cr}\Bigg]$$
$$+\,2\sum_{c\neq i,j,k}\bar{\mathcal{E}}^{(k)(ijr)}_{jc}\bar B^{(ijr,kcj)}_{\mu\nu,jc}\Bigg\}\,. \tag{C.35}$$

The improved limits $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR$, $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR$ and their combination $\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR$ appear in the sector topology $\mathcal{W}_{ijjk}$ only, and are given by ($j\neq i$ and $r=r_{ijk}\neq i,j,k$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}RR \equiv -\mathcal{N}_1^{2}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg\{\frac{P_{ij(r)}}{s_{ij}}\,\bar{\mathcal{E}}^{(j)(ijr)}_{cd}+\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\Bigg[\frac{\bar k^{(ijr)}_{c,\mu}}{\bar s^{(ijr)}_{jc}}-\frac{\bar k^{(ijr)}_{d,\mu}}{\bar s^{(ijr)}_{jd}}\Bigg]\Bigg[\frac{\bar k^{(ijr)}_{c,\nu}}{\bar s^{(ijr)}_{jc}}-\frac{\bar k^{(ijr)}_{d,\nu}}{\bar s^{(ijr)}_{jd}}\Bigg]\Bigg\}\bar B^{(ijr,jcd)}_{cd}, \tag{C.36}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{S}}_{ji}\,RR \;\equiv\; -2\,\mathcal{N}_1^{\,2}\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\bar{\mathcal{E}}^{(j)(ijr)}_{cd}\,\bar B^{(ijr,jcd)}_{cd}\,, \tag{C.37}$$

$$\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR \equiv \overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\right)\overline{\mathbf{S}}_{ij}RR \tag{C.38}$$
$$= -\mathcal{N}_1^{2}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\left[\frac{P^{\mathrm{hc}}_{ij(r)}}{s_{ij}}\,\bar{\mathcal{E}}^{(j)(ijr)}_{cd}+\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\left(\frac{\bar k^{(ijr)}_{c,\mu}}{\bar s^{(ijr)}_{jc}}-\frac{\bar k^{(ijr)}_{d,\mu}}{\bar s^{(ijr)}_{jd}}\right)\left(\frac{\bar k^{(ijr)}_{c,\nu}}{\bar s^{(ijr)}_{jc}}-\frac{\bar k^{(ijr)}_{d,\nu}}{\bar s^{(ijr)}_{jd}}\right)\right]\bar B^{(ijr,jcd)}_{cd}.$$

For the strongly-ordered double-unresolved limits involving $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}$, we have ($j\neq i$, $k\neq i,j$, $r=r_{ijk}\neq i,j,k$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR \equiv \mathcal{N}_1^2\Bigg\{\frac{P_{ij(r)}}{s_{ij}}\frac{\bar{P}^{(ijr)\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\bar{B}^{(ijr,jkr)}_{\mu\nu}+2\,C_A\,\bar{\mathcal{E}}^{(k)(ijr)}_{jr}\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\,\bar{B}^{(ijr,jkr)}_{\mu\nu} \tag{C.39}$$
$$-2\,C_{f_k}\,\bar{\mathcal{E}}^{(j)(ijr)}_{kr}\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\frac{\tilde{\bar{k}}^{(ijr)}_{\mu}\tilde{\bar{k}}^{(ijr)}_{\nu}}{\big(\tilde{\bar{k}}^{(ijr)}\big)^2}\bar{B}^{(ijr,jkr)}\Bigg\},$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{C}}_{jik}\,RR \;\equiv\; 2\mathcal{N}_1^2\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\,\frac{\bar{P}^{(ijr)\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\,\bar{B}^{(ijr,jkr)}_{\mu\nu}\,, \tag{C.40}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}RR \equiv 2\mathcal{N}_1^2\,C_{f_k}\,\bar{\mathcal{E}}^{(j)(ijr)}_{kr}\Bigg\{\frac{P_{ij(r)}}{s_{ij}}-\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\frac{\tilde{\bar{k}}^{(ijr)}_{\mu}\tilde{\bar{k}}^{(ijr)}_{\nu}}{\big(\tilde{\bar{k}}^{(ijr)}\big)^2}\Bigg\}\bar{B}^{(ijr,jkr)}\,, \tag{C.41}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{S}}_{ji}\,\overline{\mathbf{C}}_{jik}\,RR \;\equiv\; 4\mathcal{N}_1^2\,C_{f_j}\,C_{f_k}\,\mathcal{E}^{(i)}_{jr}\,\bar{\mathcal{E}}^{(j)(ijr)}_{kr}\,\bar{B}^{(ijr,jkr)}\,, \tag{C.42}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,RR \equiv 2\mathcal{N}_1^2\,C_{f_{[ij]}}\,\bar{\mathcal{E}}^{(k)(ijr)}_{jr}\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\bar{B}^{(ijr,kjr)}_{\mu\nu}\,, \tag{C.43}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{C}}_{jik}\,\overline{\mathbf{SC}}_{kji}\,RR \;\equiv\; 4\mathcal{N}_1^2\,C^2_{f_j}\,\mathcal{E}^{(i)}_{jr}\,\bar{\mathcal{E}}^{(k)(ijr)}_{jr}\,\bar{B}^{(ijr,kjr)}\,, \tag{C.44}$$

$$\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{HC}}^{(\mathbf{c})}_{ijk}\,RR\,\mathcal{Z}_{ijk} \equiv \overline{\mathbf{C}}_{ij}\big(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\big)\,\overline{\mathbf{C}}_{ijk}\,\big(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{SC}}_{kij}\big)\,RR \tag{C.45}$$
$$=\mathcal{N}_1^2\,\frac{P^{\rm hc}_{ij(r)}}{s_{ij}}\,\frac{\bar{P}^{(ijr){\rm hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\,\bar{B}^{(ijr,jkr)}_{\mu\nu}\,.$$

Finally the limits involving $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}$ are given by ($j\neq i$, $k\neq i,j$, $l\neq i,j,k$ and $r=r_{ijkl}\neq i,j,k,l$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,RR \equiv \mathcal{N}_1^2\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\frac{\bar{P}^{(ijr)\rho\sigma}_{kl(r)}}{\bar{s}^{(ijr)}_{kl}}\,\bar{B}^{(ijr,klr)}_{\mu\nu\rho\sigma}\,, \tag{C.46}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{C}}_{jikl}\,RR \;\equiv\; 2\mathcal{N}_1^2\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\,\frac{\bar{P}^{(ijr)\rho\sigma}_{kl(r)}}{\bar{s}^{(ijr)}_{kl}}\bar{B}^{(ijr,klr)}_{\rho\sigma}\,, \tag{C.47}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,RR \equiv \overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijlk}\,RR \;\equiv\; 2\mathcal{N}_1^2\,C_{f_l}\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\bar{\mathcal{E}}^{(k)(ijr)}_{lr}\,\bar{B}^{(ijr,klr)}_{\mu\nu}\,, \tag{C.48}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{SC}}_{kji}\,\overline{\mathbf{C}}_{jikl}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijlk}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{SC}}_{kji}\,\overline{\mathbf{C}}_{jilk}\,RR$$
$$\equiv 4\mathcal{N}_1^2\,C_{f_j}C_{f_l}\,\mathcal{E}^{(i)}_{jr}\,\mathcal{E}^{(k)}_{lr}\,\bar{B}^{(ijr,klr)}\,, \tag{C.49}$$

$$\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{HC}}^{(\mathbf{c})}_{ijkl}\,RR \equiv \overline{\mathbf{C}}_{ij}\,\big(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\big)\,\overline{\mathbf{C}}_{ijkl}\,\big(1-\overline{\mathbf{SC}}_{kij}-\overline{\mathbf{SC}}_{lij}\big)\,RR \tag{C.50}$$
$$=\mathcal{N}_1^2\,\frac{P^{{\rm hc},\mu\nu}_{ij(r)}}{s_{ij}}\,\frac{\bar{P}^{(ijr){\rm hc},\rho\sigma}_{kl(r)}}{\bar{s}^{(ijr)}_{kl}}\,\bar{B}^{(ijr,klr)}_{\mu\nu\rho\sigma}\,.$$

C.2 Improved limits of $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$, $\mathcal{W}_{ijkl}$

Single-unresolved improved limits

For the single-unresolved improved limits we have ($j\neq i$, $k\neq i$, $l\neq i,k$ and $r=r_{ijk}\neq i,j,k$)

$$\overline{\mathbf{S}}_i\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{kl}\,\mathcal{W}^{(\alpha)}_{{\rm s},\,ij}\,, \tag{C.51}$$
$$\overline{\mathbf{C}}_{ij}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{kl}\,\mathcal{W}^{(\alpha)}_{{\rm c},\,ij(r)}\,, \tag{C.52}$$
$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{kl}\,. \tag{C.53}$$

Uniform double-unresolved improved limits

The double-soft improved limit is given by ($j \neq i$, $k \neq i$, $l \neq i, k$)

$$\overline{\mathbf{S}}_{ik}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ijkl}}{\sum_{b\neq i}\sum_{d\neq i,k}\sigma_{ibkd}+\sum_{b\neq k}\sum_{d\neq k,i}\sigma_{kbid}}\,. \tag{C.54}$$

The soft-collinear improved limits $\overline{\mathbf{SC}}_{ikl}$ and $\overline{\mathbf{SC}}_{kij}$ as well as their double-soft versions $\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}$ and $\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{kij}$ read ($j \neq i$, $k \neq i$, $l \neq i, k$)

$$\overline{\mathbf{SC}}_{ikl}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ij}^{(\alpha)}\frac{\sigma_{kl}}{w_{kr}}}{\sum_{b\neq i}\sigma_{ib}^{(\alpha)}\left(\frac{\sigma_{kl}}{w_{kr}}+\frac{\sigma_{lk}}{w_{lr}}\right)+\frac{\sigma_{kl}^{(\alpha)}}{w_{kr}}\sum_{d\neq i,k}\sigma_{id}+\frac{\sigma_{lk}^{(\alpha)}}{w_{lr}}\sum_{d\neq i,l}\sigma_{id}}\,, \qquad r = r_{ikl}\,, \tag{C.55}$$

$$\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkl} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}}\sigma_{kl}}{\sum_{b\neq k}\sigma_{kb}^{(\alpha)}\left(\frac{\sigma_{ij}}{w_{ir}}+\frac{\sigma_{ji}}{w_{jr}}\right)+\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}}\sum_{d\neq i,k}\sigma_{kd}+\frac{\sigma_{ji}^{(\alpha)}}{w_{jr}}\sum_{d\neq k,j}\sigma_{kd}}\,, \qquad r = r_{ijk}\,, \tag{C.56}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ij}^{(\alpha)}\sigma_{kl}}{\sum_{b\neq i}\sigma_{ib}^{(\alpha)}\sigma_{kl}+\sigma_{kl}^{(\alpha)}\sum_{d\neq i,k}\sigma_{id}}\,, \qquad r = r_{ikl}\,, \tag{C.57}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ij}^{(\alpha)}\sigma_{kl}}{\sum_{b\neq k}\sigma_{kb}^{(\alpha)}\sigma_{ij}+\sigma_{ij}^{(\alpha)}\sum_{d\neq i,k}\sigma_{kd}}\,, \qquad r = r_{ijk}\,. \tag{C.58}$$

For the uniform double-unresolved limits involving $\overline{\mathbf{C}}_{ijk}$, we have ($j \neq i$, $k \neq i, j$ and $r = r_{ijk} \neq i, j, k$)

$$\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijjk} \equiv \frac{\hat{\sigma}_{ijjk(r)}}{\hat{\sigma}_{\{ijk\}(r)}}\,, \qquad \overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijkj} \equiv \frac{\hat{\sigma}_{ijkj(r)}}{\hat{\sigma}_{\{ijk\}(r)}}\,; \tag{C.59}$$

$$\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijjk} \equiv \frac{\hat{\sigma}_{ijjk(r)}}{\hat{\sigma}_{ijjk(r)}+\hat{\sigma}_{ikjk(r)}+\hat{\sigma}_{jiik(r)}+\hat{\sigma}_{jkik(r)}}\,, \tag{C.60}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijkj} \equiv \frac{\hat{\sigma}_{ijkj(r)}}{\hat{\sigma}_{ijkj(r)}+\hat{\sigma}_{ikkj(r)}+\hat{\sigma}_{kiij(r)}+\hat{\sigma}_{kjij(r)}}\,; \tag{C.61}$$

$$\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,\mathcal{W}_{ijjk} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{jk}}{w_{jr}}}{\frac{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}{w_{ir}^{\alpha}}\left(\frac{\sigma_{jk}}{w_{jr}}+\frac{\sigma_{kj}}{w_{kr}}\right)+\frac{\sigma_{jk}^{(\alpha)}}{w_{jr}^{\alpha}}\frac{\sigma_{ik}}{w_{ir}}+\frac{\sigma_{kj}^{(\alpha)}}{w_{kr}^{\alpha}}\frac{\sigma_{ij}}{w_{ir}}}\,, \tag{C.62}$$

$$\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,\mathcal{W}_{ijkj} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}{\frac{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}{w_{ir}^{\alpha}}\left(\frac{\sigma_{jk}}{w_{jr}}+\frac{\sigma_{kj}}{w_{kr}}\right)+\frac{\sigma_{jk}^{(\alpha)}}{w_{jr}^{\alpha}}\frac{\sigma_{ik}}{w_{ir}}+\frac{\sigma_{kj}^{(\alpha)}}{w_{kr}^{\alpha}}\frac{\sigma_{ij}}{w_{ir}}}\,, \tag{C.63}$$

$$\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkj} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}{\frac{\sigma_{kj}^{(\alpha)}+\sigma_{ki}^{(\alpha)}}{w_{kr}^{\alpha}}\left(\frac{\sigma_{ji}}{w_{jr}}+\frac{\sigma_{ij}}{w_{ir}}\right)+\frac{\sigma_{ji}^{(\alpha)}}{w_{jr}^{\alpha}}\frac{\sigma_{ki}}{w_{kr}}+\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}\,; \tag{C.64}$$

$$\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,\mathcal{W}_{ijjk} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{jk}}{w_{jr}}}{\frac{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{jk}}{w_{jr}}+\frac{\sigma_{jk}^{(\alpha)}}{w_{jr}^{\alpha}}\frac{\sigma_{ik}}{w_{ir}}}\,, \tag{C.65}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,\mathcal{W}_{ijkj} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}{\frac{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}+\frac{\sigma_{kj}^{(\alpha)}}{w_{kr}^{\alpha}}\frac{\sigma_{ij}}{w_{ir}}}\,, \tag{C.66}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkj} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}{\frac{\sigma_{kj}^{(\alpha)}+\sigma_{ki}^{(\alpha)}}{w_{kr}^{\alpha}}\frac{\sigma_{ij}}{w_{ir}}+\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}\,. \tag{C.67}$$

Finally the limits involving $\overline{\mathbf{C}}_{ijkl}$ are given by ($j \neq i$, $k \neq i,j$, $l \neq i,j,k$ and $r = r_{ijkl} \neq i,j,k,l$)

$$\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \frac{\frac{\sigma_{ijkl}}{w_{ir}w_{kr}}}{\frac{\sigma_{ijkl}+\sigma_{klij}}{w_{ir}w_{kr}}+\frac{\sigma_{ijlk}+\sigma_{lkij}}{w_{ir}w_{lr}}+\frac{\sigma_{jikl}+\sigma_{klji}}{w_{jr}w_{kr}}+\frac{\sigma_{jilk}+\sigma_{lkji}}{w_{jr}w_{lr}}}\,, \tag{C.68}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ij}^{(\alpha)}\sigma_{kl}}{\sigma_{ij}^{(\alpha)}\sigma_{kl}+\sigma_{kl}^{(\alpha)}\sigma_{ij}}\,, \tag{C.69}$$

$$\overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ij}^{(\alpha)}\frac{\sigma_{kl}}{w_{kr}}}{\sigma_{ij}^{(\alpha)}\left(\frac{\sigma_{kl}}{w_{kr}}+\frac{\sigma_{lk}}{w_{lr}}\right)+\left(\frac{\sigma_{kl}^{(\alpha)}}{w_{kr}}+\frac{\sigma_{lk}^{(\alpha)}}{w_{lr}}\right)\sigma_{ij}}\,, \tag{C.70}$$

$$\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}}\sigma_{kl}}{\sigma_{kl}^{(\alpha)}\left(\frac{\sigma_{ij}}{w_{ir}}+\frac{\sigma_{ji}}{w_{jr}}\right)+\left(\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}}+\frac{\sigma_{ji}^{(\alpha)}}{w_{jr}}\right)\sigma_{kl}}\,. \tag{C.71}$$

Strongly-ordered double-unresolved improved limits

The improved limit $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}$ is given by ($j \neq i$, $k \neq i$, $l \neq i,k$)

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{\mathrm{s},\,kl}\,\mathcal{W}^{(\alpha)}_{\mathrm{s},\,ij}\,. \tag{C.72}$$

For $\overline{\mathbf{S}}_i\,\overline{\mathbf{SC}}_{ikl}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}$ we have ($j \neq i$, $k \neq i$, $l \neq i,k$, and $r = r_{ikl} \neq i,k,l$)

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{SC}}_{ikl}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{\mathrm{c},\,kl(r)}\,\mathcal{W}^{(\alpha)}_{\mathrm{s},\,ij}\,, \tag{C.73}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}\,\mathcal{W}_{ijkl} \equiv \mathcal{W}^{(\alpha)}_{\mathrm{s},\,ij}\,. \tag{C.74}$$

For the strongly-ordered double-unresolved limits involving $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk}$, we have ($j \neq i$, $k \neq i,j$, $r = r_{ijk} \neq i,j,k$ and $\tau = jk, kj$)

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})\mathcal{W}_{ij\tau} \equiv \overline{\mathcal{W}}_{\mathrm{c},\tau(r)}\,\frac{\sigma_{ij}^{(\alpha)}}{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}\,, \tag{C.75}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})\mathcal{W}_{ijjk} \equiv \frac{\sigma_{ij}^{(\alpha)}}{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}\,, \tag{C.76}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})\mathcal{W}_{ijkj} \;\equiv\; \frac{\sigma_{ij}^{(\alpha)}}{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}\,. \tag{C.77}$$

For $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}$ we have ($j\neq i$, $k\neq i$, $l\neq i,k$, and $r=r_{ijk}\neq i,j,k$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkl} \;\equiv\; \mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\,\overline{\mathcal{W}}_{\mathrm{s},\,kl}\,; \tag{C.78}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkl} \;\equiv\; \overline{\mathcal{W}}_{\mathrm{s},\,kl}\,. \tag{C.79}$$

The improved limits $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR\,\mathcal{W}_{ijjk}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR\,\mathcal{W}_{ijjk}$ read ($j\neq i$, $k\neq i,j$ and $r=r_{ijk}\neq i,j,k$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\mathcal{W}_{ijjk} \;\equiv\; \mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\,\overline{\mathcal{W}}_{\mathrm{s},\,jk}\,; \tag{C.80}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\mathcal{W}_{ijjk} \;\equiv\; \overline{\mathcal{W}}_{\mathrm{s},\,jk}\,. \tag{C.81}$$

For the strongly-ordered double-unresolved limits involving $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}$, we have ($j\neq i$, $k\neq i,j$, $r=r_{ijk}\neq i,j,k$, and $\tau=jk,kj$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ij\tau} \;\equiv\; \mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\,\overline{\mathcal{W}}_{\mathrm{c},\,\tau(r)}\,; \tag{C.82}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ij\tau} \;\equiv\; \overline{\mathcal{W}}_{\mathrm{c},\,\tau(r)}\,; \tag{C.83}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijjk} \;\equiv\; \mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\,; \tag{C.84}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijjk} \;\equiv\; 1\,; \tag{C.85}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkj} \;\equiv\; \mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\,; \tag{C.86}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkj} \;\equiv\; 1\,. \tag{C.87}$$

Finally the limits involving $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}$ are given by ($j\neq i$, $k\neq i,j$, $l\neq i,j,k$ and $r=r_{ijkl}\neq i,j,k,l$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \;\equiv\; \mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\,\overline{\mathcal{W}}_{\mathrm{c},\,kl(r)}\,; \tag{C.88}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \;\equiv\; \overline{\mathcal{W}}_{\mathrm{c},\,kl(r)}\,; \tag{C.89}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \;\equiv\; \mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\,; \tag{C.90}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \;\equiv\; 1\,. \tag{C.91}$$

C.3 Improved limits of $\mathcal{Z}_{ijk}$, $\mathcal{Z}_{ijkl}$

Single-unresolved improved limits

For the single-unresolved improved limits in $K^{(\mathbf{1})}_{\{ijk\}}$ we have ($j\neq i$, $k\neq i,j$)

$$\overline{\mathbf{S}}_i\,\mathcal{Z}_{ijk} \;=\; \bar{\mathcal{Z}}_{jk}\Big(\mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij}+\mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ik}\Big)\,, \qquad \overline{\mathbf{HC}}_{ij}\,\mathcal{Z}_{ijk} \;=\; \bar{\mathcal{Z}}_{jk}\,; \tag{C.92}$$

while for $K^{(\mathbf{1})}_{\{ijkl\}}$ we have ($j\neq i$, $k\neq i,j$)

$$\overline{\mathbf{S}}_i\,\mathcal{Z}_{ijkl} \;=\; \bar{\mathcal{Z}}_{kl}\,\mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij}\,, \qquad \overline{\mathbf{HC}}_{ij}\,\mathcal{Z}_{ijkl} \;=\; \bar{\mathcal{Z}}_{kl}\,. \tag{C.93}$$

Uniform double-unresolved improved limits

For $K^{(\mathbf{2})}_{\{ijk\}}$ we have ($j \neq i$, $k \neq i, j$, and $r = r_{ijk} \neq i, j, k$)

$$\begin{aligned}
\overline{\mathbf{S}}_{ik}\, \mathcal{Z}_{ijk} &= \frac{\sigma_{ikkj} + \sigma_{ijkj} + \sigma_{kiij} + \sigma_{kjij}}{\sum_{b\neq i}\sum_{d\neq i,k}\sigma_{ibkd} + \sum_{b\neq k}\sum_{d\neq k,i}\sigma_{kbid}}\,, \\
\overline{\mathbf{SC}}_{ijk}\, \mathcal{Z}_{ijk} &= \frac{\left(\sigma^{(\alpha)}_{ij} + \sigma^{(\alpha)}_{ik}\right)\left(\frac{\sigma_{jk}}{w_{jr}} + \frac{\sigma_{kj}}{w_{kr}}\right) + \frac{\sigma^{(\alpha)}_{jk}}{w_{jr}}\sigma_{ik} + \frac{\sigma^{(\alpha)}_{kj}}{w_{kr}}\sigma_{ij}}{\sum_{b\neq i}\sigma^{(\alpha)}_{ib}\left(\frac{\sigma_{jk}}{w_{jr}} + \frac{\sigma_{kj}}{w_{kr}}\right) + \frac{\sigma^{(\alpha)}_{jk}}{w_{jr}}\sum_{d\neq i,j}\sigma_{id} + \frac{\sigma^{(\alpha)}_{kj}}{w_{kr}}\sum_{d\neq i,k}\sigma_{id}}\,, \\
\overline{\mathbf{S}}_{ij}\, \overline{\mathbf{SC}}_{ijk}\, \mathcal{Z}_{ijk} &= \frac{\left(\sigma^{(\alpha)}_{ij} + \sigma^{(\alpha)}_{ik}\right)\sigma_{jk} + \sigma^{(\alpha)}_{jk}\sigma_{ik}}{\sum_{b\neq i}\sigma^{(\alpha)}_{ib}\sigma_{jk} + \sigma^{(\alpha)}_{jk}\sum_{d\neq i,j}\sigma_{id}}\,, \\
\overline{\mathbf{HC}}_{ijk}\, \mathcal{Z}_{ijk} &= 1\,, \\
\overline{\mathbf{C}}_{ijk}\, \overline{\mathbf{SHC}}_{ijk}\, \mathcal{Z}_{ijk} &= 1\,.
\end{aligned} \tag{C.94}$$

For $K^{(\mathbf{2})}_{\{ijkl\}}$ one has ($j \neq i$, $k \neq i, j$, $l \neq i, j, k$, and $r = r_{ikl} \neq i, k, l$)

$$\begin{aligned}
\overline{\mathbf{S}}_{ik}\, \mathcal{Z}_{ijkl} &= \frac{\sigma_{ijkl} + \sigma_{klij}}{\sum_{b\neq i}\sum_{d\neq i,k}\sigma_{ibkd} + \sum_{b\neq k}\sum_{d\neq k,i}\sigma_{kbid}}\,, \\
\overline{\mathbf{SC}}_{ikl}\, \mathcal{Z}_{ijkl} &= \frac{\sigma^{(\alpha)}_{ij}\left(\frac{\sigma_{kl}}{w_{kr}} + \frac{\sigma_{lk}}{w_{lr}}\right) + \left(\frac{\sigma^{(\alpha)}_{kl}}{w_{kr}} + \frac{\sigma^{(\alpha)}_{lk}}{w_{lr}}\right)\sigma_{ij}}{\sum_{b\neq i}\sigma^{(\alpha)}_{ib}\left(\frac{\sigma_{kl}}{w_{kr}} + \frac{\sigma_{lk}}{w_{lr}}\right) + \frac{\sigma^{(\alpha)}_{kl}}{w_{kr}}\sum_{d\neq i,k}\sigma_{id} + \frac{\sigma^{(\alpha)}_{lk}}{w_{lr}}\sum_{d\neq i,l}\sigma_{id}}\,, \\
\overline{\mathbf{S}}_{ik}\, \overline{\mathbf{SC}}_{ikl}\, \mathcal{Z}_{ijkl} &= \frac{\sigma^{(\alpha)}_{ij}\sigma_{kl} + \sigma^{(\alpha)}_{kl}\sigma_{ij}}{\sum_{b\neq i}\sigma^{(\alpha)}_{ib}\sigma_{kl} + \sigma^{(\alpha)}_{kl}\sum_{d\neq i,k}\sigma_{id}}\,, \\
\overline{\mathbf{HC}}_{ijkl}\, \mathcal{Z}_{ijkl} &= 1\,.
\end{aligned} \tag{C.95}$$

Strongly-ordered double-unresolved improved limits

For $K^{(\mathbf{12})}_{\{ijk\}}$ one has ($j \neq i$, $k \neq i, j$)

$$\begin{aligned}
\overline{\mathbf{S}}_{i}\, \overline{\mathbf{S}}_{ij}\, \mathcal{Z}_{ijk} &= \bar{\mathcal{Z}}_{\mathrm{s},\,jk}\left(\mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij} + \mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ik}\right), \\
\overline{\mathbf{S}}_{i}\, \overline{\mathbf{SHC}}_{ijk}\, \mathcal{Z}_{ijk} &= \mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij} + \mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ik}\,, \\
\overline{\mathbf{S}}_{i}\, \overline{\mathbf{HC}}^{(\mathbf{s})}_{ijk}\, \mathcal{Z}_{ijk} &= 1\,, \\
\overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{S}}_{ij}\, \mathcal{Z}_{ijk} &= \bar{\mathcal{Z}}_{\mathrm{s},\,jk}\,, \\
\overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{SC}}_{kij}\, \mathcal{Z}_{ijk} &= \bar{\mathcal{Z}}_{\mathrm{s},\,kj}\,, \\
\overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{HC}}^{(\mathbf{c})}_{ijk}\, \mathcal{Z}_{ijk} &= 1\,.
\end{aligned} \tag{C.96}$$

For $K^{(\mathbf{12})}_{\{ijkl\}}$ one has ($j \neq i$, $k \neq i, j$, $l \neq i, j, k$)

$$\begin{aligned}
\overline{\mathbf{S}}_{i}\, \overline{\mathbf{S}}_{ik}\, \mathcal{Z}_{ijkl} &= \bar{\mathcal{Z}}_{\mathrm{s},\,kl}\,\mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij}\,, \\
\overline{\mathbf{S}}_{i}\, \overline{\mathbf{SHC}}_{ikl}\, \mathcal{Z}_{ijkl} &= \mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij}\,, \\
\overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{SC}}_{kij}\, \mathcal{Z}_{ijkl} &= \bar{\mathcal{Z}}_{\mathrm{s},\,kl}\,, \\
\overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{HC}}^{(\mathbf{c})}_{ijkl}\, \mathcal{Z}_{ijkl} &= 1\,.
\end{aligned} \tag{C.97}$$

D Integration of azimuthal contributions

The azimuthal parts of the collinear kernels $Q^{\mu\nu}_{ij(r)}$, $\tilde{Q}^{\mu\nu}_{ij(r)}$ and $Q^{\mu\nu}_{ijk(r)}$, defined in Appendix B, contain $\tilde{k}^\mu_a \tilde{k}^\nu_a$, where $a = i$ for $P^{\mu\nu}_{ij(r)}$, $\tilde{P}^{\mu\nu}_{ij(r)}$ and $a = i, j, k$ for $P^{\mu\nu}_{ijk(r)}$. In all counterterms, $Q^{\mu\nu}_{ij(r)}$ has to be integrated in the single-radiative phase space $d\Phi^{(ijr)}_{\text{rad}}$, $d\Phi^{(irj)}_{\text{rad}}$ or $d\Phi^{(jri)}_{\text{rad}}$, while $\tilde{Q}^{\mu\nu}_{ij(r)}$ and $Q^{\mu\nu}_{ijk(r)}$ are always integrated in $d\Phi^{(ijr)}_{\text{rad}}$ and $d\Phi^{(ijkr)}_{\text{rad},2}$, respectively. In all cases, when integrating $Q^{\mu\nu}_{ij(r)}$ and $\tilde{Q}^{\mu\nu}_{ij(r)}$ in their single-radiative phase space, or $Q^{\mu\nu}_{ijk(r)}$ in its double-radiative phase space, the integral of the tensor structure $\tilde{k}^\mu_a \tilde{k}^\nu_a$ must be a symmetric rank-2 tensor constructed combining $g^{\mu\nu}$ and mapped momenta, see [99]. Thus

$$\int d\Phi^{(\tau)}_{\text{rad}}\, f(\{k\})\, \tilde{k}^\mu_a \tilde{k}^\nu_a = A\, g^{\mu\nu} + B\, \bar{k}^{(\tau)\mu} \bar{k}^{(\tau)\nu} + C\Big(\bar{k}^{(\tau)\mu} \bar{k}^{(\tau)\nu}_q + \bar{k}^{(\tau)\mu}_q \bar{k}^{(\tau)\nu}\Big) + D\, \bar{k}^{(\tau)\mu}_q \bar{k}^{(\tau)\nu}_q\,, \tag{D.1}$$

where $\tau = ijr, irj, jri, ijkr$, $q = r$ if $\tau = ijr, irj, jri$, $q = r$ if $\tau = ijkr$, and

$$\bar{k}^{(ijr)} = \bar{k}^{(ijr)}_j\,, \qquad \bar{k}^{(irj)} = \bar{k}^{(irj)}_j\,, \qquad \bar{k}^{(jri)} = \bar{k}^{(jri)}_i\,, \qquad \bar{k}^{(ijkr)} = \bar{k}^{(ijkr)}_k\,. \tag{D.2}$$

Since $\tilde{k}_a$ is orthogonal to $\bar{k}^{(\tau)\mu}$ and $\bar{k}^{(\tau)\mu}_q$, so must be also its integrals. This leads to the conditions $D = 0$ and $A + C\, \bar{k}^{(\tau)} \cdot \bar{k}^{(\tau)}_q = 0$. We have

$$\int d\Phi^{(\tau)}_{\text{rad}}\, f(\{k\})\, \tilde{k}^\mu_a \tilde{k}^\nu_a = A\left[g^{\mu\nu} - \frac{\bar{k}^{(\tau)\mu} \bar{k}^{(\tau)\nu}_q + \bar{k}^{(\tau)\mu}_q \bar{k}^{(\tau)\nu}}{\bar{k}^{(\tau)} \cdot \bar{k}^{(\tau)}_q}\right] + B\, \bar{k}^{(\tau)\mu} \bar{k}^{(\tau)\nu}\,. \tag{D.3}$$

In all counterterms this tensor is contracted with either

$$\bar{R}^{(\tau)}_{\mu\nu}\,, \qquad \bar{B}^{(\tau)}_{\mu\nu}\,, \qquad \bar{B}^{(\tau,\dots)}_{\mu\nu}\,, \qquad \text{or} \qquad \left[\frac{\bar{k}^{(\tau)}_{c,\mu}}{\bar{s}^{(\tau)}_{jc}} - \frac{\bar{k}^{(\tau)}_{d,\mu}}{\bar{s}^{(\tau)}_{jd}}\right]\left[\frac{\bar{k}^{(\tau)}_{c,\nu}}{\bar{s}^{(\tau)}_{jc}} - \frac{\bar{k}^{(\tau)}_{d,\nu}}{\bar{s}^{(\tau)}_{jd}}\right]. \tag{D.4}$$

As a consequence, the terms proportional to $\bar{k}^{(\tau)\mu}$ or to $\bar{k}^{(\tau)\nu}$ vanish, and just $A\, g^{\mu\nu}$ contributes. On the other hand, since $\bar{k}^{(\tau)}$ is on shell, A can be obtained as follows:

$$g_{\mu\nu} \int d\Phi^{(\tau)}_{\text{rad}}\, f(\{k\})\, \tilde{k}^\mu_a \tilde{k}^\nu_a \;=\; A\,(d-2) \quad \Longrightarrow \quad A \;=\; \frac{1}{d-2} \int d\Phi^{(\tau)}_{\text{rad}}\, f(\{k\})\, \tilde{k}^2_a\,. \tag{D.5}$$

Thus in all counterterms we can subtitute

$$\int d\Phi^{(\tau)}_{\text{rad}}\, f(\{k\})\, \tilde{k}^\mu_a \tilde{k}^\nu_a \;\to\; A\, g^{\mu\nu} \;=\; \int d\Phi^{(\tau)}_{\text{rad}}\, f(\{k\})\, \frac{g^{\mu\nu}}{d-2}\, \tilde{k}^2_a\,, \tag{D.6}$$

and the integrals of $Q^{\mu\nu}_{ij(r)}$, $\tilde{Q}^{\mu\nu}_{ij(r)}$ and $Q^{\mu\nu}_{ijk(r)}$ vanish in all counterterms:

$$\begin{aligned}
\int d\Phi^{(\tau)}_{\text{rad}}\, \frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}} &= \int d\Phi^{(\tau)}_{\text{rad}}\, \frac{Q_{ij(r)}}{s_{ij}}\left[-\, g^{\mu\nu} + (d-2)\, \frac{\tilde{k}^\mu_i \tilde{k}^\nu_i}{\tilde{k}^2_i}\right] \to 0\,, \qquad \tau = ijr, irj, jri\,;\\
\int d\Phi^{(\tau)}_{\text{rad}}\, \frac{\tilde{Q}^{\mu\nu}_{ij(r)}}{s_{ij}} &= \int d\Phi^{(\tau)}_{\text{rad}}\, \frac{\tilde{Q}_{ij(r)}}{s_{ij}}\left[-\, g^{\mu\nu} + (d-2)\, \frac{\tilde{k}^\mu_i \tilde{k}^\nu_i}{\tilde{k}^2_i}\right] \to 0\,, \qquad \tau = ijr\,;\\
\int d\Phi^{(\tau)}_{\text{rad},2}\, \frac{Q^{\mu\nu}_{ijk(r)}}{s^2_{ijk}} &= \sum_{a=i,j,k} \int d\Phi^{(\tau)}_{\text{rad},2}\, \frac{Q^{(a)}_{ijk(r)}}{s^2_{ijk}}\left[-\, g^{\mu\nu} + (d-2)\, \frac{\tilde{k}^\mu_a \tilde{k}^\nu_a}{\tilde{k}^2_a}\right] \to 0\,, \qquad \tau = ijkr\,.
\end{aligned} \tag{D.7}$$

E Constituent integrals

In the following we report the constituent integrals relevant for the analytic integration of all counterterms at NNLO. Such integrals are schematically denoted as J^ℓ_t, where t indicates the type of integral, while ℓ is a set of labels whose different indices denote distinguished particles.

The soft integrated kernel is

$$J_{\rm s}^{ilm} \;\equiv\; \mathcal{N}_1 \int d\Phi_{\rm rad}^{(ilm)}\, \mathcal{E}_{lm}^{(i)} \;\equiv\; \delta_{f_i g}\, J_s(\bar{s}_{lm}^{(ilm)})\,, \tag{E.1}$$

with

$$\begin{aligned} J_{\rm s}(s) \;&=\; \frac{\alpha_s}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \\ &=\; \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\Bigg[\;\frac{1}{\epsilon^2}+\frac{2}{\epsilon}+6-\frac{7}{12}\pi^2+\left(18-\frac{7}{6}\pi^2-\frac{25}{3}\zeta_3\right)\epsilon \\ &\qquad +\left(54-\frac{7}{2}\pi^2-\frac{50}{3}\zeta_3-\frac{71}{1440}\pi^4\right)\epsilon^2+\mathcal{O}(\epsilon^3)\Bigg]. \end{aligned} \tag{E.2}$$

The double-soft integrated kernels read

$$\begin{aligned} J_{\rm s\otimes s}^{ijcdef} &\equiv \mathcal{N}_1^2\int d\Phi_{\rm rad,2}^{(icd,jef)}\,\mathcal{E}_{cd}^{(i)}\,\mathcal{E}_{ef}^{(j)} \;\equiv\; J_{\rm s\otimes s}^{(4)}\left(\bar{s}_{cd}^{(icd,jef)},\bar{s}_{ef}^{(icd,jef)}\right) f_{ij}^{gg}\,, \\ J_{\rm s\otimes s}^{ijcde} &\equiv \mathcal{N}_1^2\int d\Phi_{\rm rad,2}^{(icd,jed)}\,\mathcal{E}_{cd}^{(i)}\,\mathcal{E}_{ed}^{(j)} \;\equiv\; J_{\rm s\otimes s}^{(3)}\left(\bar{s}_{cd}^{(icd,jed)},\bar{s}_{ed}^{(icd,jed)}\right) f_{ij}^{gg}\,, \\ J_{\rm s\otimes s}^{ijcd} &\equiv \mathcal{N}_1^2\int d\Phi_{\rm rad,2}^{(ijcd)}\,\mathcal{E}_{cd}^{(i)}\,\mathcal{E}_{cd}^{(j)} \;\equiv\; J_{\rm s\otimes s}^{(2)}\left(\bar{s}_{cd}^{(ijcd)}\right) f_{ij}^{gg}\,, \\ J_{\rm ss}^{ijcd} &\equiv \mathcal{N}_1^2\int d\Phi_{\rm rad,2}^{(ijcd)}\,\mathcal{E}_{cd}^{(ij)} \;\equiv\; 2\,T_R\,J_{\rm ss}^{(\rm q\bar q)}\left(\bar{s}_{cd}^{(ijcd)}\right) f_{ij}^{q\bar q} - 2\,C_A\,J_{\rm ss}^{(\rm gg)}\left(\bar{s}_{cd}^{(ijcd)}\right) f_{ij}^{gg}\,, \end{aligned} \tag{E.3}$$

with

$$\begin{aligned} J_{\rm s\otimes s}^{(4)}(s,s') \;&=\; \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon}\Bigg[\;\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(16-\frac{7}{6}\pi^2\right)\frac{1}{\epsilon^2}+\left(60-\frac{14}{3}\pi^2-\frac{50}{3}\zeta_3\right)\frac{1}{\epsilon} \\ &\qquad +216-\frac{56}{3}\pi^2-\frac{200}{3}\zeta_3+\frac{29}{120}\pi^4+\mathcal{O}(\epsilon)\Bigg], \\ J_{\rm s\otimes s}^{(3)}(s,s') \;&=\; \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon}\Bigg[\;\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(17-\frac{4}{3}\pi^2\right)\frac{1}{\epsilon^2}+\left(70-\frac{16}{3}\pi^2-\frac{68}{3}\zeta_3\right)\frac{1}{\epsilon} \\ &\qquad +284-\frac{68}{3}\pi^2-\frac{272}{3}\zeta_3+\frac{13}{90}\pi^4+\mathcal{O}(\epsilon)\Bigg], \\ J_{\rm s\otimes s}^{(2)}(s) \;&=\; \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\;\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(18-\frac{3}{2}\pi^2\right)\frac{1}{\epsilon^2}+\left(76-6\pi^2-\frac{74}{3}\zeta_3\right)\frac{1}{\epsilon} \\ &\qquad +312-27\pi^2-\frac{308}{3}\zeta_3+\frac{49}{120}\pi^4+\mathcal{O}(\epsilon)\Bigg], \\ J_{\rm ss}^{(\rm q\bar q)}(s) \;&=\; \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\;\frac{1}{6}\,\frac{1}{\epsilon^3}+\frac{17}{18}\,\frac{1}{\epsilon^2}+\left(\frac{116}{27}-\frac{7}{36}\,\pi^2\right)\frac{1}{\epsilon}+\frac{1474}{81}-\frac{131}{108}\,\pi^2-\frac{19}{9}\,\zeta_3+\mathcal{O}(\epsilon)\Bigg], \\ J_{\rm ss}^{(\rm gg)}(s) \;&=\; \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\;\frac{1}{2}\,\frac{1}{\epsilon^4}+\frac{35}{12}\,\frac{1}{\epsilon^3}+\left(\frac{487}{36}-\frac{2}{3}\,\pi^2\right)\frac{1}{\epsilon^2}+\left(\frac{1562}{27}-\frac{269}{72}\,\pi^2-\frac{77}{6}\,\zeta_3\right)\frac{1}{\epsilon} \\ &\qquad +\frac{19351}{81}-\frac{3829}{216}\,\pi^2-\frac{1025}{18}\,\zeta_3-\frac{23}{240}\,\pi^4+\mathcal{O}(\epsilon)\;\Bigg]. \end{aligned} \tag{E.4}$$

The soft real-virtual integrated kernels are

$$
\begin{aligned}
\tilde{J}_{\rm s}^{icd} &\equiv \mathcal{N}_1 \int d\Phi_{\rm rad}^{(icd)}\, \tilde{\mathcal{E}}_{cd}^{(i)} \,\equiv\, \delta_{f_i g}\, C_A\, \tilde{J}_{\rm s}\Big(\bar{s}_{cd}^{(icd)}\Big)\,, \\
J_{\Delta{\rm s}}^{icd(e)} &\equiv \mathcal{N}_1 \frac{2}{\epsilon^2} \int d\Phi_{\rm rad}^{(icd)}\, \mathcal{E}_{cd}^{(i)} \left[\left(\frac{s_{ed}}{\bar{s}_{ed}^{(icd)}} \right)^{-\epsilon} - 1 \right] \,\equiv\, f_i^g\, J_{\Delta{\rm s}}^{(3)}\Big(\bar{s}_{cd}^{(icd)}\Big)\,, \\
J_{\Delta{\rm s}}^{icd} &\equiv \mathcal{N}_1 \frac{1}{\epsilon^2} \int d\Phi_{\rm rad}^{(icd)}\, \mathcal{E}_{cd}^{(i)} \left[\left(\frac{s_{cd}}{\bar{s}_{cd}^{(icd)}} \right)^{-\epsilon} - 1 \right] \,\equiv\, f_i^g\, J_{\Delta{\rm s}}^{(2)}\Big(\bar{s}_{cd}^{(icd)}\Big)\,, \\
\tilde{J}_{\rm s}^{icde} &\equiv \mathcal{N}_1 \int d\Phi_{\rm rad}^{(icd)}\, \tilde{\mathcal{E}}_{cde}^{(i)}\,,
\end{aligned}
\tag{E.5}
$$

with

$$
\begin{aligned}
\tilde{J}_{\rm s}(s) &= \frac{\alpha_s}{2\pi} \left(\frac{s}{e^{\gamma_E} \mu^2} \right)^{-2\epsilon} \frac{\Gamma^3(1+\epsilon)\Gamma^3(1-\epsilon)}{4\,\epsilon^4\,\Gamma(1+2\epsilon)\Gamma(2-4\epsilon)} \qquad \text{(E.6)} \\
&= \frac{\alpha_s}{2\pi} \left(\frac{s}{\mu^2} \right)^{-2\epsilon} \bigg[\frac{1}{4\epsilon^4} + \frac{1}{\epsilon^3} + \left(4 - \frac{7}{24}\pi^2 \right) \frac{1}{\epsilon^2} + \left(16 - \frac{7}{6}\pi^2 - \frac{14}{3}\zeta_3 \right) \frac{1}{\epsilon} \\
&\qquad + 64 - \frac{14}{3}\pi^2 - \frac{56}{3}\zeta_3 - \frac{7}{480}\pi^4 + \mathcal{O}(\epsilon) \bigg]\,, \\
J_{\Delta{\rm s}}^{(3)}(s) &= \frac{\alpha_s}{2\pi} \left(\frac{s}{\mu^2} \right)^{-\epsilon} \left[\left(2 - \frac{\pi^2}{3} \right) \frac{1}{\epsilon^2} + \left(16 - \frac{2}{3}\pi^2 - 12\,\zeta_3 \right) \frac{1}{\epsilon} + 92 - \frac{7}{2}\pi^2 - 24\,\zeta_3 - \frac{7}{18}\pi^4 + \mathcal{O}(\epsilon) \right], \\
J_{\Delta{\rm s}}^{(2)}(s) &= \frac{\alpha_s}{2\pi} \left(\frac{s}{\mu^2} \right)^{-\epsilon} \left[\left(2 - \frac{\pi^2}{3} \right) \frac{1}{\epsilon^2} + \left(14 - \frac{2}{3}\pi^2 - 10\,\zeta_3 \right) \frac{1}{\epsilon} + 74 - \frac{23}{6}\pi^2 - 20\,\zeta_3 - \frac{7}{36}\pi^4 + \mathcal{O}(\epsilon) \right],
\end{aligned}
$$

$$
\sum_{\substack{c\neq i, d\neq i,c \\ e\neq i,c,d}} \tilde{J}_{\rm s}^{icde}\, B_{cde} \,=\, -\, f_i^g\, \frac{\alpha_s}{2\pi} \sum_{\substack{c\neq i, d\neq i,c \\ e\neq i,c,d}} B_{cde} \left[\frac{1}{2} \ln\frac{\bar{s}_{ce}}{\bar{s}_{de}} \ln^2\frac{\bar{s}_{cd}}{\mu^2} + \frac{1}{6} \ln^3\frac{\bar{s}_{ce}}{\bar{s}_{de}} + {\rm Li}_3\left(-\frac{\bar{s}_{ce}}{\bar{s}_{de}} \right) + \mathcal{O}(\epsilon) \right].
$$

The hard-collinear integrated kernels are given by

$$
\begin{aligned}
J_{\rm hc}^{ijr} &\equiv \mathcal{N}_1 \int d\Phi_{\rm rad}^{(ijr)}\, \frac{P_{ij(r)}^{\rm hc}}{s_{ij}} \\
&\equiv J_{\rm hc}^{(0{\rm g})}\Big(\bar{s}_{jr}^{(ijr)}\Big)\, f_{ij}^{q\bar{q}} + J_{\rm hc}^{(1{\rm g})}\Big(\bar{s}_{jr}^{(ijr)}\Big)\, (f_{ij}^{gq} + f_{ij}^{g\bar{q}}) + J_{\rm hc}^{(2{\rm g})}\Big(\bar{s}_{jr}^{(ijr)}\Big)\, f_{ij}^{gg}\,,
\end{aligned}
\tag{E.7}
$$

where

$$
\begin{aligned}
J_{\rm hc}^{(0{\rm g})}(s) &= \frac{\alpha_s}{2\pi} \left(\frac{s}{e^{\gamma_E} \mu^2} \right)^{-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)}\, T_R\, \frac{-2}{3-2\epsilon} \\
&= \frac{\alpha_s}{2\pi} \left(\frac{s}{\mu^2} \right)^{-\epsilon} T_R \bigg[-\frac{2}{3}\frac{1}{\epsilon} - \frac{16}{9} - \left(\frac{140}{27} - \frac{7}{18}\pi^2 \right) \epsilon - \left(\frac{1252}{81} - \frac{28}{27}\pi^2 - \frac{50}{9}\zeta_3 \right) \epsilon^2 + \mathcal{O}(\epsilon^3) \bigg], \\
J_{\rm hc}^{(1{\rm g})}(s) &= \frac{\alpha_s}{2\pi} \left(\frac{s}{e^{\gamma_E} \mu^2} \right)^{-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)}\, C_F \left(-\frac{1}{2} \right) \\
&= \frac{\alpha_s}{2\pi} \left(\frac{s}{\mu^2} \right)^{-\epsilon} C_F \bigg[-\frac{1}{2}\frac{1}{\epsilon} - 1 - \left(3 - \frac{7}{24}\pi^2 \right) \epsilon - \left(9 - \frac{7}{12}\pi^2 - \frac{25}{6}\zeta_3 \right) \epsilon^2 + \mathcal{O}(\epsilon^3) \bigg], \\
J_{\rm hc}^{(2{\rm g})}(s) &= \frac{\alpha_s}{2\pi} \left(\frac{s}{e^{\gamma_E} \mu^2} \right)^{-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)}\, C_A \left(-\frac{1}{3-2\epsilon} \right) \qquad \text{(E.8)} \\
&= \frac{\alpha_s}{2\pi} \left(\frac{s}{\mu^2} \right)^{-\epsilon} C_A \bigg[-\frac{1}{3}\frac{1}{\epsilon} - \frac{8}{9} - \left(\frac{70}{27} - \frac{7}{36}\pi^2 \right) \epsilon - \left(\frac{626}{81} - \frac{14}{27}\pi^2 - \frac{25}{9}\zeta_3 \right) \epsilon^2 + \mathcal{O}(\epsilon^3) \bigg].
\end{aligned}
$$

A useful combination of these constituent integrals is

$$
\begin{aligned}
J^{k}_{\rm hc}(s) \;&=\; (f^{q}_{k}+f^{\bar q}_{k})\,J^{(1{\rm g})}_{\rm hc}(s) + f^{g}_{k}\left[N_f\,J^{(0{\rm g})}_{\rm hc}(s) + \frac{1}{2}\,J^{(2{\rm g})}_{\rm hc}(s)\right] \\
&=\; \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[\frac{\gamma^{\rm hc}_{k}}{\epsilon} + \phi^{\rm hc}_{k} + \mathcal{O}(\epsilon)\right].
\end{aligned} \tag{E.9}
$$

The hard double-collinear integrated kernels are given by

$$
\begin{aligned}
J^{ijkr}_{\rm hcc} \;&\equiv\; \mathcal{N}_1^{\,2}\int d\Phi^{(ijkr)}_{\rm rad,2}\,\frac{P^{\rm hc}_{ijk(r)}}{s^2_{ijk}} \\
&\equiv\; J^{(0{\rm g})}_{\rm hcc}\Big(\bar s^{(ijkr)}_{kr}\Big)\,(f^{q\bar q q'}_{ijk} + f^{q\bar q\bar q'}_{ijk}) + J^{(0{\rm g,id})}_{\rm hcc}\Big(\bar s^{(ijkr)}_{kr}\Big)\,(f^{q\bar q q}_{ijk} + f^{q\bar q\bar q}_{ijk}) \\
&\quad + J^{(1{\rm g})}_{\rm hcc}\Big(\bar s^{(ijkr)}_{kr}\Big)\,f^{q\bar q g}_{ijk} + J^{(2{\rm g})}_{\rm hcc}\Big(\bar s^{(ijkr)}_{kr}\Big)\,(f^{ggq}_{ijk} + f^{gg\bar q}_{ijk}) + J^{(3{\rm g})}_{\rm hcc}\Big(\bar s^{(ijkr)}_{kr}\Big)\,f^{ggg}_{ijk}\,,
\end{aligned} \tag{E.10}
$$

with

$$
\begin{aligned}
J^{(0{\rm g})}_{\rm hcc}(s) \;&=\; \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_F T_R\left[\frac{1}{6}\frac{1}{\epsilon^2} + \left(\frac{13}{36}+\frac{1}{9}\pi^2\right)\frac{1}{\epsilon} - \frac{119}{216} + \frac{17}{108}\,\pi^2 + \frac{14}{3}\,\zeta_3 + \mathcal{O}(\epsilon)\right], \\
J^{(0{\rm g,id})}_{\rm hcc}(s) \;&=\; \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_F\,\big(2C_F - C_A\big) \\
&\quad\times\left[-\left(\frac{13}{8} - \frac{1}{4}\,\pi^2 + \zeta_3\right)\frac{1}{\epsilon} - \frac{227}{16} + \pi^2 + \frac{17}{2}\,\zeta_3 - \frac{11}{120}\,\pi^4 + \mathcal{O}(\epsilon)\right], \\
J^{(1{\rm g})}_{\rm hcc}(s) \;&=\; \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} \\
&\quad\times\Bigg\{C_F T_R\left[-\frac{2}{3}\frac{1}{\epsilon^3} - \frac{31}{9}\frac{1}{\epsilon^2} - \left(\frac{889}{54} - \pi^2\right)\frac{1}{\epsilon} - \frac{23833}{324} + \frac{31}{6}\,\pi^2 + \frac{160}{9}\,\zeta_3 + \mathcal{O}(\epsilon)\right] \\
&\qquad + C_A T_R\left[-\frac{1}{\epsilon^3} - \frac{89}{18}\frac{1}{\epsilon^2} - \left(\frac{1211}{54} - \frac{3}{2}\pi^2\right)\frac{1}{\epsilon} - \frac{2620}{27} + \frac{89}{12}\pi^2 + \frac{80}{3}\zeta_3 + \mathcal{O}(\epsilon)\right]\Bigg\}, \\
J^{(2{\rm g})}_{\rm hcc}(s) \;&=\; \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg\{C_F^2\bigg[-\frac{2}{\epsilon^3} - \frac{37}{4}\frac{1}{\epsilon^2} - \left(\frac{307}{8} - 3\,\pi^2 + 4\,\zeta_3\right)\frac{1}{\epsilon} \\
&\qquad - \frac{2361}{16} + \frac{111}{8}\,\pi^2 + \frac{136}{3}\,\zeta_3 - \frac{\pi^4}{3} + \mathcal{O}(\epsilon)\bigg] \\
&\qquad + C_F C_A\bigg[-\frac{1}{2}\frac{1}{\epsilon^3} - \frac{23}{12}\frac{1}{\epsilon^2} - \left(\frac{241}{36} - \frac{1}{18}\,\pi^2 - 4\,\zeta_3\right)\frac{1}{\epsilon} \\
&\qquad - \frac{4609}{216} + \frac{53}{216}\,\pi^2 - \frac{47}{6}\,\zeta_3 + \frac{7}{20}\,\pi^4 + \mathcal{O}(\epsilon)\bigg]\Bigg\}, \\
J^{(3{\rm g})}_{\rm hcc}(s) \;&=\; \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A^2\bigg[-\frac{5}{2}\frac{1}{\epsilon^3} - \frac{77}{6}\frac{1}{\epsilon^2} - \left(48 - \frac{11}{4}\,\pi^2 + 3\,\zeta_3\right)\frac{1}{\epsilon} \\
&\qquad - \frac{16943}{108} + \frac{61}{4}\,\pi^2 + \frac{56}{3}\,\zeta_3 - \frac{9}{40}\,\pi^4 + \mathcal{O}(\epsilon)\bigg].
\end{aligned} \tag{E.11}
$$

For the hard-collinear times hard-collinear integrated kernels we have

$$
\begin{aligned}
J^{ijklr}_{\rm hc\otimes hc} &\equiv \mathcal{N}_1^{\,2} \int d\Phi^{(ijr,klr)}_{\rm rad,2} \, \frac{P^{\rm hc}_{ij(r)}(s_{ir},s_{jr})}{s_{ij}} \, \frac{P^{\rm hc}_{kl(r)}(s_{kr},s_{lr})}{s_{kl}} \\
&\equiv J^{\rm qqqq}_{\rm hc\otimes hc}\Big(\bar{s}^{(ijr,klr)}_{jr}\,\bar{s}^{(ijr,klr)}_{lr}\Big)\, f^{q\bar{q}}_{ij}\, f^{q'\bar{q}'}_{kl} \\
&\quad + J^{\rm qqqg}_{\rm hc\otimes hc}\Big(\bar{s}^{(ijr,klr)}_{jr}\,\bar{s}^{(ijr,klr)}_{lr}\Big)\Big[f^{q\bar{q}}_{ij}(f^{gq'}_{kl}+f^{g\bar{q}'}_{kl})+(f^{gq'}_{ij}+f^{g\bar{q}'}_{ij})f^{q\bar{q}}_{kl}\Big] \\
&\quad + J^{\rm qqgg}_{\rm hc\otimes hc}\Big(\bar{s}^{(ijr,klr)}_{jr}\,\bar{s}^{(ijr,klr)}_{lr}\Big)\,(f^{q\bar{q}}_{ij}\, f^{gg}_{kl} + f^{gg}_{ij}\, f^{q\bar{q}}_{kl}) \\
&\quad + J^{\rm qgqg}_{\rm hc\otimes hc}\Big(\bar{s}^{(ijr,klr)}_{jr}\,\bar{s}^{(ijr,klr)}_{lr}\Big)\,(f^{gq}_{ij} + f^{g\bar{q}}_{ij})(f^{gq'}_{kl} + f^{g\bar{q}'}_{kl}) \\
&\quad + J^{\rm qggg}_{\rm hc\otimes hc}\Big(\bar{s}^{(ijr,klr)}_{jr}\,\bar{s}^{(ijr,klr)}_{lr}\Big)\Big[(f^{gq}_{ij}+f^{g\bar{q}}_{ij})f^{gg}_{kl}+f^{gg}_{ij}(f^{gq}_{kl}+f^{g\bar{q}}_{kl})\Big] \\
&\quad + J^{\rm gggg}_{\rm hc\otimes hc}\Big(\bar{s}^{(ijr,klr)}_{jr}\,\bar{s}^{(ijr,klr)}_{lr}\Big)\, f^{gg}_{ij}\, f^{gg}_{kl}\,,
\end{aligned}
\tag{E.12}
$$

with

$$
\begin{aligned}
J^{\rm qqqq}_{\rm hc\otimes hc}(ss') &= \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} T_R^2 \left[\frac{4}{9}\,\frac{1}{\epsilon^2} + \frac{64}{27}\,\frac{1}{\epsilon} + \frac{284}{27} - \frac{16}{27}\,\pi^2 + \mathcal{O}(\epsilon)\right], \\
J^{\rm qqqg}_{\rm hc\otimes hc}(ss') &= \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} T_R\, C_F \left[\frac{1}{3}\,\frac{1}{\epsilon^2} + \frac{14}{9}\,\frac{1}{\epsilon} + \frac{181}{27} - \frac{4}{9}\,\pi^2 + \mathcal{O}(\epsilon)\right], \\
J^{\rm qqgg}_{\rm hc\otimes hc}(ss') &= \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} T_R\, C_A \left[\frac{2}{9}\,\frac{1}{\epsilon^2} + \frac{32}{27}\,\frac{1}{\epsilon} + \frac{142}{27} - \frac{8}{27}\,\pi^2 + \mathcal{O}(\epsilon)\right], \\
J^{\rm qgqg}_{\rm hc\otimes hc}(ss') &= \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_F^2 \left[\frac{1}{4}\,\frac{1}{\epsilon^2} + \frac{1}{\epsilon} + \frac{17}{4} - \frac{1}{3}\,\pi^2 + \mathcal{O}(\epsilon)\right], \\
J^{\rm qggg}_{\rm hc\otimes hc}(ss') &= \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_A\, C_F \left[\frac{1}{6}\,\frac{1}{\epsilon^2} + \frac{7}{9}\,\frac{1}{\epsilon} + \frac{181}{54} - \frac{2}{9}\,\pi^2 + \mathcal{O}(\epsilon)\right], \\
J^{\rm gggg}_{\rm hc\otimes hc}(ss') &= \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_A^2 \left[\frac{1}{9}\,\frac{1}{\epsilon^2} + \frac{16}{27}\,\frac{1}{\epsilon} + \frac{71}{27} - \frac{4}{27}\,\pi^2 + \mathcal{O}(\epsilon)\right].
\end{aligned}
\tag{E.13}
$$

The soft-times-hard-collinear integrated kernels read

$$
\begin{aligned}
J^{jkricd}_{\rm s\otimes hc} &\equiv \mathcal{N}_1^{\,2} \int d\Phi^{(jkr,icd)}_{\rm rad,2} \, \frac{P^{\rm hc}_{jk(r)}}{s_{jk}}\, \mathcal{E}^{(i)}_{cd} \\
&\equiv f^g_i \Big[J^{4(1{\rm g})}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{kr},\bar{s}^{(\mu)}_{cd}\Big) f^{q\bar{q}}_{jk} + J^{4(2{\rm g})}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{kr},\bar{s}^{(\mu)}_{cd}\Big)(f^{gq}_{jk}+f^{g\bar{q}}_{jk}) + J^{4(3{\rm g})}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{kr},\bar{s}^{(\mu)}_{cd}\Big) f^{gg}_{jk}\Big]_{\mu=jkr,icd}, \\
J^{jkricr}_{\rm s\otimes hc} &\equiv \mathcal{N}_1^{\,2} \int d\Phi^{(jkr,icr)}_{\rm rad,2} \, \frac{P^{\rm hc}_{jk(r)}}{s_{jk}}\, \mathcal{E}^{(i)}_{cr} \\
&\equiv f^g_i \Big[J^{3(1{\rm g})}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{kr},\bar{s}^{(\mu)}_{cr}\Big) f^{q\bar{q}}_{jk} + J^{3(2{\rm g})}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{kr},\bar{s}^{(\mu)}_{cr}\Big)(f^{gq}_{jk}+f^{g\bar{q}}_{jk}) + J^{3(3{\rm g})}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{kr},\bar{s}^{(\mu)}_{cr}\Big) f^{gg}_{jk}\Big]_{\mu=jkr,icr}, \\
J^{krjic}_{\rm s\otimes hc} &\equiv \mathcal{N}_1^{\,2} \int d\Phi^{(krj,icj)}_{\rm rad,2} \, \frac{P^{\rm hc}_{jk(r)}}{s_{jk}}\, \mathcal{E}^{(i)}_{jc} \\
&\equiv f^g_i \Big[J^{3(1{\rm g})}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{jr},\bar{s}^{(\mu)}_{jc}\Big) f^{q\bar{q}}_{jk} + J^{3(2{\rm g})}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{jr},\bar{s}^{(\mu)}_{jc}\Big)(f^{gq}_{jk}+f^{g\bar{q}}_{jk}) + J^{3(3{\rm g})}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{jr},\bar{s}^{(\mu)}_{jc}\Big) f^{gg}_{jk}\Big]_{\mu=krj,icj}, \\
J^{krjir}_{\rm s\otimes hc} &\equiv \mathcal{N}_1^{\,2} \int d\Phi^{(\mu)}_{\rm rad,2} \, \frac{P^{\rm hc}_{jk(r)}}{s_{jk}}\, \mathcal{E}^{(i)}_{jr} \\
&\equiv f^g_i \Big[\, J^{gqq}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{jr}\Big) f^{q\bar{q}}_{jk} + J^{ggq}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{jr}\Big) f^{g}_{j}(f^{q}_{k}+f^{\bar{q}}_{k}) \\
&\qquad + J^{gqg}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{jr}\Big)(f^{q}_{j}+f^{\bar{q}}_{j})f^{g}_{k} + J^{ggg}_{\rm s\otimes hc}\Big(\bar{s}^{(\mu)}_{jr}\Big) f^{gg}_{jk}\Big]_{\mu=\{krj,irj;\; krj,ijr\}},
\end{aligned}
\tag{E.14}
$$

with

$$
\begin{aligned}
J^{4(1g)}_{s\otimes hc}(s,s') &= \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon} T_R\left[-\frac{2}{3}\frac{1}{\epsilon^3}-\frac{28}{9}\frac{1}{\epsilon^2}-\left(\frac{344}{27}-\frac{7}{9}\pi^2\right)\frac{1}{\epsilon}-\frac{3928}{81}+\frac{98}{27}\pi^2+\frac{100}{9}\zeta(3)+\mathcal{O}(\epsilon)\right],\\
J^{4(2g)}_{s\otimes hc}(s,s') &= \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_F\left[-\frac{1}{2}\frac{1}{\epsilon^3}-\frac{2}{\epsilon^2}-\left(8-\frac{7}{12}\pi^2\right)\frac{1}{\epsilon}-30+\frac{7}{3}\pi^2+\frac{25}{3}\zeta(3)+\mathcal{O}(\epsilon)\right],\\
J^{4(3g)}_{s\otimes hc}(s,s') &= \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_A\left[-\frac{1}{3}\frac{1}{\epsilon^3}-\frac{14}{9}\frac{1}{\epsilon^2}-\left(\frac{172}{27}-\frac{7}{18}\pi^2\right)\frac{1}{\epsilon}-\frac{1964}{81}+\frac{49}{27}\pi^2+\frac{50}{9}\zeta(3)+\mathcal{O}(\epsilon)\right],
\end{aligned}
$$

$$
\begin{aligned}
J^{3(1g)}_{s\otimes hc}(s,s') &= \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon} T_R\left[-\frac{2}{3}\frac{1}{\epsilon^3}-\frac{28}{9}\frac{1}{\epsilon^2}-\left(\frac{362}{27}-\frac{8}{9}\pi^2\right)\frac{1}{\epsilon}-\frac{4504}{81}+\frac{112}{27}\pi^2+\frac{136}{9}\zeta(3)+\mathcal{O}(\epsilon)\right],\\
J^{3(2g)}_{s\otimes hc}(s,s') &= \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_F\left[-\frac{1}{2}\frac{1}{\epsilon^3}-\frac{2}{\epsilon^2}-\left(\frac{17}{2}-\frac{2}{3}\pi^2\right)\frac{1}{\epsilon}-35+\frac{8}{3}\pi^2+\frac{34}{3}\zeta(3)+\mathcal{O}(\epsilon)\right],\\
J^{3(3g)}_{s\otimes hc}(s,s') &= \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_A\left[-\frac{1}{3}\frac{1}{\epsilon^3}-\frac{14}{9}\frac{1}{\epsilon^2}-\left(\frac{181}{27}-\frac{4}{9}\pi^2\right)\frac{1}{\epsilon}-\frac{2252}{81}+\frac{56}{27}\pi^2+\frac{68}{9}\zeta(3)+\mathcal{O}(\epsilon)\right],
\end{aligned}
$$

$$
\begin{aligned}
J^{gq\bar{q}}_{s\otimes hc}(s) &= \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} T_R\left[-\frac{2}{3}\frac{1}{\epsilon^3}-\frac{28}{9}\frac{1}{\epsilon^2}-\left(\frac{344}{27}-\frac{17}{18}\pi^2\right)\frac{1}{\epsilon}-\frac{4225}{81}+\frac{128}{27}\pi^2+\frac{139}{9}\zeta(3)+\mathcal{O}(\epsilon)\right],\\
J^{gqg}_{s\otimes hc}(s) &= \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_F\left[-\frac{1}{2}\frac{1}{\epsilon^3}-\frac{2}{\epsilon^2}-\left(9-\frac{5}{6}\pi^2\right)\frac{1}{\epsilon}-38+\frac{19}{6}\pi^2+\frac{101}{6}\zeta(3)+\mathcal{O}(\epsilon)\right],\\
J^{ggq}_{s\otimes hc}(s) &= \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_F\left[-\frac{1}{2}\frac{1}{\epsilon^3}-\frac{2}{\epsilon^2}-\left(8-\frac{2}{3}\pi^2\right)\frac{1}{\epsilon}-32+\frac{17}{6}\pi^2+\frac{59}{6}\zeta(3)+\mathcal{O}(\epsilon)\right], \qquad (E.15)\\
J^{ggg}_{s\otimes hc}(s) &= \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A\left[-\frac{1}{3}\frac{1}{\epsilon^3}-\frac{14}{9}\frac{1}{\epsilon^2}-\left(\frac{199}{27}-\frac{5}{9}\pi^2\right)\frac{1}{\epsilon}-\frac{2477}{81}+\frac{119}{54}\pi^2+\frac{101}{9}\zeta(3)+\mathcal{O}(\epsilon)\right].
\end{aligned}
$$

Finally the hard-collinear real-virtual integrated kernels read

$$
\begin{aligned}
\tilde{J}^{ijr}_{hc} &\equiv \mathcal{N}_1\int d\Phi^{(ijr)}_{rad}\,\frac{\tilde{P}^{hc}_{ij(r)}}{s_{ij}} \;\equiv\; \tilde{J}^{(0g)}_{hc}\!\left(\bar{s}^{(ijr)}_{jr}\right)f^{q\bar{q}}_{ij}+\tilde{J}^{(1g)}_{hc}\!\left(\bar{s}^{(ijr)}_{jr}\right)(f^{gq}_{ij}+f^{g\bar{q}}_{ij})+\tilde{J}^{(2g)}_{hc}\!\left(\bar{s}^{(ijr)}_{jr}\right)f^{gg}_{ij},\\
J^{ijr}_{\Delta hc} &\equiv \mathcal{N}_1\,\frac{2}{\epsilon^2}\int d\Phi^{(ijr)}_{rad}\,\frac{P^{hc}_{ij(r)}}{s_{ij}}\left[\left(\frac{s_{cr}}{\bar{s}^{(ijr)}_{cr}}\right)^{-\epsilon}-1\right]\\
&\equiv J^{(0g)}_{\Delta hc}\left(\bar{s}^{(ijr)}_{jr}\right)f^{q\bar{q}}_{ij}+J^{(1g)}_{\Delta hc}\left(\bar{s}^{(ijr)}_{jr}\right)(f^{gq}_{ij}+f^{g\bar{q}}_{ij})+J^{(2g)}_{\Delta hc}\left(\bar{s}^{(ijr)}_{jr}\right)f^{gg}_{ij},\\
J^{ijrc}_{\Delta hc} &\equiv \mathcal{N}_1\,\frac{2}{\epsilon^2}\int d\Phi^{(ijr)}_{rad}\left\{\frac{P^{hc}_{ij(r)}}{s_{ij}}\left[1-\left(\frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}}\right)^{-\epsilon}\right]+2C_{f_j}\mathcal{E}^{(i)}_{jr}\left[1-\left(\frac{s_{jr}}{s_{[ij]r}}\right)^{-\epsilon}\right]+2C_{f_i}\mathcal{E}^{(j)}_{ir}\left[1-\left(\frac{s_{ir}}{s_{[ij]r}}\right)^{-\epsilon}\right]\right\}\\
&\equiv \left[J^{(0g)}_{\Delta hc,A}\!\left(\bar{s}^{(ijr)}_{jr}\right)+J^{(0g)}_{\Delta hc,B}\!\left(\bar{s}^{(ijr)}_{jc}\right)\right]f^{q\bar{q}}_{ij}+\left[J^{(1g)}_{\Delta hc,A}\!\left(\bar{s}^{(ijr)}_{jr}\right)+J^{(1g)}_{\Delta hc,B}\!\left(\bar{s}^{(ijr)}_{jc}\right)\right](f^{gq}_{ij}+f^{g\bar{q}}_{ij})\\
&\quad+\left[J^{(2g)}_{\Delta hc,A}\!\left(\bar{s}^{(ijr)}_{jr}\right)+J^{(2g)}_{\Delta hc,B}\!\left(\bar{s}^{(ijr)}_{jc}\right)\right]f^{gg}_{ij},\\
J^{jri,c}_{\Delta hc} &\equiv \mathcal{N}_1\,\frac{\rho^{(C)}_{ij}}{\epsilon^2}\int d\Phi^{(jri)}_{rad}\,\frac{P^{hc}_{ij(r)}}{s_{ij}}\left[\left(\frac{\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}}\right)^{-\epsilon}-\left(\frac{s_{ir}\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}\,s_{ic}}\right)^{-\epsilon}\right]\\
&\equiv \left[J^{qq}_{\Delta hc,A}\!\left(\bar{s}^{(jri)}_{ir}\right)+J^{qq}_{\Delta hc,B}\!\left(\bar{s}^{(jri)}_{ic}\right)\right]f^{q\bar{q}}_{ij}+\left[J^{qg}_{\Delta hc,A}\!\left(\bar{s}^{(jri)}_{ir}\right)+J^{qg}_{\Delta hc,B}\!\left(\bar{s}^{(jri)}_{ic}\right)\right](f^{q}_{i}+f^{\bar{q}}_{i})f^{g}_{j}\\
&\quad+\left[J^{gq}_{\Delta hc,A}\!\left(\bar{s}^{(jri)}_{ir}\right)+J^{gq}_{\Delta hc,B}\!\left(\bar{s}^{(jri)}_{ic}\right)\right]f^{g}_{i}(f^{q}_{j}+f^{\bar{q}}_{j})+\left[J^{gg}_{\Delta hc,A}\!\left(\bar{s}^{(jri)}_{ir}\right)+J^{gg}_{\Delta hc,B}\!\left(\bar{s}^{(jri)}_{ic}\right)\right]f^{gg}_{ij},\\
\tilde{J}^{jri,c}_{\Delta hc} &\equiv \frac{\mathcal{N}_1}{\epsilon^2}\int d\Phi^{(jri)}_{rad}\,\frac{T_R}{s_{ij}}\left(1-\frac{2}{1-\epsilon}\frac{s_{ir}s_{jr}}{s^2_{[ij]r}}\right)\left[\left(\frac{\bar{s}^{(jri)}_{ic}}{\mu^2}\right)^{-\epsilon}-\left(\frac{\bar{s}^{(jri)}_{ic}}{s_{ic}}\right)^{-\epsilon}\right]\equiv\tilde{J}^{c}_{\Delta hc}\left(\bar{s}^{(jri)}_{ir},\bar{s}^{(jri)}_{ic}\right), \qquad (E.16)
\end{aligned}
$$

where

$$\begin{aligned}
\tilde{J}^{(0g)}_{\rm hc}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-2\epsilon} T_R\,\Bigg\{\; N_f T_R\left[\frac{4}{9}\frac{1}{\epsilon^2}+\frac{64}{27}\frac{1}{\epsilon}+\frac{284}{27}-\frac{2}{3}\pi^2+\mathcal{O}(\epsilon)\right]\\
&+C_F\left[\frac{2}{3}\frac{1}{\epsilon^3}+\frac{31}{9}\frac{1}{\epsilon^2}+\left(\frac{431}{27}-\pi^2\right)\frac{1}{\epsilon}+\frac{5506}{81}-\frac{31}{6}\pi^2-\frac{124}{9}\zeta_3+\mathcal{O}(\epsilon)\right]\\
&+C_A\left[-\frac{1}{3\epsilon^3}-\frac{31}{18}\frac{1}{\epsilon^2}-\left(\frac{211}{27}-\frac{1}{2}\pi^2\right)\frac{1}{\epsilon}-\frac{5281}{162}+\frac{31}{12}\pi^2+\frac{62}{9}\zeta_3+\mathcal{O}(\epsilon)\right]\Bigg\},\\
\tilde{J}^{(1g)}_{\rm hc}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_F\,\Bigg\{\; C_F\Bigg[-\left(\frac{5}{4}-\frac{\pi^2}{3}\right)\frac{1}{\epsilon^2}-\left(\frac{15}{2}-\frac{2}{3}\pi^2-10\zeta_3\right)\frac{1}{\epsilon}\\
&-\frac{141}{4}+\frac{109}{24}\pi^2+20\zeta_3-\frac{7}{45}\pi^4+\mathcal{O}(\epsilon)\Bigg]\\
&+C_A\left[\frac{1}{4\epsilon^3}+\frac{1}{2\epsilon^2}+\left(1-\frac{\pi^2}{24}-4\zeta_3\right)\frac{1}{\epsilon}+\frac{9}{4}+\frac{7}{12}\pi^2-\frac{67}{6}\zeta_3-\frac{11}{90}\pi^4+\mathcal{O}(\epsilon)\right]\Bigg\},\\
\tilde{J}^{(2g)}_{\rm hc}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A\,\Bigg\{\; N_f T_R\left[\frac{1}{3}\frac{1}{\epsilon}+\frac{25}{9}+\mathcal{O}(\epsilon)\right]\\
&+C_A\Bigg[\frac{1}{6}\frac{1}{\epsilon^3}-\left(\frac{28}{9}-\frac{2}{3}\pi^2\right)\frac{1}{\epsilon^2}-\left(\frac{61}{3}-\frac{7}{4}\pi^2-12\,\zeta_3\right)\frac{1}{\epsilon}\\
&-\frac{15805}{162}+\frac{38}{3}\pi^2+\frac{221}{9}\zeta_3-\frac{5}{9}\pi^4+\mathcal{O}(\epsilon)\Bigg]\Bigg\}\,;
\end{aligned} \tag{E.17}$$

$$\begin{aligned}
J^{(0g)}_{\Delta\rm hc}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[-\left(\frac{4}{3}-\frac{2}{9}\pi^2\right)\frac{1}{\epsilon}-\frac{104}{9}+\frac{16}{27}\pi^2+8\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{(1g)}_{\Delta\rm hc}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_F\left[-\left(1-\frac{\pi^2}{6}\right)\frac{1}{\epsilon}-8+\frac{\pi^2}{3}+6\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{(2g)}_{\Delta\rm hc}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[-\left(\frac{2}{3}-\frac{\pi^2}{9}\right)\frac{1}{\epsilon}-\frac{52}{9}+\frac{8}{27}\pi^2+4\zeta_3+\mathcal{O}(\epsilon)\right];
\end{aligned} \tag{E.18}$$

$$\begin{aligned}
J^{(0g)}_{\Delta{\rm hc,A}}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[-\frac{4}{3}\frac{1}{\epsilon^3}-\frac{32}{9}\frac{1}{\epsilon^2}-\left(\frac{280}{27}-\frac{7}{9}\pi^2\right)\frac{1}{\epsilon}-\frac{2504}{81}+\frac{56}{27}\pi^2+\frac{100}{9}\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{(0g)}_{\Delta{\rm hc,B}}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[\frac{4}{3}\frac{1}{\epsilon^3}+\frac{32}{9}\frac{1}{\epsilon^2}+\left(\frac{244}{27}-\frac{5}{9}\pi^2\right)\frac{1}{\epsilon}+\frac{1784}{81}-\frac{40}{27}\pi^2-\frac{52}{9}\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{(1g)}_{\Delta{\rm hc,A}}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_F\Bigg[-\frac{1}{\epsilon^3}-\left(6-\frac{2}{3}\pi^2\right)\frac{1}{\epsilon^2}-\left(30-\frac{23}{12}\pi^2-16\zeta_3\right)\frac{1}{\epsilon}\\
&-126+\frac{49}{6}\pi^2+\frac{121}{3}\zeta_3+\frac{\pi^4}{9}+\mathcal{O}(\epsilon)\Bigg],
\end{aligned} \tag{E.19}$$

$$\begin{aligned}
J^{(1g)}_{\Delta{\rm hc,B}}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_F\left[\frac{1}{\epsilon^3}+\frac{2}{\epsilon^2}+\left(5-\frac{5}{12}\pi^2\right)\frac{1}{\epsilon}+12-\frac{5}{6}\pi^2-\frac{13}{3}\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{(2g)}_{\Delta{\rm hc,A}}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\Bigg[-\frac{2}{3}\frac{1}{\epsilon^3}-\left(\frac{88}{9}-\frac{4}{3}\pi^2\right)\frac{1}{\epsilon^2}+-\left(\frac{1436}{27}-\frac{55}{18}\pi^2-32\zeta_3\right)\frac{1}{\epsilon}\\
&-\frac{18748}{81}+\frac{406}{27}\pi^2+\frac{626}{9}\zeta_3+\frac{2}{9}\pi^4+\mathcal{O}(\epsilon)\Bigg],\\
J^{(2g)}_{\Delta{\rm hc,B}}(s) \;=\; & \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[\frac{2}{3}\frac{1}{\epsilon^3}+\frac{16}{9}\frac{1}{\epsilon^2}+\left(\frac{122}{27}-\frac{5}{18}\pi^2\right)\frac{1}{\epsilon}+\frac{892}{81}-\frac{20}{27}\pi^2-\frac{26}{9}\zeta_3+\mathcal{O}(\epsilon)\right];
\end{aligned}$$

$$
\begin{aligned}
J^{\mathrm{qq}}_{\Delta\mathrm{hc,A}}(s) &= \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[\frac{2}{3}\frac{1}{\epsilon^3}+\frac{16}{9}\frac{1}{\epsilon^2}+\left(\frac{122}{27}-\frac{4}{9}\pi^2\right)\frac{1}{\epsilon}+\frac{1108}{81}-\frac{44}{27}\pi^2-\frac{47}{9}\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{\mathrm{qq}}_{\Delta\mathrm{hc,B}}(s) &= \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[-\frac{2}{3}\frac{1}{\epsilon^3}-\frac{16}{9}\frac{1}{\epsilon^2}-\left(\frac{140}{27}-\frac{7}{18}\pi^2\right)\frac{1}{\epsilon}-\frac{1252}{81}+\frac{28}{27}\pi^2+\frac{50}{9}\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{\mathrm{qg}}_{\Delta\mathrm{hc,A}}(s) &= \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}(2C_F-C_A)\left[\frac{1}{2}\frac{1}{\epsilon^3}+\frac{1}{\epsilon^2}+\left(\frac{7}{2}-\frac{11}{24}\pi^2\right)\frac{1}{\epsilon}+\frac{23}{2}-\frac{2}{3}\pi^2-\frac{32}{3}\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{\mathrm{qg}}_{\Delta\mathrm{hc,B}}(s) &= \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}(2C_F-C_A)\left[-\frac{1}{2\epsilon^3}-\frac{1}{\epsilon^2}-\left(3-\frac{7}{24}\pi^2\right)\frac{1}{\epsilon}-9+\frac{7}{12}\pi^2+\frac{25}{6}\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{\mathrm{gq}}_{\Delta\mathrm{hc,A}}(s) &= \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[\frac{1}{2\epsilon^3}+\frac{1}{\epsilon^2}+\left(\frac{5}{2}-\frac{7}{24}\pi^2\right)\frac{1}{\epsilon}+\frac{13}{2}-\frac{5}{6}\pi^2-\frac{5}{3}\zeta_3+\mathcal{O}(\epsilon)\right], \qquad \text{(E.20)}\\
J^{\mathrm{gq}}_{\Delta\mathrm{hc,B}}(s) &= \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[-\frac{1}{2\epsilon^3}-\frac{1}{\epsilon^2}-\left(3-\frac{7}{24}\pi^2\right)\frac{1}{\epsilon}-9+\frac{7}{12}\pi^2+\frac{25}{6}\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{\mathrm{gg}}_{\Delta\mathrm{hc,A}}(s) &= \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[\frac{1}{3\epsilon^3}+\frac{8}{9}\frac{1}{\epsilon^2}+\left(\frac{88}{27}-\frac{11}{36}\pi^2\right)\frac{1}{\epsilon}+\frac{716}{81}-\frac{17}{54}\pi^2-\frac{64}{9}\zeta_3+\mathcal{O}(\epsilon)\right],\\
J^{\mathrm{gg}}_{\Delta\mathrm{hc,B}}(s) &= \frac{\alpha_s}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[-\frac{1}{3\epsilon^3}-\frac{8}{9}\frac{1}{\epsilon^2}-\left(\frac{70}{27}-\frac{7}{36}\pi^2\right)\frac{1}{\epsilon}-\frac{626}{81}+\frac{14}{27}\pi^2+\frac{25}{9}\zeta_3+\mathcal{O}(\epsilon)\right].
\end{aligned}
$$

References

[1] A. Huss, J. Huston, S. Jones and M. Pellen, *Les Houches 2021: Physics at TeV Colliders: Report on the Standard Model Precision Wishlist*, 2207.02122.

[2] M. Czakon, A. Mitov and R. Poncelet, *Next-to-Next-to-Leading Order Study of Three-Jet Production at the LHC*, Phys. Rev. Lett. **127** (2021) 152001 [2106.05331].

[3] H.B. Hartanto, R. Poncelet, A. Popescu and S. Zoia, *Next-to-next-to-leading order QCD corrections to Wbb production at the LHC*, Phys. Rev. D **106** (2022) 074016 [2205.01687].

[4] L. Buonocore, S. Devoto, S. Kallweit, J. Mazzitelli, L. Rottoli and C. Savoini, *Associated production of a W boson and massive bottom quarks at next-to-next-to-leading order in QCD*, 2212.04954.

[5] S. Catani, S. Devoto, M. Grazzini, S. Kallweit, J. Mazzitelli and C. Savoini, *$t\bar{t}H$ production in NNLO QCD*, 2210.07846.

[6] J. Baglio, C. Duhr, B. Mistlberger and R. Szafron, *Inclusive Production Cross Sections at N3LO*, 2209.06138.

[7] F. Caola, W. Chen, C. Duhr, X. Liu, B. Mistlberger, F. Petriello et al., *The Path forward to N^3LO*, in *2022 Snowmass Summer Study*, 3, 2022 [2203.06730].

[8] N. Agarwal, L. Magnea, C. Signorile-Signorile and A. Tripathi, *The infrared structure of perturbative gauge theories*, Phys. Rept. **994** (2023) 1 [2112.07099].

[9] F. Bloch and A. Nordsieck, *Note on the Radiation Field of the electron*, Phys. Rev. **52** (1937) 54.

[10] T. Kinoshita, *Mass singularities of Feynman amplitudes*, *J. Math. Phys.* **3** (1962) 650.

[11] T.D. Lee and M. Nauenberg, *Degenerate Systems and Mass Singularities*, Phys. Rev. **133** (1964) B1549.

[12] J. Grammer, G. and D. Yennie, *Improved treatment for the infrared divergence problem in quantum electrodynamics*, Phys.Rev. **D8** (1973) 4332.

[13] J.C. Collins, D.E. Soper and G.F. Sterman, *Factorization of Hard Processes in QCD*, Adv. Ser. Direct. High Energy Phys. **5** (1989) 1 [hep-ph/0409313].

[14] A. Sen, *Asymptotic Behavior of the Wide Angle On-Shell Quark Scattering Amplitudes in Nonabelian Gauge Theories*, Phys. Rev. **D28** (1983) 860.

[15] J.C. Collins, *Sudakov form-factors*, *Adv. Ser. Direct. High Energy Phys.* **5** (1989) 573 [hep-ph/0312336].

[16] L. Magnea and G.F. Sterman, *Analytic continuation of the Sudakov form-factor in QCD*, *Phys.Rev.* **D42** (1990) 4222.

[17] G.F. Sterman, *Partons, factorization and resummation, TASI 95*, in *QCD and beyond. Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI-95, Boulder, USA, June 4-30, 1995*, pp. 327–408, 1995 [hep-ph/9606312].

[18] S. Catani, *The Singular behavior of QCD amplitudes at two loop order*, *Phys. Lett.* **B427** (1998) 161 [hep-ph/9802439].

[19] G.F. Sterman and M.E. Tejeda-Yeomans, *Multiloop amplitudes and resummation*, *Phys. Lett.* **B552** (2003) 48 [hep-ph/0210130].

[20] L.J. Dixon, L. Magnea and G.F. Sterman, *Universal structure of subleading infrared poles in gauge theory amplitudes*, *JHEP* **0808** (2008) 022 [0805.3515].

[21] E. Gardi and L. Magnea, *Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes*, *JHEP* **0903** (2009) 079 [0901.1091].

[22] E. Gardi and L. Magnea, *Infrared singularities in QCD amplitudes*, *Nuovo Cim.* **C32N5-6** (2009) 137 [0908.3273].

[23] T. Becher and M. Neubert, *Infrared singularities of scattering amplitudes in perturbative QCD*, *Phys.Rev.Lett.* **102** (2009) 162001 [0901.0722].

[24] T. Becher and M. Neubert, *On the Structure of Infrared Singularities of Gauge-Theory Amplitudes*, *JHEP* **06** (2009) 081 [0903.1126].

[25] I. Feige and M.D. Schwartz, *Hard-Soft-Collinear Factorization to All Orders*, *Phys. Rev.* **D90** (2014) 105020 [1403.6472].

[26] S. Catani and M. Grazzini, *Collinear factorization and splitting functions for next-to-next-to-leading order QCD calculations*, *Phys. Lett.* **B446** (1999) 143 [hep-ph/9810389].

[27] S. Catani and M. Grazzini, *Infrared factorization of tree level QCD amplitudes at the next-to-next-to-leading order and beyond*, *Nucl. Phys.* **B570** (2000) 287 [hep-ph/9908523].

[28] D.A. Kosower, *All order collinear behavior in gauge theories*, *Nucl. Phys.* **B552** (1999) 319 [hep-ph/9901201].

[29] Ø. Almelid, C. Duhr and E. Gardi, *Three-loop corrections to the soft anomalous dimension in multileg scattering*, *Phys. Rev. Lett.* **117** (2016) 172002 [1507.00047].

[30] Ø. Almelid, C. Duhr, E. Gardi, A. McLeod and C.D. White, *Bootstrapping the QCD soft anomalous dimension*, *JHEP* **09** (2017) 073 [1706.10162].

[31] J.M. Campbell and E.W.N. Glover, *Double unresolved approximations to multiparton scattering amplitudes*, *Nucl. Phys.* **B527** (1998) 264 [hep-ph/9710255].

[32] Z. Bern, V. Del Duca, W.B. Kilgore and C.R. Schmidt, *The infrared behavior of one loop QCD amplitudes at next-to-next-to leading order*, *Phys. Rev.* **D60** (1999) 116001 [hep-ph/9903516].

[33] S. Catani and M. Grazzini, *The soft gluon current at one loop order*, *Nucl. Phys.* **B591** (2000) 435 [hep-ph/0007142].

[34] V. Del Duca, A. Frizzo and F. Maltoni, *Factorization of tree QCD amplitudes in the high-energy limit and in the collinear limit*, *Nucl. Phys.* **B568** (2000) 211 [hep-ph/9909464].

[35] S.D. Badger and E.W.N. Glover, *Two loop splitting functions in QCD*, *JHEP* **07** (2004) 040 [hep-ph/0405236].

[36] C. Duhr and T. Gehrmann, *The two-loop soft current in dimensional regularization*, *Phys. Lett.* **B727** (2013) 452 [1309.4393].

[37] Y. Li and H.X. Zhu, *Single soft gluon emission at two loops*, *JHEP* **11** (2013) 080 [1309.4391].

[38] P. Banerjee, P.K. Dhani and V. Ravindran, *Gluon jet function at three loops in QCD*, *Phys. Rev. D* **98** (2018) 094016 [1805.02637].

[39] R. Brüser, Z.L. Liu and M. Stahlhofen, *Three-Loop Quark Jet Function*, *Phys. Rev. Lett.* **121** (2018) 072003 [1804.09722].

[40] L.J. Dixon, E. Herrmann, K. Yan and H.X. Zhu, *Soft gluon emission at two loops in full color*, *JHEP* **05** (2020) 135 [1912.09370].

[41] S. Catani, D. Colferai and A. Torrini, *Triple (and quadruple) soft-gluon radiation in QCD hard scattering*, *JHEP* **01** (2020) 118 [1908.01616].

[42] V. Del Duca, C. Duhr, R. Haindl, A. Lazopoulos and M. Michel, *Tree-level splitting amplitudes for a gluon into four collinear partons*, *JHEP* **10** (2020) 093 [2007.05345].

[43] S. Catani and L. Cieri, *Multiple soft radiation at one-loop order and the emission of a soft quark–antiquark pair*, *Eur. Phys. J. C* **82** (2022) 97 [2108.13309].

[44] V. Del Duca, C. Duhr, R. Haindl and Z. Liu, *Tree-level soft emission of a quark pair in association with a gluon*, 2206.01584.

[45] M. Czakon, F. Eschment and T. Schellenberger, *Revisiting the double-soft asymptotics of one-loop amplitudes in massless QCD*, 2211.06465.

[46] M. Czakon and S. Sapeta, *Complete collection of one-loop triple-collinear splitting operators for dimensionally-regulated QCD*, *JHEP* **07** (2022) 052 [2204.11801].

[47] S. Catani, L. Cieri, D. Colferai and F. Coradeschi, *Soft gluon-quark-antiquark emission in QCD hard scattering*, 2210.09397.

[48] W.T. Giele, E.W.N. Glover and D.A. Kosower, *Higher order corrections to jet cross-sections in hadron colliders*, *Nucl. Phys.* **B403** (1993) 633 [hep-ph/9302225].

[49] W.T. Giele, E.W.N. Glover and D.A. Kosower, *The inclusive two jet triply differential cross-section*, *Phys. Rev.* **D52** (1995) 1486 [hep-ph/9412338].

[50] S. Frixione, Z. Kunszt and A. Signer, *Three jet cross-sections to next-to-leading order*, *Nucl. Phys.* **B467** (1996) 399 [hep-ph/9512328].

[51] S. Catani and M.H. Seymour, *A General algorithm for calculating jet cross-sections in NLO QCD*, *Nucl. Phys.* **B485** (1997) 291 [hep-ph/9605323].

[52] Z. Nagy and D.E. Soper, *General subtraction method for numerical calculation of one loop QCD matrix elements*, *JHEP* **09** (2003) 055 [hep-ph/0308127].

[53] G. Bevilacqua, M. Czakon, M. Kubocz and M. Worek, *Complete Nagy-Soper subtraction for next-to-leading order calculations in QCD*, *JHEP* **10** (2013) 204 [1308.5605].

[54] R.M. Prisco and F. Tramontano, *Dual subtractions*, *JHEP* **06** (2021) 089 [2012.05012].

[55] J.M. Campbell and R.K. Ellis, *An Update on vector boson pair production at hadron colliders*, *Phys. Rev.* **D60** (1999) 113006 [hep-ph/9905386].

[56] T. Gleisberg and F. Krauss, *Automating dipole subtraction for QCD NLO calculations*, *Eur. Phys. J.* **C53** (2008) 501 [0709.2881].

[57] R. Frederix, T. Gehrmann and N. Greiner, *Automation of the Dipole Subtraction Method in MadGraph/MadEvent*, *JHEP* **09** (2008) 122 [0808.2128].

[58] M. Czakon, C.G. Papadopoulos and M. Worek, *Polarizing the Dipoles*, *JHEP* **08** (2009) 085 [0905.0883].

[59] K. Hasegawa, S. Moch and P. Uwer, *AutoDipole: Automated generation of dipole subtraction terms*, *Comput. Phys. Commun.* **181** (2010) 1802 [0911.4371].

[60] R. Frederix, S. Frixione, F. Maltoni and T. Stelzer, *Automation of next-to-leading order computations in QCD: The FKS subtraction*, *JHEP* **10** (2009) 003 [0908.4272].

[61] S. Alioli, P. Nason, C. Oleari and E. Re, *A general framework for implementing NLO calculations in shower Monte Carlo programs: the POWHEG BOX*, *JHEP* **06** (2010) 043 [1002.2581].

[62] S. Platzer and S. Gieseke, *Dipole Showers and Automated NLO Matching in Herwig++*, *Eur. Phys. J.* **C72** (2012) 2187 [1109.6256].

[63] J. Reuter, B. Chokoufe, A. Hoang, W. Kilian, M. Stahlhofen, T. Teubner et al., *Automation of NLO processes and decays and POWHEG matching in WHIZARD*, *J. Phys. Conf. Ser.* **762** (2016) 012059 [1602.06270].

[64] S. Frixione and M. Grazzini, *Subtraction at NNLO*, *JHEP* **06** (2005) 010 [hep-ph/0411399].

[65] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, *Antenna subtraction at NNLO*, *JHEP* **09** (2005) 056 [hep-ph/0505111].

[66] J. Currie, E.W.N. Glover and S. Wells, *Infrared Structure at NNLO Using Antenna Subtraction*, *JHEP* **04** (2013) 066 [1301.4693].

[67] G. Somogyi, Z. Trócsányi and V. Del Duca, *Matching of singly- and doubly-unresolved limits of tree-level QCD squared matrix elements*, *JHEP* **06** (2005) 024 [hep-ph/0502226].

[68] G. Somogyi, Z. Trócsányi and V. Del Duca, *A Subtraction scheme for computing QCD jet cross sections at NNLO: Regularization of doubly-real emissions*, *JHEP* **01** (2007) 070 [hep-ph/0609042].

[69] G. Somogyi and Z. Trócsányi, *A Subtraction scheme for computing QCD jet cross sections at NNLO: Regularization of real-virtual emission*, *JHEP* **01** (2007) 052 [hep-ph/0609043].

[70] M. Czakon, *A novel subtraction scheme for double-real radiation at NNLO*, *Phys. Lett.* **B693** (2010) 259 [1005.0274].

[71] M. Czakon, *Double-real radiation in hadronic top quark pair production as a proof of a certain concept*, *Nucl. Phys.* **B849** (2011) 250 [1101.0642].

[72] C. Anastasiou, K. Melnikov and F. Petriello, *A new method for real radiation at NNLO*, *Phys. Rev. D* **69** (2004) 076010 [hep-ph/0311311].

[73] F. Caola, K. Melnikov and R. Röntsch, *Nested soft-collinear subtractions in NNLO QCD computations*, *Eur. Phys. J.* **C77** (2017) 248 [1702.01352].

[74] S. Catani and M. Grazzini, *An NNLO subtraction formalism in hadron collisions and its application to Higgs boson production at the LHC*, *Phys. Rev. Lett.* **98** (2007) 222002 [hep-ph/0703012].

[75] M. Grazzini, S. Kallweit and M. Wiesemann, *Fully differential NNLO computations with MATRIX*, *Eur. Phys. J. C* **78** (2018) 537 [1711.06631].

[76] R. Boughezal, K. Melnikov and F. Petriello, *A subtraction scheme for NNLO computations*, *Phys. Rev.* **D85** (2012) 034025 [1111.7041].

[77] R. Boughezal, C. Focke, X. Liu and F. Petriello, *W-boson production in association with a jet at next-to-next-to-leading order in perturbative QCD*, *Phys. Rev. Lett.* **115** (2015) 062002 [1504.02131].

[78] R. Boughezal, C. Focke, W. Giele, X. Liu and F. Petriello, *Higgs boson production in association with a jet at NNLO using jettiness subtraction*, *Phys. Lett.* **B748** (2015) 5 [1505.03893].

[79] J. Gaunt, M. Stahlhofen, F.J. Tackmann and J.R. Walsh, *N-jettiness Subtractions for NNLO QCD Calculations*, *JHEP* **09** (2015) 058 [1505.04794].

[80] M. Cacciari, F.A. Dreyer, A. Karlberg, G.P. Salam and G. Zanderighi, *Fully Differential Vector-Boson-Fusion Higgs Production at Next-to-Next-to-Leading Order*, *Phys. Rev. Lett.* **115** (2015) 082002 [1506.02660].

[81] G.F.R. Sborlini, F. Driencourt-Mangin and G. Rodrigo, *Four-dimensional unsubtraction with massive particles*, *JHEP* **10** (2016) 162 [1608.01584].

[82] F. Herzog, *Geometric IR subtraction for final state real radiation*, *JHEP* **08** (2018) 006 [1804.07949].

[83] L. Magnea, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, *Local analytic sector subtraction at NNLO*, *JHEP* **12** (2018) 107 [1806.09570].

[84] L. Magnea, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, *Factorisation and Subtraction beyond NLO*, *JHEP* **12** (2018) 062 [1809.05444].

[85] Z. Capatti, V. Hirschi, D. Kermanschah and B. Ruijl, *Loop-Tree Duality for Multiloop Numerical Integration*, *Phys. Rev. Lett.* **123** (2019) 151602 [1906.06138].

[86] W.J. Torres Bobadilla et al., *May the four be with you: Novel IR-subtraction methods to tackle NNLO calculations*, *Eur. Phys. J. C* **81** (2021) 250 [2012.02567].

[87] H.A. Chawdhry, M.L. Czakon, A. Mitov and R. Poncelet, *NNLO QCD corrections to three-photon production at the LHC*, *JHEP* **02** (2020) 057 [1911.00479].

[88] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *NNLO QCD corrections to diphoton production with an additional jet at the LHC*, *JHEP* **09** (2021) 093 [2105.06940].

[89] G. Bertolotti, P. Torrielli, S. Uccirati and M. Zaro, *Local analytic sector subtraction for initial- and final-state radiation at NLO in massless QCD*, 2209.09123.

[90] F. Febres Cordero, A. von Manteuffel and T. Neumann, *Computational Challenges for Multi-loop Collider Phenomenology: A Snowmass 2021 White Paper*, *Comput. Softw. Big Sci.* **6** (2022) 14 [2204.04200].

[91] L. Magnea, C. Milloy, C. Signorile-Signorile, P. Torrielli and S. Uccirati, *Strongly-ordered infrared limits for subtraction counterterms from factorisation*, in *16th DESY Workshop on Elementary Particle Physics: Loops and Legs in Quantum Field Theory 2022*, 9, 2022 [2209.06102].

[92] L. Magnea, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, *Analytic integration of soft and collinear radiation in factorised QCD cross sections at NNLO*, *JHEP* **02** (2021) 037 [2010.14493].

[93] V. Del Duca, N. Deutschmann and S. Lionetti, *Momentum mappings for subtractions at higher orders in QCD*, *JHEP* **12** (2019) 129 [1910.01024].

[94] V. Del Duca, C. Duhr, A. Kardos, G. Somogyi and Z. Trócsányi, *Three-Jet Production in Electron-Positron Collisions at Next-to-Next-to-Leading Order Accuracy*, *Phys. Rev. Lett.* **117** (2016) 152004 [1603.08927].

[95] V. Del Duca, C. Duhr, A. Kardos, G. Somogyi, Z. Ször, Z. Trócsányi et al., *Jet production in the CoLoRFulNNLO method: event shapes in electron-positron collisions*, *Phys. Rev.* **D94** (2016) 074019 [1606.03453].

[96] A. Grozin, J. Henn and M. Stahlhofen, *On the Casimir scaling violation in the cusp anomalous dimension at small angle*, *JHEP* **10** (2017) 052 [1708.01221].

[97] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*, *Phys. Lett. B* **782** (2018) 627 [1805.09638].

[98] S. Weinzierl, *Does one need the $O(\epsilon)-$ and $O(\epsilon^2)$-terms of one-loop amplitudes in an NNLO calculation ?*, *Phys. Rev. D* **84** (2011) 074007 [1107.5131].

[99] S. Catani and M. Seymour, *The Dipole formalism for the calculation of QCD jet cross-sections at next-to-leading order*, *Phys.Lett.* **B378** (1996) 287 [hep-ph/9602277].