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The massless three-loop Wilson coefficients for the deep-inelastic structure functions F_2, F_L, xF_3 and g_1

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Abstract

We calculate the massless unpolarized Wilson coefficients for deeply inelastic scattering for the structure functions $F_2(x, Q^2), F_L(x, Q^2), xF_3(x, Q^2)$ in the $\overline{\text{MS}}$ scheme and the polarized Wilson coefficients of the structure function $g_1(x, Q^2)$ in the Larin scheme up to three–loop order in QCD in a fully automated way based on the method of arbitrary high Mellin moments. We work in the Larin scheme in the case of contributing axial–vector couplings or polarized nucleons. For the unpolarized structure functions we compare to results given in the literature. The polarized three–loop Wilson coefficients are calculated for the first time. As a by–product we also obtain the quarkonic three–loop anomalous dimensions from the $O(1/\varepsilon)$ terms of the unrenormalized forward Compton amplitude. Expansions for small and large values of the Bjorken variable x are provided.

1 Introduction

The scaling violations of the deep–inelastic structure functions $F_k(x, Q^2)$ [1–4] in the massless limit and twist 2 approximation are described by those of the twist 2 parton distribution functions f_i and massless Wilson coefficients $\mathbb{C}_j$ in a perturbative expansion in the strong coupling constant $a_s(Q^2) = \alpha_s(Q^2)/(4\pi)$, where $Q^2 = -q^2$ denotes the space–like virtuality of the scattering process and $x = Q^2/(2p.q)$ is the Bjorken variable, with p the nucleon momentum. In the present paper we calculate the Wilson coefficients in the case of virtual photon exchange in the unpolarized and polarized case to three–loop order, where the latter results are new. For the structure function xF_3 we consider the exchange of both the weak $W^\pm$ bosons. The present results are of importance for the experimental data analysis of the deep–inelastic scattering data at HERA and future lepton–nucleon colliders, such as the EIC [5] and the LHeC–project [6] for precision measurements in the unpolarized and polarized case, to extract the parton distribution functions [7] and to measure the strong coupling constant at highest possible precision [8].

The calculation is based on massless virtual forward Compton amplitudes for on–shell partonic states, which are gauge–invariant quantities. The structure functions are described by

$$F_k(x, Q^2) = \sum_i \mathbb{C}_{k,i}\left(x, \frac{Q^2}{\mu^2}\right) \otimes f_i(x, \mu^2) \tag{1}$$

with

$$\mathbb{C}_{k,i}\left(x, \frac{Q^2}{\mu^2}\right) = \delta_{iq} + \sum_{l=1}^{\infty} a_s^l \sum_{n=0}^{l} \ln^n\left(\frac{Q^2}{\mu^2}\right) C_{k,i}^{(l,n)}(x), \tag{2}$$

where μ^2 denotes the factorization scale and $\otimes$ the Mellin convolution

$$A(x) \otimes B(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2). \tag{3}$$

The Mellin transform

$$\mathbf{M}[A(x)](N) = \int_0^1 dx\, x^{N-1} A(x) \tag{4}$$

diagonalizes the expression in Eq. (3) into the product $\mathbf{M}[A](N) \cdot \mathbf{M}[B](N)$ and will therefore be often used in the subsequent calculation. The dependence on the factorization scale diminishes for higher and higher orders in the coupling constant. One may also derive representations in Mellin N space which are scheme–invariant order by order, see e.g. [9–11]. Here the preferred scale of the deep–inelastic process is $\mu^2 = Q^2$.

The massless Wilson coefficients for the structure functions $F_2(x, Q^2), F_L(x, Q^2)$ and $xF_3(x, Q^2)$ have been calculated at one– [12], two– [13, 14] and three–loop order [15–17]. Furthermore, also their Mellin moments have been computed at 2– [18] and 3–loop order [19]. In the polarized massless case the corresponding 1– and 2–loop Wilson coefficients for the structure function $g_1(x, Q^2)$ have been obtained in [20–24].

In the present paper we re–calculate the massless Wilson coefficients for the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ as well as for the structure function $xF_3(x, Q^2) \equiv xF_3^{W^+}(x, Q^2) + xF_3^{W^-}(x, Q^2)$ and calculate newly those for the polarized structure function $g_1(x, Q^2)$ in the Larin scheme [25,26]. The Wilson coefficients are calculated using the method of the forward Compton amplitude. Here the parameter $x = Q^2/(2p.q)$ is the variable on which the master integrals and

the associated differential equations depend. The Mellin moments of the structure functions correspond to the expansion coefficients of the forward Compton amplitude in the unphysical limit $\omega = 1/x \ll 1$ which is frequently exploited to compute low Mellin moments. Therefore, the variable ω corresponds to the parameter t used in Refs. [27–29] and the subsequent technical steps are quite similar to those in the calculation of the three–loop anomalous dimensions.

The paper is organized as follows. In Section 2 we discuss the basic formalism, including the renormalization, and summarize the technical details of the calculation in Section 3. In Section 4 we calculate the one– and two–loop Wilson coefficients to the order in the dimensional parameter $\varepsilon = D - 4$, needed for the present calculation at three–loop order and present the 2–loop Wilson coefficients. In Section 5 the three–loop Wilson coefficients for the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ are given and in Section 6 those for the structure function $xF_3(x, Q^2)$. Finally, we present in Section 7 the three–loop Wilson coefficients for the polarized structure function $g_1(x, Q^2)$ and also remark that the quarkonic anomalous dimensions being obtained as a by–product of the present calculation from the $O(1/\varepsilon)$ terms of the unrenormalized Compton amplitude, agree with the results in the literature. In Section 8 we present the small–x and large–x expansions of the respective Wilson coefficients and compare to the large N_F behaviour predicted in the literature. Here N_F denotes the number of massless flavors. Section 9 contains the conclusions. To shorten the presentation we will identify the factorization scale $\mu^2 = Q^2$ and only print the Mellin N space representation for the complete expressions. In an ancillary file to this paper we present all Wilson coefficients in Mellin N and z–space, in full form, including their scale dependence. Here z denotes the parton momentum fraction in the nucleon, $z \in [0, 1]$, and is, for twist–2, identical to the Bjorken variable x.

2 Basic Formalism

We consider the forward Compton amplitude for neutral current photon exchange in the unpolarized case for the deep–inelastic structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$, as well as for the structure functions $g_1(x, Q^2)$ in the polarized case in the twist–2 approximation to three–loop order. Furthermore, we study the charged current structure function $xF_3(x, Q^2)$.

The hadronic tensor has the principal structure [4, 30]

$$
\begin{aligned}
W_{\mu\nu} &= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(x, Q^2) + \frac{\hat{p}_\mu \hat{p}_\nu}{p.q} F_2(x, Q^2) - i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda p^\sigma}{2p.q} F_3(x, Q^2) \\
&\quad + i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda S^\sigma}{p.q} g_1(x, Q^2) + \cdots .
\end{aligned}
\tag{5}
$$

Here the ellipses denote contributions from other structure functions not considered in the present paper, p denotes the nucleon 4–momentum and S the polarization vector of the nucleon, which can be taken in the longitudinal case in the nucleon rest frame

$$
S_L = (M, 0, 0, 0) ,
\tag{6}
$$

where only the energy component is non–vanishing and M denotes the nucleon mass. The vector $\hat{p}$ is given by

$$
\hat{p} = p_\mu - \frac{p.q}{q^2} q_\mu .
\tag{7}
$$

All structure functions can be isolated by using corresponding projectors in $D = 4 + \varepsilon$ space–time dimensions. In the polarized case one furthermore considers the polarization asymmetry,

i.e. the difference for $S = S_L$ and $-S_L$. In the following we will consider the structure functions $F_{2,L}(x, Q^2)$ and $g_1(x, Q^2)$ in the case of pure photon exchange, while $xF_3(x, Q^2)$ is measured for the arithmetic mean of $W^{\pm}$ boson exchange.

The associated forward Compton amplitudes are given by the Fourier transform of the time–ordered product of current operators

$$T_{\mu\nu} = i \int dz \; e^{iqz} \langle p | T \left(J_\mu^\dagger(z) J_\nu(0) \right) | p \rangle \;, \tag{8}$$

and

$$W_{\mu\nu} = \frac{1}{\pi} \text{Im} T_{\mu\nu} \;. \tag{9}$$

We consider the deep–inelastic scattering cross section for pure photon exchange [31]

$$
\begin{aligned}
\frac{d^2\sigma^{\gamma^*}(\lambda, \pm S_L)}{dxdy} &= 2\pi S \frac{\alpha^2}{Q^4} \Bigg\{ y^2 2x F_1(x, Q^2) + 2 \left(1 - y - \frac{xyM^2}{S} \right) F_2(x, Q^2) \\
&\pm \left[-2\lambda y \left(2 - y - \frac{2xyM^2}{S} \right) xg_1(x, Q^2) + 8\lambda \frac{yx^2 M^2}{S} g_2(x, Q^2) \right] \Bigg\} \;,
\end{aligned}
\tag{10}
$$

with λ the longitudinal charged lepton polarization, M the nucleon mass, $y = Q^2/Sx$ a Bjorken variable, and S the cms energy squared. We will use the collinear parton model [32] in the present calculation, where the parton momentum is given by zp, with p the nucleon momentum and $z \in [0, 1]$. Due to this the Wilson coefficients of the structure function $g_2(x, Q^2)$ cannot be calculated. It requires the use of the covariant parton model [33], already at tree level.

One furthermore has

$$F_L(x, Q^2) = F_2(x, Q^2) - 2x F_1(x, Q^2) \;. \tag{11}$$

The structure functions are related to the parton densities for pure photon exchange at tree level as [4]

$$F_2(x, Q^2) = x \sum_{f=1}^{N_F} e_f^2 [q_f(x, Q^2) + \bar{q}_f(x, Q^2)] \;, \tag{12}$$

$$g_1(x, Q^2) = \frac{1}{2} \sum_{f=1}^{N_F} e_f^2 [\Delta q_f(x, Q^2) + \Delta\bar{q}_f(x, Q^2)] \;, \tag{13}$$

for pure photon exchange, with $q_i(\bar{q}_i)$ the unpolarized and $\Delta q_i(\Delta\bar{q}_i)$ the polarized parton densities and $F_L = 0$.

In the charged current case we consider the massless Wilson coefficients of the structure function $xF_3(x, Q^2)$. One should notice, that here also strange–charm transitions occur, which imply heavy flavor corrections [34–37]. They are of importance in quantitative analyses. Here we consider all contributing quark flavors as massless. For incoming charged leptons the scattering cross section reads, cf. [38–40],

$$\frac{d^2\sigma^{cc}}{dxdy} = \frac{G_\mu^2 S}{8\pi} \left[\frac{M_W^2}{Q^2 + M_W^2} \right]^2 \left\{ Y_+ W_2(x, Q^2) + Y_- W_3(x, Q^2) - y^2 W_L(x, Q^2) \right\} \;, \tag{14}$$

4

where G_μ denotes the Fermi constant and $Y_\pm = 1 \pm (1-y)^2$ and both unpolarized leptons and nucleons are considered. The structure functions are given by

$$
\begin{aligned}
W_2(x, Q^2) &= F_2^{cc,Q_l}(x, Q^2) \,, & (15) \\
W_3(x, Q^2) &= -\text{sign}(Q_l)x F_3^{cc,Q_l}(x, Q^2) \,, & (16)
\end{aligned}
$$

where Q_l is the charge of the incoming charged lepton. At tree level these structure functions have the following quark flavor decomposition [39]

$$
F_2^{cc,+}(x, Q^2) = 2x \sum_i \left[d_i(x, Q^2) + \bar{u}_i(x, Q^2) \right] \,, \tag{17}
$$

$$
F_2^{cc,-}(x, Q^2) = 2x \sum_i \left[u_i(x, Q^2) + \bar{d}_i(x, Q^2) \right] \,, \tag{18}
$$

$$
x F_3^{cc,+}(x, Q^2) = 2x \sum_i \left[d_i(x, Q^2) - \bar{u}_i(x, Q^2) \right] \,, \tag{19}
$$

$$
x F_3^{cc,-}(x, Q^2) = 2x \sum_i \left[u_i(x, Q^2) - \bar{d}_i(x, Q^2) \right] \,, \tag{20}
$$

e.g. for four flavors[1], and $W_L = F_L^{cc,Q_l}$ denotes the longitudinal structure function with $F_L^{cc,\pm} = F_2^{cc,\pm} - 2x F_1^{cc,\pm}$.

In the present paper we consider in the charged current case the flavor non–singlet structure function combination $x F_3^{cc,+}(x, Q^2) + x F_3^{cc,-}(x, Q^2)$. At the experimental side one might consider the scattering off deuteron targets, obeying the $SU(2)$ isospin flavor symmetry

$$
u(x, Q^2) = d(x, Q^2) \,, \qquad \bar{u}(x, Q^2) = \bar{d}(x, Q^2) \,, \tag{21}
$$

after deuteron wave function corrections. Through this one obtains at tree level

$$
\begin{aligned}
F_2^{cc,d,+} &= x[u + d + 2s + \bar{u} + \bar{d} + 2\bar{c}] \,, & (22) \\
F_2^{cc,d,-} &= x[u + d + 2c + \bar{u} + \bar{d} + 2\bar{s}] \,, & (23) \\
F_2^{cc,d,+} + F_2^{cc,d,-} &= 2x\Sigma \,, & (24) \\
x F_3^{cc,d,+} &= x[u + d + 2s - \bar{u} - \bar{d} - 2\bar{c}] \,, & (25) \\
x F_3^{cc,d,-} &= x[u + d + 2sc - \bar{u} - \bar{d} - 2\bar{s}] \,, & (26) \\
x F_3^{cc,d,+} + x F_3^{cc,d,-} &= 2x[u + d - \bar{u} - \bar{d} + s - \bar{s} + c - \bar{c}] \,, \\
&\equiv 2x[u_v + d_v], & (27)
\end{aligned}
$$

and

$$
x F_3 = -\frac{8\pi}{Y_- G_\mu^2 S} \left[\frac{d^2\sigma^{cc,d,+}}{dx\,dy} - \frac{d^2\sigma^{cc,d,-}}{dx\,dy} \right] + \frac{Y_+}{Y_-} \left(F_2^{cc,d,+} - F_2^{cc,d,+} \right) - \frac{y^2}{Y_-} \left(W_L^{+,d} - W_L^{-d} \right) \tag{28}
$$

Here the non–singlet combinations for F_2^d and $W_L^d \propto Y_+, y^2$ correspond to the quark flavor combination

$$
f^{\text{NS},+} = 2x[(s - \bar{s}) - (c - \bar{c})] \,, \tag{29}
$$

---

[1]More generally, one has to account for the Cabibbo–Kobayashi–Maskawa mixing, cf. e.g. [35], effectively redefining the down–quark densities in the charged current case.

which vanishes under the assumption of symmetric sea–quarks of the same flavor and one obtains the direct projection on xF_3^d by the weighted differential cross section difference. In other cases one has to perform fits in Bjorken y to separate the Y_- from the Y_+ and y^2 contributions.

The projectors of the hadronic tensor allowing to isolate the Wilson coefficients of the different structure functions are given by

$$P_L^{\mu\nu} = \frac{8x^3}{Q^2} p^\mu p^\nu \, , \tag{30}$$

$$P_2^{\mu\nu} = -\frac{2x}{D-2} \left(g^{\mu\nu} - (D-1) \frac{4x^2}{Q^2} p^\mu p^\nu \right) \, , \tag{31}$$

$$P_3^{\mu\nu} = \frac{-i}{(D-2)(D-3)} \frac{4x}{Q^2} \varepsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma \, , \tag{32}$$

$$P_{g_1}^{\mu\nu} = \frac{i}{(D-2)(D-3)} \frac{2x}{Q^2} \varepsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma \, , \tag{33}$$

applied to $W_{\mu\nu}$, cf. also [41]. The twist–2 contributions to the structure functions in the massless case are given by Eq. (1).

The perturbative expansion of the structure functions

$$F_i(x, Q^2) = \sum_{k=0}^{\infty} a_s^k(Q^2) F_{i,k}(x) \tag{34}$$

obey the following renormalization group equation (RGE) in the massless limit [42, 43], see also [9],

$$[\mathcal{D} + \gamma_{J_1} + \gamma_{J_2} - n_\psi \gamma_\psi - n_A \gamma_A] F_i(N, Q^2) = 0 \tag{35}$$

with

$$\mathcal{D} = \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s) \frac{\partial}{\partial a_s} \tag{36}$$

in Mellin space with $a_s = a_s(\mu^2)$, $\beta(a_s)$ the β function of Quantum Chromodynamics (QCD), and setting $\mu \equiv \mu_F = \mu_R$, where $\mu_{F,R}$ are the factorization and renormalization scale. γ_{J_i} are the anomalous dimensions of the currents forming the forward Compton amplitude and $n_{\psi(A)}$ and $\gamma_{\psi(A)}$ are the numbers of external quark (gluon fields) and their anomalous dimensions. One may now split (35) for those of the renormalized Wilson coefficients and renormalized parton densities,

$$\sum_j \left[\mathcal{D}(\mu^2)\delta_{ij} + \gamma_{ij}^{S,NS} - n_\psi \gamma_\psi - n_A \gamma_A \right] f_j(N, \mu^2) = 0 \, , \tag{37}$$

$$\sum_j \left[\mathcal{D}(\mu^2)\delta_{ij} + \gamma_{J_1} + \gamma_{J_2} - \gamma_{ij}^{S,NS} \right] C_i \left(N, \frac{Q^2}{\mu^2} \right) = 0 \, . \tag{38}$$

In this way the Z factors of the massless Wilson coefficients and the massless parton densities are related.[2]

Before we turn to the renormalized Wilson coefficients we will study the unrenormalized ones, as those emerge in the calculation. They have the following representation. Here and in the following we work in Mellin N space.

$$\hat{C}_{i,q}^{NS} = 1 + \hat{a}_s \left\{ \frac{1}{\varepsilon} c_{i,q}^{NS,(1,-1)} + c_{i,q}^{NS,(1,0)} + \varepsilon c_{i,q}^{NS,(1,1)} + \varepsilon^2 c_{i,q}^{NS,(1,2)} + O(\varepsilon^3) \right\}$$

---

[2]This is different in the massive case, cf. [44], and even in treating heavy quarks as light at asymptotic scales.

$$+\hat{a}_s^2\left\{\frac{1}{\varepsilon^2}c_{i,q}^{\mathrm{NS},(2,-2)} + \frac{1}{\varepsilon}c_{i,q}^{\mathrm{NS},(2,-1)} + c_{i,q}^{\mathrm{NS},(2,0)} + \varepsilon c_{i,q}^{\mathrm{NS},(2,1)} + O(\varepsilon^2)\right\}$$

$$+\hat{a}_s^3\left\{\frac{1}{\varepsilon^3}c_{i,q}^{\mathrm{NS},(3,-3)} + \frac{1}{\varepsilon^2}c_{i,q}^{\mathrm{NS},(3,-2)} + \frac{1}{\varepsilon}c_{i,q}^{\mathrm{NS},(3,-1)} + c_{i,q}^{\mathrm{NS},(3,0)} + O(\varepsilon)\right\} + O\left(\hat{a}_s^4\right) , \quad (39)$$

$$\hat{C}_{i,q}^{\mathrm{PS}} = \hat{a}_s^2\left\{\frac{1}{\varepsilon^2}c_{i,q}^{\mathrm{PS},(2,-2)} + \frac{1}{\varepsilon}c_{i,q}^{\mathrm{PS},(2,-1)} + c_{i,q}^{\mathrm{PS},(2,0)} + \varepsilon c_{i,q}^{\mathrm{PS},(2,1)} + O(\varepsilon^2)\right\}$$

$$+\hat{a}_s^3\left\{\frac{1}{\varepsilon^3}c_{i,q}^{\mathrm{PS},(3,-3)} + \frac{1}{\varepsilon^2}c_{i,q}^{\mathrm{PS},(3,-2)} + \frac{1}{\varepsilon}c_{i,q}^{\mathrm{PS},(3,-1)} + c_{i,q}^{\mathrm{PS},(3,0)} + O(\varepsilon)\right\} + O\left(\hat{a}_s^4\right) , \quad (40)$$

$$\hat{C}_{i,g} = \hat{a}_s\left\{\frac{1}{\varepsilon}c_{i,g}^{(1,-1)} + c_{i,g}^{(1,0)} + \varepsilon c_{i,g}^{(1,1)} + \varepsilon^2 c_{i,g}^{(1,2)} + O(\varepsilon^3)\right\}$$

$$+\hat{a}_s^2\left\{\frac{1}{\varepsilon^2}c_{i,g}^{(2,-2)} + \frac{1}{\varepsilon}c_{i,g}^{(2,-1)} + c_{i,g}^{(2,0)} + \varepsilon c_{i,g}^{(2,1)} + O(\varepsilon^2)\right\}$$

$$+\hat{a}_s^3\left\{\frac{1}{\varepsilon^3}c_{i,g}^{(3,-3)} + \frac{1}{\varepsilon^2}c_{i,g}^{(3,-2)} + \frac{1}{\varepsilon}c_{i,g}^{(3,-1)} + c_{i,g}^{(3,0)} + O(\varepsilon)\right\} + O\left(\hat{a}_s^4\right) , \quad (41)$$

with the bare coupling constant $\hat{a}_s$. These relations apply synonymously to the structure functions $i = F_2, F_3, g_1$.

For those of the structure function F_L one obtains

$$\hat{C}_{F_L,q}^{\mathrm{NS}} = \hat{a}_s\left\{c_{F_L,q}^{\mathrm{NS},(1,0)} + \varepsilon c_{F_L,q}^{\mathrm{NS},(1,1)} + \varepsilon^2 c_{F_L,q}^{\mathrm{NS},(1,2)} + O(\varepsilon^3)\right\}$$

$$+\hat{a}_s^2\left\{\frac{1}{\varepsilon}c_{F_L,q}^{\mathrm{NS},(2,-1)} + c_{F_L,q}^{\mathrm{NS},(2,0)} + \varepsilon c_{F_L,q}^{\mathrm{NS},(2,1)} + O(\varepsilon^2)\right\}$$

$$+\hat{a}_s^3\left\{\frac{1}{\varepsilon^2}c_{F_L,q}^{\mathrm{NS},(3,-2)} + \frac{1}{\varepsilon}c_{F_L,q}^{\mathrm{NS},(3,-1)} + c_{F_L,q}^{\mathrm{NS},(3,0)} + O(\varepsilon)\right\} + O\left(\hat{a}_s^4\right) , \quad (42)$$

$$\hat{C}_{F_L,q}^{\mathrm{PS}} = \hat{a}_s^2\left\{\frac{1}{\varepsilon}c_{F_L,q}^{\mathrm{PS},(2,-1)} + c_{F_L,q}^{\mathrm{PS},(2,0)} + \varepsilon c_{F_L,q}^{\mathrm{PS},(2,1)} + O(\varepsilon^2)\right\}$$

$$+\hat{a}_s^3\left\{\frac{1}{\varepsilon^2}c_{F_L,q}^{\mathrm{PS},(3,-2)} + \frac{1}{\varepsilon}c_{F_L,q}^{\mathrm{PS},(3,-1)} + c_{F_L,q}^{\mathrm{PS},(3,0)} + O(\varepsilon)\right\} + O\left(\hat{a}_s^4\right) , \quad (43)$$

$$\hat{C}_{F_L,g} = \hat{a}_s\left\{c_{F_L,g}^{(1,0)} + \varepsilon c_{F_L,g}^{(1,1)} + \varepsilon^2 c_{F_L,g}^{(1,2)} + O(\varepsilon^3)\right\}$$

$$+\hat{a}_s.^2\left\{\frac{1}{\varepsilon}c_{F_L,g}^{(2,-1)} + c_{F_L,g}^{(2,0)} + \varepsilon c_{F_L,g}^{(2,1)} + O(\varepsilon^2)\right\}$$

$$+\hat{a}_s^3\left\{\frac{1}{\varepsilon^2}c_{F_L,g}^{(3,-2)} + \frac{1}{\varepsilon}c_{F_L,g}^{(3,-1)} + c_{F_L,g}^{(3,0)} + O(\varepsilon)\right\} . \quad (44)$$

The renormalization proceeds in the following way. The coupling constant is renormalized in the $\overline{\mathrm{MS}}$ scheme

$$\hat{a}_s = a_s\left(1 + \frac{2\beta_0}{\varepsilon}a_s + \left[\frac{4\beta_0^2}{\varepsilon^2} + \frac{\beta_1}{\varepsilon}\right]a_s^2 + O(a_s^3)\right) . \quad (45)$$

Here β_i denote the expansion coefficients of the QCD β–function, [45].

$$\beta_0(N_F) = \frac{11}{3}C_A - \frac{4}{3}T_F N_F \ , \tag{46}$$

$$\beta_1(N_F) = \frac{34}{3}C_A^2 - 4\left(\frac{5}{3}C_A + C_F\right)T_F N_F \ . \tag{47}$$

The color factors are $C_F = (N_C^2 - 1)/(2N_C), C_A = N_C, T_F = 1/2$ for $SU(N_C)$ and $N_C = 3$ for QCD; N_F denotes the number of massless quark flavors. Later we will also need the color factor[3]

$$d_{abc}d^{abc} = \frac{(N_C^2 - 1)(N_C^2 - 4)}{16N_C} \tag{48}$$

and the dimension of the adjoint representation $N_A = N_C^2 - 1$. We are carrying out a partial renormalization of the Wilson coefficient of the structure function xF_3 for the axial vector coupling, which is performed by multiplying the unrenormalized Wilson coefficient by Z_A, [17], see also [25, 46],

$$
\begin{aligned}
Z_A = \ & 1 + \frac{\hat{a}_s^2}{\varepsilon}\left[\frac{22}{3}C_A C_F - \frac{8}{3}C_F T_F N_F\right] + \hat{a}_s^3\left[\frac{1}{\varepsilon^2}\left(-\frac{484}{27}C_A^2 C_F + \frac{352}{27}C_A C_F T_F N_F\right.\right. \\
& \left. -\frac{64}{27}C_F T_F^2 N_F^2\right) + \frac{1}{\varepsilon}\left(\frac{3578}{81}C_A^2 C_F - \frac{308}{9}C_A C_F^2 - \frac{1664}{81}C_A C_F T_F N_F + \frac{64}{9}C_F^2 T_F N_F\right. \\
& \left.\left. +\frac{32}{81}C_F T_F^2 N_F^2\right)\right] \ .
\end{aligned}
\tag{49}
$$

The non–singlet Wilson coefficients are renormalized via

$$C_{i,q}^{\mathrm{NS}} = Z_{qq}^{\mathrm{NS}}\hat{C}_{i,q}^{\mathrm{NS}} \tag{50}$$

and the singlet Wilson coefficients via

$$\begin{pmatrix} C_{i,q}^{\mathrm{S}} \\ C_{i,g} \end{pmatrix} = Z^{\mathrm{S}\,T} \cdot \begin{pmatrix} \hat{C}_{i,q}^{\mathrm{S}} \\ \hat{C}_{i,g} \end{pmatrix} \ , \tag{51}$$

with

$$
\begin{aligned}
Z_{qq}^{\mathrm{NS}} = \ & 1 + a_s\frac{\gamma_{qq}^{(0),\mathrm{NS}}}{\varepsilon} + a_s^2\left[\frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{qq}^{(0),\mathrm{NS}\,2} + \beta_0\gamma_{qq}^{(0),\mathrm{NS}}\right) + \frac{1}{2\varepsilon}\gamma_{qq}^{(0),\mathrm{NS}}\right] + a_s^3\left[\frac{1}{\varepsilon^3}\left(\frac{1}{6}\gamma_{qq}^{(0),\mathrm{NS}\,3}\right.\right. \\
& \left. +\beta_0\gamma_{qq}^{(0),\mathrm{NS}\,2} + \frac{4}{3}\beta_0^2\gamma_{qq}^{(0),\mathrm{NS}}\right) + \frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{qq}^{(0),\mathrm{NS}}\gamma_{qq}^{(1),\mathrm{NS}} + \frac{2}{3}\beta_0\gamma_{qq}^{(1),\mathrm{NS}} + \frac{2}{3}\beta_1\gamma_{qq}^{(0),\mathrm{NS}}\right) \\
& \left. +\frac{1}{3\varepsilon}\gamma_{qq}^{(2),\mathrm{NS}}\right] + O(a_s^4) \ , \\
Z_{ij}^{\mathrm{S}} = \ & \delta_{ij} + a_s\frac{\gamma_{ij}^{(0)}}{\varepsilon} + a_s^2\left[\frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{il}^{(0)}\gamma_{lj}^{(0)} + \beta_0\gamma_{ij}^{(0)}\right) + \frac{1}{2\varepsilon}\gamma_{ij}^{(1)}\right]
\end{aligned}
\tag{52}
$$

---

[3]We follow the notation of `COLOR`, see Ref. [56].

$$+a_s^3 \left[\frac{1}{\varepsilon^3} \left(\frac{1}{6}\gamma_{il}^{(0)}\gamma_{lk}^{(0)}\gamma_{kj}^{(0)} + \beta_0\gamma_{il}^{(0)}\gamma_{lj}^{(0)} + \frac{4}{3}\beta_0^2\gamma_{ij}^{(0)} \right) \right.$$

$$\left. + \frac{1}{\varepsilon^2} \left(\frac{1}{6}\gamma_{il}^{(1)}\gamma_{lj}^{(0)} + \frac{1}{3}\gamma_{il}^{(0)}\gamma_{lj}^{(1)} + \frac{2}{3}\beta_0\gamma_{ij}^{(1)} + \frac{2}{3}\beta_1\gamma_{ij}^{(0)} \right) + \frac{1}{3\varepsilon}\gamma_{ij}^{(2)} \right] + O(a_s^4) \, . \tag{53}$$

The pole terms of the Wilson coefficients are predicted by the anomalous dimensions and the expansion coefficients of the unrenormalized Wilson coefficients in lower orders, where in the polarized case the anomalous dimensions γ_{ij} have to be replaced by $\Delta\gamma_{ij}$.

$$c_{i,q}^{\mathrm{NS},(1,-1)} = -\gamma_{qq}^{\mathrm{NS},(0)} \, , \tag{54}$$

$$c_{i,q}^{\mathrm{NS},(2,-2)} = \gamma_{qq}^{\mathrm{NS},(0)} \left(\beta_0 + \frac{1}{2}\gamma_{qq}^{\mathrm{NS},(0)} \right) \, , \tag{55}$$

$$c_{i,q}^{\mathrm{NS},(2,-1)} = -c_{i,q}^{\mathrm{NS},(1,0)} \left(\gamma_{qq}^{\mathrm{NS},(0)} + 2\beta_0 \right) - \frac{1}{2}\gamma_{qq}^{\mathrm{NS},(1)} \, , \tag{56}$$

$$c_{i,q}^{\mathrm{NS},(3,-3)} = -\gamma_{qq}^{\mathrm{NS},(0)} \left(\frac{4}{3}\beta_0^2 + \beta_0\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{6}\gamma_{qq}^{\mathrm{NS},(0)\,2} \right) \, , \tag{57}$$

$$c_{i,q}^{\mathrm{NS},(3,-2)} = 4\beta_0^2 c_{i,q}^{\mathrm{NS},(1,0)} + \frac{1}{3}\beta_1\gamma_{qq}^{\mathrm{NS},(0)}$$

$$+ 3\beta_0 c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)\,2} + \frac{4}{3}\beta_0\gamma_{qq}^{\mathrm{NS},(1)} + \frac{1}{2}\gamma_{qq}^{\mathrm{NS},(0)}\gamma_{qq}^{\mathrm{NS},(1)} \, , \tag{58}$$

$$c_{i,q}^{\mathrm{NS},(3,-1)} = -\beta_1 c_{i,q}^{\mathrm{NS},(1,0)} - 4\beta_0^2 c_{i,q}^{\mathrm{NS},(1,1)} - 4\beta_0 c_{i,q}^{\mathrm{NS},(2,0)} - 3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)} - c_{i,q}^{\mathrm{NS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)}$$

$$- \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)\,2} - \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(1)} - \frac{1}{3}\gamma_{qq}^{\mathrm{NS},(2)} \, , \tag{59}$$

$$c_{i,q}^{\mathrm{PS},(2,-2)} = \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} \, , \tag{60}$$

$$c_{i,q}^{\mathrm{PS},(2,-1)} = -\gamma_{gq}^{(0)}c_{i,g}^{(1,0)} - \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(1)} \, , \tag{61}$$

$$c_{i,q}^{\mathrm{PS},(3,-3)} = -\beta_0\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} - \frac{1}{6}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} - \frac{1}{3}\gamma_{qq}^{\mathrm{NS},(0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} \, , \tag{62}$$

$$c_{i,q}^{\mathrm{PS},(3,-2)} = 3\beta_0\gamma_{gq}^{(0)}c_{i,g}^{(1,0)} + \frac{1}{2}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}c_{i,g}^{(1,0)} + \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}c_{i,q}^{\mathrm{NS},(1,0)} + \frac{1}{6}\gamma_{gq}^{(1)}\gamma_{qg}^{(0)} + \frac{1}{3}\gamma_{gq}^{(0)}\gamma_{qg}^{(1)}$$

$$+ \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}c_{i,g}^{(1,0)} + \frac{4}{3}\beta_0\gamma_{qq}^{\mathrm{PS},(1)} + \frac{1}{2}\gamma_{qq}^{\mathrm{NS},(0)}\gamma_{qq}^{\mathrm{PS},(1)} \, , \tag{63}$$

$$c_{i,q}^{\mathrm{PS},(3,-1)} = -4\beta_0 c_{i,q}^{\mathrm{PS},(2,0)} - 3\beta_0\gamma_{gq}^{(0)}c_{i,g}^{(1,1)} - \gamma_{gq}^{(0)}c_{i,g}^{(2,0)} - \frac{1}{2}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}c_{i,g}^{(1,1)} - \frac{1}{2}\gamma_{gq}^{(1)}c_{i,g}^{(1,0)}$$

$$- \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}c_{i,q}^{\mathrm{NS},(1,1)} - \gamma_{qq}^{\mathrm{NS},(0)}c_{i,q}^{\mathrm{PS},(2,0)} - \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}c_{i,g}^{(1,1)} - \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(1)}c_{i,q}^{\mathrm{NS},(1,0)}$$

$$- \frac{1}{3}\gamma_{qq}^{\mathrm{PS},(2)} \, , \tag{64}$$

and

$$c_{i,g}^{(1,-1)} = -\gamma_{qg}^{(0)} \, , \tag{65}$$

$$c_{i,g}^{(2,-2)} = \beta_0\gamma_{qg}^{(0)} + \frac{1}{2}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} \, , \tag{66}$$

$$c_{i,g}^{(2,-1)} = -2\beta_0 c_{i,g}^{(1,0)} - \gamma_{gg}^{(0)}c_{i,g}^{(1,0)} - \gamma_{qg}^{(0)}c_{i,q}^{\mathrm{NS},(1,0)} - \frac{1}{2}\gamma_{qg}^{(1)} \, , \tag{67}$$

$$c_{i,g}^{(3,-3)} = -\frac{4}{3}\beta_0^2\gamma_{qg}^{(0)} - \beta_0\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} - \frac{1}{6}\gamma_{gg}^{(0)\,2}\gamma_{qg}^{(0)} - \frac{1}{6}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)\,2} - \beta_0\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}$$
$$-\frac{1}{6}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} - \frac{1}{6}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)\,2} \,, \tag{68}$$

$$c_{i,g}^{(3,-2)} = 4\beta_0^2 c_{i,g}^{(1,0)} + 3\beta_0 c_{i,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gg}^{(0)\,2} + \frac{1}{3}\beta_1\gamma_{qg}^{(0)} + 3\beta_0 c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}$$
$$+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{6}\gamma_{gg}^{(1)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{4}{3}\beta_0\gamma_{qg}^{(1)} + \frac{1}{3}\gamma_{gg}^{(0)}\gamma_{qg}^{(1)}$$
$$+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{6}\gamma_{qg}^{(1)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{3}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(1)} + \frac{1}{3}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{PS},(1)} \,, \tag{69}$$

$$c_{i,g}^{(3,-1)} = -\beta_1 c_{i,g}^{(1,0)} - 4\beta_0^2 c_{i,g}^{(1,1)} - 4\beta_0 c_{i,g}^{(2,0)} - 3\beta_0 c_{i,g}^{(1,1)}\gamma_{gg}^{(0)} - c_{i,g}^{(2,0)}\gamma_{gg}^{(0)}$$
$$-\frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gg}^{(0)\,2} - \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gg}^{(1)} - 3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)} - c_{i,q}^{\mathrm{NS},(2,0)}\gamma_{qg}^{(0)} - c_{i,q}^{\mathrm{PS},(2,0)}\gamma_{qg}^{(0)}$$
$$-\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} - \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} - \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(1)} - \frac{1}{3}\gamma_{qg}^{(2)}$$
$$-\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} \,. \tag{70}$$

For the longitudinal structure function F_L the poles are predicted by

$$c_{F_L,q}^{\mathrm{NS},(2,-1)} = -c_{F_L,q}^{\mathrm{NS},(1,0)}\left(\gamma_{qq}^{\mathrm{NS},(0)} + 2\beta_0\right) \,, \tag{71}$$

$$c_{F_L,q}^{\mathrm{NS},(3,-2)} = 4\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,0)} + 3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)\,2} \,, \tag{72}$$

$$c_{F_L,q}^{\mathrm{NS},(3,-1)} = -\beta_1 c_{F_L,q}^{\mathrm{NS},(1,0)} - 4\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,1)} - 4\beta_0 c_{F_L,q}^{\mathrm{NS},(2,0)} - 3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)} - c_{F_L,q}^{\mathrm{NS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)}$$
$$-\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)\,2} - \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(1)} \,, \tag{73}$$

$$c_{F_L,q}^{\mathrm{PS},(2,-1)} = -\gamma_{gq}^{(0)}c_{F_L,g}^{(1,0)} \,, \tag{74}$$

$$c_{F_L,q}^{\mathrm{PS},(3,-2)} = 3\beta_0\gamma_{gq}^{(0)}c_{F_L,g}^{(1,0)} + \frac{1}{2}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}c_{F_L,g}^{(1,0)} + \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}c_{F_L,q}^{\mathrm{NS},(1,0)} + \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}c_{F_L,g}^{(1,0)} \,, \tag{75}$$

$$c_{F_L,q}^{\mathrm{PS},(3,-1)} = -4\beta_0 c_{F_L,q}^{\mathrm{PS},(2,0)} - 3\beta_0\gamma_{gq}^{(0)}c_{F_L,g}^{(1,1)} - \gamma_{gq}^{(0)}c_{F_L,g}^{(2,0)} - \frac{1}{2}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}c_{F_L,g}^{(1,1)} - \frac{1}{2}\gamma_{gq}^{(1)}c_{F_L,g}^{(1,0)}$$
$$-\frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}c_{F_L,q}^{\mathrm{NS},(1,1)} - \gamma_{qq}^{\mathrm{NS},(0)}c_{F_L,q}^{\mathrm{PS},(2,0)} - \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}c_{F_L,g}^{(1,1)} - \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(1)}c_{F_L,q}^{\mathrm{NS},(1,0)} \,, \tag{76}$$

$$c_{F_L,g}^{(2,-1)} = -2\beta_0 c_{F_L,g}^{(1,0)} - \gamma_{gg}^{(0)}c_{F_L,g}^{(1,0)} - \gamma_{qg}^{(0)}c_{F_L,q}^{\mathrm{NS},(1,0)} \,, \tag{77}$$

$$c_{F_L,g}^{(3,-2)} = 4\beta_0^2 c_{F_L,g}^{(1,0)} + 3\beta_0 c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)\,2} + 3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}$$
$$+\frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} \,, \tag{78}$$

$$c_{F_L,g}^{(3,-1)} = -\beta_1 c_{F_L,g}^{(1,0)} - 4\beta_0^2 c_{F_L,g}^{(1,1)} - 4\beta_0 c_{F_L,g}^{(2,0)} - 3\beta_0 c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)} - c_{F_L,g}^{(2,0)}\gamma_{gg}^{(0)} - \frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)\,2}$$
$$-\frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(1)} - 3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)} - c_{F_L,q}^{\mathrm{NS},(2,0)}\gamma_{qg}^{(0)} - c_{F_L,q}^{\mathrm{PS},(2,0)}\gamma_{qg}^{(0)} - \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}$$
$$-\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} - \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(1)} - \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} \,. \tag{79}$$

The renormalized Wilson coefficients for the structure functions F_2, F_3 and g_1 have the following structure

$$C_{i,q}^{\mathrm{NS}} = 1 + a_s\left\{\ln\left(\frac{Q^2}{\mu^2}\right)\left[-\frac{1}{2}\gamma_{qq}^{\mathrm{NS},(0)}\right] + c_{i,q}^{\mathrm{NS},(1,0)}\right\} + a_s^2\left\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{4}\beta_0\gamma_{qq}^{\mathrm{NS},(0)}\right.\right.$$

$$+\frac{1}{8}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}] - \ln\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0 c_{i,q}^{\mathrm{NS},(1,0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}\gamma_{qq}^{\mathrm{NS},(1)}\right] + 2\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}$$

$$+c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)} + c_{i,q}^{\mathrm{NS},(2,0)}\bigg\} + a_s^3\bigg\{ -\ln^3\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{6}\beta_0^2\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{8}\beta_0\,\gamma_{qq}^{\mathrm{NS},(0)\,2}\right.$$

$$\left.+\frac{1}{48}\,\gamma_{qq}^{\mathrm{NS},(0)\,3}\right] + \ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0^2 c_{i,q}^{\mathrm{NS},(1,0)} + \frac{1}{4}\beta_1\gamma_{qq}^{\mathrm{NS},(0)} + \frac{3}{4}\beta_0 c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)}\right.$$

$$\left.+\frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2} + \frac{1}{2}\beta_0\gamma_{qq}^{\mathrm{NS},(1)} + \frac{1}{4}\gamma_{qq}^{\mathrm{NS},(0)}\gamma_{qq}^{\mathrm{NS},(1)}\right] - \ln\left(\frac{Q^2}{\mu^2}\right)\left[\beta_1 c_{i,q}^{\mathrm{NS},(1,0)}\right.$$

$$\left.+4\beta_0^2 c_{i,q}^{\mathrm{NS},(1,1)} + 2\beta_0 c_{i,q}^{\mathrm{NS},(2,0)} + 3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}\right.$$

$$\left.+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(1)} + \frac{1}{2}\gamma_{qq}^{\mathrm{NS},(2)}\right] + \beta_1 c_{i,q}^{\mathrm{NS},(1,1)} + 4\beta_0^2 c_{i,q}^{\mathrm{NS},(1,2)} + 4\beta_0 c_{i,q}^{\mathrm{NS},(2,1)}$$

$$+3\beta_0 c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{qq}^{\mathrm{NS},(0)} + c_{i,q}^{\mathrm{NS},(2,1)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(1)} + c_{i,q}^{\mathrm{NS},(3,0)}\bigg\}$$

$$+O(a_s^4)\,, \tag{80}$$

$$C_{i,q}^{\mathrm{PS}} = a_s^2\bigg\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{8}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\right] - \ln\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{2}(c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}) + \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(1)}\right] + c_{i,g}^{(1,1)}\gamma_{gq}^{(0)}$$

$$+c_{i,q}^{\mathrm{PS},(2,0)}\bigg\} + a_s^3\bigg\{ -\ln^3\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{8}(\beta_0\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}) + \frac{1}{48}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{24}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\right]$$

$$+\ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\frac{3}{4}\beta_0 c_{i,g}^{(1,0)}\gamma_{gq}^{(0)} + \frac{1}{8}c_{i,g}^{(1,0)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)} + \frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{8}\gamma_{gq}^{(1)}\gamma_{qg}^{(0)}\right.$$

$$\left.+\frac{1}{8}\gamma_{gq}^{(0)}\gamma_{qg}^{(1)} + \frac{1}{8}c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}\beta_0\gamma_{qq}^{\mathrm{PS},(1)} + \frac{1}{4}\gamma_{qq}^{\mathrm{NS},(0)}\gamma_{qq}^{\mathrm{PS},(1)}\right] - \ln\left(\frac{Q^2}{\mu^2}\right)$$

$$\times\left[2\beta_0 c_{i,q}^{\mathrm{PS},(2,0)} + 3\beta_0 c_{i,g}^{(1,1)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(2,0)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gq}^{(1)}\right.$$

$$\left.+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{PS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{PS},(1)} + \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(2)}\right]$$

$$+4\beta_0 c_{i,q}^{\mathrm{PS},(2,1)} + 3\beta_0 c_{i,g}^{(1,2)}\gamma_{gq}^{(0)} + c_{i,g}^{(2,1)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,2)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(1)}$$

$$+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + c_{i,q}^{\mathrm{PS},(2,1)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,g}^{(1,2)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{PS},(1)} + c_{i,q}^{\mathrm{PS},(3,0)}\bigg\}$$

$$+O(a_s^4)\,, \tag{81}$$

$$C_{i,g} = a_s\bigg\{ -\ln\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{2}\gamma_{qg}^{(0)}\right] + c_{i,g}^{(1,0)}\bigg\} + a_s^2\bigg\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{4}\beta_0\gamma_{qg}^{(0)} + \frac{1}{8}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}\right.$$

$$\left.+\frac{1}{8}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\right] - \ln\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0 c_{i,g}^{(1,0)} + \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)} + \frac{1}{2}\gamma_{qg}^{(1)}\right]$$

$$+2\beta_0 c_{i,g}^{(1,1)} + c_{i,g}^{(1,1)}\gamma_{gg}^{(0)} + c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)} + c_{i,g}^{(2,0)}\bigg\} + a_s^3\bigg\{ -\ln^3\left(\frac{Q^2}{\mu^2}\right)\left[\frac{1}{6}\beta_0^2\gamma_{qg}^{(0)}\right.$$

$$\left.+\frac{1}{8}\beta_0\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{48}\,\gamma_{gg}^{(0)\,2}\,\gamma_{qg}^{(0)} + \frac{1}{48}\gamma_{gq}^{(0)}\,\gamma_{qg}^{(0)\,2} + \frac{1}{8}\beta_0\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{48}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\right.$$

$$\left.+\frac{1}{48}\gamma_{qg}^{(0)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}\right] + \ln^2\left(\frac{Q^2}{\mu^2}\right)\left[\beta_0^2 c_{i,g}^{(1,0)} + \frac{3}{4}\beta_0 c_{i,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{8}c_{i,g}^{(1,0)}\,\gamma_{gg}^{(0)\,2} + \frac{1}{4}\beta_1\gamma_{qg}^{(0)}\right.$$

$$\left.+\frac{3}{4}\beta_0 c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)} + \frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{8}\gamma_{gg}^{(1)}\gamma_{qg}^{(0)} + \frac{1}{8}c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}\beta_0\gamma_{qg}^{(1)} + \frac{1}{8}\gamma_{gg}^{(0)}\gamma_{qg}^{(1)}\right.$$

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$$
\begin{aligned}
&+\frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{8}\gamma_{qg}^{(1)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{8}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(1)}+\frac{1}{8}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{PS},(1)}\Big]\\
&-\ln\left(\frac{Q^2}{\mu^2}\right)\Big[(\beta_1 c_{i,g}^{(1,0)})+4\beta_0^2 c_{i,g}^{(1,1)}+2\beta_0 c_{i,g}^{(2,0)}+3\beta_0 c_{i,g}^{(1,1)}\gamma_{gg}^{(0)}+\frac{1}{2}c_{i,g}^{(2,0)}\gamma_{gg}^{(0)}\\
&+\frac{1}{2}c_{i,g}^{(1,1)}\;\gamma_{gg}^{(0)\,2}+\frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gg}^{(1)}+3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(2,0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,q}^{\mathrm{PS},(2,0)}\gamma_{qg}^{(0)}\\
&+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(1)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}\gamma_{qg}^{(2)}\Big]\\
&+\beta_1 c_{i,g}^{(1,1)}+4\beta_0^2 c_{i,g}^{(1,2)}+4\beta_0 c_{i,g}^{(2,1)}+3\beta_0 c_{i,g}^{(1,2)}\gamma_{gg}^{(0)}+c_{i,g}^{(2,1)}\gamma_{gg}^{(0)}+\frac{1}{2}c_{i,g}^{(1,2)}\;\gamma_{gg}^{(0)\,2}\\
&+\frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gg}^{(1)}+3\beta_0 c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{qg}^{(0)}+c_{i,q}^{\mathrm{NS},(2,1)}\gamma_{qg}^{(0)}+c_{i,q}^{\mathrm{PS},(2,1)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}\\
&+\frac{1}{2}c_{i,g}^{(1,2)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(1)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}+c_{i,g}^{(3,0)}\bigg\}+O(a_s^4)\;. \qquad (82)
\end{aligned}
$$

Accordingly, one obtains for the structure function F_L,

$$
\begin{aligned}
C_{F_L,q}^{\mathrm{NS}} &= a_s\bigg\{c_{F_L,q}^{\mathrm{NS},(1,0)}\bigg\}+a_s^2\bigg\{-\ln\left(\frac{Q^2}{\mu^2}\right)\Big[\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)}\Big]+2\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}\\
&\quad+c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)}+c_{F_L,q}^{\mathrm{NS},(2,0)}\bigg\}+a_s^3\bigg\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\Big[\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,0)}+\frac{3}{4}\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)}\\
&\quad+\frac{1}{8}c_{F_L,q}^{\mathrm{NS},(1,0)}\;\gamma_{qq}^{\mathrm{NS},(0)\,2}\Big]-\ln\left(\frac{Q^2}{\mu^2}\right)\Big[\beta_1 c_{F_L,q}^{\mathrm{NS},(1,0)}+4\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,1)}+2\beta_0 c_{F_L,q}^{\mathrm{NS},(2,0)}\\
&\quad+3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\;\gamma_{qq}^{\mathrm{NS},(0)\,2}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(1)}\Big]\\
&\quad+\beta_1 c_{F_L,q}^{\mathrm{NS},(1,1)}+4\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,2)}+4\beta_0 c_{F_L,q}^{\mathrm{NS},(2,1)}+3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,2)}\gamma_{qq}^{\mathrm{NS},(0)}+c_{F_L,q}^{\mathrm{NS},(2,1)}\gamma_{qq}^{\mathrm{NS},(0)}\\
&\quad+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,2)}\;\gamma_{qq}^{\mathrm{NS},(0)\,2}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(1)}+c_{F_L,q}^{\mathrm{NS},(3,0)}\bigg\}+O(a_s^4)\;, \qquad (83)\\[6pt]
C_{F_L,q}^{\mathrm{PS}} &= a_s^2\bigg\{-\ln\left(\frac{Q^2}{\mu^2}\right)\Big[\frac{1}{2}(c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)})\Big]+c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}+c_{F_L,q}^{\mathrm{PS},(2,0)}\bigg\}\\
&\quad+a_s^3\bigg\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\Big[\frac{3}{4}\beta_0 c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)}+\frac{1}{8}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}+\frac{1}{8}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\\
&\quad+\frac{1}{8}c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\Big]-\ln\left(\frac{Q^2}{\mu^2}\right)\Big[2\beta_0 c_{F_L,q}^{\mathrm{PS},(2,0)}+3\beta_0 c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}+\frac{1}{2}c_{F_L,g}^{(2,0)}\gamma_{gq}^{(0)}\\
&\quad+\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}+\frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gq}^{(1)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{PS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)}\\
&\quad+\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{PS},(1)}\Big]+4\beta_0 c_{F_L,q}^{\mathrm{PS},(2,1)}+3\beta_0 c_{F_L,g}^{(1,2)}\gamma_{gq}^{(0)}\\
&\quad+c_{F_L,g}^{(2,1)}\gamma_{gq}^{(0)}+\frac{1}{2}c_{F_L,g}^{(1,2)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}+\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gq}^{(1)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,2)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\\
&\quad+c_{F_L,q}^{\mathrm{PS},(2,1)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,g}^{(1,2)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{PS},(1)}+c_{q,(3,0)}^{\mathrm{PS},(L)}\bigg\}+O(a_s^4)\;, \qquad (84)\\[6pt]
C_{F_L,g} &= a_s\bigg\{c_{F_L,g}^{(1,0)}\bigg\}+a_s^2\bigg\{-\ln\left(\frac{Q^2}{\mu^2}\right)\Big[\beta_0 c_{F_L,g}^{(1,0)}+\frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\Big]\\
&\quad+2\beta_0 c_{F_L,g}^{(1,1)}+c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)}+c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}+c_{F_L,g}^{(2,0)}\bigg\}+a_s^3\bigg\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\Big[\beta_0^2 c_{F_L,g}^{(1,0)}
\end{aligned}
$$

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$$
+\frac{3}{4}\beta_0 c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{8}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)\,2} + \frac{3}{4}\beta_0 c_{F_L,q}^{\text{NS},(1,0)}\gamma_{qg}^{(0)} + \frac{1}{8}c_{F_L,q}^{\text{NS},(1,0)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}
$$

$$
+\frac{1}{8}c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{8}c_{F_L,q}^{\text{NS},(1,0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\text{NS},(0)}\Big] - \ln\left(\frac{Q^2}{\mu^2}\right)\Big[\beta_1 c_{F_L,g}^{(1,0)} + 4\beta_0^2 c_{F_L,g}^{(1,1)}
$$

$$
+2\beta_0 c_{F_L,g}^{(2,0)} + 3\beta_0 c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{F_L,g}^{(2,0)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)\,2} + \frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(1)}
$$

$$
+3\beta_0 c_{F_L,q}^{\text{NS},(1,1)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\text{NS},(2,0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\text{PS},(2,0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\text{NS},(1,1)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}
$$

$$
+\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\text{NS},(1,0)}\gamma_{qg}^{(1)} + \frac{1}{2}c_{F_L,q}^{\text{NS},(1,1)}\gamma_{qg}^{(0)}\gamma_{qq}^{\text{NS},(0)}\Big] + \beta_1 c_{F_L,g}^{(1,1)} + 4\beta_0^2 c_{F_L,g}^{(1,2)}
$$

$$
+4\beta_0 c_{F_L,g}^{(2,1)} + 3\beta_0 c_{F_L,g}^{(1,2)}\gamma_{gg}^{(0)} + c_{F_L,g}^{(2,1)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{F_L,g}^{(1,2)}\gamma_{gg}^{(0)\,2} + \frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gg}^{(1)}
$$

$$
+3\beta_0 c_{F_L,q}^{\text{NS},(1,2)}\gamma_{qg}^{(0)} + c_{F_L,q}^{\text{NS},(2,1)}\gamma_{qg}^{(0)} + c_{F_L,q}^{\text{PS},(2,1)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\text{NS},(1,2)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}
$$

$$
+\frac{1}{2}c_{F_L,g}^{(1,2)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\text{NS},(1,1)}\gamma_{qg}^{(1)} + \frac{1}{2}c_{F_L,q}^{\text{NS},(1,2)}\gamma_{qg}^{(0)}\gamma_{qq}^{\text{NS},(0)} + c_{F_L,g}^{(3,0)}\Big\} + O(a_s^4) \ . \tag{85}
$$

The Wilson coefficients for the different deep–inelastic structure functions are split w.r.t. the parton contents they are mapping to, the observables and the specific couplings of the gauge bosons exchanged, see Ref. [47].

For pure photon exchange there are two contributions in the flavor non–singlet case. One corresponds to a through-flowing external fermion line to which the photons are coupling to, which is of relative weight $\mathsf{w}_1 = 1$. The second case corresponds to a photon coupling to one through-flowing line and another to a closed internal fermion line, with relative weight

$$
\mathsf{w}_2 = \frac{\text{tr}(\hat{Q}_f\lambda_\alpha)}{\text{tr}(\hat{Q}_f^2\lambda_\alpha)}\frac{1}{N_F}\sum_{f=1}^{N_F}e_f, \qquad \frac{\text{tr}(\hat{Q}_f\lambda_\alpha)}{\text{tr}(\hat{Q}_f^2\lambda_\alpha)} = 3 \ , \tag{86}
$$

for $SU(N_F)$ and λ_a the generalized Pauli–Gell–Mann matrices. e_f denotes the fermion charge and $\hat{Q}_f = \text{diag}(2/3, -1/3, -1/3, 2/3, -1/3)$ the quark charge matrix of the electromagnetic current.

The flavor non–singlet structure functions for photon exchange read [14]

$$
F_i^{\text{NS},+}(x,Q^2) = \sum_{f=1}^{N_F}e_f^2\left[C_{i,q}^{\text{NS}}\left(x,\frac{Q^2}{\mu^2}\right) + \mathsf{w}_2 C_{i,q}^{d_{abc}}\left(x,\frac{Q^2}{\mu^2}\right)\right]\otimes f_{q,f}^{\text{NS},+}(x,\mu^2) \ , \tag{87}
$$

with

$$
\mathsf{w}_2 = \frac{\text{tr}(\hat{Q}_f)}{N_F}\frac{\text{tr}(\hat{Q}_f\lambda_\alpha)}{\text{tr}(\hat{Q}_f^2\lambda_\alpha)} \ , \quad \mathsf{w}_2(N_F=3) = 0 \ , \quad \mathsf{w}_2(N_F=4) = \frac{1}{2} \ , \quad \mathsf{w}_2(N_F=5) = \frac{1}{5} \ . \tag{88}
$$

The non–singlet distribution function is given by

$$
f_{q,f}^{\text{NS},+}(x,\mu^2) = f_{q,f}(x,\mu^2) + \bar{f}_{q,f}(x,\mu^2) - \frac{1}{N_F}\Sigma(x,\mu^2) \ , \tag{89}
$$

with the quark singlet distribution

$$
\Sigma(x,\mu^2) = \sum_{f=1}^{N_F}f_{q,f}(x,\mu^2) + \bar{f}_{q,f}(x,\mu^2) \ . \tag{90}
$$

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In the flavor singlet case the weight factor w_3 appears

$$\mathsf{w}_3 = \frac{1}{N_F} \frac{\left(\sum_{f=1}^{N_F} e_f\right)^2}{\sum_{f=1}^{N_F} e_f^2} \ , \quad \mathsf{w}_3(N_F = 3) = 0 \ , \quad \mathsf{w}_3(N_F = 4) = \frac{1}{10} \ , \quad \mathsf{w}_3(N_F = 5) = \frac{1}{55} \ , \quad (91)$$

because of the overall normalization to the sum of the quark charge squares for all diagrams in which the electromagnetic current couples to two different fermion lines.

The flavor singlet structure functions for photon exchange are given by [14]

$$F_i^{\mathrm{S},+}(x,Q^2) = \left(\frac{1}{N_F} \sum_{f=1}^{N_F} e_f^2\right) \left[C_{i,q}^{\mathrm{S}}\left(x, \frac{Q^2}{\mu^2}\right) \otimes \Sigma(x, \mu^2) + C_{i,g}\left(x, \frac{Q^2}{\mu^2}\right) \otimes G(x, \mu^2) \right] \ , \quad (92)$$

with

$$C_{i,q}^{\mathrm{S}} = C_{i,q}^{\mathrm{NS}} + \mathsf{w}_3 C_{i,q}^{d_{abc}} + C_{i,q}^{\mathrm{PS}} \ , \quad (93)$$

$$C_{i,g} = C_{i,g}^a + \mathsf{w}_3 C_{i,g}^{d_{abc}} \ , \quad (94)$$

and $G(x, \mu^2)$ denotes the gluon distribution and $C_{i,q}^{\mathrm{S},1}$ the non–weighted contributions and $C_{i,q}^{\mathrm{S},2}$ the one corresponding to the charge weight factor w_3. Synonymous relations apply to the structure function $g_1(x, Q^2)$ by replacing the unpolarized quantities by the corresponding polarized ones.

Usually the deep–inelastic structure functions are represented referring to $N_F = 3$ massless Wilson coefficients and massive Wilson coefficients [36, 37, 44, 48–53] due to charm and bottom quark corrections. The scaling violations of the heavy flavor Wilson coefficients are very different compared to those in the massless case.

The renormalization of the forward Compton amplitude is performed by renormalizing the strong coupling constant and removing the collinear singularities into the running of the parton distribution functions. In the case of the non–singlet structure function $xF_3(x, Q^2)$ the original calculation is performed in the Larin scheme and we finally switch to the $\overline{\mathrm{MS}}$ scheme by a finite renormalization. In the case of the structure function $g_1(x, Q^2)$ this last step requires the knowledge of the polarized four–loop anomalous dimensions in the singlet case. Because of this we present the corresponding Wilson coefficients in the Larin scheme and switch to the $\overline{\mathrm{MS}}$ in addition only for the non–singlet Wilson coefficient, since here the respective Ward identity is known in explicit form. Thereby, we also derive the Z-factor $Z_5^{\mathrm{NS}}(N, a_s)$ from the ratio of off–shell massless non–singlet operator matrix elements to three–loop order.

3   Details of the calculation

The Feynman diagrams for the different massless Wilson coefficients are generated by `QGRAF` [54] using the forward Compton amplitude.

We maintain all contributions up to the first order in the R_ξ gauge parameter ξ, which is canceling already for the unrenormalized result. For the Compton amplitude the corresponding crossing relations have to be observed [3, 31]. We decompose the Wilson coefficients into their flavor non–singlet $\mathbb{C}_{i,q}^{\mathrm{NS}}$, pure singlet $\mathbb{C}_{i,q}^{\mathrm{PS}}$ and gluonic $\mathbb{C}_{i,g}$ parts. At one–loop order the contributions up to $O(\varepsilon^2)$ and two–loop order up to $O(\varepsilon)$ are required to extract the three–loop Wilson coefficients. We perform the calculation keeping the gauge parameter ξ to first order as a test for gauge invariance and show that the corresponding terms disappear. The Dirac and Lorentz algebra is

performed by `FORM` [55] and the color algebra is performed by using `Color` [56]. The crossing relations [3, 31] imply that for the Wilson coefficients contributing to the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ only the even moments contribute and to $xF_3(x, Q^2)$ and $g_1(x, Q^2)$ only the odd moments. The corresponding generating variable $\omega = 1/x$ appears therefore dominantly quadratically, except of a factor ω in the case of $xF_3(x, Q^2)$ and $g_1(x, Q^2)$.

The irreducible three–loop diagrams are reduced to 293 master integrals using the code `Crusher` [57] by applying the integration–by–parts relations [58, 59]. The master integrals are mapped to ω^2 dependent structures and a sufficient number of their Mellin moments is calculated and inserted into the forward Compton amplitude. The required numbers of moments are summarized in Table 1. For the gluonic and non-singlet structures we do not map to a manifestly ω^2 dependent structure a priori.

The number of moments in Table 1 in the lower loop orders includes also the terms needed to renormalize the Wilson coefficients at three–loop order. As has been observed before in Ref. [27] there is a small number of master–integral relations which are difficult to prove for vanishing powers in ω. We have verified them explicitly for their moments up to a much higher number than requested and to the respective contributing power in ε, cf. Table 1. We determine recurrence relations for the corresponding color and ζ–value projections [60] of the Wilson coefficients using the method of arbitrary large moments [61] implemented within the package `SolveCoupledSystem` [62].

This is done by using the method of guessing [63, 64] and its implementation in `Sage` [65, 66]. For the calculation of the necessary initial values for the difference equations we use the results given in [59, 67]. In the three–loop case the calculation is based on 5000 even or odd moments. We have explicitly checked, that the other moments vanish, which is implied by the amplitude crossing relations. The difference equations are solved by using methods from difference field theory [68] implemented in the package `Sigma` [69, 70] utilizing functions from `HarmonicSums` [71–79], to obtain the three–loop Wilson coefficients. The largest difference equation for the individual color, gauge parameter, and ζ_k–factors contributing in the present case has order `o` = 25 and degree `d` = 778 and needed 4300 moments. In parallel, we calculated the Wilson coefficients by using the differential equations for the master integrals directly in the variable ω, using the method presented in Ref. [80]. The depth of the initial conditions in the dimensional parameter ε is the same as in the method described previously. The corresponding systems were decoupled using the formalisms of Ref. [81] implemented in the package `ORESYS` [82] and we further proceeded to find the N-space solution by using algorithms contained in the package `HarmonicSums`.

Comparing to the reconstruction of the anomalous dimensions and unpolarized Wilson coefficients out of their moments performed in Ref. [64] in 2008 the largest difference equation had order `o` = 35 and degree `d` = 938 requiring 5114 moments. Here, however, even and odd moments had been used. The overall computation time using the automated chain of codes described amounted to about one year on `Intel(R) Xeon(R) CPU E5-2643 v4` processors, using also parallelizations.

Working in the variable ω^2 rather than ω we obtain the results of the recurrences first for only the even or odd moments expressed in terms of cyclotomic harmonic sums at argument $2N$. For a systematic study of this class of nested sums see [75]. These objects can be algorithmically reduced to simple harmonic sums at argument N using `HarmonicSums`.

Therefore, all Wilson coefficients can be expressed by harmonic sums [71, 72]

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_\emptyset = 1, \quad b, a_i \in \mathbb{Z}\backslash\{0\}, N \in \mathbb{N}\backslash\{0\}. \qquad (95)$$

If it is clear from the context we will write $S_{\vec{a}}$ instead of $S_{\vec{a}}(N)$.

Their Mellin inversion to momentum fraction z–space

$$C(N) = \int_0^1 dz\, z^{N-1} \bar{C}(z) \tag{96}$$

can be performed using routines of the packages `HarmonicSums` and is expressed in terms of harmonic polylogarithms [73] given by

$$\mathrm{H}_{b,\vec{a}}(z) = \int_0^z dx\, f_b(x) \mathrm{H}_{\vec{a}}(x), \quad \mathrm{H}_\emptyset = 1, \quad b, a_i \in \{-1, 0, 1\}, \tag{97}$$

with the alphabet of letters

$$\mathfrak{A}_\mathrm{H} = \left\{ f_0(z) = \frac{1}{z}, \quad f_{-1}(z) = \frac{1}{1+z}, \quad f_1(z) = \frac{1}{1-z} \right\}. \tag{98}$$

In z–space one distinguishes three contributions to the individual Wilson coefficients because of their different treatment in Mellin convolutions,

$$\bar{C}(z) = \bar{C}^\delta(z) + \bar{C}^\mathrm{plu}(z) + \bar{C}^\mathrm{reg}(z), \tag{99}$$

where $\bar{C}^\delta(z) = c_0 \delta(1-z)$, $\bar{C}^\mathrm{reg}(z)$ is a regular function in $z \in [0,1]$ and $\bar{C}^\mathrm{plu}(z)$ denotes the remaining genuine +-distribution, the Mellin transformation of which is given by

$$C^\mathrm{plu}(N) = \int_0^1 dz\, (z^{N-1} - 1) \bar{C}^\mathrm{plu}(z). \tag{100}$$

We will use this representation later on.

Wilson coefficient	1 loop	2 loop	3 loop
F_1^NS	126	1219	4300
F_1^PS	0	374	1708
F_1^g	104	960	3534
F_L^NS	48	560	2387
F_L^PS	0	175	774
F_L^g	54	434	2046
xF_3^NS	126	1219	4171
g_1^NS	126	1219	4171
g_1^PS	0	175	1458
g_1^g	84	1166	2998

Table 1: The necessary maximal number of non–vanishing even (resp. odd) Mellin moments for $F_1, F_L, (xF_3, g_1)$ to determine the Wilson coefficients.

In the Mellin N space representation it can technically occur that there are factors of up to $1/(N-2)^2$ contributing and structures of $1/(N-1)$, in the cases that they are physically not

allowed. However, these are all tractable poles, which can be shown by expanding at $N = 2$ and $N = 1$ using `HarmonicSums`. No Kronecker symbols have to be introduced for this case and the usual analytic continuation, described in Ref. [78], can be applied to the N–space expressions directly.

The Wilson coefficients can be represented in N space by harmonic sums weighted by rational functions in N. Here the degree of the numerator needs not to be smaller than that of the denominator, which needs special care in performing the inverse Mellin transform to z–space. After applying the algebraic relations between harmonic sums, c.f. e.g. [79], one obtains the following set of 60 harmonic sums

$$\{S_1; S_2, S_{-2}; S_3, S_{-3}, S_{2,1}, S_{-2,1}; S_4, S_{-4}, S_{-2,2}, S_{3,1}, S_{-3,1}, S_{2,1,1}, S_{-2,1,1}; S_5, S_{-5}, S_{-2,3}, S_{2,3},$$
$$S_{2,-3}, S_{-2,-3}, S_{2,2,1}, S_{-2,1,-2}, S_{-2,2,1}S_{4,1}, S_{-4,1}, S_{2,1,-2}, S_{3,1,1}, S_{-3,1,1}, S_{2,1,1,1}, S_{-2,1,1,1}; S_6, S_{-6},$$
$$S_{-3,3}, S_{4,2}, S_{4,-2}, S_{-4,2}, S_{-4,-2}, S_{5,1}, S_{-5,1}, S_{-2,2,-2}, S_{-2,2,2}, S_{2,-3,1}, S_{-2,3,1}, S_{-3,1,-2}, S_{-3,-2,1},$$
$$S_{-3,2,1}, S_{-4,1,1}, S_{2,3,1}, S_{3,1,-2}, S_{3,2,1}, S_{4,1,1}, S_{-2,-2,1,1}, S_{-2,1,1,2}, S_{-2,2,1,1}, S_{2,-2,1,1}, S_{2,2,1,1}, S_{3,1,1,1},$$
$$S_{-3,1,1,1}, S_{2,1,1,1,1}, S_{-2,1,1,1,1}\}. \tag{101}$$

One may, furthermore, apply also the structural relations [78,83] and obtains the following 31 harmonic sums.

$$\{S_1, S_{-1,1}, S_{-2,1}, S_{-3,1}, S_{-4,1}, S_{-5,1}, S_{2,1}, S_{4,1}, S_{-1,1,1}, S_{2,1,1}, S_{1,2,-1}, S_{2,1,-1}, S_{-2,1,-2}, S_{2,1,-2},$$
$$S_{-3,1,1}, S_{3,1,1}, S_{-2,2,-2}, S_{-3,-2,1}, S_{3,1,-2}, S_{-4,1,1}, S_{4,1,1}, S_{-2,1,1,1}, S_{2,1,1,1}, S_{-3,1,1,1}, S_{-2,-2,1,1},$$
$$S_{-2,1,1,2}, S_{-2,2,1,1}, S_{2,-2,1,1}, S_{2,2,1,1}, S_{-2,1,1,1,1}, S_{2,1,1,1,1}\} \tag{102}$$

spanning all quantities. In z space, the number of harmonic polylogarithms is, usually, higher if compared to the objects needed in Mellin N space. Here 68 harmonic polylogarithms of up to weight $\mathtt{w} = \mathtt{5}$, weighted by one more letter, contribute

$$\{ H_{-1}, H_0, H_1, H_{0,-1}, H_{0,1}, H_{0,-1,-1}, H_{0,-1,1}, H_{0,0,-1}, H_{0,0,1}, H_{0,1,-1}, H_{0,1,1}, H_{0,-1,-1,-1}, H_{0,-1,-1,1},$$
$$H_{0,-1,0,1}, H_{0,-1,1,-1}, H_{0,-1,1,1}, H_{0,0,-1,-1}, H_{0,0,-1,1}, H_{0,0,0,-1}, H_{0,0,0,1}, H_{0,0,1,-1}, H_{0,0,1,1}, H_{0,1,-1,-1},$$
$$H_{0,1,-1,1}, H_{0,1,1,-1}, H_{0,1,1,1}, H_{0,-1,-1,-1,-1}, H_{0,-1,-1,-1,1}, H_{0,-1,-1,0,1}, H_{0,-1,-1,1,-1}, H_{0,-1,-1,1,1},$$
$$H_{0,-1,0,-1,-1}, H_{0,-1,0,-1,1}, H_{0,-1,0,1,-1}, H_{0,-1,0,1,1}, H_{0,-1,1,-1,-1}, H_{0,-1,1,-1,1}, H_{0,-1,1,0,1}, H_{0,-1,1,1,-1},$$
$$H_{0,-1,1,1,1}, H_{0,0,-1,-1,-1}, H_{0,0,-1,-1,1}, H_{0,0,-1,0,-1}, H_{0,0,-1,0,1}, H_{0,0,-1,1,-1}, H_{0,0,-1,1,1}, H_{0,0,0,-1,-1},$$
$$H_{0,0,0,-1,1}, H_{0,0,0,0,-1}, H_{0,0,0,0,1}, H_{0,0,0,1,-1}, H_{0,0,0,1,1}, H_{0,0,1,-1,-1}, H_{0,0,1,-1,1}, H_{0,0,1,0,-1}, H_{0,0,1,0,1},$$
$$H_{0,0,1,1,-1}, H_{0,0,1,1,1}, H_{0,1,-1,-1,-1}, H_{0,1,-1,-1,1}, H_{0,1,-1,1,-1}, H_{0,1,-1,1,1}, H_{0,1,0,1,-1}, H_{0,1,0,1,1},$$
$$H_{0,1,1,-1,-1}, H_{0,1,1,-1,1}, H_{0,1,1,1,-1}, H_{0,1,1,1,1} \} \tag{103}$$

after algebraic reduction.

4 The one– and two–loop Wilson coefficients

In the following we present the results for the massless Wilson coefficients contributing to the structure functions F_2, F_L, xF_3 and g_1 at one- and two–loop order, together with the higher expansion coefficients in ε of the corresponding forward Compton amplitudes in the unrenormalized case, see Section 2. These contributions are needed for the calculation of the three–loop Wilson coefficients. We will work in Mellin N space and also present the renormalized one- and two–loop Wilson coefficients.

4.1 One-Loop Order

At one loop–order one obtains the following expansion coefficients, cf. (39–41),

$$c_{F_2,q}^{(1,0)} = C_F \left\{ \frac{2 + 5N - 2N^2 - 9N^3}{N^2(1+N)} + \frac{-2 + 3N + 3N^2}{N(1+N)} S_1 + 2S_1^2 - 2S_2 \right\}, \tag{104}$$

$$c_{F_2,q}^{(1,1)} = C_F \left\{ \frac{P_3}{2N^3(1+N)} + \left(\frac{-2 - 5N - 14N^2 - 7N^3}{2N^2(1+N)} + S_2 + \zeta_2 \right) S_1 - \frac{1}{3} S_1^3 \right.$$
$$+ \frac{\left(2 - 3N - 3N^2\right)}{4N(1+N)} S_1^2 + \frac{\left(-2 + 3N + 3N^2\right)}{4N(1+N)} S_2 + \frac{\left(-2 - 3N - 3N^2\right)}{4N(1+N)} \zeta_2$$
$$\left. - \frac{2}{3} S_3 \right\}, \tag{105}$$

$$c_{F_2,q}^{(1,2)} = C_F \left\{ \frac{P_5}{4N^4(1+N)} + \left(\frac{P_2}{4N^3(1+N)} + \frac{\left(2 - 3N - 3N^2\right)}{8N(1+N)} [S_2 + \zeta_2] + \frac{1}{3} S_3 - \frac{7}{3} \zeta_3 \right) S_1 \right.$$
$$+ \left(\frac{2 + 5N + 14N^2 + 7N^3}{8N^2(1+N)} - \frac{1}{4} S_2 - \frac{1}{4} \zeta_2 \right) S_1^2 + \frac{\left(-2 + 3N + 3N^2\right)}{24N(1+N)} S_1^3 + \frac{1}{24} S_1^4$$
$$+ \left(\frac{-2 - 5N - 14N^2 - 7N^3}{8N^2(1+N)} + \frac{1}{4} \zeta_2 \right) S_2 + \frac{1}{8} S_2^2 + \frac{\left(-2 + 3N + 3N^2\right)}{12N(1+N)} S_3 - \frac{1}{4} S_4$$
$$\left. + \frac{\left(-2 - 5N + 2N^2 + 9N^3\right)}{8N^2(1+N)} \zeta_2 + \frac{7\left(2 + 3N + 3N^2\right)}{12N(1+N)} \zeta_3 \right\}, \tag{106}$$

$$c_{F_2,g}^{(1,0)} = T_F N_F \left\{ - \frac{4\left(-2 - N - 4N^2 + N^3\right)}{N^2(1+N)(2+N)} - \frac{4\left(2 + N + N^2\right)}{N(1+N)(2+N)} S_1 \right\}, \tag{107}$$

$$c_{F_2,g}^{(1,1)} = T_F N_F \left\{ - \frac{2P_1}{N^3(1+N)(2+N)} + \frac{2\left(-2 - N - 4N^2 + N^3\right)}{N^2(1+N)(2+N)} S_1 \right.$$
$$\left. + \frac{\left(2 + N + N^2\right)}{N(1+N)(2+N)} [S_1^2 - S_2 - \zeta_2] \right\}, \tag{108}$$

$$c_{F_2,g}^{(1,2)} = T_F N_F \left\{ \frac{P_6}{N^4(1+N)(2+N)} + \left(\frac{P_1}{N^3(1+N)(2+N)} + \frac{\left(2 + N + N^2\right)}{2N(1+N)(2+N)} \right. \right.$$
$$\left. \times [S_2 + \zeta_2] \right) S_1 + \frac{\left(2 + N + 4N^2 - N^3\right)}{2N^2(1+N)(2+N)} [S_1^2 - S_2 - \zeta_2] + \frac{\left(-2 - N - N^2\right)}{6N(1+N)(2+N)} S_1^3$$
$$\left. + \frac{\left(-2 - N - N^2\right)}{3N(1+N)(2+N)} S_3 + \frac{7\left(2 + N + N^2\right)}{3N(1+N)(2+N)} \zeta_3 \right\}, \tag{109}$$

$$c_{F_L,q}^{(1,0)} = C_F \frac{4}{1+N}, \tag{110}$$

$$c_{F_L,q}^{(1,1)} = C_F \left\{ -\frac{2}{1+N} [1 + S_1] \right\}, \tag{111}$$

$$c_{F_L,q}^{(1,2)} = C_F \left\{ \frac{2}{1+N} + \frac{1}{1+N} S_1 + \frac{1}{2(1+N)} [S_1^2 - S_2 - \zeta_2] \right\}, \tag{112}$$

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$$c_{F_L,g}^{(1,0)} = T_F N_F \frac{16}{(1+N)(2+N)} \, , \qquad (113)$$

$$c_{F_L,g}^{(1,1)} = T_F N_F \left\{ -\frac{16}{(1+N)(2+N)} - \frac{8}{(1+N)(2+N)} S_1 \right\} \, , \qquad (114)$$

$$c_{F_L,g}^{(1,2)} = T_F N_F \left\{ \frac{16}{(1+N)(2+N)} + \frac{8}{(1+N)(2+N)} S_1 + \frac{2}{(1+N)(2+N)} \right.$$
$$\left. [S_1^2 - S_2 - \zeta_2] \right\} \, , \qquad (115)$$

$$c_{F_3,q}^{(1,0),\mathrm{L}} = C_F \left\{ \frac{2 + 3N - 2N^2 - 5N^3}{N^2(1+N)} + \frac{(-2 + 3N + 3N^2)}{N(1+N)} S_1 + 2S_1^2 - 2S_2 \right\} \, , \qquad (116)$$

$$c_{F_3,q}^{(1,1),\mathrm{L}} = C_F \left\{ \frac{2 + 3N + 3N^3 + 8N^4}{2N^3(1+N)} + \left(\frac{-2 - 3N - 10N^2 - 7N^3}{2N^2(1+N)} + S_2 \right) S_1 \right.$$
$$+ \frac{(2 - 3N - 3N^2)}{4N(1+N)} S_1^2 + \frac{(-2 + 3N + 3N^2)}{4N(1+N)} S_2 - \frac{1}{3} S_1^3 - \frac{2}{3} S_3$$
$$\left. + \left(\frac{-2 - 3N - 3N^2}{4N(1+N)} + S_1 \right) \zeta_2 \right\} \, , \qquad (117)$$

$$c_{F_3,q}^{(1,2),\mathrm{L}} = C_F \left\{ \frac{P_4}{4N^4(1+N)} + \left(\frac{2 + 3N + 10N^2 + 7N^3}{8N^2(1+N)} - \frac{1}{4} S_2 \right) S_1^2 \right.$$
$$+ \left(\frac{-2 - 3N + 19N^3 + 14N^4}{4N^3(1+N)} + \frac{(2 - 3N - 3N^2)}{8N(1+N)} S_2 + \frac{1}{3} S_3 \right) S_1$$
$$+ \frac{(-2 + 3N + 3N^2)}{24N(1+N)} S_1^3 + \frac{1}{24} S_1^4 + \frac{(-2 - 3N - 10N^2 - 7N^3)}{8N^2(1+N)} S_2 + \frac{1}{8} S_2^2$$
$$+ \frac{(-2 + 3N + 3N^2)}{12N(1+N)} S_3 - \frac{1}{4} S_4 + \left(\frac{-2 - 3N + 2N^2 + 5N^3}{8N^2(1+N)} \right.$$
$$\left. + \frac{(2 - 3N - 3N^2) S_1}{8N(1+N)} - \frac{1}{4} S_1^2 + \frac{1}{4} S_2 \right) \zeta_2 + \left(\frac{7(2 + 3N + 3N^2)}{12N(1+N)} - \frac{7}{3} S_1 \right) \zeta_3 \right\} \, , \quad (118)$$

$$c_{g_1,q}^{(1,0),\mathrm{NS,L}} = C_F \left\{ \frac{2 - 5N - 6N^2 - 9N^3}{N^2(1+N)} + \frac{(-2 + 3N + 3N^2)}{N(1+N)} S_1 + 2S_1^2 - 2S_2 \right\} \, , \qquad (119)$$

$$c_{g_1,q}^{(1,1),\mathrm{NS,L}} = C_F \left\{ \frac{P_8}{2N^3(1+N)} + \left(\frac{-2 + 5N - 10N^2 - 7N^3}{2N^2(1+N)} + S_2 \right) S_1 + \frac{(2 - 3N - 3N^2)}{4N(1+N)} S_1^2 \right.$$
$$\left. - \frac{1}{3} S_1^3 + \frac{(-2 + 3N + 3N^2)}{4N(1+N)} S_2 - \frac{2}{3} S_3 + \left(\frac{-2 - 3N - 3N^2}{4N(1+N)} + S_1 \right) \zeta_2 \right\} \, , \qquad (120)$$

$$c_{g_1,q}^{(1,2),\mathrm{NS,L}} = C_F \left\{ \frac{P_9}{4N^4(1+N)} + \left(\frac{P_7}{4N^3(1+N)} + \frac{(2 - 3N - 3N^2)}{8N(1+N)} S_2 + \frac{1}{3} S_3 \right) S_1 \right.$$
$$+ \left(\frac{2 - 5N + 10N^2 + 7N^3}{8N^2(1+N)} - \frac{1}{4} S_2 \right) S_1^2 + \frac{(-2 + 3N + 3N^2)}{24N(1+N)} S_1^3 + \frac{1}{24} S_1^4$$

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$$+\frac{\left(-2+5N-10N^2-7N^3\right)S_2}{8N^2(1+N)}+\frac{1}{8}S_2^2+\frac{\left(-2+3N+3N^2\right)}{12N(1+N)}S_3-\frac{1}{4}S_4$$

$$+\left(\frac{-2+5N+6N^2+9N^3}{8N^2(1+N)}+\frac{\left(2-3N-3N^2\right)}{8N(1+N)}S_1-\frac{1}{4}S_1^2+\frac{1}{4}S_2\right)\zeta_2$$

$$+\left(\frac{7\left(2+3N+3N^2\right)}{12N(1+N)}-\frac{7}{3}S_1\right)\zeta_3\Bigg\}\,, \tag{121}$$

$$c_{g1,g}^{(1,0)}\;=\;T_F N_F(N-1)\left\{-\frac{4(N-1)}{N^2(1+N)}-\frac{4}{N(1+N)}S_1\right\}\,, \tag{122}$$

$$c_{g1,g}^{(1,1)}\;=\;T_F N_F(N-1)\left\{\frac{2\left(1-N+2N^2\right)}{N^3(1+N)}+\frac{2(N-1)}{N^2(1+N)}S_1+\frac{1}{N(1+N)}S_1^2\right.$$

$$\left.-\frac{1}{N(1+N)}S_2-\frac{1}{N(1+N)}\zeta_2\right\}\,, \tag{123}$$

$$c_{g1,g}^{(1,2)}\;=\;T_F N_F(N-1)\left\{\frac{\left(1-N+2N^2-4N^3\right)}{N^4(1+N)}+\left(-\frac{\left(1-N+2N^2\right)}{N^3(1+N)}\right.\right.$$

$$\left.+\frac{1}{2N(1+N)}S_2\right)S_1-\frac{(N-1)}{2N^2(1+N)}S_1^2-\frac{1}{6N(1+N)}S_1^3+\frac{(N-1)}{2N^2(1+N)}S_2$$

$$\left.-\frac{1}{3N(1+N)}S_3+\left(\frac{(N-1)}{2N^2(1+N)}+\frac{1}{2N(1+N)}S_1\right)\zeta_2+\frac{7}{3N(1+N)}\zeta_3\right\}\,, \tag{124}$$

with

$$P_1\;=\;-3N^4+6N^3-4N^2-N-2, \tag{125}$$

$$P_2\;=\;14N^4+27N^3+2N^2-5N-2, \tag{126}$$

$$P_3\;=\;18N^4+5N^3-2N^2+5N+2, \tag{127}$$

$$P_4\;=\;-14N^5-6N^4-N^3+3N+2, \tag{128}$$

$$P_5\;=\;-36N^5-10N^4+5N^3-2N^2+5N+2, \tag{129}$$

$$P_6\;=\;-7N^5+12N^4-6N^3+4N^2+N+2, \tag{130}$$

$$P_7\;=\;14N^4+23N^3-8N^2+5N-2, \tag{131}$$

$$P_8\;=\;18N^4+9N^3+8N^2-5N+2, \tag{132}$$

$$P_9\;=\;-36N^5-18N^4-15N^3+8N^2-5N+2. \tag{133}$$

The one–loop Wilson coefficients at $\mu^2=Q^2$ are given by the expansion coefficients $c_{F_k,q(g)}^{(1,0)}$, $k=2,L$. There are finite renormalizations for the Wilson coefficients of the structure functions xF_3 and g_1. The respective Wilson coefficients are given by

$$C_{F3,q}^{(1),\text{NS}}\;=\;C_F\left[-\frac{(2+3N)\left(-1+3N^2\right)}{N^2(1+N)}+\frac{\left(-2+3N+3N^2\right)S_1}{N(1+N)}+2S_1^2-2S_2\right]\,, \tag{134}$$

$$C_{g1,q}^{(1),\text{NS,L}}\;=\;C_F\left[\frac{2-5N-6N^2-9N^3}{N^2(1+N)}+\frac{\left(-2+3N+3N^2\right)S_1}{N(1+N)}+2S_1^2-2S_2\right]\,, \tag{135}$$

$$C_{g1,q}^{(1),\text{NS}}\;=\;C_{g1,q}^{(1),\text{NS,L}}-z_{qq}^{(1)}=C_{g1,q}^{(1),\text{NS,L}}+\frac{8C_F}{N(N+1)}=C_{F3,q}^{(1),\text{NS}}\,, \tag{136}$$

where the superscript L denotes the Wilson coefficient obtained in the Larin scheme.

4.2 Two-Loop Order

The corresponding expansion coefficients at two–loop order read, cf. (39–41),

$$
\begin{aligned}
c_{F_2,q}^{\mathrm{NS},(2,0)} = \ & C_F\Bigg\{ N_F T_F\Bigg[\frac{P_{78}}{54N^3(1+N)^3} + \left(-\frac{2P_{32}}{27N^2(1+N)^2} + \frac{16}{3}S_2 + \frac{8}{3}\zeta_2\right)S_1 \\
& -\frac{4\left(-6+19N+19N^2\right)S_1^2}{9N(1+N)} - \frac{16}{9}S_1^3 + \frac{4\left(-6+47N+47N^2\right)S_2}{9N(1+N)} - \frac{104}{9}S_3 \\
& +\frac{16}{3}S_{2,1} - \frac{2\left(2+3N+3N^2\right)\zeta_2}{3N(1+N)}\Bigg] + C_A\Bigg[\frac{P_{89}}{216(N-2)N^3(1+N)^3(3+N)} \\
& +\left(\frac{P_{47}}{54N^2(1+N)^3} - \frac{4\left(3+11N+11N^2\right)S_2}{3N(1+N)} + 24S_3 - 16S_{2,1} - 16S_{-2,1} - \frac{22}{3}\zeta_2\right. \\
& \left.-48\zeta_3\right)S_1 + \left(\frac{-66+233N+233N^2}{9N(1+N)} + 4S_2\right)S_1^2 + \frac{44}{9}S_1^3 - 4S_2^2 - 8S_4 \\
& +\frac{\left(66-481N-1166N^2-583N^3\right)S_2}{9N(1+N)^2} + \frac{2\left(-72+215N+143N^2\right)S_3}{9N(1+N)} \\
& +\left(\frac{4P_{61}}{(N-2)N^2(1+N)^2(3+N)} + \frac{8(-3+4N)S_1}{N(1+N)} + 16S_1^2 - 8S_2\right)S_{-2} - 12S_{-2}^2 \\
& +\left(\frac{16}{1+N} - 8S_1\right)S_{-3} - 20S_{-4} - \frac{4\left(-6+11N+11N^2\right)S_{2,1}}{3N(1+N)} - \frac{16(-1+2N)S_{-2,1}}{N(1+N)} \\
& -24S_{3,1} + 8S_{-2,2} + 16S_{-3,1} + 24S_{2,1,1} + \frac{11\left(2+3N+3N^2\right)\zeta_2}{6N(1+N)} \\
& +\frac{6\left(6+N+9N^2\right)\zeta_3}{N(1+N)}\Bigg]\Bigg\} + C_F^2\Bigg[\frac{2S_2 P_{29}}{N^2(1+N)^2} + \frac{P_{106}}{8(N-2)N^4(1+N)^4(3+N)} \\
& +\left(\frac{P_{49}}{2N^3(1+N)^3} - \frac{2\left(-14+9N+9N^2\right)S_2}{N(1+N)} - \frac{56}{3}S_3 + 16S_{2,1} + 32S_{-2,1}\right. \\
& \left.+\frac{4\left(2+3N+3N^2\right)\zeta_2}{N(1+N)} + 48\zeta_3\right)S_1 + \left(\frac{2P_{23}}{N^2(1+N)^2} - 28S_2 - 8\zeta_2\right)S_1^2 \\
& +\frac{2\left(-14+15N+15N^2\right)S_1^3}{3N(1+N)} + \frac{14}{3}S_1^4 + 6S_2^2 - \frac{2\left(-26+81N+33N^2\right)S_3}{3N(1+N)} + 12S_4 \\
& +\left(-\frac{8P_{61}}{(N-2)N^2(1+N)^2(3+N)} - \frac{16(-3+4N)S_1}{N(1+N)} - 32S_1^2 + 16S_2\right)S_{-2} + 24S_{-2}^2 \\
& +\left(-\frac{32}{1+N} + 16S_1\right)S_{-3} + 40S_{-4} + \frac{4\left(-2+3N+3N^2\right)S_{2,1}}{N(1+N)} + 40S_{3,1} \\
& +\frac{32(-1+2N)S_{-2,1}}{N(1+N)} - 16S_{-2,2} - 32S_{-3,1} - 24S_{2,1,1} - \frac{\left(2+3N+3N^2\right)^2\zeta_2}{2N^2(1+N)^2}
\end{aligned}
$$

$$
-\frac{24\big(2-N+3N^2\big)\zeta_3}{N(1+N)}\Bigg],
\tag{137}
$$

$$
\begin{aligned}
c_{F_2,q}^{\mathrm{NS},(2,1)} \;=\; & C_F\Bigg\{C_A\Bigg[\frac{\zeta_2 P_{10}}{72N^2(1+N)^2}+\frac{2S_{2,1}P_{45}}{9N^2(1+N)^2(3+N)}-\frac{2\zeta_3 P_{77}}{9(N-2)N^2(1+N)^2(3+N)}\\[2mm]
& +\frac{P_{111}}{2592(N-2)^2N^4(1+N)^4(3+N)^2}+\Bigg(\frac{S_2 P_{44}}{9N^2(1+N)^2(3+N)}+8S_2^2+32S_4\\[2mm]
& +\frac{P_{93}}{648(N-2)N^3(1+N)^4(3+N)}-\frac{8\big(-27+56N+11N^2\big)S_3}{9N(1+N)}+\frac{16(N-1)S_{2,1}}{N(1+N)}\\[2mm]
& -4S_{3,1}+\frac{16(-1+2N)S_{-2,1}}{N(1+N)}-8S_{-2,2}+\frac{\big(-66+233N+233N^2\big)\zeta_2}{18N(1+N)}\\[2mm]
& -16S_{-3,1}-20S_{2,1,1}+\frac{72}{5}\zeta_2^2+\frac{2\big(-216+689N+257N^2\big)\zeta_3}{9N(1+N)}\Bigg)S_1\\[2mm]
& +\Bigg(\frac{P_{34}}{108N^2(1+N)^3}+\frac{\big(15+10N+22N^2\big)S_2}{3N(1+N)}-14S_3+12S_{2,1}+8S_{-2,1}+\frac{11}{3}\zeta_2\\[2mm]
& +24\zeta_3\Bigg)S_1^2+\Bigg(\frac{66-233N-233N^2}{27N(1+N)}-\frac{8}{3}S_2\Bigg)S_1^3+\Bigg(\frac{P_{80}}{108N^2(1+N)^3(3+N)}\\[2mm]
& -\frac{22}{3}S_3-8S_{2,1}-8S_{-2,1}-\frac{22}{3}\zeta_2-48\zeta_3\Bigg)S_2-\frac{11}{9}S_1^4+\frac{\big(-6-9N-17N^2\big)S_2^2}{N(1+N)}\\[2mm]
& +\Bigg(\frac{P_{48}}{27(N-2)N^2(1+N)^2(3+N)}+2\zeta_2\Bigg)S_3+\frac{\big(-63+119N+59N^2\big)S_4}{3N(1+N)}-22S_5\\[2mm]
& +\Bigg(-\frac{4P_{98}}{(N-2)^2N^3(1+N)^3(3+N)^2}+\Bigg(-\frac{4P_{62}}{(N-2)N^2(1+N)^2(3+N)}+16S_2\\[2mm]
& +4\zeta_2\Bigg)S_1-\frac{4(-5+8N)S_1^2}{N(1+N)}-8S_1^3+\frac{4(-3+4N)S_2}{N(1+N)}-32S_3-8S_{-2,1}-\frac{2\zeta_2}{N(1+N)}\\[2mm]
& -72\zeta_3\Bigg)S_{-2}+\Bigg(\frac{8(-2+3N)}{N(1+N)}+20S_1\Bigg)S_{-2}^2+\Bigg(\frac{4P_{60}}{(N-2)N^2(1+N)^2(3+N)}\\[2mm]
& +\frac{16(N-1)S_1}{N(1+N)}+12S_1^2+4S_2-16S_{-2}+2\zeta_2\Bigg)S_{-3}+\Bigg(\frac{8(-2+5N)}{N(1+N)}+12S_1\Bigg)S_{-4}\\[2mm]
& -38S_{-5}+20S_{2,3}-8S_{2,-3}-\frac{2(N-1)(-2+5N)S_{3,1}}{N(1+N)}-44S_{4,1}+\Bigg(\frac{16}{N^2}-4\zeta_2\Bigg)\\[2mm]
& \times S_{-2,1}-\frac{8(-1+2N)S_{-2,2}}{N(1+N)}+44S_{-2,3}-\frac{16(-1+2N)S_{-3,1}}{N(1+N)}+8S_{-4,1}\\[2mm]
& +\frac{4\big(12-7N+11N^2\big)S_{2,1,1}}{3N(1+N)}+4S_{2,2,1}+40S_{3,1,1}-8S_{-2,1,-2}+4S_{2,1,1,1}\\[2mm]
& -\frac{9\big(6+N+9N^2\big)\zeta_2^2}{5N(1+N)}\Bigg]+N_F T_F\Bigg[\frac{\zeta_2 P_{30}}{18N^2(1+N)^2}+\frac{P_{88}}{648N^4(1+N)^4}
\end{aligned}
$$

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$$+ \left(\frac{P_{79}}{162 N^3 (1+N)^3} - \frac{4 \left(-6 + 19N + 19N^2 \right) S_2}{9 N (1+N)} - \frac{2 \left(-6 + 19N + 19N^2 \right) \zeta_2}{9 N (1+N)} \right.$$

$$+ \frac{32}{9} S_3 - \frac{128}{9} \zeta_3 \bigg) S_1 + \left(\frac{P_{32}}{27 N^2 (1+N)^2} - \frac{8}{3} S_2 - \frac{4}{3} \zeta_2 \right) S_1^2 + \frac{4}{9} S_1^4 + 4 S_2^2 - \frac{28}{3} S_4$$

$$+ \frac{4 \left(-6 + 19N + 19N^2 \right) S_1^3}{27 N (1+N)} + \left(\frac{P_{11}}{27 N^2 (1+N)^2} + \frac{8 \zeta_2}{3} \right) S_2 - \frac{112}{9} S_{2,1} + 8 S_{3,1}$$

$$\left. - \frac{16}{3} S_{2,1,1} + \frac{16 \left(-3 + 41N + 41N^2 \right) S_3}{27 N (1+N)} + \frac{32 \left(2 + 3N + 3N^2 \right) \zeta_3}{9 N (1+N)} \right] \right\}$$

$$+ C_F^2 \left[- \frac{2 S_{2,1} P_{73}}{N^2 (1+N)^2 (2+N)(3+N)} + \frac{4 \zeta_3 P_{87}}{3 (N-2) N^2 (1+N)^2 (2+N)(3+N)} \right.$$

$$+ \frac{\zeta_2 P_{76}}{8 N^3 (1+N)^3} + \frac{P_{112}}{32 (N-2)^2 N^5 (1+N)^5 (3+N)^2}$$

$$+ \left(\frac{P_{108}}{8 (N-2) N^4 (1+N)^4 (2+N)(3+N)} + \left(\frac{P_{75}}{2 N^2 (1+N)^2 (2+N)(3+N)} \right. \right.$$

$$\left. - 8 \zeta_2 \right) S_2 - 23 S_2^2 - \frac{4 \left(8 - 15N + 5N^2 \right) S_3}{N (1+N)} - 38 S_4 + \frac{4 \left(6 - 5N + 3N^2 \right) S_{2,1}}{N (1+N)} - 8 S_{3,1}$$

$$- \frac{32 (-1+2N) S_{-2,1}}{N (1+N)} + 32 S_{-3,1} + 32 S_{2,1,1} + \frac{\left(16 + 20N - 17N^2 - 45N^3 \right) \zeta_2}{2 N^2 (1+N)}$$

$$+ 16 S_{-2,2} - \frac{72}{5} \zeta_2^2 - \frac{4 \left(-40 + 165N + 21N^2 \right) \zeta_3}{3 N (1+N)} \bigg) S_1 + \left(\frac{\left(-42 + 43N + 27N^2 \right) S_2}{2 N (1+N)} \right.$$

$$+ \frac{P_{50}}{4 N^3 (1+N)^3} + 12 S_3 - 16 S_{2,1} - 16 S_{-2,1} + \frac{3 (N-1)(2+N) \zeta_2}{N (1+N)} - \frac{16}{3} \zeta_3 \bigg) S_1^2$$

$$+ \left(\frac{P_{12}}{6 N^2 (1+N)^2} + \frac{38}{3} S_2 + 4 \zeta_2 \right) S_1^3 + \frac{\left(10 - 13N - 13N^2 \right) S_1^4}{4 N (1+N)} - S_1^5$$

$$+ \left(\frac{P_{91}}{4 N^3 (1+N)^3 (2+N)(3+N)} + \frac{28}{3} S_3 + 16 S_{2,1} + 16 S_{-2,1} + \frac{2 \left(2 + 3N + 3N^2 \right) \zeta_2}{N (1+N)} \right.$$

$$+ 48 \zeta_3 \bigg) S_2 + \frac{\left(62 + 17N + 81N^2 \right) S_2^2}{4 N (1+N)} + \left(\frac{2 P_{86}}{3 (N-2) N^2 (1+N)^2 (2+N)(3+N)} \right.$$

$$\left. - 4 \zeta_2 \right) S_3 + \frac{\left(58 - 83N - 3N^2 \right) S_4}{2 N (1+N)} + 36 S_5 + \left(\frac{8 P_{98}}{(N-2)^2 N^3 (1+N)^3 (3+N)^2} \right.$$

$$+ \left(\frac{8 P_{62}}{(N-2) N^2 (1+N)^2 (3+N)} - 32 S_2 - 8 \zeta_2 \right) S_1 + \frac{8 (-5+8N) S_1^2}{N (1+N)} + 16 S_1^3$$

$$- \frac{8 (-3+4N) S_2}{N (1+N)} + 64 S_3 + 16 S_{-2,1} + \frac{4 \zeta_2}{N (1+N)} + 144 \zeta_3 \bigg) S_{-2} + \left(- \frac{16 (-2+3N)}{N (1+N)} \right.$$

$$\left. - 40 S_1 \right) S_{-2}^2 + \left(- \frac{8 P_{60}}{(N-2) N^2 (1+N)^2 (3+N)} - \frac{32 (N-1) S_1}{N (1+N)} - 24 S_1^2 - 8 S_2 \right.$$

$$+32S_{-2} - 4\zeta_2\bigg)S_{-3} + \left(-\frac{16(-2+5N)}{N(1+N)} - 24S_1\right)S_{-4} + 76S_{-5} - 32S_{2,3} + 16S_{2,-3}$$

$$-\frac{16(-1+3N)S_{3,1}}{N(1+N)} + 76S_{4,1} + \left(-\frac{32}{N^2} + 8\zeta_2\right)S_{-2,1} + \frac{16(-1+2N)S_{-2,2}}{N(1+N)} - 88S_{-2,3}$$

$$+\frac{32(-1+2N)S_{-3,1}}{N(1+N)} - 16S_{-4,1} - \frac{2\big(14-9N+15N^2\big)S_{2,1,1}}{N(1+N)} - 48S_{3,1,1} + 16S_{-2,1,-2}$$

$$-20S_{2,1,1,1} + \frac{36\big(2-N+3N^2\big)\zeta_2^2}{5N(1+N)}\Bigg] , \tag{138}$$

$$c_{F_2,q}^{\mathrm{PS},(2,0)} = C_F N_F T_F \Bigg[\frac{8S_1 P_{83}}{(N-1)N^3(1+N)^3(2+N)^2} + \frac{4P_{100}}{(N-1)N^4(1+N)^4(2+N)^3}$$

$$+\frac{8\big(2+N+N^2\big)^2 S_1^2}{(N-1)N^2(1+N)^2(2+N)} - \frac{8\big(2+N+N^2\big)^2 S_2}{(N-1)N^2(1+N)^2(2+N)}$$

$$+\frac{64S_{-2}}{(N-1)N(1+N)(2+N)} - \frac{4\big(2+N+N^2\big)^2\zeta_2}{(N-1)N^2(1+N)^2(2+N)}\Bigg] , \tag{139}$$

$$c_{F_2,q}^{\mathrm{PS},(2,1)} = C_F N_F T_F \Bigg[-\frac{16S_3 P_{16}}{3(N-1)N^2(1+N)^2(2+N)} + \frac{64\zeta_3 P_{17}}{3(N-1)N^2(1+N)^2(2+N)}$$

$$-\frac{4S_1^2 P_{83}}{(N-1)N^3(1+N)^3(2+N)^2} + \frac{4S_2 P_{83}}{(N-1)N^3(1+N)^3(2+N)^2}$$

$$+\frac{2\zeta_2 P_{83}}{(N-1)N^3(1+N)^3(2+N)^2} - \frac{2P_{114}}{(N-2)(N-1)N^5(1+N)^5(2+N)^4(3+N)}$$

$$+\left(-\frac{4P_{100}}{(N-1)N^4(1+N)^4(2+N)^3} + \frac{8\big(2+N+N^2\big)^2 S_2}{(N-1)N^2(1+N)^2(2+N)}\right.$$

$$\left.+\frac{4\big(2+N+N^2\big)^2\zeta_2}{(N-1)N^2(1+N)^2(2+N)}\right)S_1 - \frac{8\big(2+N+N^2\big)^2 S_1^3}{3(N-1)N^2(1+N)^2(2+N)}$$

$$+\left(\frac{8P_{82}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)} - \frac{64S_1}{(N-1)N(1+N)(2+N)}\right)$$

$$\times S_{-2} + \frac{64S_{-3}}{(N-1)N(1+N)(2+N)}\Bigg] , \tag{140}$$

$$c_{F_2,g}^{(2,0)} = C_A N_F T_F \Bigg[\frac{8S_2 P_{51}}{(N-1)N^2(1+N)^2(2+N)^2} - \frac{8S_1^2 P_{63}}{(N-1)N^2(1+N)^2(2+N)^2}$$

$$-\frac{4P_{109}}{(N-1)N^4(1+N)^4(2+N)^4} + \left(-\frac{4P_{96}}{(N-1)N^3(1+N)^3(2+N)^3}\right.$$

$$\left.+\frac{28\big(2+N+N^2\big)S_2}{N(1+N)(2+N)} + \frac{8\big(2+N+N^2\big)\zeta_2}{N(1+N)(2+N)}\right)S_1 - \frac{28\big(2+N+N^2\big)S_1^3}{3N(1+N)(2+N)}$$

$$+\frac{8\big(-2+5N+5N^2\big)S_3}{3N(1+N)(2+N)} + \left(\frac{16P_{38}}{(N-1)N(1+N)^2(2+N)^2} + \frac{16(N-1)S_1}{N(1+N)}\right)S_{-2}$$

$$-\frac{16\left(4+N+N^2\right)S_{-3}}{N(1+N)(2+N)}-\frac{16\left(2+N+N^2\right)S_{2,1}}{N(1+N)(2+N)}+\frac{64S_{-2,1}}{N(1+N)(2+N)}$$

$$-\frac{16\left(1+N+N^2\right)\left(2+N+N^2\right)\zeta_2}{(N-1)N^2(1+N)^2(2+N)^2}-\frac{24(N-1)\zeta_3}{N(1+N)}\Bigg]$$

$$+C_F N_F T_F\Bigg[\frac{4S_2 P_{24}}{N^2(1+N)^2(2+N)}+\frac{2P_{103}}{(N-2)N^4(1+N)^4(2+N)(3+N)}$$

$$-\frac{4S_1^2 P_{25}}{N^2(1+N)^2(2+N)}+\left(-\frac{4P_{64}}{N^3(1+N)^3(2+N)}+\frac{20\left(2+N+N^2\right)S_2}{N(1+N)(2+N)}\right.$$

$$\left.+\frac{8\left(2+N+N^2\right)\zeta_2}{N(1+N)(2+N)}\right)S_1-\frac{28\left(2+N+N^2\right)S_1^3}{3N(1+N)(2+N)}-\frac{16\left(-2+5N+5N^2\right)S_3}{3N(1+N)(2+N)}$$

$$+\left(\frac{16P_{59}}{(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{128S_1}{N(1+N)(2+N)}\right)S_{-2}$$

$$+\frac{64S_{-3}}{N(1+N)(2+N)}+\frac{16\left(2+N+N^2\right)S_{2,1}}{N(1+N)(2+N)}-\frac{128S_{-2,1}}{N(1+N)(2+N)}$$

$$-\frac{2\left(2+N+N^2\right)\left(2+3N+3N^2\right)\zeta_2}{N^2(1+N)^2(2+N)}+\frac{48(N-1)\zeta_3}{N(1+N)}\Bigg]\;, \tag{141}$$

$$c_{F_2,g}^{(2,1)}\;=\;C_F N_F T_F\Bigg[-\frac{8S_{2,1}P_{14}}{N(1+N)^2(2+N)(3+N)}+\frac{2S_1^3 P_{27}}{3N^2(1+N)^2(2+N)}$$

$$-\frac{16\zeta_3 P_{67}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{4S_3 P_{72}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$+\frac{\zeta_2 P_{68}}{N^3(1+N)^3(2+N)}+\frac{P_{113}}{(N-2)^2 N^5(1+N)^5(2+N)(3+N)^2}$$

$$+\left(-\frac{2\zeta_2 P_{20}}{N^2(1+N)^2(2+N)}+\frac{2P_{102}}{(N-2)N^4(1+N)^4(2+N)(3+N)}-\frac{16(N-1)S_{2,1}}{N(1+N)}\right.$$

$$-\frac{2S_2 P_{42}}{N^2(1+N)^2(2+N)(3+N)}+\frac{20(N-2)(3+N)S_3}{N(1+N)(2+N)}+\frac{128S_{-2,1}}{N(1+N)(2+N)}$$

$$\left.-\frac{128\left(-7+N+N^2\right)\zeta_3}{3N(1+N)(2+N)}\right)S_1+\left(\frac{P_{74}}{N^3(1+N)^3(2+N)}+\frac{\left(-34-9N-9N^2\right)S_2}{N(1+N)(2+N)}\right.$$

$$\left.-\frac{6\left(2+N+N^2\right)\zeta_2}{N(1+N)(2+N)}\right)S_1^2+\frac{5\left(2+N+N^2\right)S_1^4}{2N(1+N)(2+N)}+\left(\frac{P_{81}}{N^3(1+N)^3(2+N)(3+N)}\right.$$

$$\left.+\frac{6\left(2+N+N^2\right)\zeta_2}{N(1+N)(2+N)}\right)S_2+\frac{\left(78+7N+7N^2\right)S_2^2}{2N(1+N)(2+N)}-\frac{5\left(-6+5N+5N^2\right)S_4}{N(1+N)(2+N)}$$

$$+\left(\frac{16S_1 P_{54}}{(N-2)N^2(1+N)^2(2+N)(3+N)}-\frac{8P_{99}}{(N-2)^2 N^3(1+N)^3(2+N)(3+N)^2}\right.$$

$$\left.-\frac{128S_1^2}{N(1+N)(2+N)}+\frac{64S_2}{N(1+N)(2+N)}\right)S_{-2}+\frac{96S_{-2}^2}{N(1+N)(2+N)}$$

$$
+\left(\frac{32(N-1)P_{18}}{(N-2)N^2(1+N)^2(3+N)}+\frac{64S_1}{N(1+N)(2+N)}\right)S_{-3}+\frac{160S_{-4}}{N(1+N)(2+N)}
$$

$$
+\frac{16\left(8+N+N^2\right)S_{3,1}}{N(1+N)(2+N)}-\frac{32\left(4+4N-N^2+N^3\right)S_{-2,1}}{N^2(1+N)(2+N)}-\frac{64S_{-2,2}}{N(1+N)(2+N)}
$$

$$
-\frac{128S_{-3,1}}{N(1+N)(2+N)}+\frac{8\left(-10+N+N^2\right)S_{2,1,1}}{N(1+N)(2+N)}-\frac{72(N-1)\zeta_2^2}{5N(1+N)}\Bigg]
$$

$$
+C_A N_F T_F\Bigg[\frac{16S_{2,1}P_{35}}{N^2(1+N)^2(2+N)^2}-\frac{8S_3 P_{40}}{3(N-1)N(1+N)^2(2+N)^2}
$$

$$
+\frac{4S_1^3 P_{65}}{3(N-1)N^2(1+N)^2(2+N)^2}+\frac{8\zeta_3 P_{70}}{3(N-1)N^2(1+N)^2(2+N)^2}
$$

$$
-\frac{2\zeta_2 P_{95}}{(N-1)N^3(1+N)^3(2+N)^3}+\frac{2P_{115}}{(N-2)(N-1)N^5(1+N)^5(2+N)^5(3+N)}
$$

$$
+\left(-\frac{8\zeta_2 P_{37}}{(N-1)N(1+N)^2(2+N)^2}-\frac{4S_2 P_{66}}{(N-1)N^2(1+N)^2(2+N)^2}\right.
$$

$$
+\frac{4\left(22+N+N^2\right)S_3}{N(1+N)(2+N)}+\frac{16S_{2,1}}{2+N}-\frac{64S_{-2,1}}{N(1+N)(2+N)}-\frac{64\left(13+2N+2N^2\right)\zeta_3}{3N(1+N)(2+N)}\Bigg)S_1
$$

$$
+\left(\frac{P_{92}}{(N-1)N^2(1+N)^3(2+N)^3}-\frac{\left(30+19N+19N^2\right)S_2}{N(1+N)(2+N)}-\frac{6\left(2+N+N^2\right)\zeta_2}{N(1+N)(2+N)}\right)
$$

$$
\times S_1^2+\frac{5\left(2+N+N^2\right)S_1^4}{2N(1+N)(2+N)}+\left(\frac{P_{94}}{(N-1)N^3(1+N)^3(2+N)^3}+\frac{2\left(2+N+N^2\right)\zeta_2}{N(1+N)(2+N)}\right)
$$

$$
\times S_2+\frac{\left(-2+15N+15N^2\right)S_2^2}{2N(1+N)(2+N)}+\frac{\left(2+21N+21N^2\right)S_4}{N(1+N)(2+N)}+\left(\frac{8(N-1)S_2}{N(1+N)}\right.
$$

$$
-\frac{32S_1 P_{52}}{(N-1)N^2(1+N)^2(2+N)^2}-\frac{8P_{107}}{(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}
$$

$$
-\frac{8(N-2)(3+N)S_1^2}{N(1+N)(2+N)}+\frac{4\left(2+N+N^2\right)\zeta_2}{N(1+N)(2+N)}\Bigg)S_{-2}+\frac{8\left(-4+N+N^2\right)S_{-2}^2}{N(1+N)(2+N)}
$$

$$
+\left(\frac{8P_{55}}{(N-1)N^2(1+N)^2(2+N)^2}+\frac{16S_1}{2+N}\right)S_{-3}-\frac{8\left(12+N+N^2\right)S_{-4}}{N(1+N)(2+N)}
$$

$$
-\frac{16\left(5+N+N^2\right)S_{3,1}}{N(1+N)(2+N)}+\frac{16\left(4+4N-N^2+N^3\right)S_{-2,1}}{N^2(1+N)(2+N)}+\frac{32S_{-2,2}}{N(1+N)(2+N)}
$$

$$
+\frac{64S_{-3,1}}{N(1+N)(2+N)}-\frac{8\left(-4+N+N^2\right)S_{2,1,1}}{N(1+N)(2+N)}+\frac{36(N-1)\zeta_2^2}{5N(1+N)}\Bigg], \tag{142}
$$

$$
c_{F_L,q}^{\mathrm{NS},(2,0)} = C_F\Bigg\{N_F T_F\Bigg[-\frac{8\left(-6+13N+25N^2\right)}{9N(1+N)^2}-\frac{32S_1}{3(1+N)}\Bigg]+C_A\Bigg[\frac{136S_1}{3(1+N)}
$$

$$
+\frac{2P_{31}}{9(N-2)N(1+N)^2(3+N)}+\frac{16S_3}{1+N}+\left(-\frac{32P_{15}}{(N-2)N(1+N)^2(3+N)}\right.
$$

$$+\frac{32S_1}{1+N}\Bigg)S_{-2}+\frac{16S_{-3}}{1+N}-\frac{32S_{-2,1}}{1+N}-\frac{48\zeta_3}{1+N}\Bigg]\Bigg\}+C_F^2\Bigg[-\frac{16(1+2N)S_1}{N(1+N)}+\frac{24S_1^2}{1+N}$$

$$-\frac{2P_{71}}{(N-2)N^2(1+N)^3(3+N)}-\frac{8S_2}{1+N}-\frac{32S_3}{1+N}+\Bigg(\frac{64P_{15}}{(N-2)N(1+N)^2(3+N)}$$

$$-\frac{64S_1}{1+N}\Bigg)S_{-2}-\frac{32S_{-3}}{1+N}+\frac{64S_{-2,1}}{1+N}+\frac{96\zeta_3}{1+N}\Bigg]\,, \tag{143}$$

$$c_{F_L,q}^{\mathrm{NS},(2,1)} = C_F\Bigg\{N_FT_F\Bigg[\frac{2P_{33}}{27N^2(1+N)^3}+\frac{8\big(-6+13N+25N^2\big)S_1}{9N(1+N)^2}+\frac{16S_1^2}{3(1+N)}-\frac{16S_2}{3(1+N)}$$

$$-\frac{8\zeta_2}{3(1+N)}\Bigg]+C_A\Bigg[\frac{24\zeta_3P_{13}}{(N-2)N(1+N)^2(3+N)}+\frac{8S_3P_{21}}{(N-2)N(1+N)^2(3+N)}$$

$$+\frac{P_{90}}{54(N-2)^2N^2(1+N)^3(3+N)^2}+\Bigg(-\frac{2P_{46}}{9(N-2)N(1+N)^3(3+N)}-\frac{40S_3}{1+N}$$

$$+\frac{8(N-1)(3+2N)S_2}{(1+N)^2(3+N)}+\frac{16S_{2,1}}{1+N}+\frac{32S_{-2,1}}{1+N}+\frac{96\zeta_3}{1+N}\Bigg)S_1-\Bigg(\frac{80}{3(1+N)}+\frac{4S_2}{1+N}\Bigg)$$

$$\times S_1^2-\frac{16\big(-6-8N+N^2\big)S_2}{3(1+N)^2(3+N)}+\frac{8S_2^2}{1+N}+\frac{20S_4}{1+N}+\Bigg(\frac{32S_1P_{15}}{(N-2)N(1+N)^2(3+N)}$$

$$-\frac{16P_{69}}{(N-2)^2N^2(1+N)^3(3+N)^2}-\frac{32S_1^2}{1+N}+\frac{16S_2}{1+N}\Bigg)S_{-2}+\frac{24S_{-2}^2}{1+N}+\frac{40S_{-4}}{1+N}$$

$$-\Bigg(\frac{32P_{15}}{(N-2)N(1+N)^2(3+N)}-\frac{16S_1}{1+N}\Bigg)S_{-3}-\frac{16(N-1)(3+2N)S_{2,1}}{(1+N)^2(3+N)}$$

$$+\frac{24S_{3,1}}{1+N}-\frac{16S_{-2,2}}{1+N}-\frac{32S_{-3,1}}{1+N}-\frac{24S_{2,1,1}}{1+N}+\frac{22\zeta_2}{3(1+N)}+\frac{72\zeta_2^2}{5(1+N)}\Bigg]\Bigg\}$$

$$+C_F^2\Bigg[\frac{2S_2P_{28}}{N(1+N)^2(2+N)(3+N)}-\frac{48\zeta_3P_{36}}{(N-2)N(1+N)^2(2+N)(3+N)}$$

$$-\frac{16S_3P_{43}}{3(N-2)N(1+N)^2(2+N)(3+N)}+\frac{P_{105}}{2(N-2)^2N^3(1+N)^4(3+N)^2}$$

$$+\Bigg(\frac{2P_{85}}{(N-2)N^2(1+N)^3(2+N)(3+N)}+\frac{4\big(46+41N-2N^2-5N^3\big)S_2}{(1+N)^2(2+N)(3+N)}$$

$$+\frac{80S_3}{1+N}-\frac{32S_{2,1}}{1+N}-\frac{64S_{-2,1}}{1+N}+\frac{8\zeta_2}{1+N}-\frac{192\zeta_3}{1+N}\Bigg)S_1+\Bigg(\frac{2(1+3N)(4+3N)}{N(1+N)^2}$$

$$+\frac{8S_2}{1+N}\Bigg)S_1^2-\frac{28S_1^3}{3(1+N)}-\frac{16S_2^2}{1+N}-\frac{40S_4}{1+N}+\Bigg(-\frac{64S_1P_{15}}{(N-2)N(1+N)^2(3+N)}$$

$$+\frac{64S_1^2}{1+N}-\frac{32S_2}{1+N}+\frac{32P_{69}}{(N-2)^2N^2(1+N)^3(3+N)^2}\Bigg)S_{-2}-\frac{48S_{-2}^2}{1+N}+\Bigg(-\frac{32S_1}{1+N}$$

$$+\frac{64P_{15}}{(N-2)N(1+N)^2(3+N)}\Bigg)S_{-3}-\frac{80S_{-4}}{1+N}+\frac{16\big(-8+7N+16N^2+5N^3\big)S_{2,1}}{(1+N)^2(2+N)(3+N)}$$

$$-\frac{48S_{3,1}}{1+N} + \frac{32S_{-2,2}}{1+N} + \frac{64S_{-3,1}}{1+N} + \frac{48S_{2,1,1}}{1+N} - \frac{2\big(2+3N+3N^2\big)\zeta_2}{N(1+N)^2} - \frac{144\zeta_2^2}{5(1+N)}\Bigg] \ , \quad (144)$$

$$c_{F_L,q}^{\mathrm{PS},(2,0)} = C_F N_F T_F \Bigg[-\frac{32P_{39}}{(N-1)N^2(1+N)^3(2+N)^2} - \frac{64\big(2+N+N^2\big)S_1}{(N-1)N(1+N)^2(2+N)} \Bigg] \ , \quad (145)$$

$$c_{F_L,q}^{\mathrm{PS},(2,1)} = C_F N_F T_F \Bigg[\frac{16P_{101}}{(N-2)(N-1)N^3(1+N)^4(2+N)^3(3+N)}$$

$$+\frac{32\big(2+N+N^2\big)S_1^2}{(N-1)N(1+N)^2(2+N)} + \frac{32S_1 P_{39}}{(N-1)N^2(1+N)^3(2+N)^2}$$

$$-\frac{32\big(2+N+N^2\big)S_2}{(N-1)N(1+N)^2(2+N)} - \frac{32\big(6+N+N^2\big)S_{-2}}{(N-2)(N-1)(1+N)(2+N)(3+N)}$$

$$-\frac{16\big(2+N+N^2\big)\zeta_2}{(N-1)N(1+N)^2(2+N)} \Bigg] \ , \quad (146)$$

$$c_{F_L,g}^{(2,0)} = C_A N_F T_F \Bigg[\frac{64S_1 P_{22}}{(N-1)N(1+N)^2(2+N)^2} - \frac{32P_{41}}{(N-1)N^2(1+N)^3(2+N)^3}$$

$$+\frac{96S_1^2}{(1+N)(2+N)} - \frac{32S_2}{(1+N)(2+N)} - \frac{64S_{-2}}{(1+N)(2+N)} \Bigg]$$

$$+C_F N_F T_F \Bigg[-\frac{16P_{56}}{(N-2)N^2(1+N)^3(2+N)(3+N)} - \frac{64\big(1+N+N^2\big)S_1}{N(1+N)^2(2+N)}$$

$$+\frac{64(N-1)S_{-2}}{(N-2)(1+N)(3+N)} \Bigg] \ , \quad (147)$$

$$c_{F_L,g}^{(2,1)} = C_A N_F T_F \Bigg[\frac{16S_2 P_{19}}{(N-1)N(1+N)^2(2+N)^2} - \frac{16S_1^2 P_{26}}{(N-1)N(1+N)^2(2+N)^2}$$

$$+\frac{16P_{104}}{(N-2)(N-1)N^3(1+N)^4(2+N)^4(3+N)} - \frac{112S_1^3}{3(1+N)(2+N)}$$

$$+\left(-\frac{32P_{84}}{(N-1)N^2(1+N)^3(2+N)^3} + \frac{112S_2}{(1+N)(2+N)} + \frac{32\zeta_2}{(1+N)(2+N)} \right)S_1$$

$$+\left(-\frac{16P_{58}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)} + \frac{96S_1}{(1+N)(2+N)} \right)S_{-2}$$

$$-\frac{48S_{-3}}{(1+N)(2+N)} - \frac{64S_{2,1}}{(1+N)(2+N)} - \frac{64\big(1+N+N^2\big)\zeta_2}{(N-1)N(1+N)^2(2+N)^2}$$

$$+\frac{208S_3}{3(1+N)(2+N)} - \frac{32S_{-2,1}}{(1+N)(2+N)} - \frac{144\zeta_3}{(1+N)(2+N)} \Bigg]$$

$$+C_F N_F T_F \Bigg[\frac{8P_{97}}{(N-2)^2N^3(1+N)^4(2+N)(3+N)^2} + \left(-\frac{16(5+3N)S_2}{(1+N)(2+N)(3+N)} \right.$$

$$\left. \frac{16P_{53}}{(N-2)N^2(1+N)^3(2+N)(3+N)} \right)S_1 + \frac{8(N-1)\big(12+13N+5N^2\big)S_2}{N(1+N)^2(2+N)(3+N)}$$

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$$
+ \frac{8\left(4 + 5N + 5N^2\right)S_1^2}{N(1+N)^2(2+N)} - \frac{16\left(-26 + 3N + 7N^2\right)S_3}{(N-2)(1+N)(2+N)(3+N)}
$$

$$
+ \left(\frac{128 P_{57}}{(N-2)^2 N(1+N)^2(2+N)(3+N)^2} - \frac{128\left(-4 + N + N^2\right)S_1}{(N-2)(1+N)(2+N)(3+N)} \right) S_{-2}
$$

$$
+ \frac{32\left(2 + N + N^2\right)S_{-3}}{(N-2)(1+N)(2+N)(3+N)} + \frac{32(5+3N)S_{2,1}}{(1+N)(2+N)(3+N)} + \frac{64 S_{-2,1}}{(1+N)(2+N)}
$$

$$
\left. - \frac{8\left(2 + N + N^2\right)\zeta_2}{N(1+N)^2(2+N)} + \frac{192\left(-6 + 3N + N^2\right)\zeta_3}{(N-2)(1+N)(2+N)(3+N)} \right] \, , \tag{148}
$$

with the polynomials

$$
\begin{aligned}
P_{10} &= -1239N^4 - 1554N^3 - 259N^2 + 584N + 396 \, , & (149) \\
P_{11} &= -1079N^4 - 2410N^3 - 1379N^2 - 192N - 108 \, , & (150) \\
P_{12} &= -75N^4 - 248N^3 - 225N^2 - 108N - 50 \, , & (151) \\
P_{13} &= N^4 - 12N^3 - 11N^2 + 14N + 24 \, , & (152) \\
P_{14} &= N^4 - 7N^3 - 7N^2 + 27N + 18 \, , & (153) \\
P_{15} &= N^4 + 2N^3 - N^2 - 2N - 6 \, , & (154) \\
P_{16} &= N^4 + 2N^3 + 11N^2 + 10N + 4 \, , & (155) \\
P_{17} &= N^4 + 2N^3 + 14N^2 + 13N + 4 \, , & (156) \\
P_{18} &= N^4 + 3N^3 - 6N^2 - 2N - 6 \, , & (157) \\
P_{19} &= 2N^4 + 6N^3 - 9N^2 - 15N - 8 \, , & (158) \\
P_{20} &= 3N^4 + 4N^3 - N^2 - 6N - 8 \, , & (159) \\
P_{21} &= 4N^4 + N^3 - 7N^2 + 2N - 12 \, , & (160) \\
P_{22} &= 4N^4 + 6N^3 - 9N^2 - 9N - 4 \, , & (161) \\
P_{23} &= 5N^4 + 31N^3 + 38N^2 + 24N + 11 \, , & (162) \\
P_{24} &= 6N^4 + 7N^3 + 8N^2 + 5N - 6 \, , & (163) \\
P_{25} &= 6N^4 + 9N^3 + 4N^2 - 5N - 10 \, , & (164) \\
P_{26} &= 12N^4 + 18N^3 - 23N^2 - 23N - 8 \, , & (165) \\
P_{27} &= 15N^4 + 23N^3 + 13N^2 - 9N - 22 \, , & (166) \\
P_{28} &= 15N^4 + 28N^3 - 21N^2 - 38N - 24 \, , & (167) \\
P_{29} &= 26N^4 + 45N^3 + 11N^2 - 8N - 5 \, , & (168) \\
P_{30} &= 111N^4 + 138N^3 + 11N^2 - 64N - 36 \, , & (169) \\
P_{31} &= 281N^4 + 430N^3 - 1927N^2 - 1284N + 396 \, , & (170) \\
P_{32} &= 373N^4 + 998N^3 + 673N^2 + 192N + 108 \, , & (171) \\
P_{33} &= 541N^4 + 782N^3 + 199N^2 - 42N + 36 \, , & (172) \\
P_{34} &= -4541N^5 - 17529N^4 - 20631N^3 - 8099N^2 - 1968N - 1188 \, , & (173) \\
P_{35} &= N^5 - 5N^3 - 6N^2 - 8N - 4 \, , & (174) \\
P_{36} &= N^5 - 10N^4 - 37N^3 - 6N^2 + 56N + 48 \, , & (175) \\
P_{37} &= N^5 - 2N^4 - 10N^3 - 5N^2 - 6N - 2 \, , & (176) \\
P_{38} &= N^5 - N^4 - 5N^3 + 3N^2 + 14N + 12 \, , & (177) \\
P_{39} &= 3N^5 + 10N^4 + 16N^3 + 11N^2 - 4N - 4 \, , & (178)
\end{aligned}
$$

$$P_{40} = 8N^5 - 7N^4 - 11N^3 + 68N^2 + 78N + 32 , \tag{179}$$

$$P_{41} = 13N^5 + 40N^4 + 45N^3 + 18N^2 - 12N - 8 , \tag{180}$$

$$P_{42} = 13N^5 + 80N^4 + 94N^3 - 2N^2 - 51N - 54 , \tag{181}$$

$$P_{43} = 14N^5 + 35N^4 - 20N^3 - 65N^2 - 42N - 72 , \tag{182}$$

$$P_{44} = 179N^5 + 1219N^4 + 1691N^3 + 111N^2 - 396N - 108 , \tag{183}$$

$$P_{45} = 229N^5 + 821N^4 + 1081N^3 + 795N^2 + 198N + 108 , \tag{184}$$

$$P_{46} = 461N^5 + 999N^4 - 2037N^3 - 4507N^2 - 1320N + 396 , \tag{185}$$

$$P_{47} = 4541N^5 + 17151N^4 + 20199N^3 + 8477N^2 + 2292N + 1188 , \tag{186}$$

$$P_{48} = -2311N^6 - 5529N^5 + 6417N^4 + 19097N^3 + 15042N^2 + 4284N + 4536 , \tag{187}$$

$$P_{49} = -237N^6 - 699N^5 - 683N^4 - 321N^3 - 316N^2 - 272N - 72 , \tag{188}$$

$$P_{50} = -37N^6 - 171N^5 - 231N^4 + 7N^3 + 388N^2 + 352N + 96 , \tag{189}$$

$$P_{51} = N^6 - 12N^4 - 13N^3 - 12N^2 - 8N - 4 , \tag{190}$$

$$P_{52} = N^6 - 2N^4 + 7N^3 + 10N^2 - 4 , \tag{191}$$

$$P_{53} = N^6 - 6N^5 - 22N^4 - 2N^3 + 5N^2 + 16N + 12 , \tag{192}$$

$$P_{54} = N^6 - 5N^5 + N^4 + 61N^3 - 26N^2 - 128N - 96 , \tag{193}$$

$$P_{55} = N^6 - 3N^5 - 11N^4 - 5N^3 + 22N^2 + 36N + 8 , \tag{194}$$

$$P_{56} = N^6 + N^5 - 18N^4 - 27N^3 - 17N^2 + 16N + 12 , \tag{195}$$

$$P_{57} = N^6 + 3N^5 - 2N^4 - 7N^3 + 3N^2 + 6N + 24 , \tag{196}$$

$$P_{58} = N^6 + 3N^5 + 13N^4 + 21N^3 - 2N^2 - 12N + 72 , \tag{197}$$

$$P_{59} = N^6 + 7N^5 - 7N^4 - 39N^3 + 14N^2 + 40N + 48 , \tag{198}$$

$$P_{60} = 2N^6 - 2N^5 - 5N^4 + 20N^3 - 39N^2 - 12N - 36 , \tag{199}$$

$$P_{61} = 2N^6 - 2N^5 - 3N^4 + 26N^3 - 45N^2 - 34N - 48 , \tag{200}$$

$$P_{62} = 2N^6 - 2N^5 + N^4 + 38N^3 - 57N^2 - 78N - 72 , \tag{201}$$

$$P_{63} = 3N^6 - 6N^5 - 28N^4 - 11N^3 - 10N^2 + 4 , \tag{202}$$

$$P_{64} = 6N^6 + 35N^5 + 57N^4 + 73N^3 + 77N^2 + 56N + 20 , \tag{203}$$

$$P_{65} = 7N^6 - 14N^5 - 64N^4 - 23N^3 - 18N^2 + 4N + 12 , \tag{204}$$

$$P_{66} = 7N^6 - 10N^5 - 58N^4 - 27N^3 - 24N^2 + 4N + 12 , \tag{205}$$

$$P_{67} = 12N^6 - 9N^5 - 133N^4 + 295N^3 + 99N^2 - 524N - 492 , \tag{206}$$

$$P_{68} = 13N^6 + 25N^5 + 16N^4 - 27N^3 - 55N^2 - 40N - 12 , \tag{207}$$

$$P_{69} = 15N^6 + 53N^5 - 7N^4 - 137N^3 + 40N^2 + 204N + 144 , \tag{208}$$

$$P_{70} = 18N^6 - 18N^5 - 31N^4 + 244N^3 + 479N^2 + 168N - 44 , \tag{209}$$

$$P_{71} = 23N^6 + 57N^5 - 171N^4 - 357N^3 - 120N^2 + 80N + 24 , \tag{210}$$

$$P_{72} = 24N^6 - 25N^5 - 21N^4 + 433N^3 - 403N^2 - 1012N - 516 , \tag{211}$$

$$P_{73} = 27N^6 + 107N^5 + 139N^4 + 169N^3 + 238N^2 + 136N + 48 , \tag{212}$$

$$P_{74} = 31N^6 + 130N^5 + 187N^4 + 192N^3 + 176N^2 + 128N + 48 , \tag{213}$$

$$P_{75} = 67N^6 + 375N^5 + 963N^4 + 1625N^3 + 1656N^2 + 918N + 324 , \tag{214}$$

$$P_{76} = 105N^6 + 231N^5 + 159N^4 - 59N^3 - 172N^2 - 112N - 24 , \tag{215}$$

$$P_{77} = 834N^6 + 1773N^5 - 2738N^4 - 7729N^3 - 2150N^2 + 2550N + 2916 , \tag{216}$$

$$P_{78} = 2667N^6 + 5469N^5 + 3269N^4 + 463N^3 + 860N^2 + 1296N + 504 , \tag{217}$$

$$P_{79} = 7081N^6 + 28839N^5 + 35439N^4 + 13693N^3 - 2580N^2 - 3888N - 1512 , \tag{218}$$

$$P_{80} = 14311N^6 + 85344N^5 + 179550N^4 + 159592N^3 + 57195N^2 + 9036N + 3564 , \qquad (219)$$

$$P_{81} = -37N^7 - 157N^6 - 335N^5 - 639N^4 - 924N^3 - 708N^2 - 320N - 96 , \qquad (220)$$

$$P_{82} = N^7 + N^6 - N^5 + 3N^4 + 8N^3 - 60N^2 - 208N - 192 , \qquad (221)$$

$$P_{83} = 2N^7 - 25N^5 - 84N^4 - 135N^3 - 114N^2 - 68N - 24 , \qquad (222)$$

$$P_{84} = 13N^7 + 56N^6 + 52N^5 - 60N^4 - 121N^3 - 56N^2 + 12N + 8 , \qquad (223)$$

$$P_{85} = 55N^7 + 219N^6 - 41N^5 - 1119N^4 - 1418N^3 - 320N^2 + 184N + 48 , \qquad (224)$$

$$P_{86} = 108N^7 + 362N^6 - 6N^5 - 908N^4 - 1447N^3 - 2073N^2 - 1604N - 924 , \qquad (225)$$

$$P_{87} = 198N^7 + 765N^6 + 114N^5 - 2865N^4 - 3904N^3 - 324N^2 + 1984N + 1632 , \qquad (226)$$

$$P_{88} = -56163N^8 - 166680N^7 - 175082N^6 - 69468N^5 - 1323N^4 + 15676N^3 + 27792N^2$$
$$+22608N + 6480 , \qquad (227)$$

$$P_{89} = -30651N^8 - 94248N^7 + 83888N^6 + 331530N^5 + 210723N^4 + 5534N^3 + 17376N^2$$
$$+67032N + 33264 , \qquad (228)$$

$$P_{90} = -5885N^8 - 20168N^7 + 51756N^6 + 169186N^5 - 154415N^4 - 399114N^3$$
$$-107208N^2 + 34344N - 14256 , \qquad (229)$$

$$P_{91} = -397N^8 - 2944N^7 - 8680N^6 - 12850N^5 - 10191N^4 - 5186N^3 - 2760N^2$$
$$-1392N - 288 , \qquad (230)$$

$$P_{92} = 55N^8 + 111N^7 - 227N^6 - 263N^5 + 1108N^4 + 2112N^3$$
$$+1632N^2 + 688N + 160 , \qquad (231)$$

$$P_{93} = -86393N^9 - 545545N^8 - 734144N^7 + 1327702N^6 + 4117751N^5 + 3541963N^4$$
$$+1033098N^3 - 171576N^2 - 254232N - 99792 , \qquad (232)$$

$$P_{94} = -17N^9 - 85N^8 - 63N^7 - 3N^6 - 552N^5 - 1424N^4 - 1584N^3 - 1040N^2$$
$$-480N - 128 , \qquad (233)$$

$$P_{95} = N^9 + 6N^8 + 7N^7 + 25N^6 + 124N^5 + 285N^4 + 384N^3 + 304N^2 + 160N + 48 , \qquad (234)$$

$$P_{96} = 19N^9 + 41N^8 - 71N^7 - 71N^6 + 452N^5 + 894N^4 + 800N^3 + 432N^2$$
$$+160N + 32 , \qquad (235)$$

$$P_{97} = 2N^{10} + 8N^9 - 2N^8 + 8N^7 + 271N^6 + 571N^5 + 491N^4 + 117N^3 + 98N^2$$
$$+156N + 72 , \qquad (236)$$

$$P_{98} = 3N^{10} + 15N^9 - 11N^8 - 93N^7 - 24N^6 + 52N^5 + 304N^4 + 998N^3 - 356N^2$$
$$-1272N - 720 , \qquad (237)$$

$$P_{99} = 7N^{10} + 19N^9 - 60N^8 - 186N^7 + 139N^6 + 571N^5 - 542N^4 - 1940N^3$$
$$+1224N^2 + 3456N + 1728 , \qquad (238)$$

$$P_{100} = 9N^{10} + 44N^9 + 83N^8 + 107N^7 + 291N^6 + 879N^5 + 1579N^4 + 1672N^3$$
$$+1128N^2 + 496N + 112 , \qquad (239)$$

$$P_{101} = 9N^{10} + 66N^9 + 147N^8 - 41N^7 - 787N^6 - 1627N^5 - 1655N^4 - 904N^3$$
$$-296N^2 - 112N - 48 , \qquad (240)$$

$$P_{102} = 18N^{10} + 114N^9 + 109N^8 - 331N^7 - 718N^6 - 904N^5 - 623N^4 + 545N^3$$
$$+1386N^2 + 988N + 264 , \qquad (241)$$

$$P_{103} = 34N^{10} + 136N^9 + 10N^8 - 372N^7 - 447N^6 - 99N^5 - 215N^4 - 921N^3$$
$$-1210N^2 - 716N - 168 , \qquad (242)$$

$$P_{104} = 36N^{10} + 248N^9 + 451N^8 - 523N^7 - 3189N^6 - 5209N^5 - 4382N^4$$

$$-2104N^3 - 704N^2 - 272N - 96 \ , \tag{243}$$

$$P_{105} = 133N^{10} + 573N^9 - 1052N^8 - 5646N^7 + 2901N^6 + 21505N^5 + 18130N^4$$
$$+2320N^3 - 160N^2 + 1200N + 288 \ , \tag{244}$$

$$P_{106} = 763N^{10} + 3027N^9 - 128N^8 - 9930N^7 - 11309N^6 - 2945N^5 - 350N^4$$
$$-5224N^3 - 7296N^2 - 3824N - 672 \ , \tag{245}$$

$$P_{107} = 2N^{11} + 17N^{10} + 34N^9 - 34N^8 - 190N^7 - 255N^6 - 102N^5 + 648N^4$$
$$+1592N^3 + 1136N^2 + 32N - 192 \ , \tag{246}$$

$$P_{108} = 1203N^{11} + 9097N^{10} + 19614N^9 - 6786N^8 - 83057N^7 - 122227N^6$$
$$-74000N^5 - 9540N^4 + 26544N^3 + 34608N^2 + 19072N + 3648 \ , \tag{247}$$

$$P_{109} = 4N^{12} + 34N^{11} + 76N^{10} - 4N^9 - 257N^8 - 653N^7 - 1597N^6$$
$$-3349N^5 - 4822N^4 - 4520N^3 - 2784N^2 - 1104N - 224 \ , \tag{248}$$

$$P_{110} = 68N^{12} + 360N^{11} + 367N^{10} - 1033N^9 - 2581N^8 - 2317N^7$$
$$-3258N^6 - 7354N^5 - 9852N^4 - 7648N^3 - 3776N^2 - 1184N - 192 \ , \tag{249}$$

$$P_{111} = 648699N^{12} + 3241902N^{11} - 1257119N^{10} - 24261820N^9 - 20205047N^8$$
$$+35553886N^7 + 60343151N^6 + 28400368N^5 + 5406396N^4 + 1441440N^3$$
$$-4831920N^2 - 7008768N - 2566080 \ , \tag{250}$$

$$P_{112} = -4521N^{14} - 26467N^{13} - 9301N^{12} + 178761N^{11} + 272257N^{10} - 131725N^9$$
$$-554287N^8 - 465045N^7 - 251812N^6 - 157892N^5 + 36656N^4 + 261568N^3$$
$$+265824N^2 + 114048N + 17280 \ , \tag{251}$$

$$P_{113} = -92N^{14} - 560N^{13} - 355N^{12} + 3542N^{11} + 7046N^{10} + 469N^9 - 11572N^8$$
$$-16160N^7 - 14914N^6 - 9595N^5 + 5423N^4 + 21784N^3 + 22728N^2 + 11088N$$
$$+2160 \ , \tag{252}$$

$$P_{114} = 28N^{15} + 246N^{14} + 715N^{13} + 259N^{12} - 3044N^{11} - 7282N^{10} - 6158N^9 + 2882N^8$$
$$+18427N^7 + 44383N^6 + 79244N^5 + 99004N^4 + 83536N^3 + 47056N^2 + 16704N$$
$$+2880 \ , \tag{253}$$

$$P_{115} = 14N^{17} + 174N^{16} + 731N^{15} + 818N^{14} - 2448N^{13} - 7521N^{12} - 3298N^{11} + 11946N^{10}$$
$$+15938N^9 - 10959N^8 - 71053N^7 - 162458N^6 - 255636N^5 - 282952N^4$$
$$-216304N^3 - 111584N^2 - 36288N - 5760. \tag{254}$$

The expansion coefficients for the Compton amplitudes of $xF_3(x, Q^2)$ and $g_1(x, Q^2)$ are

$$c_{F_3,q}^{(2,0),\text{L}} = C_F \Bigg\{ T_F N_F \Bigg[\frac{P_{136}}{54N^3(1+N)^3} - \frac{4\big(-6 + 19N + 19N^2\big)}{9N(1+N)} S_1^2 + \Bigg(-\frac{2P_{122}}{27N^2(1+N)^2}$$

$$+\frac{16}{3}S_2 + \frac{8}{3}\zeta_2 \Bigg) S_1 - \frac{16}{9}S_1^3 + \frac{4\big(-6 + 47N + 47N^2\big)S_2}{9N(1+N)} - \frac{104}{9}S_3 + \frac{16}{3}S_{2,1}$$

$$-\frac{2\big(2 + 3N + 3N^2\big)\zeta_2}{3N(1+N)} \Bigg] + C_A \Bigg[\frac{P_{144}}{216(N-1)N^4(1+N)^4(2+N)} + \Bigg(\frac{P_{137}}{54N^3(1+N)^3}$$

$$-\frac{4\big(3 + 11N + 11N^2\big)S_2}{3N(1+N)} + 24S_3 - 16S_{2,1} - 16S_{-2,1} - \frac{22}{3}\zeta_2 - 48\zeta_3 \Bigg) S_1$$

$$+\Bigg(\frac{-66 + 233N + 233N^2}{9N(1+N)} + 4S_2 \Bigg) S_1^2 + \frac{\big(66 - 481N - 1166N^2 - 583N^3\big)}{9N(1+N)^2} S_2$$

$$
\begin{aligned}
&+\frac{44}{9}S_1^3 - 4S_2^2 + \frac{2\left(-36 + 143N + 143N^2\right)}{9N(1+N)}S_3 + \left(\frac{4P_{123}}{(N-1)N(1+N)^2(2+N)}\right.\\
&\left.-\frac{8S_1}{N(1+N)} + 16S_1^2 - 8S_2\right)S_{-2} - 8S_4 - 12S_{-2}^2 + \left(\frac{8}{N(1+N)} - 8S_1\right)S_{-3}\\
&-20S_{-4} - \frac{4\left(-6 + 11N + 11N^2\right)}{3N(1+N)}S_{2,1} - 24S_{3,1} + 8S_{-2,2} + 16S_{-3,1} + 24S_{2,1,1}\\
&\left.\left.+\frac{11\left(2 + 3N + 3N^2\right)}{6N(1+N)}\zeta_2 + \frac{6(1+3N)(2+3N)}{N(1+N)}\zeta_3\right]\right\} + C_F^2\left[\frac{2S_2 P_{119}}{N^2(1+N)^2}\right.\\
&+\frac{P_{145}}{8(N-1)N^4(1+N)^4(2+N)} + \left(\frac{P_{130}}{2N^3(1+N)^3} - \frac{2\left(-14 + 9N + 9N^2\right)}{N(1+N)}S_2\right.\\
&\left.-\frac{56}{3}S_3 + 16S_{2,1} + 32S_{-2,1} + \frac{4\left(2 + 3N + 3N^2\right)}{N(1+N)}\zeta_2 + 48\zeta_3\right)S_1 + \left(\frac{2P_{118}}{N^2(1+N)^2}\right.\\
&\left.-28S_2 - 8\zeta_2\right)S_1^2 + \frac{2\left(-14 + 15N + 15N^2\right)}{3N(1+N)}S_1^3 - \frac{2\left(-2 + 33N + 33N^2\right)}{3N(1+N)}S_3\\
&+\frac{14}{3}S_1^4 + 6S_2^2 + 12S_4 + \left(-\frac{8P_{123}}{(N-1)N(1+N)^2(2+N)} + \frac{16S_1}{N(1+N)} - 32S_1^2\right.\\
&\left.+16S_2\right)S_{-2} + \left(-\frac{16}{N(1+N)} + 16S_1\right)S_{-3} + \frac{4\left(-2 + 3N + 3N^2\right)}{N(1+N)}S_{2,1}\\
&+40S_{-4} + 24S_{-2}^2 + 40S_{3,1} - 16S_{-2,2} - 32S_{-3,1} - 24S_{2,1,1} - \frac{\left(2 + 3N + 3N^2\right)^2}{2N^2(1+N)^2}\zeta_2\\
&\left.-72\zeta_3\right], \hspace{8cm}(255)\\[2mm]
c_{F3,q}^{(2,1),L} &= C_F\left\{C_A\left[\frac{2S_{2,1}P_{121}}{9N^2(1+N)(2+N)} + \frac{P_{129}}{72N^3(1+N)^3}\zeta_2 - \frac{2P_{135}}{9(N-1)N^2(1+N)^2(2+N)}\zeta_3\right.\right.\\
&+\frac{P_{148}}{2592(N-1)^2N^5(1+N)^5(2+N)^2} + \left(\frac{P_{143}}{648(N-1)N^4(1+N)^4(2+N)}\right.\\
&+\frac{S_2 P_{126}}{9N^2(1+N)^2(2+N)} + 8S_2^2 - \frac{4\left(-9 + 22N + 22N^2\right)S_3}{9N(1+N)} + 32S_4 - \frac{8S_{2,1}}{N(1+N)}\\
&\left.-4S_{3,1} - 8S_{-2,2} - 16S_{-3,1} - 20S_{2,1,1} + \frac{\left(-66 + 233N + 233N^2\right)\zeta_2}{18N(1+N)} + \frac{72}{5}\zeta_2^2\right.\\
&\left.+\frac{514}{9}\zeta_3\right)S_1 + \left(\frac{P_{127}}{108N^3(1+N)^3} + \frac{\left(9 + 22N + 22N^2\right)S_2}{3N(1+N)} - 14S_3 + 12S_{2,1}\right.\\
&\left.+8S_{-2,1} + \frac{11}{3}\zeta_2 + 24\zeta_3\right)S_1^2 + \left(\frac{66 - 233N - 233N^2}{27N(1+N)} - \frac{8}{3}S_2\right)S_1^3 - \frac{11}{9}S_1^4\\
&+\left(\frac{P_{140}}{108N^3(1+N)^3(2+N)} - \frac{22}{3}S_3 - 8S_{2,1} - 8S_{-2,1} - \frac{22}{3}\zeta_2 - 48\zeta_3\right)S_2
\end{aligned}
$$

$$+\frac{\left(-2-17N-17N^2\right)S_2^2}{N(1+N)}+\left(\frac{P_{128}}{27(N-1)N^2(1+N)^2(2+N)}+2\zeta_2\right)S_3$$

$$+\frac{\left(-33+59N+59N^2\right)S_4}{3N(1+N)}-22S_5+\left(-\frac{4P_{142}}{(N-1)^2N(1+N)^3(2+N)^2}\right.$$

$$+\left(-\frac{4P_{123}}{(N-1)N(1+N)^2(2+N)}+16S_2+4\zeta_2\right)S_1+\frac{4S_1^2}{N(1+N)}-8S_1^3-\frac{4S_2}{N(1+N)}$$

$$-32S_3-8S_{-2,1}-\frac{2\zeta_2}{N(1+N)}-72\zeta_3\right)S_{-2}+\left(-\frac{4}{N(1+N)}+20S_1\right)S_{-2}^2$$

$$+\left(\frac{4P_{123}}{(N-1)N(1+N)^2(2+N)}-\frac{8S_1}{N(1+N)}+12S_1^2+4S_2-16S_{-2}+2\zeta_2\right)S_{-3}$$

$$+\left(\frac{4}{N(1+N)}+12S_1\right)S_{-4}-38S_{-5}+20S_{2,3}-\frac{2\left(-4+5N+5N^2\right)}{N(1+N)}S_{3,1}$$

$$-8S_{2,-3}-44S_{4,1}+44S_{-2,3}+8S_{-4,1}+\frac{4\left(3+11N+11N^2\right)}{3N(1+N)}S_{2,1,1}$$

$$+4S_{2,2,1}+40S_{3,1,1}-8S_{-2,1,-2}+4S_{2,1,1,1}-4S_{-2,1}\zeta_2-\frac{9(1+3N)(2+3N)\zeta_2^2}{5N(1+N)}\right]$$

$$+T_FN_F\left[\frac{P_{120}}{18N^2(1+N)^2}\zeta_2+\frac{P_{141}}{648N^4(1+N)^4}+\left(\frac{P_{138}}{162N^3(1+N)^3}\right.\right.$$

$$-\frac{4\left(-6+19N+19N^2\right)S_2}{9N(1+N)}+\frac{32}{9}S_3-\frac{2\left(-6+19N+19N^2\right)\zeta_2}{9N(1+N)}-\frac{128}{9}\zeta_3\right)S_1$$

$$+\left(\frac{P_{122}}{27N^2(1+N)^2}-\frac{8}{3}S_2-\frac{4}{3}\zeta_2\right)S_1^2+\frac{4\left(-6+19N+19N^2\right)S_1^3}{27N(1+N)}+\frac{4}{9}S_1^4$$

$$+\left(\frac{P_{116}}{27N^2(1+N)^2}+\frac{8\zeta_2}{3}\right)S_2+4S_2^2+\frac{16\left(-3+41N+41N^2\right)S_3}{27N(1+N)}-\frac{28}{3}S_4$$

$$\left.\left.-\frac{112}{9}S_{2,1}+8S_{3,1}-\frac{16}{3}S_{2,1,1}+\frac{32\left(2+3N+3N^2\right)\zeta_3}{9N(1+N)}\right]\right\}$$

$$+C_F^2\left[-\frac{2S_{2,1}P_{124}}{N^2(1+N)^2(2+N)}+\frac{P_{132}}{8N^3(1+N)^3}\zeta_2+\frac{4P_{134}}{3(N-1)N^2(1+N)^2(2+N)}\zeta_3\right.$$

$$+\frac{P_{147}}{32(N-1)^2N^5(1+N)^5(2+N)^2}+\left(\frac{P_{146}}{8(N-1)N^4(1+N)^4(2+N)}\right.$$

$$+\left(\frac{P_{125}}{2N^2(1+N)^2(2+N)}-8\zeta_2\right)S_2-23S_2^2-\frac{4\left(-2+5N+5N^2\right)S_3}{N(1+N)}-38S_4$$

$$+\frac{4\left(2+3N+3N^2\right)S_{2,1}}{N(1+N)}-8S_{3,1}+16S_{-2,2}+32S_{-3,1}+32S_{2,1,1}$$

$$+\frac{\left(16+12N-17N^2-29N^3\right)}{2N^2(1+N)}\zeta_2-\frac{72}{5}\zeta_2^2-\frac{4\left(32+21N+21N^2\right)}{3N(1+N)}\zeta_3\right)S_1$$

$$+\left(\frac{P_{131}}{4N^3(1+N)^3} + \frac{\left(-34+27N+27N^2\right)S_2}{2N(1+N)} + 12S_3 - 16S_{2,1} - 16S_{-2,1}\right.$$

$$+\frac{3(N-1)(2+N)}{N(1+N)}\zeta_2 - \frac{16}{3}\zeta_3\left)S_1^2 + \left(\frac{P_{117}}{6N^2(1+N)^2} + \frac{38}{3}S_2 + 4\zeta_2\right)S_1^3\right.$$

$$+\frac{\left(10-13N-13N^2\right)}{4N(1+N)}S_1^4 - S_1^5 + \left(\frac{P_{139}}{4N^3(1+N)^3(2+N)} + \frac{28}{3}S_3 + 16S_{2,1}\right.$$

$$+16S_{-2,1} + \frac{2\left(2+3N+3N^2\right)}{N(1+N)}\zeta_2 + 48\zeta_3\left)S_2 + \frac{3\left(10+27N+27N^2\right)}{4N(1+N)}S_2^2\right.$$

$$+\left(\frac{2P_{133}}{3(N-1)N^2(1+N)^2(2+N)} - 4\zeta_2\right)S_3 + \frac{3(2-N)(3+N)S_4}{2N(1+N)} + 36S_5$$

$$+\left(\frac{8P_{142}}{(N-1)^2N(1+N)^3(2+N)^2} + \left(\frac{8P_{123}}{(N-1)N(1+N)^2(2+N)} - 32S_2\right.\right.$$

$$-8\zeta_2\left)S_1 - \frac{8}{N(1+N)}S_1^2 + 16S_1^3 + \frac{8}{N(1+N)}S_2 + 64S_3 + 16S_{-2,1}\right.$$

$$+\frac{4}{N(1+N)}\zeta_2 + 144\zeta_3\left)S_{-2} + \left(-40S_1 + \frac{8}{N(1+N)}\right)S_{-2}^2 + \left(\frac{16}{N(1+N)}S_1\right.\right.$$

$$-\frac{8P_{123}}{(N-1)N(1+N)^2(2+N)} - 24S_1^2 - 8S_2 + 32S_{-2} - 4\zeta_2\left)S_{-3} - \left(24S_1\right.\right.$$

$$+\frac{8}{N(1+N)}\left)S_{-4} + 76S_{-5} - 32S_{2,3} + 16S_{2,-3} - \frac{8}{N(1+N)}S_{3,1} + 76S_{4,1}\right.$$

$$-88S_{-2,3} - 16S_{-4,1} - \frac{2\left(2+15N+15N^2\right)}{N(1+N)}S_{2,1,1} - 48S_{3,1,1} + 16S_{-2,1,-2}$$

$$-20S_{2,1,1,1} + 8S_{-2,1}\zeta_2 + \frac{108}{5}\zeta_2^2\Bigg] \,, \tag{256}$$

with

$$P_{116} = -1007N^4 - 2122N^3 - 1091N^2 - 120N - 108 \,, \tag{257}$$

$$P_{117} = -83N^4 - 208N^3 - 149N^2 - 80N - 50 \,, \tag{258}$$

$$P_{118} = 9N^4 + 27N^3 + 24N^2 + 18N + 11 \,, \tag{259}$$

$$P_{119} = 22N^4 + 41N^3 + 13N^2 - 6N - 5 \,, \tag{260}$$

$$P_{120} = 87N^4 + 138N^3 + 59N^2 - 40N - 36 \,, \tag{261}$$

$$P_{121} = 229N^4 + 507N^3 + 188N^2 + 72 \,, \tag{262}$$

$$P_{122} = 373N^4 + 854N^3 + 457N^2 + 120N + 108 \,, \tag{263}$$

$$P_{123} = 2N^5 + 6N^4 + 5N^3 + 2N^2 + 7N + 2 \,, \tag{264}$$

$$P_{124} = 27N^5 + 66N^4 + 33N^3 - 2N^2 + 24N + 16 \,, \tag{265}$$

$$P_{125} = 75N^5 + 246N^4 + 285N^3 + 202N^2 + 198N + 108 \,, \tag{266}$$

$$P_{126} = 179N^5 + 896N^4 + 1261N^3 + 394N^2 - 204N - 72 \,, \tag{267}$$

$$P_{127} = -4541N^6 - 14649N^5 - 16023N^4 - 7523N^3 - 2256N^2 - 324N + 432 \,, \tag{268}$$

$$P_{128} = -2311N^6 - 6393N^5 - 1423N^4 + 6333N^3 + 2654N^2 - 372N + 216 \ , \tag{269}$$

$$P_{129} = -975N^6 - 2529N^5 - 2341N^4 - 467N^3 + 428N^2 + 108N - 144 \ , \tag{270}$$

$$P_{130} = -133N^6 - 355N^5 - 379N^4 - 281N^3 - 228N^2 - 160N - 40 \ , \tag{271}$$

$$P_{131} = -89N^6 - 351N^5 - 331N^4 + 83N^3 + 308N^2 + 224N + 64 \ , \tag{272}$$

$$P_{132} = 57N^6 + 135N^5 + 103N^4 - 43N^3 - 68N^2 - 32N + 8 \ , \tag{273}$$

$$P_{133} = 104N^6 + 246N^5 + 14N^4 - 208N^3 - 3N^2 + N - 10 \ , \tag{274}$$

$$P_{134} = 180N^6 + 351N^5 - 120N^4 - 447N^3 - 232N^2 - 76N - 88 \ , \tag{275}$$

$$P_{135} = 726N^6 + 1557N^5 - 320N^4 - 1921N^3 - 1108N^2 - 122N - 108 \ , \tag{276}$$

$$P_{136} = 1923N^6 + 4725N^5 + 3893N^4 + 679N^3 + 20N^2 + 864N + 504 \ , \tag{277}$$

$$P_{137} = 4541N^6 + 14703N^5 + 15807N^4 + 7037N^3 + 1932N^2 + 324N - 432 \ , \tag{278}$$

$$P_{138} = 7081N^6 + 24375N^5 + 26871N^4 + 10813N^3 - 60N^2 - 2592N - 1512 \ , \tag{279}$$

$$P_{139} = -345N^7 - 1629N^6 - 2849N^5 - 2367N^4 - 1142N^3 - 456N^2 - 208N - 32 \ , \tag{280}$$

$$P_{140} = 13519N^7 + 67649N^6 + 122235N^5 + 96259N^4 + 31370N^3 + 3540N^2$$
$$+216N - 864 \ , \tag{281}$$

$$P_{141} = -38715N^8 - 131784N^7 - 168338N^6 - 89028N^5 - 11355N^4 + 4564N^3$$
$$+11808N^2 + 16560N + 6480 \ , \tag{282}$$

$$P_{142} = 3N^8 + 15N^7 + 13N^6 - 41N^5 - 94N^4 - 78N^3 - 50N^2 - 36N - 20 \ , \tag{283}$$

$$P_{143} = -86393N^{10} - 469657N^9 - 848380N^8 - 376714N^7 + 589247N^6 + 803483N^5$$
$$+303838N^4 - 78072N^3 - 102168N^2 + 432N + 15552 \ , \tag{284}$$

$$P_{144} = -20163N^{10} - 89115N^9 - 115804N^8 + 11018N^7 + 141341N^6 + 95849N^5$$
$$+15002N^4 + 21272N^3 + 28872N^2 - 144N - 5184 \ , \tag{285}$$

$$P_{145} = 187N^{10} + 707N^9 + 468N^8 - 1258N^7 - 2037N^6 - 289N^5 - 202N^4 - 1848N^3$$
$$-1856N^2 - 176N + 160 \ , \tag{286}$$

$$P_{146} = 619N^{10} + 3003N^9 + 4988N^8 + 2654N^7 - 2525N^6 - 5553N^5 - 3274N^4$$
$$+1480N^3 + 3136N^2 + 1392N + 224 \ , \tag{287}$$

$$P_{147} = -41N^{14} + 461N^{13} + 4075N^{12} + 10649N^{11} + 5585N^{10} - 28621N^9 - 66255N^8$$
$$-64133N^7 - 59540N^6 - 74900N^5 - 47952N^4 + 10560N^3 + 16480N^2 + 384N$$
$$-1664 \ , \tag{288}$$

$$P_{148} = 370659N^{14} + 2352657N^{13} + 4903159N^{12} + 1576653N^{11} - 7789003N^{10}$$
$$-9288369N^9 + 1194501N^8 + 8087991N^7 + 5995676N^6 + 3609228N^5$$
$$+2013520N^4 - 526896N^3 - 789120N^2 + 88128N + 145152. \tag{289}$$

and

$$c_{g_1,q}^{\mathrm{NS},(2,0),\mathrm{L}} = C_F \Bigg\{ T_F N_F \Bigg[\frac{P_{190}}{54N^3(1+N)^3} + \left(-\frac{2P_{159}}{27N^2(1+N)^2} + \frac{16}{3}S_2 + \frac{8}{3}\zeta_2 \right) S_1 - \frac{16}{9}S_1^3$$
$$-\frac{4(-6+19N+19N^2)}{9N(1+N)}S_1^2 + \frac{4(-6+47N+47N^2)}{9N(1+N)}S_2 - \frac{104}{9}S_3 + \frac{16}{3}S_{2,1}$$
$$-\frac{2(2+3N+3N^2)}{3N(1+N)}\zeta_2 \Bigg] + C_A \Bigg[\frac{P_{202}}{216(N-1)N^4(1+N)^4(2+N)} + \Bigg(24S_3 - 16S_{2,1}$$

$$+\frac{P_{191}}{54N^3(1+N)^3} - \frac{4\big(3+11N+11N^2\big)}{3N(1+N)}S_2 - 16S_{-2,1} - \frac{22}{3}\zeta_2 - 48\zeta_3\Bigg)S_1$$

$$+\left(\frac{-66+233N+233N^2}{9N(1+N)} + 4S_2\right)S_1^2 + \frac{\big(66-481N-1166N^2-583N^3\big)}{9N(1+N)^2}S_2$$

$$+\frac{44}{9}S_1^3 - 4S_2^2 + \frac{2\big(-36+143N+143N^2\big)}{9N(1+N)}S_3 - 8S_4 - 12S_{-2}^2 + \left(\frac{8}{N(1+N)}\right.$$

$$\left.-8S_1\right)S_{-3} + \left(\frac{4P_{165}}{(N-1)N(1+N)^2(2+N)} - \frac{8}{N(1+N)}S_1 + 16S_1^2 - 8S_2\right)S_{-2}$$

$$-20S_{-4} - \frac{4\big(-6+11N+11N^2\big)}{3N(1+N)}S_{2,1} - 24S_{3,1} + 8S_{-2,2} + 16S_{-3,1} + 24S_{2,1,1}$$

$$+\frac{11\big(2+3N+3N^2\big)}{6N(1+N)}\zeta_2 + \frac{6(1+3N)(2+3N)}{N(1+N)}\zeta_3\Bigg]\Bigg\} + C_F^2\Bigg[\frac{2P_{156}}{N^2(1+N)^2}S_2$$

$$+\frac{P_{207}}{8(N-1)N^4(1+N)^4(2+N)} + \left(\frac{P_{174}}{2N^3(1+N)^3} - \frac{2\big(-14+9N+9N^2\big)}{N(1+N)}S_2\right.$$

$$\left.-\frac{56}{3}S_3 + 16S_{2,1} + 32S_{-2,1} + \frac{4\big(2+3N+3N^2\big)\zeta_2}{N(1+N)} + 48\zeta_3\right)S_1 + \left(\frac{2P_{154}}{N^2(1+N)^2}\right.$$

$$\left.-28S_2 - 8\zeta_2\right)S_1^2 + \frac{2\big(-14+15N+15N^2\big)}{3N(1+N)}S_1^3 - \frac{2\big(-2+33N+33N^2\big)}{3N(1+N)}S_3$$

$$+\frac{14}{3}S_1^4 + 6S_2^2 + 12S_4 + \left(-\frac{8P_{165}}{(N-1)N(1+N)^2(2+N)} + \frac{16}{N(1+N)}S_1\right.$$

$$\left.-32S_1^2 + 16S_2\right)S_{-2} + 24S_{-2}^2 + \left(-\frac{16}{N(1+N)} + 16S_1\right)S_{-3} + 40S_{-4}$$

$$+\frac{4\big(-2+3N+3N^2\big)}{N(1+N)}S_{2,1} + 40S_{3,1} - 16S_{-2,2} - 32S_{-3,1} - 24S_{2,1,1}$$

$$-\frac{\big(2+3N+3N^2\big)^2}{2N^2(1+N)^2}\zeta_2 - 72\zeta_3\Bigg]\,, \tag{290}$$

$$c_{g_1,q}^{\mathrm{NS},(2,1),\mathrm{L}} = C_F\Bigg\{C_A\Bigg[\frac{2P_{158}}{9N^2(1+N)(2+N)}S_{2,1} - \frac{2P_{189}}{9(N-1)N^2(1+N)^2(2+N)}\zeta_3$$

$$+\frac{P_{173}}{72N^3(1+N)^3}\zeta_2 + \frac{P_{213}}{2592(N-1)^2N^5(1+N)^5(2+N)^2} + \left(8S_2^2 + 32S_4\right.$$

$$\left.-\frac{8}{N(1+N)}S_{2,1} + \frac{P_{170}}{9N^2(1+N)^2(2+N)}S_2 + \frac{P_{201}}{648(N-1)N^4(1+N)^4(2+N)}\right.$$

$$+\frac{4\big(9-22N-22N^2\big)}{9N(1+N)}S_3 - 4S_{3,1} - 8S_{-2,2} - \frac{\big(66-233N-233N^2\big)}{18N(1+N)}\zeta_2 - 16S_{-3,1}$$

$$\left.-20S_{2,1,1} + \frac{514}{9}\zeta_3 + \frac{72}{5}\zeta_2^2\right)S_1 + \left(\frac{\big(9+22N+22N^2\big)}{3N(1+N)}S_2 - 14S_3 + 12S_{2,1}\right.$$

$$
+ \frac{P_{171}}{108 N^3 (1+N)^3} + 8 S_{-2,1} + \frac{11}{3} \zeta_2 + 24 \zeta_3 \Bigg) S_1^2 + \left(\frac{66 - 233 N - 233 N^2}{27 N (1+N)} - \frac{8}{3} S_2 \right)
$$

$$
\times S_1^3 - \frac{11}{9} S_1^4 + \left(\frac{P_{196}}{108 N^3 (1+N)^3 (2+N)} - \frac{22}{3} S_3 - 8 S_{2,1} - 8 S_{-2,1} - \frac{22}{3} \zeta_2 \right.
$$

$$
\left. - 48 \zeta_3 \right) S_2 + \frac{(-2 - 17 N - 17 N^2)}{N (1+N)} S_2^2 + \left(\frac{P_{172}}{27 (N-1) N^2 (1+N)^2 (2+N)} + 2 \zeta_2 \right)
$$

$$
\times S_3 + \frac{(-33 + 59 N + 59 N^2) S_4}{3 N (1+N)} - 22 S_5 + \left(- \frac{4 P_{199}}{(N-1)^2 N (1+N)^3 (2+N)^2} \right.
$$

$$
+ \left(- \frac{4 P_{165}}{(N-1) N (1+N)^2 (2+N)} + 16 S_2 + 4 \zeta_2 \right) S_1 + \frac{4}{N (1+N)} S_1^2 - 8 S_1^3
$$

$$
- \frac{4}{N (1+N)} S_2 - 32 S_3 - 8 S_{-2,1} - \frac{2}{N (1+N)} \zeta_2 - 72 \zeta_3 \Bigg) S_{-2} + \left(- \frac{4}{N (1+N)} \right.
$$

$$
+ 20 S_1 \Bigg) S_{-2}^2 + \left(\frac{4 P_{165}}{(N-1) N (1+N)^2 (2+N)} - \frac{8}{N (1+N)} S_1 + 12 S_1^2 + 4 S_2 \right.
$$

$$
\left. - 16 S_{-2} + 2 \zeta_2 \right) S_{-3} + \left(\frac{4}{N (1+N)} + 12 S_1 \right) S_{-4} - 38 S_{-5} + 20 S_{2,3} - 8 S_{2,-3}
$$

$$
- \frac{2 (-4 + 5 N + 5 N^2)}{N (1+N)} S_{3,1} - 44 S_{4,1} + 44 S_{-2,3} + 8 S_{-4,1} + \frac{4 (3 + 11 N + 11 N^2)}{3 N (1+N)}
$$

$$
\times S_{2,1,1} + 4 S_{2,2,1} + 40 S_{3,1,1} - 8 S_{-2,1,-2} + 4 S_{2,1,1,1} - 4 S_{-2,1} \zeta_2
$$

$$
- \frac{9 (1 + 3 N)(2 + 3 N)}{5 N (1+N)} \zeta_2^2 \Bigg] + T_F N_F \left[\frac{P_{157}}{18 N^2 (1+N)^2} \zeta_2 + \frac{P_{197}}{648 N^4 (1+N)^4} \right.
$$

$$
+ \left(\frac{P_{192}}{162 N^3 (1+N)^3} - \frac{4 (-6 + 19 N + 19 N^2)}{9 N (1+N)} S_2 - \frac{2 (-6 + 19 N + 19 N^2)}{9 N (1+N)} \zeta_2 \right.
$$

$$
\left. + \frac{32}{9} S_3 - \frac{128}{9} \zeta_3 \right) S_1 + \frac{4 (-6 + 19 N + 19 N^2)}{27 N (1+N)} S_1^3 + \left(\frac{P_{159}}{27 N^2 (1+N)^2} - \frac{8}{3} S_2 \right.
$$

$$
\left. - \frac{4}{3} \zeta_2 \right) S_1^2 + \frac{4}{9} S_1^4 + \left(\frac{P_{149}}{27 N^2 (1+N)^2} + \frac{8}{3} \zeta_2 \right) S_2 + \frac{16 (-3 + 41 N + 41 N^2)}{27 N (1+N)} S_3
$$

$$
+ 4 S_2^2 - \frac{28}{3} S_4 - \frac{112}{9} S_{2,1} + 8 S_{3,1} - \frac{16}{3} S_{2,1,1} + \frac{32 (2 + 3 N + 3 N^2)}{9 N (1+N)} \zeta_3 \Bigg] \Bigg\}
$$

$$
+ C_F^2 \Bigg[- \frac{2 P_{168}}{N^2 (1+N)^2 (2+N)} S_{2,1} + \frac{4 P_{188}}{3 (N-1) N^2 (1+N)^2 (2+N)} \zeta_3
$$

$$
+ \frac{P_{186}}{8 N^3 (1+N)^3} \zeta_2 + \frac{P_{210}}{32 (N-1)^2 N^5 (1+N)^5 (2+N)^2}
$$

$$
+ \left(\left(\frac{P_{169}}{2 N^2 (1+N)^2 (2+N)} - 8 \zeta_2 \right) S_2 + \frac{(16 - 20 N - 33 N^2 - 45 N^3)}{2 N^2 (1+N)} \zeta_2 \right.
$$

$$+\frac{P_{208}}{8(N-1)N^4(1+N)^4(2+N)} - 23S_2^2 - \frac{4\left(-2+5N+5N^2\right)}{N(1+N)}S_3 - 38S_4$$

$$+\frac{4\left(2+3N+3N^2\right)}{N(1+N)}S_{2,1} - 8S_{3,1} + 16S_{-2,2} + 32S_{-3,1} + 32S_{2,1,1} - \frac{72}{5}\zeta_2^2$$

$$-\frac{4\left(32+21N+21N^2\right)}{3N(1+N)}\zeta_3\Bigg)S_1 + \left(\frac{P_{150}}{6N^2(1+N)^2} + \frac{38}{3}S_2 + 4\zeta_2\right)S_1^3$$

$$+\Bigg(\frac{P_{175}}{4N^3(1+N)^3} + \frac{\left(-34+27N+27N^2\right)}{2N(1+N)}S_2 + 12S_3 - 16S_{2,1}$$

$$-16S_{-2,1} + \frac{3(N-1)(2+N)\zeta_2}{N(1+N)} - \frac{16}{3}\zeta_3\Bigg)S_1^2$$

$$+\frac{\left(10-13N-13N^2\right)}{4N(1+N)}S_1^4 - S_1^5 + \Bigg(\frac{P_{193}}{4N^3(1+N)^3(2+N)} + \frac{28}{3}S_3 + 16S_{2,1}$$

$$+16S_{-2,1} + \frac{2\left(2+3N+3N^2\right)}{N(1+N)}\zeta_2 + 48\zeta_3\Bigg)S_2 + \frac{3\left(10+27N+27N^2\right)}{4N(1+N)}S_2^2$$

$$+\left(\frac{2P_{187}}{3(N-1)N^2(1+N)^2(2+N)} - 4\zeta_2\right)S_3 + \frac{3(2-N)(3+N)}{2N(1+N)}S_4 + 36S_5$$

$$+\Bigg(\frac{8P_{199}}{(N-1)^2N(1+N)^3(2+N)^2} + \Big(\frac{8P_{165}}{(N-1)N(1+N)^2(2+N)} - 32S_2$$

$$-8\zeta_2\Big)S_1 - \frac{8}{N(1+N)}S_1^2 + 16S_1^3 + \frac{8}{N(1+N)}S_2 + 64S_3 + 16S_{-2,1}$$

$$+\frac{4}{N(1+N)}\zeta_2 + 144\zeta_3\Bigg)S_{-2} + \left(\frac{8}{N(1+N)} - 40S_1\right)S_{-2}^2 + \Bigg(\frac{16}{N(1+N)}S_1$$

$$-\frac{8P_{165}}{(N-1)N(1+N)^2(2+N)} - 24S_1^2 - 8S_2 + 32S_{-2} - 4\zeta_2\Bigg)S_{-3} - \Bigg(24S_1$$

$$+\frac{8}{N(1+N)}\Bigg)S_{-4} + 76S_{-5} - 32S_{2,3} + 16S_{2,-3} - \frac{8}{N(1+N)}S_{3,1} + 76S_{4,1}$$

$$-88S_{-2,3} - 16S_{-4,1} - \frac{2\left(2+15N+15N^2\right)}{N(1+N)}S_{2,1,1} - 48S_{3,1,1} + 16S_{-2,1,-2}$$

$$-20S_{2,1,1,1} + 8S_{-2,1}\zeta_2 + \frac{108}{5}\zeta_2^2\Bigg]\,, \tag{291}$$

$$c_{g_1,q}^{\mathrm{PS},(2,0),L} = C_F T_F N_F\Bigg[\frac{4P_{200}}{(N-1)N^4(1+N)^4(2+N)} + \frac{8(2+N)\left(3-2N^2+3N^3\right)}{N^3(1+N)^3}S_1$$

$$+\frac{8(N-1)(2+N)}{N^2(1+N)^2}S_1^2 - \frac{8(N-1)(2+N)}{N^2(1+N)^2}S_2 - \frac{64}{(N-1)N(1+N)(2+N)}S_{-2}$$

$$-\frac{4(N-1)(2+N)}{N^2(1+N)^2}\zeta_2\Bigg]\,, \tag{292}$$

$$c_{g_1,q}^{\mathrm{PS},(2,1),L} = C_F T_F N_F \left[\frac{64 P_{151}}{3(N-1)N^2(1+N)^2(2+N)}\zeta_3 - \frac{16 P_{152}}{3(N-1)N^2(1+N)^2(2+N)}S_3 \right.$$

$$- \frac{2 P_{209}}{(N-1)^2 N^5(1+N)^5(2+N)^2} + \left(-\frac{4 P_{200}}{(N-1)N^4(1+N)^4(2+N)} \right.$$

$$+ \frac{8(N-1)(2+N)}{N^2(1+N)^2}S_2 + \frac{4(N-1)(2+N)}{N^2(1+N)^2}\zeta_2 \right) S_1 - \frac{4(2+N)\left(3-2N^2+3N^3\right)}{N^3(1+N)^3}S_1^2$$

$$- \frac{64}{(N-1)N(1+N)(2+N)}S_{-3} + \frac{4(2+N)\left(3-2N^2+3N^3\right)}{N^3(1+N)^3}S_2$$

$$+ \frac{2(2+N)\left(3-2N^2+3N^3\right)}{N^3(1+N)^3}\zeta_2 - \frac{8(N-1)(2+N)}{3N^2(1+N)^2}S_1^3$$

$$\left. + \left(\frac{64}{(N-1)N(1+N)(2+N)}S_1 - \frac{32\left(4+13N+6N^2+N^3\right)}{(N-1)^2N(1+N)^2(2+N)^2} \right)S_{-2} \right], \quad (293)$$

$$c_{g_1,g}^{(2,0)} = C_A T_F N_F \left[-\frac{4 P_{203}}{(N-1)N^4(1+N)^4(2+N)^2} + \left(-\frac{4 P_{181}}{N^3(1+N)^3(2+N)} \right. \right.$$

$$+ \frac{28(N-1)}{N(1+N)}S_2 + \frac{8(N-1)\zeta_2}{N(1+N)} \right) S_1 - \frac{8\left(7-4N-3N^2+3N^3\right)}{N^2(1+N)^2}S_1^2$$

$$- \frac{28(N-1)}{3N(1+N)}S_1^3 + \frac{8\left(5-4N+N^2+N^3\right)}{N^2(1+N)^2}S_2 + \frac{8\left(2+5N+5N^2\right)}{3N(1+N)(2+N)}S_3$$

$$+ \left(\frac{16 P_{163}}{(N-1)N(1+N)^2(2+N)^2} + \frac{16\left(2+N+N^2\right)}{N(1+N)(2+N)}S_1 \right)S_{-2} - \frac{16(N-1)}{N(1+N)}S_{2,1}$$

$$- \frac{16\left(-4+N+N^2\right)}{N(1+N)(2+N)}S_{-3} - \frac{64}{N(1+N)(2+N)}S_{-2,1} - \frac{16(N-1)}{N^2(1+N)^2}\zeta_2$$

$$\left. - \frac{24\left(2+N+N^2\right)}{N(1+N)(2+N)}\zeta_3 \right] + C_F T_F N_F \left[\frac{2 P_{205}}{(N-1)N^4(1+N)^4(2+N)^2} \right.$$

$$+ \left(-\frac{4 P_{178}}{N^3(1+N)^3(2+N)} + \frac{20(N-1)}{N(1+N)}S_2 + \frac{8(N-1)}{N(1+N)}\zeta_2 \right) S_1 - \frac{28(N-1)}{3N(1+N)}$$

$$\times S_1^3 - \frac{4(N-1)\left(-5+3N+6N^2\right)}{N^2(1+N)^2}S_1^2 + \frac{4\left(3-12N-5N^2+6N^3\right)}{N^2(1+N)^2}S_2$$

$$- \frac{16\left(2+5N+5N^2\right)}{3N(1+N)(2+N)}S_3 + \left(\frac{16\left(10+N+N^2\right)}{(N-1)(2+N)^2} - \frac{128}{N(1+N)(2+N)}S_1 \right)S_{-2}$$

$$- \frac{64}{N(1+N)(2+N)}S_{-3} + \frac{16(N-1)}{N(1+N)}S_{2,1} + \frac{128}{N(1+N)(2+N)}S_{-2,1}$$

$$\left. - \frac{2(N-1)\left(2+3N+3N^2\right)}{N^2(1+N)^2}\zeta_2 + \frac{48\left(2+N+N^2\right)}{N(1+N)(2+N)}\zeta_3 \right], \quad (294)$$

$$c_{g_1,g}^{(2,1)} = C_F T_F N_F \left[-\frac{8 P_{153}}{N(1+N)^2(2+N)^2}S_{2,1} - \frac{16 P_{180}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3 \right.$$

$$+\frac{4P_{183}}{3(N-1)N^2(1+N)^2(2+N)^2}S_3 + \frac{P_{211}}{(N-1)^2N^5(1+N)^5(2+N)^3}$$

$$+\frac{(N-1)P_{155}}{N^3(1+N)^3}\zeta_2 - \left(\frac{2P_{167}}{N^2(1+N)^2(2+N)^2}S_2 - \frac{2P_{204}}{(N-1)N^4(1+N)^4(2+N)^2}\right.$$

$$-\frac{20\left(6+N+N^2\right)}{N(1+N)(2+N)}S_3 + \frac{16\left(2+N+N^2\right)}{N(1+N)(2+N)}S_{2,1} + \frac{128}{N(1+N)(2+N)}S_{-2,1}$$

$$+\frac{2(N-1)^2(4+3N)}{N^2(1+N)^2}\zeta_2 + \frac{128\left(7+N+N^2\right)}{3N(1+N)(2+N)}\zeta_3\right)S_1 + \left(\frac{P_{184}}{N^3(1+N)^3(2+N)}\right.$$

$$+\frac{\left(34-9N-9N^2\right)}{N(1+N)(2+N)}S_2 - \frac{6(N-1)}{N(1+N)}\zeta_2\right)S_1^2 + \frac{2(N-1)\left(-11+8N+15N^2\right)}{3N^2(1+N)^2}$$

$$\times S_1^3 + \frac{5(N-1)}{2N(1+N)}S_1^4 + \left(\frac{P_{194}}{N^3(1+N)^3(2+N)^2} + \frac{6(N-1)}{N(1+N)}\zeta_2\right)S_2$$

$$+\frac{\left(-78+7N+7N^2\right)}{2N(1+N)(2+N)}S_2^2 - \frac{5\left(6+5N+5N^2\right)}{N(1+N)(2+N)}S_4 - \frac{96}{N(1+N)(2+N)}S_{-2}^2$$

$$+\left(\frac{8(2-N)\left(60+49N+28N^2+7N^3\right)}{(N-1)^2(2+N)^3} + \frac{16P_{161}}{(N-1)N^2(1+N)(2+N)^2}S_1\right.$$

$$+\frac{128}{N(1+N)(2+N)}S_1^2 - \frac{64}{N(1+N)(2+N)}S_2\right)S_{-2} + \left(-\frac{64}{N(1+N)(2+N)}\right.$$

$$\times S_1$$

$$+\frac{32P_{162}}{(N-1)N^2(1+N)(2+N)^2}\right)S_{-3} + \frac{64}{N(1+N)(2+N)}S_{-2,2}$$

$$-\frac{160}{N(1+N)(2+N)}S_{-4} - \frac{16\left(8-N-N^2\right)}{N(1+N)(2+N)}S_{3,1} + \frac{32\left(4+2N-N^2-N^3\right)}{N^2(1+N)(2+N)}$$

$$\times S_{-2,1} + \frac{128}{N(1+N)(2+N)}S_{-3,1} + \frac{8\left(10+N+N^2\right)}{N(1+N)(2+N)}S_{2,1,1}$$

$$-\frac{72\left(2+N+N^2\right)}{5N(1+N)(2+N)}\zeta_2^2\right] + C_AT_FN_F\left[\frac{16P_{164}}{N^2(1+N)^2(2+N)^2}S_{2,1}\right.$$

$$-\frac{2P_{160}}{N^3(1+N)^3}\zeta_2 - \frac{8P_{179}}{3(N-1)N^2(1+N)^2(2+N)^2}S_3$$

$$+\frac{8P_{182}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3 + \frac{2P_{212}}{(N-1)^2N^5(1+N)^5(2+N)^3}$$

$$+\left(-\frac{4P_{166}}{N^2(1+N)^2(2+N)^2}S_2 + \frac{2P_{206}}{(N-1)N^4(1+N)^4(2+N)^2}\right.$$

$$+\frac{4\left(-22+N+N^2\right)}{N(1+N)(2+N)}S_3 + \frac{16}{2+N}S_{2,1} + \frac{64}{N(1+N)(2+N)}S_{-2,1}$$

$$-\frac{8\left(3-2N-N^2+N^3\right)}{N^2(1+N)^2}\zeta_2 - \frac{64\left(-13+2N+2N^2\right)}{3N(1+N)(2+N)}\zeta_3\right)S_1$$

$$+\left(\frac{P_{185}}{N^3(1+N)^3(2+N)} + \frac{\left(30 - 19N - 19N^2\right)}{N(1+N)(2+N)}S_2 - \frac{6(N-1)}{N(1+N)}\zeta_2\right)S_1^2$$

$$+\frac{5(N-1)}{2N(1+N)}S_1^4 + \frac{4\left(15 - 8N - 7N^2 + 7N^3\right)}{3N^2(1+N)^2}S_1^3 + \left(\frac{P_{195}}{N^3(1+N)^3(2+N)^2}\right.$$

$$\left.+\frac{2(N-1)}{N(1+N)}\zeta_2\right)S_2 + \frac{\left(2 + 15N + 15N^2\right)}{2N(1+N)(2+N)}S_2^2 + \frac{\left(-2 + 21N + 21N^2\right)}{N(1+N)(2+N)}S_4$$

$$+\left(-\frac{32P_{177}}{(N-1)N^2(1+N)^2(2+N)^2}S_1 - \frac{8P_{198}}{(N-1)^2N(1+N)^3(2+N)^3}\right.$$

$$-\frac{8\left(6 + N + N^2\right)}{N(1+N)(2+N)}S_1^2 + \frac{8\left(2 + N + N^2\right)}{N(1+N)(2+N)}S_2 + \left.\frac{4(N-1)}{N(1+N)}\zeta_2\right)S_{-2}$$

$$+\frac{8\left(4 + N + N^2\right)}{N(1+N)(2+N)}S_{-2}^2 + \left(\frac{8P_{176}}{(N-1)N^2(1+N)^2(2+N)^2} + \frac{16}{2+N}S_1\right)S_{-3}$$

$$-\frac{8(N-3)(4+N)}{N(1+N)(2+N)}S_{-4} - \frac{16\left(4 + 2N - N^2 - N^3\right)}{N^2(1+N)(2+N)}S_{-2,1} - \frac{32}{N(1+N)(2+N)}$$

$$\times S_{-2,2} - \frac{16\left(-5 + N + N^2\right)}{N(1+N)(2+N)}S_{3,1} - \frac{64}{N(1+N)(2+N)}S_{-3,1}$$

$$\left.-\frac{8\left(4 + N + N^2\right)}{N(1+N)(2+N)}S_{2,1,1} + \frac{36\left(2 + N + N^2\right)}{5N(1+N)(2+N)}\zeta_2^2\right] . \tag{295}$$

with

$$P_{149} = -1079N^4 - 2266N^3 - 1019N^2 + 24N - 108 , \tag{296}$$

$$P_{150} = -75N^4 - 192N^3 - 29N^2 + 32N - 50 , \tag{297}$$

$$P_{151} = N^4 + 2N^3 - 12N^2 - 13N + 4 , \tag{298}$$

$$P_{152} = N^4 + 2N^3 - 9N^2 - 10N + 4 , \tag{299}$$

$$P_{153} = N^4 + 4N^3 - 13N^2 - 40N - 20 , \tag{300}$$

$$P_{154} = 5N^4 + 19N^3 - 4N^2 - 6N + 11 , \tag{301}$$

$$P_{155} = 13N^4 + 20N^3 + 14N^2 - N - 6 , \tag{302}$$

$$P_{156} = 26N^4 + 49N^3 + 33N^2 + 10N - 5 , \tag{303}$$

$$P_{157} = 111N^4 + 186N^3 + 131N^2 + 8N - 36 , \tag{304}$$

$$P_{158} = 229N^4 + 507N^3 + 116N^2 - 144N + 72 , \tag{305}$$

$$P_{159} = 373N^4 + 854N^3 + 313N^2 - 24N + 108 , \tag{306}$$

$$P_{160} = N^5 + N^4 - 12N^3 + 11N^2 + N - 10 , \tag{307}$$

$$P_{161} = N^5 + 2N^4 - 17N^3 - 26N^2 + 16 , \tag{308}$$

$$P_{162} = N^5 + 2N^4 + 4N^3 + N^2 + 4 , \tag{309}$$

$$P_{163} = N^5 + 3N^4 - 3N^3 - 9N^2 - 8N - 8 , \tag{310}$$

$$P_{164} = N^5 + 4N^4 + N^3 - 4N^2 + 4N + 4 , \tag{311}$$

$$P_{165} = 2N^5 + 6N^4 + N^3 - 6N^2 + 11N + 10 , \tag{312}$$

$$P_{166} = 7N^5 + 25N^4 + 2N^3 - 39N^2 + 28N + 60 , \tag{313}$$

$$P_{167} = 13N^5 + 43N^4 + 27N^3 - 39N^2 - 16N + 36 , \tag{314}$$

$$P_{168} = 27N^5 + 66N^4 + 25N^3 - 26N^2 + 8N + 16 \ , \tag{315}$$

$$P_{169} = 67N^5 + 214N^4 + 133N^3 - 150N^2 - 26N + 108 \ , \tag{316}$$

$$P_{170} = 179N^5 + 896N^4 + 1333N^3 + 610N^2 - 60N - 72 \ , \tag{317}$$

$$P_{171} = -4541N^6 - 14649N^5 - 14439N^4 - 4355N^3 - 672N^2 - 324N + 432 \ , \tag{318}$$

$$P_{172} = -2311N^6 - 6393N^5 - 991N^4 + 7197N^3 + 2222N^2 - 1236N + 216 \ , \tag{319}$$

$$P_{173} = -1239N^6 - 3321N^5 - 3661N^4 - 1787N^3 - 100N^2 + 108N - 144 \ , \tag{320}$$

$$P_{174} = -237N^6 - 635N^5 - 675N^4 - 321N^3 + 12N^2 - 40 \ , \tag{321}$$

$$P_{175} = -37N^6 - 243N^5 - 135N^4 + 167N^3 + 44N^2 + 16N + 64 \ , \tag{322}$$

$$P_{176} = N^6 + 3N^5 - 5N^4 - 7N^3 - 8N^2 - 24N - 8 \ , \tag{323}$$

$$P_{177} = N^6 + 3N^5 - 2N^4 - 10N^3 - 8N^2 + 4 \ , \tag{324}$$

$$P_{178} = 6N^6 + 19N^5 + 5N^4 - 19N^3 + 9N^2 - 4N - 20 \ , \tag{325}$$

$$P_{179} = 8N^6 + 19N^5 - 28N^4 - 56N^3 - 31N^2 - 32N + 48 \ , \tag{326}$$

$$P_{180} = 12N^6 + 27N^5 - 97N^4 - 281N^3 - 177N^2 + 64N + 20 \ , \tag{327}$$

$$P_{181} = 15N^6 + 36N^5 - 19N^4 - 42N^3 + 30N^2 - 40N - 56 \ , \tag{328}$$

$$P_{182} = 18N^6 + 36N^5 - 31N^4 - 134N^3 - 357N^2 - 200N + 236 \ , \tag{329}$$

$$P_{183} = 24N^6 + 59N^5 - 211N^4 - 487N^3 - 29N^2 + 280N + 76 \ , \tag{330}$$

$$P_{184} = 31N^6 + 90N^5 + 23N^4 - 84N^3 - 8N^2 - 16N - 48 \ , \tag{331}$$

$$P_{185} = 43N^6 + 102N^5 - 37N^4 - 100N^3 + 44N^2 - 104N - 128 \ , \tag{332}$$

$$P_{186} = 105N^6 + 279N^5 + 375N^4 + 261N^3 + 124N^2 + 32N + 8 \ , \tag{333}$$

$$P_{187} = 108N^6 + 258N^5 + 14N^4 - 228N^3 - 7N^2 + 9N - 10 \ , \tag{334}$$

$$P_{188} = 198N^6 + 405N^5 - 102N^4 - 501N^3 - 268N^2 - 76N - 88 \ , \tag{335}$$

$$P_{189} = 834N^6 + 1881N^5 - 104N^4 - 2029N^3 - 1432N^2 - 338N - 108 \ , \tag{336}$$

$$P_{190} = 2667N^6 + 6669N^5 + 6605N^4 + 2095N^3 - 940N^2 + 504 \ , \tag{337}$$

$$P_{191} = 4541N^6 + 14703N^5 + 14223N^4 + 3869N^3 + 348N^2 + 324N - 432 \ , \tag{338}$$

$$P_{192} = 7081N^6 + 25239N^5 + 25431N^4 + 8797N^3 + 2820N^2 - 1512 \ , \tag{339}$$

$$P_{193} = -397N^7 - 1873N^6 - 3293N^5 - 2811N^4 - 1302N^3 - 200N^2 + 16N - 32 \ , \tag{340}$$

$$P_{194} = -37N^7 - 160N^6 - 85N^5 + 426N^4 + 612N^3 + 184N^2 + 32N + 64 \ , \tag{341}$$

$$P_{195} = -13N^7 - 80N^6 - 121N^5 + 30N^4 + 28N^3 - 96N^2 + 208N + 192 \ , \tag{342}$$

$$P_{196} = 14311N^7 + 71609N^6 + 126915N^5 + 92875N^4 + 23306N^3 + 372N^2 \\ + 216N - 864 \ , \tag{343}$$

$$P_{197} = -56163N^8 - 192648N^7 - 268610N^6 - 170340N^5 - 27651N^4 + 4084N^3 \\ - 8928N^2 + 4464N + 6480 \ , \tag{344}$$

$$P_{198} = 2N^8 + 11N^7 + 10N^6 - 26N^5 - 2N^4 + 151N^3 + 262N^2 + 160N + 8 \ , \tag{345}$$

$$P_{199} = 3N^8 + 15N^7 + 11N^6 - 45N^5 - 74N^4 - 30N^3 - 52N^2 - 80N - 36 \ , \tag{346}$$

$$P_{200} = 10N^8 + 34N^7 + 4N^6 - 64N^5 - 51N^4 - 48N^3 - 53N^2 + 12N + 28 \ , \tag{347}$$

$$P_{201} = -86393N^{10} - 479161N^9 - 842908N^8 - 273610N^7 + 665567N^6 \\ + 681947N^5 + 206494N^4 - 29400N^3 - 86616N^2 - 20304N + 15552 \ , \tag{348}$$

$$P_{202} = -30651N^{10} - 138387N^9 - 203260N^8 - 49510N^7 + 185861N^6 + 211505N^5 \\ + 73610N^4 + 8504N^3 + 23688N^2 + 6768N - 5184 \ , \tag{349}$$

$$P_{203} = 4N^{10} + 18N^9 - 18N^8 - 130N^7 - 3N^6 + 253N^5 + 131N^4 + 115N^3 + 146N^2$$

$$-44N - 88 \ , \tag{350}$$

$$P_{204} = 16N^{10} + 96N^9 + 137N^8 - 155N^7 - 584N^6 - 618N^5 - 431N^4 - 67N^3$$
$$+178N^2 - 20N - 88 \ , \tag{351}$$

$$P_{205} = 34N^{10} + 144N^9 + 134N^8 - 156N^7 - 147N^6 + 449N^5 + 825N^4 + 355N^3$$
$$-130N^2 - 28N + 56 \ , \tag{352}$$

$$P_{206} = 50N^{10} + 214N^9 + 99N^8 - 533N^7 - 355N^6 + 605N^5 + 562N^4 + 338N^3$$
$$+244N^2 - 216N - 240 \ , \tag{353}$$

$$P_{207} = 763N^{10} + 3395N^9 + 5268N^8 + 2262N^7 - 3701N^6 - 5729N^5 - 4554N^4$$
$$-2872N^3 - 1216N^2 + 80N + 160 \ , \tag{354}$$

$$P_{208} = 1203N^{10} + 5811N^9 + 9468N^8 + 4366N^7 - 5429N^6 - 8825N^5 - 3450N^4$$
$$+1256N^3 + 1152N^2 + 368N + 224 \ , \tag{355}$$

$$P_{209} = 32N^{12} + 172N^{11} + 187N^{10} - 377N^9 - 635N^8 + 560N^7 + 1634N^6$$
$$+1246N^5 + 488N^4 - 145N^3 - 294N^2 + 84N + 120 \ , \tag{356}$$

$$P_{210} = -4521N^{14} - 28595N^{13} - 64213N^{12} - 42855N^{11} + 63121N^{10} + 130099N^9$$
$$+49137N^8 - 79237N^7 - 137428N^6 - 123924N^5 - 65104N^4 - 448N^3$$
$$+11360N^2 - 640N - 1664 \ , \tag{357}$$

$$P_{211} = -84N^{14} - 520N^{13} - 1019N^{12} - 186N^{11} + 1538N^{10} + 1433N^9 + 2372N^8$$
$$+9376N^7 + 14990N^6 + 9125N^5 + 163N^4 - 868N^3 + 720N^2 + 64N - 240 \ , \tag{358}$$

$$P_{212} = 12N^{14} + 76N^{13} + 33N^{12} - 576N^{11} - 806N^{10} + 1161N^9 + 1790N^8$$
$$-2494N^7 - 5322N^6 - 3207N^5 - 715N^4 + 664N^3 + 840N^2 - 304N - 368 \ , \tag{359}$$

$$P_{213} = 648699N^{14} + 4173081N^{13} + 9225583N^{12} + 4949301N^{11} - 11882659N^{10}$$
$$-20345337N^9 - 5717331N^8 + 11889903N^7 + 12065660N^6 + 4779564N^5$$
$$+2272528N^4 + 613584N^3 - 713088N^2 - 160704N + 145152 \ . \tag{360}$$

The evanescent poles at $N = 2$ are tractable, and the rightmost singularities are at $N = 1$ for the singlet contributions and at $N = 0$ for the non–singlet contributions, as generally expected, cf. [84].

4.3 The Two–Loop Wilson Coefficients

The two–loop Wilson coefficients for $\mu^2 = Q^2$ are given by

$$C_{F_2,q}^{\mathrm{NS},(2)} = C_F^2 \Bigg[\frac{S_2 P_{218}}{2N^2(1+N)^2} + \frac{P_{233}}{8(N-2)N^4(1+N)^4(3+N)} + \Bigg(\frac{P_{225}}{2N^3(1+N)^3}$$

$$- \frac{2(-2+3N)(5+3N)S_2}{N(1+N)} - 24S_3 + 16S_{2,1} + 32S_{-2,1} + 48\zeta_3 \Bigg) S_1 + \Bigg(\frac{P_{214}}{2N^2(1+N)^2}$$

$$- 20S_2 \Bigg) S_1^2 + \frac{2(-2+3N+3N^2)S_1^3}{N(1+N)} + 2S_1^4 + 6S_2^2 - \frac{2(-10+25N+9N^2)S_3}{N(1+N)}$$

$$+ 12S_4 + \Bigg(-\frac{8P_{61}}{(N-2)N^2(1+N)^2(3+N)} - \frac{16(-3+4N)S_1}{N(1+N)} - 32S_1^2 + 16S_2 \Bigg) S_{-2}$$

$$+ 24S_{-2}^2 + \Bigg(-\frac{32}{1+N} + 16S_1 \Bigg) S_{-3} + 40S_{-4} + \frac{4(-2+3N+3N^2)S_{2,1}}{N(1+N)} + 40S_{3,1}$$

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$$+\frac{32(-1+2N)S_{-2,1}}{N(1+N)}-16S_{-2,2}-32S_{-3,1}-24S_{2,1,1}-\frac{24\big(2-N+3N^2\big)\zeta_3}{N(1+N)}\Bigg]$$

$$+C_A C_F\Bigg[\frac{P_{229}}{216(N-2)N^3(1+N)^3(3+N)}+\Bigg(-\frac{2\big(6+11N+11N^2\big)S_2}{3N(1+N)}$$

$$+\frac{P_{224}}{54N^2(1+N)^3}+24S_3-16S_{2,1}-16S_{-2,1}-48\zeta_3\Bigg)S_1+\Bigg(\frac{-66+367N+367N^2}{18N(1+N)}$$

$$+4S_2\Bigg)S_1^2+\frac{\big(66-929N-2134N^2-1067N^3\big)S_2}{18N(1+N)^2}+\frac{2\big(-72+193N+121N^2\big)S_3}{9N(1+N)}$$

$$-4S_2^2+\frac{22}{9}S_1^3-8S_4+\Bigg(\frac{4P_{61}}{(N-2)N^2(1+N)^2(3+N)}+\frac{8(-3+4N)S_1}{N(1+N)}$$

$$+16S_1^2-8S_2\Bigg)S_{-2}+\Bigg(\frac{16}{1+N}-8S_1\Bigg)S_{-3}-\frac{4\big(-6+11N+11N^2\big)S_{2,1}}{3N(1+N)}-24S_{3,1}$$

$$-20S_{-4}-12S_{-2}^2-\frac{16(-1+2N)S_{-2,1}}{N(1+N)}+8S_{-2,2}+16S_{-3,1}+24S_{2,1,1}$$

$$+\frac{6\big(6+N+9N^2\big)\zeta_3}{N(1+N)}\Bigg]+C_F T_F N_F\Bigg[\frac{P_{235}}{54N^3(1+N)^3}+\Bigg(\frac{8}{3}S_2-\frac{2P_{236}}{27N^2(1+N)^2}\Bigg)S_1$$

$$+\frac{2\big(6-29N-29N^2\big)}{9N(1+N)}S_1^2-\frac{8}{9}S_1^3-\frac{2\big(6-85N-85N^2\big)}{9N(1+N)}S_2$$

$$-\frac{88}{9}S_3+\frac{16}{3}S_{2,1}\Bigg]\,,\tag{361}$$

$$C_{F_2,q}^{\mathrm{PS},(2)}\;=\;C_F T_F N_F\Bigg[\frac{8S_1P_{228}}{(N-1)N^3(1+N)^3(2+N)^2}+\frac{4P_{232}}{(N-1)N^4(1+N)^4(2+N)^3}$$

$$+\frac{4\big(2+N+N^2\big)^2S_1^2}{(N-1)N^2(1+N)^2(2+N)}-\frac{4\big(2+N+N^2\big)^2S_2}{(N-1)N^2(1+N)^2(2+N)}$$

$$+\frac{64S_{-2}}{(N-1)N(1+N)(2+N)}\Bigg]\,,\tag{362}$$

$$C_{F_2,g}^{(2)}\;=\;C_A T_F N_F\Bigg[\frac{8S_2P_{220}}{(N-1)N(1+N)^2(2+N)^2}-\frac{8S_1^2P_{221}}{(N-1)N(1+N)^2(2+N)^2}$$

$$-\frac{4P_{234}}{(N-1)N^4(1+N)^4(2+N)^4}+\Bigg(-\frac{4P_{230}}{(N-1)N^3(1+N)^3(2+N)^3}$$

$$+\frac{20\big(2+N+N^2\big)S_2}{N(1+N)(2+N)}\Bigg)S_1-\frac{4\big(2+N+N^2\big)S_1^3}{3N(1+N)(2+N)}+\frac{8\big(-2+5N+5N^2\big)S_3}{3N(1+N)(2+N)}$$

$$+\Bigg(\frac{16P_{38}}{(N-1)N(1+N)^2(2+N)^2}+\frac{16(N-1)S_1}{N(1+N)}\Bigg)S_{-2}-\frac{16\big(4+N+N^2\big)S_{-3}}{N(1+N)(2+N)}$$

$$-\frac{16\big(2+N+N^2\big)S_{2,1}}{N(1+N)(2+N)}+\frac{64S_{-2,1}}{N(1+N)(2+N)}-\frac{24(N-1)\zeta_3}{N(1+N)}\Bigg]$$

$$
+ C_F T_F N_F \left[\frac{2 S_2 P_{216}}{N^2 (1+N)^2 (2+N)} - \frac{2 P_{231}}{(N-2) N^4 (1+N)^4 (2+N)(3+N)} \right.
$$

$$
- \frac{2 S_1^2 P_{217}}{N^2 (1+N)^2 (2+N)} + \left(\frac{4 P_{226}}{N^3 (1+N)^3 (2+N)} + \frac{12 (2 + N + N^2) S_2}{N(1+N)(2+N)} \right) S_1
$$

$$
- \frac{20 (2 + N + N^2) S_1^3}{3 N (1+N)(2+N)} - \frac{64 (-1 + N + N^2) S_3}{3 N (1+N)(2+N)} + \left(\frac{128 S_1}{N(1+N)(2+N)} \right.
$$

$$
\left. + \frac{16 P_{59}}{(N-2) N^2 (1+N)^2 (2+N)(3+N)} \right) S_{-2} + \frac{64 S_{-3}}{N(1+N)(2+N)}
$$

$$
\left. + \frac{16 (2 + N + N^2) S_{2,1}}{N(1+N)(2+N)} - \frac{128 S_{-2,1}}{N(1+N)(2+N)} + \frac{48 (N-1) \zeta_3}{N(1+N)} \right] , \tag{363}
$$

$$
C_{F_L,q}^{\mathrm{NS},(2)} = C_F \left\{ T_F N_F \left[-\frac{8 (-6 + 7N + 19 N^2)}{9 N (1+N)^2} - \frac{16 S_1}{3(1+N)} \right] + C_A \left[\frac{92 S_1}{3(1+N)} \right. \right.
$$

$$
+ \frac{2 P_{219}}{9 (N-2) N (1+N)^2 (3+N)} + \frac{16 S_3}{1+N} + \left(-\frac{32 P_{15}}{(N-2) N (1+N)^2 (3+N)} \right.
$$

$$
\left. \left. \left. + \frac{32 S_1}{1+N} \right) S_{-2} + \frac{16 S_{-3}}{1+N} - \frac{32 S_{-2,1}}{1+N} - \frac{48 \zeta_3}{1+N} \right] \right\} + C_F^2 \left\{ -\frac{4 (2 + 13 N + 9 N^2) S_1}{N (1+N)^2} \right.
$$

$$
- \frac{2 P_{227}}{(N-2) N^2 (1+N)^3 (3+N)} + \frac{8 S_1^2}{1+N} - \frac{8 S_2}{1+N} - \frac{32 S_3}{1+N} + \left(-\frac{64 S_1}{1+N} \right.
$$

$$
\left. \left. + \frac{64 P_{15}}{(N-2) N (1+N)^2 (3+N)} \right) S_{-2} - \frac{32 S_{-3}}{1+N} + \frac{64 S_{-2,1}}{1+N} + \frac{96 \zeta_3}{1+N} \right\} , \tag{364}
$$

$$
C_{F_L,q}^{\mathrm{PS}(2)} = C_F T_F N_F \left[-\frac{32 P_{222}}{(N-1) N^2 (1+N)^3 (2+N)^2} - \frac{32 (2 + N + N^2) S_1}{(N-1) N (1+N)^2 (2+N)} \right] , \tag{365}
$$

$$
C_{F_L,g}^{(2)} = C_A T_F N_F \left[-\frac{32 P_{223}}{(N-1) N^2 (1+N)^3 (2+N)^3} + \frac{64 (-1 - N - 2 N^2 + 2 N^3) S_1}{(N-1) N (1+N)^2 (2+N)} \right.
$$

$$
\left. + \frac{32 S_1^2}{(1+N)(2+N)} - \frac{32 S_2}{(1+N)(2+N)} - \frac{64 S_{-2}}{(1+N)(2+N)} \right]
$$

$$
+ C_F T_F N_F \left[\frac{16 (N-1) P_{215}}{(N-2) N^2 (1+N)^3 (2+N)(3+N)} - \frac{16 (2 + 3N + 3 N^2) S_1}{N (1+N)^2 (2+N)} \right.
$$

$$
\left. + \frac{64 (N-1) S_{-2}}{(N-2)(1+N)(3+N)} \right] , \tag{366}
$$

with

$$
P_{214} = -27 N^4 - 26 N^3 + 9 N^2 + 40 N + 24 , \tag{367}
$$

$$
P_{215} = 2 N^4 + 19 N^3 + 39 N^2 + 40 N + 12 , \tag{368}
$$

$$
P_{216} = 9 N^4 + 8 N^3 + 9 N^2 + 6 N - 8 , \tag{369}
$$

$$
P_{217} = 9 N^4 + 12 N^3 + N^2 - 14 N - 16 , \tag{370}
$$

$$
P_{218} = 95 N^4 + 162 N^3 + 35 N^2 - 32 N - 16 , \tag{371}
$$

$$P_{219} = 215N^4 + 298N^3 - 1597N^2 - 888N + 396 \ , \tag{372}$$

$$P_{220} = N^5 - 10N^3 - 9N^2 - 4N - 2 \ , \tag{373}$$

$$P_{221} = N^5 - 2N^4 - 6N^3 - 3N^2 - 12N - 2 \ , \tag{374}$$

$$P_{222} = N^5 + 2N^4 + 2N^3 - 5N^2 - 12N - 4 \ , \tag{375}$$

$$P_{223} = 5N^5 + 8N^4 - 3N^3 - 22N^2 - 28N - 8 \ , \tag{376}$$

$$P_{224} = 3155N^5 + 11607N^4 + 12279N^3 + 3329N^2 + 510N + 792 \ , \tag{377}$$

$$P_{225} = -51N^6 - 203N^5 - 207N^4 - 33N^3 - 106N^2 - 160N - 48 \ , \tag{378}$$

$$P_{226} = N^6 - 7N^5 - 3N^4 - 5N^3 - 30N^2 - 40N - 16 \ , \tag{379}$$

$$P_{227} = 17N^6 + 39N^5 - 157N^4 - 299N^3 - 64N^2 + 104N + 24 \ , \tag{380}$$

$$P_{228} = N^7 - 15N^5 - 58N^4 - 92N^3 - 76N^2 - 48N - 16 \ , \tag{381}$$

$$P_{229} = -16395N^8 - 47520N^7 + 51416N^6 + 162042N^5 + 99843N^4 - 7930N^3$$
$$-21432N^2 + 25848N + 23760 \ , \tag{382}$$

$$P_{230} = 7N^9 + 5N^8 - 43N^7 + 25N^6 + 296N^5 + 498N^4 + 524N^3 + 336N^2 + 144N$$
$$+32 \ , \tag{383}$$

$$P_{231} = 2N^{10} + 18N^9 + 98N^8 + 98N^7 - 425N^6 - 1071N^5 - 477N^4 + 651N^3$$
$$+886N^2 + 484N + 120 \ , \tag{384}$$

$$P_{232} = 3N^{10} + 14N^9 + 33N^8 + 79N^7 + 297N^6 + 849N^5 + 1373N^4 + 1312N^3$$
$$+840N^2 + 368N + 80 \ , \tag{385}$$

$$P_{233} = 331N^{10} + 1179N^9 - 848N^8 - 4754N^7 - 2157N^6 + 4247N^5 + 3474N^4$$
$$-2528N^3 - 4976N^2 - 2704N - 480 \ , \tag{386}$$

$$P_{234} = 4N^{12} + 34N^{11} + 100N^{10} + 116N^9 - 81N^8 - 637N^7 - 1677N^6$$
$$-3093N^5 - 3998N^4 - 3472N^3 - 2064N^2 - 816N - 160 \ , \tag{387}$$

$$P_{235} = 360 + 648N + 140N^2 + 31N^3 + 1397N^4 + 2517N^5 + 1371N^6 \ , \tag{388}$$

$$P_{236} = 72 + 66N + 331N^2 + 620N^3 + 247N^4. \tag{389}$$

$$
\begin{aligned}
C_{F_3,q}^{(2),\mathrm{NS}} = \ & C_F \Bigg\{ T_F N_F \Bigg[\frac{P_{241}}{54N^3(1+N)^3} + \left(-\frac{2P_{239}}{27N^2(1+N)^2} + \frac{8S_2}{3} \right) S_1 - \frac{8}{9} S_1^3 \\
& - \frac{2\left(-6 + 29N + 29N^2\right)S_1^2}{9N(1+N)} + \frac{2\left(-6 + 85N + 85N^2\right)S_2}{9N(1+N)} - \frac{88}{9} S_3 + \frac{16}{3} S_{2,1} \Bigg] \\
& + C_A \Bigg[\frac{P_{243}}{216(N-1)N^4(1+N)^4(2+N)} + \left(\frac{P_{242}}{54N^3(1+N)^3} + 24S_3 - 16S_{2,1} \right. \\
& \left. - 16S_{-2,1} - \frac{2\left(6 + 11N + 11N^2\right)S_2}{3N(1+N)} - 48\zeta_3 \right) S_1 + \left(\frac{-66 + 367N + 367N^2}{18N(1+N)} + 4S_2 \right) \\
& \times S_1^2 + \frac{22}{9} S_1^3 + \frac{\left(66 - 929N - 2134N^2 - 1067N^3\right)S_2}{18N(1+N)^2} - 4S_2^2 - 8S_4 - 12S_{-2}^2 \\
& + \frac{2\left(-36 + 121N + 121N^2\right)S_3}{9N(1+N)} + \left(\frac{4P_{123}}{(N-1)N(1+N)^2(2+N)} - \frac{8S_1}{N(1+N)} \right. \\
& \left. + 16S_1^2 - 8S_2 \right) S_{-2} + \left(\frac{8}{N(1+N)} - 8S_1 \right) S_{-3} - 20S_{-4} + \frac{4\left(6 - 11N - 11N^2\right)S_{2,1}}{3N(1+N)}
\end{aligned}
$$

$$-24S_{3,1} + 8S_{-2,2} + 16S_{-3,1} + 24S_{2,1,1} + \frac{6(1+3N)(2+3N)\zeta_3}{N(1+N)}\Bigg]\Bigg\}$$

$$+ C_F^2\Bigg[\frac{S_2 P_{238}}{2N^2(1+N)^2} + \frac{P_{244}}{8(N-1)N^4(1+N)^4(2+N)} + \Bigg(\frac{P_{240}}{2N^3(1+N)^3}$$

$$-\frac{2(-2+3N)(5+3N)S_2}{N(1+N)} - 24S_3 + 16S_{2,1} + 32S_{-2,1} + 48\zeta_3\Bigg)S_1 + \Bigg(-20S_2$$

$$+\frac{P_{237}}{2N^2(1+N)^2}\Bigg)S_1^2 + \frac{2\big(-2+3N+3N^2\big)S_1^3}{N(1+N)} + 2S_1^4 + 6S_2^2 + 12S_4$$

$$-\frac{2\big(-2+9N+9N^2\big)S_3}{N(1+N)} + \Bigg(-\frac{8P_{123}}{(N-1)N(1+N)^2(2+N)} + \frac{16S_1}{N(1+N)}$$

$$-32S_1^2 + 16S_2\Bigg)S_{-2} + 24S_{-2}^2 + \Bigg(-\frac{16}{N(1+N)} + 16S_1\Bigg)S_{-3} + 40S_{-4}$$

$$+\frac{4\big(-2+3N+3N^2\big)S_{2,1}}{N(1+N)} + 40S_{3,1} - 16S_{-2,2} - 32S_{-3,1} - 24S_{2,1,1} - 72\zeta_3\Bigg] \,, \quad (390)$$

with the polynomials

$$\begin{aligned}
P_{237} &= -27N^4 - 42N^3 - 15N^2 + 32N + 24 \,, & (391)\\
P_{238} &= 95N^4 + 178N^3 + 59N^2 - 24N - 16 \,, & (392)\\
P_{239} &= 247N^4 + 548N^3 + 223N^2 + 30N + 72 \,, & (393)\\
P_{240} &= -51N^6 - 131N^5 - 115N^4 - 49N^3 - 54N^2 - 72N - 16 \,, & (394)\\
P_{241} &= 1371N^6 + 3429N^5 + 2957N^4 + 271N^3 - 556N^2 + 360N + 360 \,, & (395)\\
P_{242} &= 3155N^6 + 9951N^5 + 9867N^4 + 3473N^3 + 546N^2 - 72N - 432 \,, & (396)\\
P_{243} &= -16395N^{10} - 74235N^9 - 101500N^8 + 5090N^7 + 133973N^6 + 105113N^5 \\
&\quad + 16970N^4 + 6224N^3 + 16200N^2 - 3312N - 5184 \,, & (397)\\
P_{244} &= 331N^{10} + 1451N^9 + 1492N^8 - 1826N^7 - 4341N^6 - 1673N^5 + 246N^4 \\
&\quad - 928N^3 - 1232N^2 + 112N + 224 & (398)
\end{aligned}$$

$$\begin{aligned}
C_{g_1,q}^{(2),\text{NS,L}} &= C_F\Bigg\{T_F N_F\Bigg[\frac{P_{251}}{54N^3(1+N)^3} + \Bigg(-\frac{2P_{239}}{27N^2(1+N)^2} + \frac{8}{3}S_2\Bigg)S_1 - \frac{8}{9}S_1^3 \\
&\quad -\frac{2\big(-6+29N+29N^2\big)}{9N(1+N)}S_1^2 + \frac{2\big(-6+85N+85N^2\big)}{9N(1+N)}S_2 - \frac{88}{9}S_3 + \frac{16}{3}S_{2,1}\Bigg] \\
&\quad + C_A\Bigg[\frac{P_{253}}{216(N-1)N^4(1+N)^4(2+N)} + \Bigg(-\frac{2\big(6+11N+11N^2\big)}{3N(1+N)}S_2 \\
&\quad +\frac{P_{242}}{54N^3(1+N)^3} + 24S_3 - 16S_{2,1} - 16S_{-2,1} - 48\zeta_3\Bigg)S_1 + \frac{22}{9}S_1^3 + \Bigg(4S_2 \\
&\quad +\frac{-66+367N+367N^2}{18N(1+N)}\Bigg)S_1^2 + \frac{\big(66-929N-2134N^2-1067N^3\big)}{18N(1+N)^2}S_2 - 4S_2^2
\end{aligned}$$

$$+\frac{2\big(-36+121N+121N^2\big)}{9N(1+N)}S_3 - 8S_4 + \Bigg(\frac{4P_{165}}{(N-1)N(1+N)^2(2+N)}$$

$$-\frac{8}{N(1+N)}S_1 + 16S_1^2 - 8S_2\Bigg)S_{-2} - 12S_{-2}^2 + \Bigg(\frac{8}{N(1+N)} - 8S_1\Bigg)S_{-3}$$

$$-20S_{-4} - \frac{4\big(-6+11N+11N^2\big)}{3N(1+N)}S_{2,1} - 24S_{3,1} + 8S_{-2,2} + 16S_{-3,1} + 24S_{2,1,1}$$

$$+\frac{6(1+3N)(2+3N)}{N(1+N)}\zeta_3\Bigg]\Bigg\} + C_F^2\Bigg[\frac{P_{256}}{8(N-1)N^4(1+N)^4(2+N)}$$

$$+\frac{P_{246}}{2N^2(1+N)^2}S_2 + \Bigg(\frac{P_{248}}{2N^3(1+N)^3} - \frac{2(-2+3N)(5+3N)S_2}{N(1+N)} - 24S_3 + 16S_{2,1}$$

$$+32S_{-2,1} + 48\zeta_3\Bigg)S_1 + \Bigg(\frac{P_{245}}{2N^2(1+N)^2} - 20S_2\Bigg)S_1^2 + \frac{2\big(-2+3N+3N^2\big)}{N(1+N)}S_1^3$$

$$+2S_1^4 + 6S_2^2 - \frac{2\big(-2+9N+9N^2\big)S_3}{N(1+N)} + 12S_4 + \Bigg(\frac{16}{N(1+N)}S_1$$

$$-\frac{8P_{165}}{(N-1)N(1+N)^2(2+N)} - 32S_1^2 + 16S_2\Bigg)S_{-2} + 24S_{-2}^2 + 16\Bigg(-\frac{1}{N(1+N)}$$

$$+S_1\Bigg)S_{-3} + 40S_{-4} + \frac{4\big(-2+3N+3N^2\big)}{N(1+N)}S_{2,1} + 40S_{3,1} - 16S_{-2,2} - 32S_{-3,1}$$

$$-24S_{2,1,1} - 72\zeta_3\Bigg]\,, \tag{399}$$

$$C_{g1,q}^{(2),\mathrm{NS,M}} = C_{g1,q}^{(2),\mathrm{NS,L}} - \Big(C_{g1,q}^{(1),\mathrm{NS,L}} - z_{qq}^{(1)}\Big)z_{qq}^{(1)} - z_{qq}^{(2),\mathrm{NS}}$$

$$= C_{g1,q}^{(2),\mathrm{NS,L}} + C_F\Bigg\{-T_F N_F \frac{16\big(-3-N+5N^2\big)}{9N^2(1+N)^2} + C_A\Bigg[\frac{4P_{247}}{9N^3(1+N)^3}$$

$$+\frac{16}{N(1+N)}S_{-2}\Bigg]\Bigg\} + C_F^2\Bigg[-\frac{8\big(11+16N+11N^2\big)}{N(1+N)^3} + \frac{8(-4+3N)}{N^2(1+N)}S_1$$

$$+\frac{16}{N(1+N)}S_1^2 - \frac{32}{N(1+N)}S_2 - \frac{32}{N(1+N)}S_{-2}\Bigg]\,, \tag{400}$$

$$C_{g1,q}^{(2),\mathrm{PS,L}} = C_F T_F N_F\Bigg[\frac{4P_{252}}{(N-1)N^4(1+N)^4(2+N)} + \frac{8(2+N)\big(2+N-N^2+2N^3\big)}{N^3(1+N)^3}S_1$$

$$+\frac{4(N-1)(2+N)}{N^2(1+N)^2}S_1^2 - \frac{4(N-1)(2+N)}{N^2(1+N)^2}S_2 - \frac{64}{(N-1)N(1+N)(2+N)}$$

$$\times S_{-2}\Bigg]\,, \tag{401}$$

$$C_{g1,q}^{(2),\mathrm{PS,M}} = C_{g1,q}^{(2),\mathrm{NS,L}} - z_{qq}^{(2),\mathrm{PS}}$$

$$= C_{g1,q}^{(2),\mathrm{NS,L}} + C_F T_F N_F \frac{8(2+N)\big(-1-N+N^2\big)}{N^3(1+N)^3}\,, \tag{402}$$

$$C_{g1,g}^{(2)} = C_A T_F N_F \left[-\frac{4P_{255}}{(N-1)N^4(1+N)^4(2+N)^2} + \left(-\frac{4P_{250}}{N^3(1+N)^3(2+N)} \right. \right.$$

$$\left. +\frac{20(N-1)}{N(1+N)}S_2 \right)S_1 - \frac{8(3-N^2+N^3)}{N^2(1+N)^2}S_1^2 + \frac{8(3-2N+N^2+N^3)}{N^2(1+N)^2}S_2$$

$$-\frac{4(N-1)}{3N(1+N)}S_1^3 + \frac{8(2+5N+5N^2)}{3N(1+N)(2+N)}S_3 + \left(\frac{16P_{163}}{(N-1)N(1+N)^2(2+N)^2} \right.$$

$$\left. +\frac{16(2+N+N^2)}{N(1+N)(2+N)}S_1 \right)S_{-2} - \frac{16(-4+N+N^2)}{N(1+N)(2+N)}S_{-3} - \frac{16(N-1)}{N(1+N)}S_{2,1}$$

$$\left. -\frac{64}{N(1+N)(2+N)}S_{-2,1} - \frac{24(2+N+N^2)}{N(1+N)(2+N)}\zeta_3 \right]$$

$$+C_F T_F N_F \left[-\frac{2P_{254}}{(N-1)N^4(1+N)^4(2+N)^2} + \left(\frac{4P_{249}}{N^3(1+N)^3(2+N)} \right. \right.$$

$$\left. +\frac{12(N-1)}{N(1+N)}S_2 \right)S_1 - \frac{2(N-1)(-8+3N+9N^2)}{N^2(1+N)^2}S_1^2 - \frac{20(N-1)}{3N(1+N)}S_1^3$$

$$+\frac{2(4-19N-10N^2+9N^3)}{N^2(1+N)^2}S_2 - \frac{64(1+N+N^2)}{3N(1+N)(2+N)}S_3$$

$$+\left(-\frac{128}{N(1+N)(2+N)}S_1 + \frac{16(10+N+N^2)}{(N-1)(2+N)^2} \right)S_{-2} - \frac{64}{N(1+N)(2+N)}S_{-3}$$

$$+\frac{16(N-1)}{N(1+N)}S_{2,1} + \frac{128}{N(1+N)(2+N)}S_{-2,1} + \frac{48(2+N+N^2)}{N(1+N)(2+N)}\zeta_3 \right] , \qquad (403)$$

with the polynomials

$$P_{245} = -27N^4 - 42N^3 - 47N^2 + 24 , \qquad (404)$$

$$P_{246} = 95N^4 + 178N^3 + 123N^2 + 40N - 16 , \qquad (405)$$

$$P_{247} = 103N^4 + 140N^3 + 58N^2 + 21N + 36 , \qquad (406)$$

$$P_{239} = 247N^4 + 548N^3 + 223N^2 + 30N + 72 , \qquad (407)$$

$$P_{248} = -51N^6 - 131N^5 - 163N^4 - 81N^3 + 26N^2 - 8N - 16 , \qquad (408)$$

$$P_{249} = N^6 + 5N^5 + 3N^4 - 13N^3 - 20N^2 + 12N + 16 , \qquad (409)$$

$$P_{250} = 7N^6 + 16N^5 - 11N^4 - 18N^3 + 6N^2 - 44N - 32 , \qquad (410)$$

$$P_{251} = 1371N^6 + 3429N^5 + 3437N^4 + 655N^3 - 940N^2 + 72N + 360 , \qquad (411)$$

$$P_{242} = 3155N^6 + 9951N^5 + 9867N^4 + 3473N^3 + 546N^2 - 72N - 432 , \qquad (412)$$

$$P_{252} = 6N^8 + 20N^7 + 2N^6 - 36N^5 - 43N^4 - 62N^3 - 47N^2 + 12N + 20 , \qquad (413)$$

$$P_{253} = -16395N^{10} - 74235N^9 - 111388N^8 - 28126N^7 + 111413N^6 + 125177N^5$$
$$+41930N^4 + 12464N^3 + 23688N^2 + 3600N - 5184 , \qquad (414)$$

$$P_{254} = 2N^{10} + 18N^9 + 26N^8 - 42N^7 - 253N^6 - 563N^5 - 633N^4 - 181N^3 + 126N^2$$
$$+4N - 40 , \qquad (415)$$

$$P_{255} = 4N^{10} + 18N^9 - 2N^8 - 74N^7 + 5N^6 + 141N^5 + 99N^4 + 171N^3 + 122N^2$$
$$-44N - 56 , \qquad (416)$$

$$P_{256} = 331N^{10} + 1451N^9 + 2196N^8 + 606N^7 - 2293N^6 - 2697N^5 - 2506N^4$$
$$-2336N^3 - 1232N^2 + 112N + 224 , \tag{417}$$

with the coefficients $z_{ij}^{(l)}$ given in [28, 85]. Up to two–loop order, the Wilson coefficients are given in different schemes [21, 23, 24, 52]. In particular the non–singlet Wilson coefficients are also given in the $\overline{\text{MS}}$ scheme [17, 53].

5 The three–loop Wilson coefficients for the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$

Before we present the three–loop Wilson coefficients we would like to remark that the unrenormalized three–loop forward Compton amplitudes depend on the complete anomalous dimensions up to two–loop order, cf. e.g. [29], and contain the following three–loop anomalous dimensions in their pole terms of $O(1/\varepsilon)$: $\gamma_{qq}^{(2),\text{NS}}, \gamma_{qq}^{(2),\text{PS}}, \Delta\gamma_{qq}^{(2),\text{NS}}, \Delta\gamma_{qq}^{(2),\text{PS}}, \gamma_{qg}^{(2)}$ and $\Delta\gamma_{qg}^{(2)}$. We confirm the previous results given in Refs. [27, 28, 85–92].

We now turn to the unpolarized three–loop Wilson coefficients for neutral–current deep–inelastic scattering for pure virtual photon exchange, $F_2(x, Q^2)$ and $F_L(x, Q^2)$. In the structure of the following expressions there are some evanescent poles at $N = 1, 2$, which are, however, all tractable. Therefore, the analytic continuation [78] to $N \in \mathbb{N}$ can be carried out directly.

The non–singlet and gluonic Wilson coefficients are split for convenience as

$$\mathbb{C}_{i,q}^{(3)} = C_{i,q}^{\text{NS},(3)} + C_{i,q}^{d_{abc},(3)} + C_{i,q}^{\text{PS},(3)} \tag{418}$$

$$\mathbb{C}_{i,g}^{(3)} = C_{i,g}^{\text{NS},(3)} + C_{i,g}^{d_{abc},(3)}. \tag{419}$$

The Wilson coefficients are given by

$$C_{F_2,q}^{\text{NS},(3)} =$$

$$C_F^2 \Bigg\{ C_A \Bigg[\frac{20 P_{262}}{3N(1+N)} \zeta_5 - \frac{8 S_{-2,1,-2} P_{277}}{3N(1+N)} - \frac{16 S_{-2,3} P_{283}}{9N(1+N)} + \frac{36 P_{284}}{5N^2(1+N)^2} \zeta_2^2 + \frac{4 S_5 P_{285}}{9N(1+N)}$$

$$+ \frac{S_2^2 P_{327}}{27N^2(1+N)^2} - \frac{4 S_{-3,1} P_{375}}{9(N-2)N^2(1+N)^2(3+N)} + \frac{8 S_{-2,2} P_{380}}{27(N-2)N^2(1+N)^2(3+N)}$$

$$- \frac{4 S_{2,1,1} P_{316}}{3N^2(1+N)^2} - \frac{8 S_{-2,1,1} P_{390}}{27(N-2)N^2(1+N)^2(3+N)} + \frac{4 S_4 P_{418}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$+ \frac{4 S_{3,1} P_{419}}{27(N-2)N^2(1+N)^2(2+N)(3+N)} + \frac{\zeta_3 P_{435}}{3(N-2)N^3(1+N)^3(2+N)(3+N)}$$

$$+ \frac{P_{472}}{648(N-2)^2 N^5(1+N)^5(3+N)} + \Bigg(\frac{4 S_{2,1} P_{323}}{9N^2(1+N)^2} + \frac{8 S_{-2,1} P_{389}}{27(N-2)N^2(1+N)^2(3+N)}$$

$$- \frac{2 S_3 P_{409}}{27(N-2)N^2(1+N)^2(2+N)(3+N)} + \frac{2\zeta_3 P_{411}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$+ \frac{P_{474}}{648(N-2)^2 N^5(1+N)^5(2+N)(3+N)} + \Bigg(\frac{P_{422}}{162 N^3(1+N)^3(2+N)(3+N)}$$

$$- \frac{2960}{3} S_3 + \frac{160}{3} S_{2,1} - \frac{15104}{3} S_{-2,1} + 480\zeta_3 \Bigg) S_2 + \frac{8(15 + 236N + 236N^2) S_2^2}{9N(1+N)}$$

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$$+\frac{8\left(297+806N+734N^2\right)S_4}{9N(1+N)}+256S_5+\frac{2080}{3}S_{2,3}-\frac{8\left(-636+1445N+869N^2\right)S_{3,1}}{9N(1+N)}$$

$$+3920S_{2,-3}+240S_{4,1}-\frac{8\left(3750-4115N+61N^2\right)S_{-2,2}}{9N(1+N)}-3840S_{-2,3}+64S_{-2,-3}$$

$$-\frac{32\left(287-310N+50N^2\right)S_{-3,1}}{3N(1+N)}-\frac{4000}{3}S_{-4,1}-\frac{40\left(6+13N+13N^2\right)S_{2,1,1}}{3N(1+N)}-\frac{11552}{3}S_{2,1,-2}$$

$$-\frac{32}{3}S_{2,2,1}+1984S_{3,1,1}-\frac{32\left(-2028+2255N+239N^2\right)S_{-2,1,1}}{9N(1+N)}-\frac{11744}{3}S_{-2,1,-2}$$

$$-\frac{12224}{3}S_{-2,2,1}-\frac{6784}{3}S_{-3,1,1}-128S_{2,1,1,1}+11776S_{-2,1,1,1}+560\zeta_5\bigg)S_1+\bigg(-\frac{4S_2P_{326}}{9N^2(1+N)^2}$$

$$+\frac{P_{451}}{36(N-2)N^4(1+N)^4(3+N)}-\frac{40}{3}S_2^2+\frac{4\left(-492+1093N+733N^2\right)S_3}{9N(1+N)}-88S_4$$

$$+\frac{16\left(30+29N+29N^2\right)S_{2,1}}{9N(1+N)}-\frac{2464}{3}S_{3,1}+\frac{16\left(-1188+1219N+247N^2\right)S_{-2,1}}{9N(1+N)}$$

$$+\frac{7216}{3}S_{-2,2}+\frac{6304}{3}S_{-3,1}+80S_{2,1,1}-\frac{16256}{3}S_{-2,1,1}+\frac{4\left(-22+75N+35N^2\right)\zeta_3}{N(1+N)}\bigg)S_1^2$$

$$+\bigg(\frac{P_{340}}{54N^2(1+N)^3}-\frac{4\left(108+491N+491N^2\right)S_2}{27N(1+N)}+\frac{1648}{9}S_3-\frac{320}{9}S_{2,1}+1120S_{-2,1}-96\zeta_3\bigg)$$

$$\times S_1^3+\bigg(\frac{-110+433N+433N^2}{9N(1+N)}+8S_2\bigg)S_1^4+\bigg(\frac{P_{467}}{324(N-2)N^4(1+N)^4(2+N)(3+N)}$$

$$-\frac{16\left(-522+1861N+1159N^2\right)S_3}{27N(1+N)}-\frac{616}{3}S_4-\frac{20\left(4+9N+9N^2\right)S_{2,1}}{3N(1+N)}+\frac{3136}{3}S_{3,1}$$

$$-\frac{8\left(-1442+2071N+79N^2\right)S_{-2,1}}{3N(1+N)}-\frac{3760}{3}S_{-2,2}+880S_{-3,1}-\frac{544}{3}S_{2,1,1}+\frac{12736}{3}S_{-2,1,1}$$

$$-\frac{12\left(22+69N+77N^2\right)\zeta_3}{N(1+N)}\bigg)S_2+\frac{44}{9}S_1^5+\frac{64}{3}S_2^3+\bigg(\frac{P_{432}}{81(N-2)N^3(1+N)^3(2+N)(3+N)}$$

$$-\frac{1072}{9}S_{2,1}+2656S_{-2,1}+256\zeta_3\bigg)S_3+\frac{3392}{9}S_3^2+224S_6+\bigg(\frac{16\zeta_3P_{263}}{3N(1+N)}+\frac{8S_{-2,1}P_{276}}{3N(1+N)}$$

$$+\frac{8S_3P_{296}}{27N(1+N)}+\frac{4S_2P_{387}}{27(N-2)N^2(1+N)^2(3+N)}-\frac{2P_{470}}{81(N-2)^2N^4(1+N)^4(2+N)(3+N)}$$

$$+\bigg(-\frac{4P_{457}}{81(N-2)^2N^3(1+N)^3(2+N)(3+N)}+\frac{8\left(60+113N+185N^2\right)S_2}{3N(1+N)}+608S_3$$

$$+\frac{11264}{3}S_{2,1}+\frac{1312}{3}S_{-2,1}+1248\zeta_3\bigg)S_1+\bigg(-\frac{4P_{382}}{9(N-2)N^2(1+N)^2(3+N)}-112S_2\bigg)S_1^2$$

$$-3136S_{3,1}-\frac{16\left(135+31N+139N^2\right)S_1^3}{27N(1+N)}+32S_1^4+\frac{128}{3}S_2^2+2224S_4$$

$$-\frac{32\left(906-1441N+71N^2\right)S_{2,1}}{9N(1+N)}-1728S_{-2,2}-2048S_{-3,1}+4480S_{-2,1,1}+\frac{288}{5}\zeta_2^2\bigg)S_{-2}$$

$$+\left(\frac{2P_{376}}{3(N-2)N^2(1+N)^2(3+N)}+\frac{4\left(-366+449N+161N^2\right)S_1}{3N(1+N)}+392S_1^2-168S_2\right)S_{-2}^2$$

$$-\frac{32}{3}S_{-2}^3+\left(\frac{4P_{449}}{81(N-2)N^3(1+N)^3(2+N)(3+N)}+\left(\frac{4P_{381}}{27(N-2)N^2(1+N)^2(3+N)}\right.\right.$$

$$\left.+\frac{2176}{3}S_2\right)S_1+\frac{8\left(1674-925N+335N^2\right)S_1^2}{9N(1+N)}-\frac{2416}{3}S_1^3-\frac{4\left(2310-5203N+773N^2\right)S_2}{9N(1+N)}$$

$$-2720S_3+\left(\frac{8\left(-296+169N+N^2\right)}{3N(1+N)}+\frac{4064}{3}S_1\right)S_{-2}-80S_{2,1}+3584S_{-2,1}-64\zeta_3\right)S_{-3}$$

$$-1184S_{-3}^2+\left(\frac{2P_{391}}{27(N-2)N^2(1+N)^2(3+N)}-\frac{4\left(-1926+3631N+31N^2\right)S_1}{9N(1+N)}-456S_1^2\right.$$

$$\left.+\frac{1400}{3}S_2+960S_{-2}\right)S_{-4}+\left(\frac{4P_{286}}{9N(1+N)}+\frac{3920S_1}{3}\right)S_{-5}-\frac{544}{3}S_{-6}$$

$$+\left(\frac{4P_{428}}{9N^3(1+N)^3(2+N)(3+N)}-288\zeta_3\right)S_{2,1}+40S_{2,1}^2+\frac{16\left(-87+647N+431N^2\right)S_{2,3}}{9N(1+N)}$$

$$-\frac{4\left(5922-5561N+487N^2\right)S_{2,-3}}{9N(1+N)}-\frac{4\left(702+2671N+1807N^2\right)S_{4,1}}{9N(1+N)}+552S_{4,2}$$

$$+\left(\frac{8P_{444}}{81(N-2)N^3(1+N)^3(2+N)(3+N)}+96S_{2,1}-640\zeta_3\right)S_{-2,1}-2080S_{4,-2}+\frac{5888}{3}S_{5,1}$$

$$-2688S_{-2,1}^2+\frac{16(N-1)(2+N)S_{-2,-3}}{N(1+N)}+3456S_{-3,3}-\frac{8\left(-16-747N+189N^2\right)S_{-4,1}}{3N(1+N)}$$

$$-2384S_{-4,2}-1216S_{-4,-2}+288S_{-5,1}+\frac{8\left(3678-5755N+293N^2\right)S_{2,1,-2}}{9N(1+N)}$$

$$-\frac{4\left(-4+65N+65N^2\right)S_{2,2,1}}{3N(1+N)}-\frac{1760}{3}S_{2,3,1}+\frac{32\left(-171+577N+361N^2\right)S_{3,1,1}}{9N(1+N)}$$

$$-2160S_{2,-3,1}+2784S_{3,1,-2}+48S_{3,2,1}+\frac{320}{3}S_{4,1,1}+704S_{-2,2,2}+64S_{-2,2,-2}$$

$$+\frac{16\left(1902-2891N+133N^2\right)S_{-2,2,1}}{9N(1+N)}+2816S_{-2,3,1}+\frac{16\left(464-881N+127N^2\right)S_{-3,1,1}}{3N(1+N)}$$

$$+960S_{-3,1,-2}-112S_{-3,2,1}-704S_{-3,-2,1}+5504S_{-4,1,1}+\frac{32\left(6+13N+13N^2\right)S_{2,1,1,1}}{3N(1+N)}$$

$$+\frac{208}{3}S_{2,2,1,1}+\frac{32\left(-2412+3139N+115N^2\right)S_{-2,1,1,1}}{9N(1+N)}-1792S_{2,-2,1,1}-\frac{6368}{3}S_{3,1,1,1}$$

$$-288S_{-2,1,1,2}+192S_{-2,2,1,1}-1408S_{-2,-2,1,1}-896S_{-3,1,1,1}+\frac{640}{3}S_{2,1,1,1,1}-9728S_{-2,1,1,1,1}\Bigg]$$

$$+T_FN_F\left[-\frac{16S_{2,1}P_{364}}{9N^2(1+N)^2(2+N)(3+N)}+\frac{4S_3P_{392}}{81N^2(1+N)^2(2+N)(3+N)}\right.$$

$$+\frac{16\zeta_3P_{406}}{9(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{P_{471}}{162(N-2)^2N^5(1+N)^5(3+N)}$$

$$-\frac{128 S_{-2,1} P_{332}}{81 N^2 (1+N)^2 (2+N)} + \left(\frac{P_{463}}{162(N-2)N^4(1+N)^4(2+N)(3+N)} \right.$$

$$+\frac{2 S_2 P_{394}}{81 N^2 (1+N)^2 (2+N)(3+N)} - \frac{752}{9} S_2^2 - \frac{8\left(-138-83N+61N^2\right)S_3}{27 N(1+N)} - \frac{1664}{9} S_4$$

$$-\frac{16\left(-34+61N+13N^2\right)S_{2,1}}{9 N(1+N)} + \frac{1088}{9} S_{3,1} - \frac{1024\left(-6+14N+5N^2\right)S_{-2,1}}{27 N(1+N)} + \frac{256}{9} S_{-2,2}$$

$$+\frac{512}{3} S_{-3,1} + \frac{320}{3} S_{2,1,1} + \frac{2048}{9} S_{-2,1,1} + \frac{64}{5} \zeta_2^2 - \frac{16\left(-78+313N+73N^2\right)\zeta_3}{9 N(1+N)} \right) S_1$$

$$+\left(\frac{P_{361}}{9 N^3 (1+N)^3} + \frac{16\left(-29+94N+82N^2\right)S_2}{9 N(1+N)} - \frac{464}{9} S_3 - \frac{352}{9} S_{2,1} - \frac{1280}{9} S_{-2,1} - 32\zeta_3 \right) S_1^2$$

$$+\left(-\frac{2 P_{324}}{27 N^2 (1+N)^2} + \frac{832 S_2}{27} \right) S_1^3 - \frac{20\left(-2+7N+7N^2\right)S_1^4}{9 N(1+N)} - \frac{16}{9} S_1^5 + \left(\frac{4976}{27} S_3 \right.$$

$$+\frac{P_{423}}{81 N^3 (1+N)^3 (2+N)(3+N)} + 32 S_{2,1} + \frac{256}{3} S_{-2,1} + \frac{352}{3} \zeta_3 \right) S_2$$

$$-\frac{4\left(-354+455N+167N^2\right)S_2^2}{27 N(1+N)} - \frac{16\left(-309+782N+170N^2\right)S_4}{27 N(1+N)}$$

$$+\left(\frac{32 P_{454}}{81(N-2)^2 N^3 (1+N)^3 (2+N)(3+N)} + \left(\frac{128 P_{400}}{81(N-2)N^2(1+N)^2(2+N)(3+N)} \right. \right.$$

$$-\frac{512}{3} S_2 \right) S_1 + \frac{256\left(-4+11N+5N^2\right)S_1^2}{9 N(1+N)} + \frac{1280}{27} S_1^3 - \frac{128\left(-27+56N+20N^2\right)S_2}{27 N(1+N)}$$

$$+\frac{3712}{27} S_3 + \frac{512}{9} S_{2,1} + \frac{896}{3} \zeta_3 \right) S_{-2} + \left(-\frac{32\left(-6+21N+13N^2\right)}{3 N(1+N)} - \frac{256}{3} S_1 \right) S_{-2}^2$$

$$+\left(-\frac{64 P_{397}}{81(N-2)N^2(1+N)^2(2+N)(3+N)} - \frac{256\left(-15+28N+10N^2\right)S_1}{27 N(1+N)} - \frac{896}{9} S_1^2 \right.$$

$$+\frac{640}{9} S_2 + \frac{512}{3} S_{-2} \right) S_{-3} + \left(-\frac{32\left(-66+413N+197N^2\right)}{27 N(1+N)} - \frac{256}{9} S_1 \right) S_{-4} + \frac{3008}{9} S_{-5}$$

$$-\frac{16\left(156+215N+431N^2\right)S_{3,1}}{27 N(1+N)} + \frac{128\left(-3+10N+10N^2\right)S_{-2,2}}{27 N(1+N)} - \frac{1984}{9} S_{2,3} + \frac{512}{9} S_{2,-3}$$

$$+\frac{3008}{9} S_{4,1} - \frac{896}{9} S_{-2,3} + \frac{128\left(-9+22N+10N^2\right)S_{-3,1}}{9 N(1+N)} + \frac{16\left(-16+29N+5N^2\right)S_{2,1,1}}{3 N(1+N)}$$

$$-\frac{512}{9} S_{2,1,-2} + \frac{64}{3} S_{2,2,1} - \frac{2944}{9} S_{3,1,1} + \frac{256\left(-21+46N+10N^2\right)S_{-2,1,1}}{27 N(1+N)} + \frac{512}{3} S_{-2,1,-2}$$

$$-\frac{512}{9} S_{-2,2,1} - \frac{512}{3} S_{-3,1,1} - \frac{256}{3} S_{2,1,1,1} - \frac{1024}{9} S_{-2,1,1,1} - \frac{16\left(2+3N+3N^2\right)\zeta_2^2}{5 N(1+N)} + \frac{2368}{9} S_5 \right] \right\}$$

$$+C_F \left\{ C_A^2 \left[-\frac{5\zeta_5 P_{269}}{3 N(1+N)} + \frac{4 S_{-2,1,-2} P_{271}}{3 N(1+N)} - \frac{2 S_5 P_{280}}{9 N(1+N)} + \frac{4 S_{-2,3} P_{282}}{9 N(1+N)} - \frac{12\zeta_2^2 P_{284}}{5 N^2 (1+N)^2} \right. \right.$$

$$+\frac{4S_{2,1,1}P_{322}}{9N^2(1+N)^2} - \frac{8S_{-2,2}P_{370}}{27(N-2)N^2(1+N)^2(3+N)} + \frac{4S_{-3,1}P_{372}}{9(N-2)N^2(1+N)^2(3+N)}$$

$$+\frac{S_2^2 P_{319}}{9N^2(1+N)^2} + \frac{8S_{-2,1,1}P_{383}}{27(N-2)N^2(1+N)^2(3+N)} - \frac{4S_{3,1}P_{404}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$-\frac{2S_4 P_{414}}{9(N-2)N^2(1+N)^2(2+N)(3+N)} + \frac{2\zeta_3 P_{429}}{27(N-2)N^3(1+N)^2(2+N)(3+N)}$$

$$+\frac{P_{458}}{5832(N-2)^2 N^4(1+N)^4(3+N)} + \Bigg[-\frac{4S_{2,1}P_{320}}{9N^2(1+N)^2} - \frac{8S_{-2,1}P_{384}}{27(N-2)N^2(1+N)^2(3+N)}$$

$$+\frac{4S_3 P_{412}}{27(N-2)N^2(1+N)^2(2+N)(3+N)} - \frac{8\zeta_3 P_{416}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$+\frac{P_{473}}{729(N-2)N^5(1+N)^5(2+N)(3+N)} + \Bigg(-32S_{2,1} + \frac{P_{395}}{27N^3(1+N)^3(3+N)} + \frac{1184}{3}S_3$$

$$+1376S_{-2,1} - 64\zeta_3 \Bigg) S_2 - \frac{4\big(-6+145N+145N^2\big)S_2^2}{9N(1+N)} - \frac{4\big(-162+1325N+893N^2\big)S_4}{9N(1+N)}$$

$$+\frac{448}{3}S_5 - \frac{1120}{3}S_{2,3} - 1040S_{2,-3} + \frac{16\big(-73+120N+48N^2\big)S_{3,1}}{3N(1+N)} - \frac{176}{3}S_{4,1}$$

$$+\frac{32\big(249-277N+11N^2\big)S_{-2,2}}{9N(1+N)} + \frac{2912}{3}S_{-2,3} - \frac{64}{3}S_{-2,-3} + \frac{64\big(36-37N+11N^2\big)S_{-3,1}}{3N(1+N)}$$

$$+\frac{1184}{3}S_{-4,1} + \frac{8\big(12+55N+55N^2\big)S_{2,1,1}}{3N(1+N)} + 992S_{2,1,-2} + 64S_{2,2,1} - \frac{2816}{3}S_{3,1,1}$$

$$+\frac{128\big(-141+166N+22N^2\big)S_{-2,1,1}}{9N(1+N)} + \frac{3424}{3}S_{-2,1,-2} + \frac{3200}{3}S_{-2,2,1} + \frac{1600}{3}S_{-3,1,1}$$

$$+128S_{2,1,1,1} - 3200S_{-2,1,1,1} + 240\zeta_5 \Bigg] S_1 + \Bigg[\frac{2S_2 P_{312}}{9N^2(1+N)^2} + \frac{P_{431}}{162N^4(1+N)^4} - \frac{8}{3}S_2^2$$

$$+\frac{4\big(240-205N+11N^2\big)S_3}{9N(1+N)} - 72S_4 - \frac{4\big(12+209N+209N^2\big)S_{2,1}}{9N(1+N)} + \frac{1168}{3}S_{3,1}$$

$$-\frac{32\big(-171+199N+55N^2\big)S_{-2,1}}{9N(1+N)} - \frac{1888}{3}S_{-2,2} - 512S_{-3,1} - 32S_{2,1,1} + \frac{4480}{3}S_{-2,1,1}$$

$$-\frac{16\big(-4+15N+11N^2\big)\zeta_3}{N(1+N)} \Bigg] S_1^2 + \Bigg[\frac{-726+4649N+4649N^2}{81N(1+N)} + \frac{616}{27}S_2 - \frac{640}{9}S_3 + \frac{32}{9}S_{2,1}$$

$$-320S_{-2,1} \Bigg] S_1^3 + \frac{121}{27}S_1^4 + \Bigg[\frac{P_{433}}{54N^4(1+N)^4(3+N)} + \frac{8\big(-666+1453N+481N^2\big)S_3}{27N(1+N)}$$

$$+\frac{424}{3}S_4 + \frac{16\big(1+3N+3N^2\big)S_{2,1}}{N(1+N)} - \frac{1168}{3}S_{3,1} + \frac{16\big(-201+310N+22N^2\big)S_{-2,1}}{3N(1+N)}$$

$$+352S_{-2,2} - \frac{1040}{3}S_{-3,1} + \frac{272}{3}S_{2,1,1} - 1088S_{-2,1,1} + \frac{32\big(1+18N+18N^2\big)\zeta_3}{N(1+N)} \Bigg] S_2 - \frac{56}{9}S_2^3$$

$$+\Bigg[\frac{2P_{450}}{81(N-2)N^3(1+N)^3(2+N)(3+N)} + \frac{880}{9}S_{2,1} - \frac{2080}{3}S_{-2,1} - 128\zeta_3 \Bigg] S_3 - \frac{1280}{9}S_3^2$$

$$+\frac{80}{9}S_6 + \Bigg[-\frac{4\zeta_3 P_{260}}{3N(1+N)} - \frac{4S_{-2,1}P_{264}}{3N(1+N)} + \frac{2P_{469}}{81(N-2)^2 N^4 (1+N)^4 (2+N)(3+N)}$$

$$-\frac{4S_3 P_{287}}{27N(1+N)} + \left(\frac{4P_{448}}{81(N-2)N^3(1+N)^3(2+N)(3+N)} - \frac{32\big(-3+22N+22N^2\big)S_2}{3N(1+N)} \right)$$

$$-\frac{320}{3}S_3 - 960 S_{2,1} - 160 S_{-2,1} - 320\zeta_3 \Bigg) S_1 + \left(\frac{4P_{321}}{9N^2(1+N)^2} - 32 S_2 \right) S_1^2 + \frac{1760}{27}S_1^3$$

$$-\frac{4\big(-72-1206N+2737N^2+1108N^3\big)S_2}{27N^2(1+N)} - \frac{16}{3}S_2^2 + \frac{32\big(243-410N+22N^2\big)S_{2,1}}{9N(1+N)}$$

$$-\frac{1648}{3}S_4 + 768 S_{3,1} + \frac{1280}{3}S_{-2,2} + \frac{1600}{3}S_{-3,1} - \frac{3584}{3}S_{-2,1,1} - \frac{96}{5}\zeta_2^2 \Bigg] S_{-2}$$

$$+ \Bigg[-\frac{8P_{353}}{3(N-2)N^2(1+N)^2(3+N)} - \frac{8\big(-62+95N+47N^2\big)S_1}{3N(1+N)} - \frac{400}{3}S_1^2 + \frac{208}{3}S_2 \Bigg] S_{-2}^2$$

$$+\frac{32}{9}S_{-2}^3 + \Bigg[-\frac{2P_{447}}{81(N-2)N^3(1+N)^3(2+N)(3+N)} + \left(-\frac{4P_{379}}{27(N-2)N^2(1+N)^2(3+N)} \right.$$

$$\left. -144 S_2 \right) S_1 - \frac{16\big(189-67N+77N^2\big)S_1^2}{9N(1+N)} + \frac{544}{3}S_1^3 + \frac{8\big(297-754N+110N^2\big)S_2}{9N(1+N)} + \frac{2080}{3}S_3$$

$$+ \left(\frac{8\big(64-35N+13N^2\big)}{3N(1+N)} - \frac{832}{3}S_1 \right) S_{-2} + \frac{80}{3}S_{2,1} - \frac{3136}{3}S_{-2,1} + 32\zeta_3 \Bigg] S_{-3} + \frac{1072}{3}S_{-3}^2$$

$$+ \left(-\frac{4P_{385}}{27(N-2)N^2(1+N)^2(3+N)} - \frac{8\big(228-379N+53N^2\big)S_1}{9N(1+N)} + \frac{320}{3}S_1^2 - \frac{320}{3}S_2 \right.$$

$$\left. -\frac{736}{3}S_{-2} \right) S_{-4} + \Bigg[-\frac{2P_{281}}{9N(1+N)} - 208 S_1 \Bigg] S_{-5} + \frac{544}{9}S_{-6} + \Bigg[-\frac{4P_{417}}{27N^3(1+N)^3(3+N)}$$

$$+96\zeta_3 \Bigg] S_{2,1} - \frac{56}{3}S_{2,1}^2 - \frac{16\big(-35+106N+52N^2\big)S_{2,3}}{3N(1+N)}$$

$$+\frac{8\big(801-758N+106N^2\big)S_{2,-3}}{9N(1+N)} + \frac{8\big(33+691N+367N^2\big)S_{4,1}}{9N(1+N)} - \frac{952}{3}S_{4,2} + \frac{1600}{3}S_{4,-2}$$

$$-\frac{2144}{3}S_{5,1} + \Bigg[-\frac{4P_{445}}{81(N-2)N^3(1+N)^3(2+N)(3+N)} - 32 S_{2,1} + 192\zeta_3 \Bigg] S_{-2,1} + \frac{2240}{3}S_{-2,1}^2$$

$$+\frac{32 S_{-2,-3}}{3N(1+N)} - \frac{2848}{3}S_{-3,3} + \frac{8\big(-8-207N+57N^2\big)S_{-4,1}}{3N(1+N)} + \frac{2096}{3}S_{-4,2} + \frac{1216}{3}S_{-4,-2}$$

$$-224 S_{-5,1} - \frac{16\big(495-820N+44N^2\big)S_{2,1,-2}}{9N(1+N)} + \frac{4\big(-24+29N+29N^2\big)S_{2,2,1}}{3N(1+N)} + \frac{640}{3}S_{2,3,1}$$

$$+\frac{2096}{3}S_{2,-3,1} - \frac{16\big(-264+611N+287N^2\big)S_{3,1,1}}{9N(1+N)} - \frac{1952}{3}S_{3,1,-2} - \frac{208}{3}S_{3,2,1} + \frac{320}{3}S_{4,1,1}$$

$$-\frac{64\big(129-205N+11N^2\big)S_{-2,2,1}}{9N(1+N)} - \frac{832}{3}S_{-2,2,2} - \frac{64}{3}S_{-2,2,-2} - \frac{1984}{3}S_{-2,3,1}$$

$$-\frac{32\left(61-122N+22N^2\right)S_{-3,1,1}}{3N(1+N)}-320S_{-3,1,-2}+\frac{112}{3}S_{-3,2,1}+\frac{704}{3}S_{-3,-2,1}-\frac{4480}{3}S_{-4,1,1}$$

$$-\frac{16\left(36+11N+11N^2\right)S_{2,1,1,1}}{9N(1+N)}-\frac{272}{3}S_{2,2,1,1}+384S_{2,-2,1,1}+\frac{2720}{3}S_{3,1,1,1}+384S_{-3,1,1,1}$$

$$-\frac{64\left(-333+454N+22N^2\right)S_{-2,1,1,1}}{9N(1+N)}+96S_{-2,1,1,2}-64S_{-2,2,1,1}+\frac{1408}{3}S_{-2,-2,1,1}$$

$$\left.-\frac{640}{3}S_{2,1,1,1,1}+2560S_{-2,1,1,1,1}\right]$$

$$+C_A T_F N_F\left[\frac{64S_{-2,1}P_{332}}{81N^2(1+N)^2(2+N)}-\frac{8S_{3,1}P_{334}}{27(N-2)N(1+N)(2+N)(3+N)}\right.$$

$$+\frac{8S_{2,1}P_{337}}{27N^2(1+N)^2(3+N)}+\frac{4S_4 P_{338}}{27(N-2)N(1+N)(2+N)(3+N)}$$

$$-\frac{4\zeta_3 P_{339}}{27(N-2)N^2(1+N)(3+N)}-\frac{8S_3 P_{413}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$+\frac{P_{466}}{729(N-2)^2N^4(1+N)^4(3+N)}+\left[-\frac{32S_3 P_{333}}{27(N-2)N(1+N)(2+N)(3+N)}\right.$$

$$+\frac{16\zeta_3 P_{336}}{27(N-2)N(1+N)(2+N)(3+N)}-\frac{4P_{465}}{729(N-2)N^4(1+N)^4(2+N)(3+N)}$$

$$+\frac{8S_2 P_{335}}{27N^2(1+N)^2(3+N)}+\frac{224}{9}S_2^2+\frac{1312}{9}S_4+\frac{16\left(-25+64N+40N^2\right)S_{2,1}}{9N(1+N)}-96S_{3,1}$$

$$+\frac{512\left(-6+14N+5N^2\right)S_{-2,1}}{27N(1+N)}-\frac{128}{9}S_{-2,2}-\frac{256}{3}S_{-3,1}-\frac{160}{3}S_{2,1,1}-\frac{1024}{9}S_{-2,1,1}-\frac{64}{5}\zeta_2^2\right]S_1$$

$$+\left[-\frac{2P_{393}}{81N^3(1+N)^3}+\frac{8\left(17-10N+2N^2\right)S_2}{9N(1+N)}-\frac{16}{9}S_3+\frac{304}{9}S_{2,1}+\frac{640}{9}S_{-2,1}+32\zeta_3\right]S_1^2$$

$$+\left[-\frac{16\left(-33+194N+194N^2\right)}{81N(1+N)}-\frac{224}{27}S_2\right]S_1^3-\frac{88}{27}S_1^4+\left[\frac{2P_{421}}{81N^3(1+N)^3(3+N)}-\frac{1360}{27}S_3\right.$$

$$\left.-16S_{2,1}-\frac{128}{3}S_{-2,1}-\frac{352}{3}\zeta_3\right]S_2-\frac{4\left(40+61N+109N^2\right)S_2^2}{9N(1+N)}-\frac{1312}{9}S_5$$

$$+\left[-\frac{16P_{454}}{81(N-2)^2N^3(1+N)^3(2+N)(3+N)}+\left(-\frac{64P_{400}}{81(N-2)N^2(1+N)^2(2+N)(3+N)}\right.\right.$$

$$\left.+\frac{256}{3}S_2\right)S_1-\frac{128\left(-4+11N+5N^2\right)S_1^2}{9N(1+N)}-\frac{640}{27}S_1^3+\frac{64\left(-27+56N+20N^2\right)S_2}{27N(1+N)}-\frac{1856}{27}S_3$$

$$\left.-\frac{256}{9}S_{2,1}-\frac{448}{3}\zeta_3\right]S_{-2}+\left[\frac{128}{3}S_1+\frac{16\left(-6+21N+13N^2\right)}{3N(1+N)}\right]S_{-2}^2$$

$$+\left[-\frac{128\left(15-28N-10N^2\right)S_1}{27N(1+N)}+\frac{32P_{397}}{81(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{448}{9}S_1^2\right.$$

$$\left.-\frac{320}{9}S_2-\frac{256}{3}S_{-2}\right]S_{-3}+\left[\frac{16\left(-66+413N+197N^2\right)}{27N(1+N)}+\frac{128}{9}S_1\right]S_{-4}-\frac{1504}{9}S_{-5}+\frac{320}{3}S_{2,3}$$

$$-\frac{64\big(-3+10N+10N^2\big)S_{-2,2}}{27N(1+N)}+\frac{448}{9}S_{-2,3}-\frac{64\big(-9+22N+10N^2\big)S_{-3,1}}{9N(1+N)}+\frac{1696}{9}S_{3,1,1}$$

$$-\frac{256}{9}S_{2,-3}-\frac{1120}{9}S_{4,1}-\frac{64\big(-6+13N+4N^2\big)S_{2,1,1}}{9N(1+N)}+\frac{256}{9}S_{2,1,-2}-\frac{64}{3}S_{2,2,1}-\frac{256}{3}S_{-2,1,-2}$$

$$-\frac{128\big(-21+46N+10N^2\big)S_{-2,1,1}}{27N(1+N)}+\frac{256}{9}S_{-2,2,1}+\frac{256}{3}S_{-3,1,1}+\frac{64}{9}S_{2,1,1,1}$$

$$+\frac{512}{9}S_{-2,1,1,1}+\frac{16\big(2+3N+3N^2\big)\zeta_2^2}{5N(1+N)}\Bigg]$$

$$+T_F^2N_F^2\Bigg[-\frac{8S_2P_{325}}{81N^2(1+N)^2}-\frac{2P_{430}}{729N^4(1+N)^4}+\Bigg[-\frac{16\big(-6+29N+29N^2\big)S_2}{27N(1+N)}$$

$$+\frac{8P_{388}}{729N^3(1+N)^3}+\frac{128}{27}S_3+\frac{128}{27}\zeta_3\Bigg]S_1+\Bigg[\frac{8P_{314}}{81N^2(1+N)^2}-\frac{32S_2}{9}\Bigg]S_1^2-\frac{992}{27}S_4-\frac{1216}{27}S_{2,1}$$

$$+\frac{16\big(-6+29N+29N^2\big)}{81N(1+N)}S_1^3+\frac{16}{27}S_1^4+\frac{80}{9}S_2^2+\frac{32\big(-6+247N+247N^2\big)S_3}{81N(1+N)}+\frac{256}{9}S_{3,1}$$

$$-\frac{128}{9}S_{2,1,1}-\frac{32\big(2+3N+3N^2\big)\zeta_3}{27N(1+N)}\Bigg]\Bigg\}+C_F^3\Bigg\{\frac{S_2^2P_{257}}{3N^2(1+N)^2}-\frac{20\zeta_5P_{259}}{3N(1+N)}-\frac{8S_5P_{261}}{3N(1+N)}$$

$$+\frac{16S_{-2,3}P_{268}}{3N(1+N)}+\frac{16S_{-2,1,-2}P_{270}}{3N(1+N)}-\frac{24\zeta_2^2P_{284}}{5N^2(1+N)^2}-\frac{8S_{-3,1}P_{350}}{3(N-2)N^2(1+N)^2(3+N)}$$

$$-\frac{16S_{-2,2}P_{356}}{3(N-2)N^2(1+N)^2(3+N)}+\frac{16S_{-2,1,1}P_{371}}{3(N-2)N^2(1+N)^2(3+N)}$$

$$-\frac{8S_{3,1}P_{402}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}-\frac{2S_4P_{405}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$-\frac{4\zeta_3P_{440}}{3(N-2)N^3(1+N)^3(2+N)(3+N)}+\frac{P_{476}}{24(N-2)^2N^6(1+N)^6(3+N)}$$

$$+\Bigg[-\frac{4S_{2,1}P_{308}}{3N^2(1+N)^2}-\frac{16S_{-2,1}P_{369}}{3(N-2)N^2(1+N)^2(3+N)}-\frac{8\zeta_3P_{398}}{(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$+\frac{2S_3P_{399}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{P_{475}}{24(N-2)^2N^5(1+N)^5(2+N)(3+N)}$$

$$+\Bigg(\frac{P_{386}}{6N^3(1+N)^3}+\frac{976}{3}S_3-\frac{64}{3}S_{2,1}+\frac{13696}{3}S_{-2,1}-416\zeta_3\Bigg)S_2+\frac{2\big(-54+25N+25N^2\big)S_2^2}{N(1+N)}$$

$$-\frac{4\big(174-75N+85N^2\big)S_4}{N(1+N)}-\frac{2560}{3}S_5-64S_{2,3}-3680S_{2,-3}+\frac{8\big(22+45N+109N^2\big)S_{3,1}}{N(1+N)}$$

$$-\frac{1696}{3}S_{4,1}-\frac{16\big(-586+633N+9N^2\big)S_{-2,2}}{3N(1+N)}+\frac{11392}{3}S_{-2,3}-\frac{128}{3}S_{-2,-3}+\frac{11200}{3}S_{2,1,-2}$$

$$+\frac{64\big(143-162N+6N^2\big)S_{-3,1}}{3N(1+N)}+1088S_{-4,1}-\frac{24\big(-2+3N+3N^2\big)S_{2,1,1}}{N(1+N)}-\frac{160}{3}S_{2,2,1}$$

$$-\frac{1984}{3}S_{3,1,1}+\frac{64\big(-100+103N+7N^2\big)S_{-2,1,1}}{N(1+N)}+3264S_{-2,1,-2}+\frac{11648}{3}S_{-2,2,1}+\frac{7168}{3}S_{-3,1,1}$$

$$-10752 S_{-2,1,1,1} - 1120\zeta_5\bigg] S_1 + \bigg[\frac{2 S_2 P_{317}}{3 N^2 (1+N)^2} + \frac{P_{455}}{12(N-2)N^4(1+N)^4(3+N)} + 108 S_2^2$$

$$-\frac{4\big(10 + 59N + 75N^2\big)S_3}{N(1+N)} + \frac{24\big(-2 + 3N + 3N^2\big)S_{2,1}}{N(1+N)} - \frac{32\big(-56 + 47N + 3N^2\big)S_{-2,1}}{N(1+N)}$$

$$+336 S_{3,1} + 344 S_4 - \frac{6880}{3} S_{-2,2} - \frac{6464}{3} S_{-3,1} - 48 S_{2,1,1} + 4864 S_{-2,1,1} - \frac{64(N-1)\zeta_3}{N(1+N)}\bigg] S_1^2$$

$$+\bigg[\frac{P_{341}}{6 N^3(1+N)^3} - \frac{12\big(-6 + 7N + 7N^2\big)S_2}{N(1+N)} - 48 S_3 + 32 S_{2,1} - 960 S_{-2,1} + 96\zeta_3\bigg] S_1^3$$

$$+\bigg[\frac{P_{258}}{N^2(1+N)^2} - 36 S_2\bigg] S_1^4 + \frac{2\big(-2 + 3N + 3N^2\big)S_1^5}{N(1+N)} + \frac{4}{3} S_1^6 + \bigg[40 S_4 - 432 S_{3,1} + \frac{3296}{3} S_{-2,2}$$

$$+\frac{P_{452}}{12(N-2)N^4(1+N)^4(3+N)} + \frac{4\big(154 + 273N + 513N^2\big)S_3}{3N(1+N)} - \frac{8\big(-4 + 15N + 15N^2\big)S_{2,1}}{3N(1+N)}$$

$$-\frac{16\big(638 - 831N + 9N^2\big)S_{-2,1}}{3N(1+N)} - \frac{1120}{3} S_{-3,1} + \frac{272}{3} S_{2,1,1} + \frac{64\big(4 + 3N + 6N^2\big)\zeta_3}{N(1+N)}$$

$$-\frac{12416}{3} S_{-2,1,1}\bigg] S_2 - \frac{148}{9} S_2^3 + \bigg[\frac{P_{443}}{3(N-2)N^3(1+N)^3(2+N)(3+N)} + \frac{64}{3} S_{2,1} - \frac{7616}{3} S_{-2,1}$$

$$-\frac{320}{3}\zeta_3\bigg] S_3 - \frac{512}{3} S_3^2 - \frac{1664}{9} S_6 + \bigg[-\frac{16 S_{-2,1} P_{266}}{3N(1+N)} - \frac{8 S_2 P_{358}}{3(N-2)N^2(1+N)^2(3+N)}$$

$$-\frac{16\zeta_3 P_{267}}{3N(1+N)} + \frac{4 P_{468}}{3(N-2)^2 N^4(1+N)^4(2+N)(3+N)} + \bigg[-\frac{11008}{3} S_{2,1} - \frac{704}{3} S_{-2,1}$$

$$-\frac{8 P_{453}}{3(N-2)^2 N^3(1+N)^3(2+N)(3+N)} - \frac{16\big(28 - 21N + 3N^2\big)S_2}{N(1+N)} - \frac{2368}{3} S_3 - 1216\zeta_3\bigg] S_1$$

$$+\bigg[\frac{8 P_{362}}{3(N-2)N^2(1+N)^2(3+N)} + 352 S_2\bigg] S_1^2 - \frac{32\big(-5 + 7N + 3N^2\big)S_1^3}{N(1+N)} - 64 S_1^4 - 64 S_2^2$$

$$-\frac{16\big(-80 + N - 46N^2 - 22N^3 + N^4\big)S_3}{3N(1+N)} - \frac{6752}{3} S_4 + \frac{64\big(140 - 207N + 9N^2\big)S_{2,1}}{3N(1+N)}$$

$$+3200 S_{3,1} + \frac{5248}{3} S_{-2,2} + \frac{5888}{3} S_{-3,1} - \frac{12544}{3} S_{-2,1,1} - \frac{192}{5}\zeta_2^2\bigg] S_{-2} + \bigg[-\frac{752}{3} S_1^2 + \frac{176}{3} S_2$$

$$-\frac{4 P_{351}}{3(N-2)N^2(1+N)^2(3+N)} + \frac{8\big(118 - 69N + 27N^2\big)S_1}{3N(1+N)}\bigg] S_{-2}^2 + \frac{64}{9} S_{-2}^3$$

$$+\bigg[-\frac{8 P_{439}}{3(N-2)N^3(1+N)^3(2+N)(3+N)} + \bigg(\frac{8 P_{348}}{3(N-2)N^2(1+N)^2(3+N)} - \frac{2624}{3} S_2\bigg) S_1$$

$$-\frac{16\big(102 - 73N + 3N^2\big)S_1^2}{N(1+N)} + \frac{8\big(374 - 729N + 111N^2\big)S_2}{3N(1+N)} + \frac{2656}{3} S_1^3 + \frac{8000}{3} S_3$$

$$+\bigg(-\frac{16\big(-56 + 33N + 9N^2\big)}{N(1+N)} - 1600 S_1\bigg) S_{-2} + \frac{160}{3} S_{2,1} - \frac{8960}{3} S_{-2,1}\bigg] S_{-3} + \frac{2816}{3} S_{-3}^2$$

$$+\left[-\frac{4P_{347}}{3(N-2)N^2(1+N)^2(3+N)}+\frac{8\left(-338+705N+81N^2\right)S_1}{3N(1+N)}+\frac{1456}{3}S_1^2-\frac{1520}{3}S_2\right.$$

$$\left.-\frac{2816}{3}S_{-2}\right]S_{-4}+\left[-\frac{8P_{265}}{3N(1+N)}-\frac{5344S_1}{3}\right]S_{-5}+\frac{1088}{9}S_{-6}+\left[-\frac{4P_{363}}{3N^3(1+N)^3}+192\zeta_3\right]$$

$$\times S_{2,1}-\frac{64}{3}S_{2,1}^2-\frac{32\left(11+5N+17N^2\right)S_{2,3}}{N(1+N)}+\frac{8\left(302-281N+7N^2\right)S_{2,-3}}{N(1+N)}$$

$$+\frac{8\left(250-9N+135N^2\right)S_{4,1}}{3N(1+N)}-\frac{1088}{3}S_{4,2}+\frac{6080}{3}S_{4,-2}-1120S_{5,1}+\left[-64S_{2,1}+512\zeta_3\right.$$

$$\left.+\frac{16P_{438}}{3(N-2)N^3(1+N)^3(2+N)(3+N)}\right]S_{-2,1}+\frac{7168}{3}S_{-2,1}^2-\frac{32\left(-2+3N+3N^2\right)S_{-2,-3}}{3N(1+N)}$$

$$-\frac{9344}{3}S_{-3,3}+\frac{16(-111+25N)S_{-4,1}}{1+N}+\frac{5920}{3}S_{-4,2}+\frac{2432}{3}S_{-4,-2}+320S_{-5,1}+\frac{80S_{2,2,1}}{3N(1+N)}$$

$$+\frac{8\left(-14-20N+10N^2+29N^3\right)S_{2,1,1}}{N^2(1+N)}-\frac{16\left(566-825N+39N^2\right)S_{2,1,-2}}{3N(1+N)}+\frac{352}{3}S_{2,3,1}$$

$$+\frac{4576}{3}S_{2,-3,1}-\frac{16\left(82+51N+195N^2\right)S_{3,1,1}}{3N(1+N)}-\frac{8896}{3}S_{3,1,-2}+\frac{64}{3}S_{3,2,1}+\frac{128}{3}S_{4,1,1}$$

$$-\frac{32\left(290-417N+15N^2\right)S_{-2,2,1}}{3N(1+N)}-\frac{896}{3}S_{-2,2,2}-\frac{128}{3}S_{-2,2,-2}-\frac{8960}{3}S_{-2,3,1}$$

$$-\frac{32\left(220-393N+39N^2\right)S_{-3,1,1}}{3N(1+N)}-640S_{-3,1,-2}+\frac{224}{3}S_{-3,2,1}+\frac{1408}{3}S_{-3,-2,1}-\frac{15104}{3}S_{-4,1,1}$$

$$+\frac{64}{3}S_{2,2,1,1}+2048S_{2,-2,1,1}+704S_{3,1,1,1}-\frac{192\left(-40+49N+N^2\right)S_{-2,1,1,1}}{N(1+N)}+192S_{-2,1,1,2}$$

$$\left.-128S_{-2,2,1,1}+\frac{2816}{3}S_{-2,-2,1,1}+256S_{-3,1,1,1}+9216S_{-2,1,1,1,1}\right\}\,,\tag{420}$$

$$C_{F_L,q}^{\mathrm{NS},(3)}=$$

$$C_F^2\left\{T_FN_F\left[\frac{8S_2P_{303}}{3N(1+N)^2(2+N)(3+N)}-\frac{64\zeta_3P_{331}}{3(N-2)(N-1)N(1+N)(2+N)(3+N)}\right.\right.$$

$$+\frac{2P_{446}}{27(N-2)^2N^3(1+N)^4(3+N)}+\left[\frac{32P_{425}}{27(N-2)(N-1)N^2(1+N)^3(2+N)(3+N)}\right.$$

$$\left.-\frac{64(4+3N)\left(-5+N^2\right)S_2}{3(1+N)^2(2+N)(3+N)}+\frac{128S_3}{3(1+N)}-\frac{256S_{2,1}}{3(1+N)}-\frac{1024S_{-2,1}}{3(1+N)}-\frac{1280\zeta_3}{3(1+N)}\right]S_1$$

$$+\left[\frac{8\left(30+83N+23N^2\right)}{9N(1+N)^2}+\frac{64S_2}{3(1+N)}\right]S_1^2+\frac{32\left(294+553N+270N^2+35N^3\right)S_3}{9(1+N)^2(2+N)(3+N)}$$

$$-\frac{128S_1^3}{9(1+N)}-\frac{128S_2^2}{3(1+N)}-\frac{1088S_4}{3(1+N)}+\left[\frac{512S_1P_{329}}{9(N-2)(N-1)N(1+N)(2+N)(3+N)}\right.$$

$$\left.-\frac{256P_{436}}{9(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)}+\frac{512S_1^2}{3(1+N)}-\frac{512S_2}{3(1+N)}\right]S_{-2}-\frac{256S_{-2}^2}{3(1+N)}$$

$$+\left[\frac{256P_{343}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)}-\frac{512S_1}{3(1+N)}\right]S_{-3}-\frac{256S_{-4}}{1+N}$$

$$+\frac{64\left(-10+25N+38N^2+11N^3\right)S_{2,1}}{3(1+N)^2(2+N)(3+N)}+\frac{128S_{3,1}}{1+N}-\frac{512\left(-16-23N+N^2+2N^3\right)S_{-2,1}}{9(N-1)(1+N)^2(2+N)}$$

$$+\frac{512S_{-3,1}}{3(1+N)}+\frac{128S_{2,1,1}}{1+N}+\frac{1024S_{-2,1,1}}{3(1+N)}\right]+C_A\left[-\frac{64S_{-2,2}P_{300}}{(N-2)N(1+N)^2(3+N)}\right.$$

$$+\frac{128S_{-2,1,1}P_{306}}{3(N-2)N(1+N)^2(3+N)}-\frac{64S_{-3,1}P_{310}}{3(N-2)N(1+N)^2(3+N)}-\frac{16S_{2,1}P_{311}}{3N(1+N)^2(2+N)(3+N)}$$

$$-\frac{64S_{3,1}P_{344}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}+\frac{32S_4P_{367}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$+\frac{16\zeta_3P_{407}}{3(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}+\frac{P_{434}}{54(N-2)^2N^3(1+N)^4(3+N)}$$

$$+\frac{32S_{-2,1}P_{408}}{9(N-2)(N-1)N(1+N)^3(2+N)(3+N)}$$

$$-\frac{8S_3P_{410}}{9(N-2)(N-1)N(1+N)^3(2+N)(3+N)}+\left(\frac{16S_2P_{294}}{3N(1+N)^2(2+N)(3+N)}\right.$$

$$-\frac{16\zeta_3P_{349}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}+\frac{64S_4}{1+N}+\frac{512S_{2,1}}{3(1+N)}-\frac{512S_{3,1}}{1+N}$$

$$+\frac{16S_3P_{360}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}-\frac{32S_{-2,1}P_{309}}{3(N-2)N(1+N)^2(3+N)}$$

$$+\frac{3712S_{-2,2}}{1+N}+\frac{3840S_{-3,1}}{1+N}-\frac{7168S_{-2,1,1}}{1+N}$$

$$-\frac{4P_{464}}{27(N-2)^2(N-1)N^3(1+N)^4(2+N)(3+N)}\right)S_1+\left(-\frac{128S_2}{3(1+N)}+\frac{160S_3}{1+N}+\frac{1728S_{-2,1}}{1+N}\right.$$

$$+\frac{160\zeta_3}{1+N}-\frac{2P_{315}}{9(N-2)N(1+N)^2(3+N)}\right)S_1^2+\frac{640S_1^3}{9(1+N)}+\left(-\frac{416S_3}{1+N}-\frac{5312S_{-2,1}}{1+N}+\frac{96\zeta_3}{1+N}\right.$$

$$-\frac{2P_{377}}{3(N-2)N(1+N)^3(2+N)(3+N)}\right)S_2+\frac{304S_2^2}{3(1+N)}-\frac{64(N-1)(2+N)S_5}{1+N}$$

$$+\left(\frac{64S_2P_{299}}{3(N-2)N(1+N)^2(3+N)}+\frac{16P_{461}}{9(N-2)^2(N-1)N^3(1+N)^4(2+N)(3+N)}\right.$$

$$+\left(-\frac{32P_{441}}{9(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)}-\frac{192S_2}{1+N}\right)S_1+\frac{64S_1^3}{1+N}$$

$$-\frac{64\left(-5+2N+2N^2\right)S_3}{1+N}-\frac{32\left(-72-108N+35N^2+23N^3\right)S_1^2}{3(N-2)N(1+N)(3+N)}+\frac{5376S_{2,1}}{1+N}$$

$$-\frac{128\left(-3+N+N^2\right)S_{-2,1}}{1+N}-\frac{64\left(-21+2N+2N^2\right)\zeta_3}{1+N}\right)S_{-2}$$

$$+\left(\frac{16P_{307}}{3(N-2)N(1+N)^2(3+N)}+\frac{384S_1}{1+N}\right)S_{-2}^2+\left(\frac{16S_1P_{318}}{3(N-2)N(1+N)^2(3+N)}\right.$$

$$-\frac{16P_{427}}{9(N-2)(N-1)N^2(1+N)^3(2+N)(3+N)} - \frac{1120S_1^2}{1+N} + \frac{2656S_2}{1+N} + \frac{448S_{-2}}{1+N}\bigg) S_{-3}$$

$$+\bigg(\frac{32P_{305}}{(N-2)N(1+N)^2(3+N)} - \frac{1600S_1}{1+N}\bigg) S_{-4} - \frac{64\big(-34+N+N^2\big)S_{-5}}{1+N} + \frac{384S_{2,3}}{1+N}$$

$$+\frac{2688S_{2,-3}}{1+N} - \frac{384S_{4,1}}{1+N} + \frac{64\big(-65+2N+2N^2\big)S_{-2,3}}{1+N} + \frac{2496S_{-4,1}}{1+N} - \frac{256S_{2,1,1}}{1+N}$$

$$-\frac{5376S_{2,1,-2}}{1+N} + \frac{768S_{3,1,1}}{1+N} + \frac{128\big(-10+N+N^2\big)S_{-2,1,-2}}{1+N} - \frac{5376S_{-2,2,1}}{1+N} - \frac{5376S_{-3,1,1}}{1+N}$$

$$+\frac{10752S_{-2,1,1,1}}{1+N} - \frac{160\big(-21+N+N^2\big)\zeta_5}{1+N}\bigg]\bigg\}$$

$$+C_F\bigg\{ T_F^2 N_F^2\bigg[\frac{32P_{313}}{81N^2(1+N)^3} + \frac{64\big(-6+7N+19N^2\big)S_1}{27N(1+N)^2} + \frac{64S_1^2}{9(1+N)} - \frac{64S_2}{9(1+N)}\bigg]$$

$$+C_A T_F N_F\bigg[\frac{64S_{3,1}P_{274}}{(N-2)(N-1)(1+N)(2+N)(3+N)}$$

$$+\frac{32S_4 P_{291}}{3(N-2)(N-1)(1+N)(2+N)(3+N)} + \frac{64S_3 P_{328}}{9(N-2)(N-1)(1+N)^2(2+N)(3+N)}$$

$$-\frac{16P_{415}}{81(N-2)^2 N^2(1+N)^3(3+N)} + \bigg(\frac{1024\zeta_3 P_{272}}{3(N-2)(N-1)(1+N)(2+N)(3+N)}$$

$$-\frac{256S_3 P_{273}}{3(N-2)(N-1)(1+N)(2+N)(3+N)} - \frac{16P_{426}}{27(N-2)(N-1)N^2(1+N)^3(2+N)(3+N)}$$

$$+\frac{64(N-1)(3+2N)S_2}{3(1+N)^2(3+N)} + \frac{128S_{2,1}}{3(1+N)} + \frac{512S_{-2,1}}{3(1+N)}\bigg) S_1 + \bigg(-\frac{640}{9(1+N)} - \frac{32S_2}{3(1+N)}\bigg) S_1^2$$

$$+\frac{64\big(21+28N+N^2\big)S_2}{9(1+N)^2(3+N)} + \frac{64S_2^2}{3(1+N)} + \bigg(-\frac{256S_1 P_{329}}{9(N-2)(N-1)N(1+N)(2+N)(3+N)}$$

$$+\frac{128P_{436}}{9(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)} - \frac{256S_1^2}{3(1+N)} + \frac{256S_2}{3(1+N)}\bigg) S_{-2} + \frac{128S_{-2}^2}{3(1+N)}$$

$$+\bigg(-\frac{128P_{343}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)} + \frac{256S_1}{3(1+N)}\bigg) S_{-3} + \frac{128S_{-4}}{1+N}$$

$$-\frac{128(N-1)(3+2N)S_{2,1}}{3(1+N)^2(3+N)} + \frac{256\big(-16-23N+N^2+2N^3\big)S_{-2,1}}{9(N-1)(1+N)^2(2+N)} - \frac{256S_{-3,1}}{3(1+N)} - \frac{64S_{2,1,1}}{1+N}$$

$$-\frac{512S_{-2,1,1}}{3(1+N)} + \frac{128\big(36-45N-22N^2+2N^3\big)\zeta_3}{3(N-2)N(1+N)(3+N)}\bigg]$$

$$+C_A^2\bigg[-\frac{1024S_{-2,1,1}P_{279}}{3(N-2)N(1+N)^2(3+N)} + \frac{64S_{-2,2}P_{288}}{(N-2)N(1+N)^2(3+N)}$$

$$+\frac{128S_{-3,1}P_{297}}{3(N-2)N(1+N)^2(3+N)} - \frac{16S_{3,1}P_{342}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$-\frac{8S_4 P_{365}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)} + \frac{16S_3 P_{368}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$-\frac{32S_{-2,1}P_{373}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)} - \frac{8\zeta_3 P_{374}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$+\frac{4P_{420}}{81(N-2)^2N^2(1+N)^3(3+N)} + \left(\frac{512S_{-2,1}P_{275}}{3(N-2)N(1+N)^2(3+N)}\right.$$

$$+\frac{16S_3 P_{352}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)} - \frac{32\zeta_3 P_{355}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$+\frac{4P_{456}}{27(N-2)(N-1)N^3(1+N)^4(2+N)(3+N)} - \frac{176(N-1)(3+2N)S_2}{3(1+N)^2(3+N)} - \frac{192S_4}{1+N}$$

$$-\frac{352S_{2,1}}{3(1+N)} + \frac{384S_{3,1}}{1+N} - \frac{1024S_{-2,2}}{1+N} - \frac{1024S_{-3,1}}{1+N} + \frac{2048S_{-2,1,1}}{1+N}\right)S_1 + \left(\frac{1276}{9(1+N)}\right.$$

$$+\frac{88S_2}{3(1+N)} - \frac{96S_3}{1+N} - \frac{512S_{-2,1}}{1+N} - \frac{64\zeta_3}{1+N}\right)S_1^2 + \left(\frac{44\left(-51-68N+7N^2\right)}{9(1+N)^2(3+N)} + \frac{288S_3}{1+N}\right.$$

$$+\frac{1536S_{-2,1}}{1+N}\right)S_2 - \frac{176S_2^2}{3(1+N)} + \frac{16\left(4+N+N^2\right)S_5}{1+N} + \left(\frac{512S_1^2}{3(1+N)} - \frac{704S_2}{3(1+N)}\right.$$

$$+\frac{32S_1 P_{366}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)} + \frac{32\left(-4+N+N^2\right)S_3}{1+N}$$

$$-\frac{16P_{442}}{9(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)} - \frac{1536S_{2,1}}{1+N} + \frac{32\left(-4+N+N^2\right)S_{-2,1}}{1+N}$$

$$+\frac{32(N-3)(4+N)\zeta_3}{1+N}\right)S_{-2} + \left(-\frac{16P_{301}}{3(N-2)N(1+N)^2(3+N)} - \frac{128S_1}{1+N}\right)S_{-2}^2$$

$$+\left(-\frac{128S_1 P_{293}}{3(N-2)N(1+N)^2(3+N)} + \frac{16P_{378}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\right.$$

$$+\frac{256S_1^2}{1+N} - \frac{768S_2}{1+N} - \frac{128S_{-2}}{1+N}\right)S_{-3} + \left(-\frac{16P_{302}}{(N-2)N(1+N)^2(3+N)} + \frac{384S_1}{1+N}\right)S_{-4}$$

$$+\frac{16\left(-40+N+N^2\right)S_{-5}}{1+N} + \frac{352(N-1)(3+2N)S_{2,1}}{3(1+N)^2(3+N)} - \frac{288S_{2,3}}{1+N} - \frac{768S_{2,-3}}{1+N} + \frac{288S_{4,1}}{1+N}$$

$$-\frac{32\left(-38+N+N^2\right)S_{-2,3}}{1+N} - \frac{704S_{-4,1}}{1+N} + \frac{176S_{2,1,1}}{1+N} + \frac{1536S_{2,1,-2}}{1+N} - \frac{576S_{3,1,1}}{1+N}$$

$$-\frac{32(N-3)(4+N)S_{-2,1,-2}}{1+N} + \frac{1536S_{-2,2,1}}{1+N} + \frac{1536S_{-3,1,1}}{1+N} - \frac{3072S_{-2,1,1,1}}{1+N}$$

$$+\frac{40\left(-16+N+N^2\right)\zeta_5}{1+N}\Bigg]\Bigg\}$$

$$+C_F^3\Bigg\{-\frac{256S_{-2,1,1}P_{289}}{(N-2)N(1+N)^2(3+N)} + \frac{128S_{-2,2}P_{290}}{(N-2)N(1+N)^2(3+N)}$$

$$+\frac{128S_{-3,1}P_{292}}{(N-2)N(1+N)^2(3+N)} + \frac{32S_{3,1}P_{357}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$-\frac{16S_4 P_{359}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)} - \frac{64S_{-2,1}P_{396}}{(N-2)(N-1)N(1+N)^3(2+N)(3+N)}$$

$$-\frac{16\zeta_3 P_{401}}{(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)} + \frac{8S_3 P_{403}}{(N-2)(N-1)N(1+N)^3(2+N)(3+N)}$$

$$+\frac{P_{462}}{6(N-2)^2 N^4(1+N)^5(3+N)} + \left(-\frac{64S_3 P_{345}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\right.$$

$$+\frac{64S_{-2,1}P_{298}}{(N-2)N(1+N)^2(3+N)} + \frac{32\zeta_3 P_{354}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$+\frac{8P_{460}}{(N-2)^2(N-1)N^3(1+N)^4(2+N)(3+N)} + \frac{8(1+3N)(10+9N)S_2}{N(1+N)^2} + \frac{640S_4}{1+N} + \frac{64S_{2,1}}{1+N}$$

$$\left.-\frac{512S_{3,1}}{1+N} - \frac{3328S_{-2,2}}{1+N} - \frac{3584S_{-3,1}}{1+N} + \frac{6144S_{-2,1,1}}{1+N}\right)S_1 + \left(-\frac{2P_{330}}{(N-2)N^2(1+N)^2(3+N)}\right.$$

$$\left.-\frac{80S_2}{1+N} + \frac{64S_3}{1+N} - \frac{1408S_{-2,1}}{1+N} - \frac{64\zeta_3}{1+N}\right)S_1^2 - \frac{8(2+13N+9N^2)S_1^3}{N(1+N)^2} + \frac{8S_1^4}{1+N}$$

$$+\left(\frac{2P_{346}}{(N-2)N^2(1+N)^3(3+N)} - \frac{320S_3}{1+N} + \frac{4480S_{-2,1}}{1+N} - \frac{192\zeta_3}{1+N}\right)S_2 + \frac{24S_2^2}{1+N}$$

$$+\frac{64(-8+N+N^2)S_5}{1+N} + \left(-\frac{32P_{459}}{(N-2)^2(N-1)N^3(1+N)^4(2+N)(3+N)}\right.$$

$$-\frac{128S_2 P_{278}}{(N-2)N(1+N)^2(3+N)} + \left(\frac{64P_{437}}{(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)} + \frac{384S_2}{1+N}\right)S_1$$

$$-\frac{64(24-28N-N^2+3N^3)S_1^2}{(N-2)N(1+N)(3+N)} - \frac{128S_1^3}{1+N} + \frac{128(-1+N+N^2)S_3}{1+N} - \frac{4608S_{2,1}}{1+N}$$

$$\left.+\frac{128(N-1)(2+N)S_{-2,1}}{1+N} + \frac{128(-9+N+N^2)\zeta_3}{1+N}\right)S_{-2} + \left(-\frac{256S_1}{1+N}\right.$$

$$\left.-\frac{32(-8+N+N^2)(-6+5N+5N^2)}{(N-2)N(1+N)^2(3+N)}\right)S_{-2}^2 + \left(-\frac{32S_1 P_{304}}{(N-2)N(1+N)^2(3+N)}\right.$$

$$\left.+\frac{32P_{424}}{(N-2)(N-1)N^2(1+N)^3(2+N)(3+N)} + \frac{1216S_1^2}{1+N} - \frac{2240S_2}{1+N} - \frac{384S_{-2}}{1+N}\right)S_{-3}$$

$$+\left(-\frac{64P_{295}}{(N-2)N(1+N)^2(3+N)} + \frac{1664S_1}{1+N}\right)S_{-4} + \frac{64(-28+N+N^2)S_{-5}}{1+N}$$

$$+\frac{16(-2+3N+3N^2)S_{2,1}}{N(1+N)^2} + \frac{384S_{2,3}}{1+N} - \frac{2304S_{2,-3}}{1+N} - \frac{384S_{4,1}}{1+N} - \frac{128(-27+N+N^2)S_{-2,3}}{1+N}$$

$$-\frac{2176S_{-4,1}}{1+N} - \frac{96S_{2,1,1}}{1+N} + \frac{4608S_{2,1,-2}}{1+N} + \frac{768S_{3,1,1}}{1+N} - \frac{128(-8+N+N^2)S_{-2,1,-2}}{1+N}$$

$$\left.+\frac{4608S_{-2,2,1}}{1+N} + \frac{4608S_{-3,1,1}}{1+N} - \frac{9216S_{-2,1,1,1}}{1+N} + \frac{160(-26+N+N^2)\zeta_5}{1+N}\right\}$$

$$(421)$$

with

$$P_{257} = -617N^4 - 1108N^3 - 281N^2 + 222N + 148, \tag{422}$$

$$P_{258} = -9N^4 - 4N^3 + 3N^2 + 26N + 20, \tag{423}$$

$$P_{259} = N^4 - 22N^3 - 325N^2 + 322N - 312, \tag{424}$$

$$P_{260} = N^4 - 22N^3 - 237N^2 + 74N - 192, \tag{425}$$

$$P_{261} = N^4 - 22N^3 - 226N^2 - 11N - 280, \tag{426}$$

$$P_{262} = N^4 - 22N^3 - 217N^2 + 310N - 240, \tag{427}$$

$$P_{263} = N^4 - 22N^3 - 137N^2 + 138N - 180, \tag{428}$$

$$P_{264} = N^4 - 22N^3 - 133N^2 - 14N - 84, \tag{429}$$

$$P_{265} = N^4 - 22N^3 - 112N^2 + 583N - 502, \tag{430}$$

$$P_{266} = N^4 - 22N^3 - 64N^2 + 7N - 34, \tag{431}$$

$$P_{267} = N^4 - 22N^3 - 37N^2 + 202N - 168, \tag{432}$$

$$P_{268} = N^4 - 22N^3 - 37N^2 + 634N - 518, \tag{433}$$

$$P_{269} = N^4 - 22N^3 - N^2 + 406N - 96, \tag{434}$$

$$P_{270} = N^4 - 22N^3 + 98N^2 + 313N - 354, \tag{435}$$

$$P_{271} = N^4 - 22N^3 + 271N^2 + 582N - 500, \tag{436}$$

$$P_{272} = N^4 + 2N^3 - 4N^2 - 5N + 9, \tag{437}$$

$$P_{273} = N^4 + 2N^3 - N^2 - 2N + 6, \tag{438}$$

$$P_{274} = N^4 + 2N^3 + 9N^2 + 8N - 4, \tag{439}$$

$$P_{275} = N^4 + 2N^3 + 16N^2 + 15N - 36, \tag{440}$$

$$P_{276} = 2N^4 - 44N^3 - 197N^2 - 7N - 118, \tag{441}$$

$$P_{277} = 2N^4 - 44N^3 + 369N^2 + 895N - 854, \tag{442}$$

$$P_{278} = 2N^4 + 4N^3 - 5N^2 - 7N - 12, \tag{443}$$

$$P_{279} = 2N^4 + 4N^3 + 11N^2 + 9N - 36, \tag{444}$$

$$P_{280} = 3N^4 - 66N^3 - 1501N^2 - 1720N + 192, \tag{445}$$

$$P_{281} = 3N^4 - 66N^3 - 1387N^2 + 1562N - 1188, \tag{446}$$

$$P_{282} = 3N^4 - 66N^3 - 221N^2 + 2584N - 1776, \tag{447}$$

$$P_{283} = 3N^4 - 66N^3 - 166N^2 + 2243N - 1665, \tag{448}$$

$$P_{284} = 5N^4 + 10N^3 + N^2 - 4N - 4, \tag{449}$$

$$P_{285} = 6N^4 - 132N^3 - 2003N^2 - 1577N - 792, \tag{450}$$

$$P_{286} = 6N^4 - 132N^3 - 1723N^2 + 3311N - 2694, \tag{451}$$

$$P_{287} = 9N^4 - 198N^3 - 1123N^2 - 52N - 576, \tag{452}$$

$$P_{288} = 9N^4 + 18N^3 + 11N^2 + 2N - 96, \tag{453}$$

$$P_{289} = 9N^4 + 18N^3 + 43N^2 + 34N - 144, \tag{454}$$

$$P_{290} = 10N^4 + 20N^3 + 39N^2 + 29N - 150, \tag{455}$$

$$P_{291} = 11N^4 + 22N^3 - 125N^2 - 136N + 180, \tag{456}$$

$$P_{292} = 11N^4 + 22N^3 + 35N^2 + 24N - 156, \tag{457}$$

$$P_{293} = 13N^4 + 26N^3 - 23N^2 - 36N - 72, \tag{458}$$

$$P_{294} = 15N^4 - 64N^3 - 366N^2 - 343N - 18, \tag{459}$$

$$P_{295} = 15N^4 + 30N^3 + 8N^2 - 7N - 138, \tag{460}$$

$$P_{296} = 18N^4 - 396N^3 - 1537N^2 - 43N - 1296, \tag{461}$$

$$P_{297} = 19N^4 + 38N^3 - 11N^2 - 30N - 144, \tag{462}$$

$$P_{298} = 23N^4 + 42N^3 + 55N^2 + 60N - 276, \tag{463}$$

$$P_{299} = 28N^4 + 56N^3 - 125N^2 - 153N - 36, \tag{464}$$

$$P_{300} = 28N^4 + 56N^3 + 61N^2 + 33N - 342, \tag{465}$$

$$P_{301} = 31N^4 + 62N^3 - 203N^2 - 234N + 144, \tag{466}$$

$$P_{302} = 43N^4 + 86N^3 - 119N^2 - 162N - 144, \tag{467}$$

$$P_{303} = 43N^4 + 118N^3 + 19N^2 - 104N - 60, \tag{468}$$

$$P_{304} = 47N^4 + 90N^3 + 7N^2 - 12N - 420, \tag{469}$$

$$P_{305} = 58N^4 + 116N^3 - 111N^2 - 169N - 282, \tag{470}$$

$$P_{306} = 59N^4 + 118N^3 + 305N^2 + 246N - 1008, \tag{471}$$

$$P_{307} = 77N^4 + 154N^3 - 529N^2 - 606N + 432, \tag{472}$$

$$P_{308} = 85N^4 + 86N^3 - 47N^2 - 136N - 84, \tag{473}$$

$$P_{309} = 101N^4 + 190N^3 + 677N^2 + 660N - 1980, \tag{474}$$

$$P_{310} = 109N^4 + 218N^3 + 61N^2 - 48N - 1044, \tag{475}$$

$$P_{311} = 121N^4 + 418N^3 + 269N^2 - 140N - 36, \tag{476}$$

$$P_{312} = 147N^4 + 426N^3 + 80N^2 - 211N - 18, \tag{477}$$

$$P_{313} = 203N^4 + 178N^3 - 31N^2 - 6N + 36, \tag{478}$$

$$P_{314} = 235N^4 + 596N^3 + 319N^2 + 66N + 72, \tag{479}$$

$$P_{315} = 265N^4 + 1478N^3 + 745N^2 - 6156N - 2844, \tag{480}$$

$$P_{316} = 277N^4 + 728N^3 + 152N^2 - 383N - 108, \tag{481}$$

$$P_{317} = 291N^4 + 368N^3 + 21N^2 - 380N - 272, \tag{482}$$

$$P_{318} = 349N^4 + 686N^3 - 347N^2 - 612N - 2412, \tag{483}$$

$$P_{319} = 423N^4 + 264N^3 + 293N^2 + 452N + 54, \tag{484}$$

$$P_{320} = 536N^4 + 1351N^3 + 498N^2 - 329N - 42, \tag{485}$$

$$P_{321} = 536N^4 + 1459N^3 + 565N^2 - 358N - 102, \tag{486}$$

$$P_{322} = 562N^4 + 1565N^3 + 649N^2 - 354N - 72, \tag{487}$$

$$P_{323} = 575N^4 + 1468N^3 + 324N^2 - 845N - 294, \tag{488}$$

$$P_{324} = 683N^4 + 1702N^3 + 767N^2 + 164N + 292, \tag{489}$$

$$P_{325} = 1055N^4 + 2236N^3 + 1139N^2 + 66N + 72, \tag{490}$$

$$P_{326} = 1124N^4 + 2320N^3 + 823N^2 - 451N - 93, \tag{491}$$

$$P_{327} = 4513N^4 + 11762N^3 + 2455N^2 - 4794N - 684, \tag{492}$$

$$P_{328} = 2N^5 - 9N^4 + 113N^3 + 273N^2 - 61N - 174, \tag{493}$$

$$P_{329} = 2N^5 + N^4 - 38N^3 - 13N^2 + 120N - 36, \tag{494}$$

$$P_{330} = 17N^5 - 62N^4 - 413N^3 + 218N^2 + 504N + 144, \tag{495}$$

$$P_{331} = 31N^5 - 34N^4 - 301N^3 + 148N^2 + 540N - 288, \tag{496}$$

$$P_{332} = 83N^5 + 404N^4 + 643N^3 - 50N^2 - 57N + 126, \tag{497}$$

$$P_{333} = 97N^5 + 460N^4 + 32N^3 - 1669N^2 - 1518N + 72, \tag{498}$$

$$P_{334} = 131N^5 + 308N^4 - 347N^3 - 3392N^2 - 2004N + 2592, \tag{499}$$

$$P_{335} = 328N^5 + 1985N^4 + 2899N^3 + 927N^2 - 231N - 180, \tag{500}$$

$$P_{336} = 361N^5 + 2020N^4 + 899N^3 - 7756N^2 - 8724N + 1728, \tag{501}$$

$$P_{337} = 1103N^5 + 4825N^4 + 6329N^3 + 2733N^2 + 66N + 360, \tag{502}$$

$$P_{338} = 1687N^5 + 7540N^4 + 539N^3 - 34066N^2 - 30444N + 9576, \tag{503}$$

$$P_{339} = 3153N^5 + 3318N^4 - 13819N^3 - 2896N^2 - 11820N + 9936, \tag{504}$$

$$P_{340} = 8425N^5 + 30483N^4 + 30615N^3 + 7625N^2 + 2712N + 3212, \tag{505}$$

$$P_{341} = -279N^6 - 1137N^5 - 1215N^4 - 63N^3 - 714N^2 - 1344N - 496, \tag{506}$$

$$P_{342} = 13N^6 + 39N^5 - 33N^4 - 131N^3 - 380N^2 - 308N + 48, \tag{507}$$

$$P_{343} = 14N^6 + 39N^5 - 46N^4 - 129N^3 + 50N^2 + 72N + 144, \tag{508}$$

$$P_{344} = 17N^6 + 51N^5 + 11N^4 - 63N^3 + 284N^2 + 324N + 48, \tag{509}$$

$$P_{345} = 18N^6 + 52N^5 - 37N^4 - 142N^3 + 101N^2 + 148N + 60, \tag{510}$$

$$P_{346} = 21N^6 + 15N^5 - 383N^4 - 463N^3 + 378N^2 + 608N + 96, \tag{511}$$

$$P_{347} = 33N^6 + 729N^5 - 997N^4 - 4225N^3 + 5052N^2 + 3240N + 6264, \tag{512}$$

$$P_{348} = 34N^6 - 450N^5 + 657N^4 + 3376N^3 - 4743N^2 - 3234N - 6192, \tag{513}$$

$$P_{349} = 35N^6 + 69N^5 + 1763N^4 + 3747N^3 + 902N^2 - 1548N + 3096, \tag{514}$$

$$P_{350} = 36N^6 - 390N^5 + 1249N^4 + 2912N^3 - 7001N^2 - 3342N - 7632, \tag{515}$$

$$P_{351} = 37N^6 + 169N^5 + 149N^4 - 749N^3 - 1170N^2 + 484N - 1032, \tag{516}$$

$$P_{352} = 41N^6 + 123N^5 - 157N^4 - 519N^3 - 484N^2 - 204N + 72, \tag{517}$$

$$P_{353} = 49N^6 + 210N^5 + 17N^4 - 848N^3 - 807N^2 + 29N - 318, \tag{518}$$

$$P_{354} = 51N^6 + 141N^5 + 15N^4 - 93N^3 + 226N^2 + 100N + 360, \tag{519}$$

$$P_{355} = 55N^6 + 165N^5 - 395N^4 - 1065N^3 + 28N^2 + 588N - 504, \tag{520}$$

$$P_{356} = 61N^6 - 50N^5 + 490N^4 + 842N^3 - 3659N^2 - 1536N - 3780, \tag{521}$$

$$P_{357} = 65N^6 + 195N^5 - 93N^4 - 511N^3 + 284N^2 + 572N + 288, \tag{522}$$

$$P_{358} = 74N^6 + 302N^5 - 307N^4 - 1512N^3 + 137N^2 + 826N + 840, \tag{523}$$

$$P_{359} = 75N^6 + 225N^5 - 131N^4 - 637N^3 + 240N^2 + 596N + 432, \tag{524}$$

$$P_{360} = 77N^6 + 219N^5 + 161N^4 + 69N^3 + 1742N^2 + 1548N + 216, \tag{525}$$

$$P_{361} = 83N^6 - 109N^5 - 77N^4 - 567N^3 - 42N^2 + 1096N + 576, \tag{526}$$

$$P_{362} = 96N^6 + 178N^5 - 595N^4 - 740N^3 - 13N^2 + 1274N + 1536, \tag{527}$$

$$P_{363} = 100N^6 + 313N^5 + 235N^4 + 31N^3 - 79N^2 - 60N - 20, \tag{528}$$

$$P_{364} = 112N^6 + 517N^5 + 773N^4 + 539N^3 + 311N^2 + 184N + 156, \tag{529}$$

$$P_{365} = 115N^6 + 345N^5 - 671N^4 - 1917N^3 + 1756N^2 + 2772N - 144, \tag{530}$$

$$P_{366} = 122N^6 + 300N^5 - 871N^4 - 1626N^3 + 2009N^2 + 1794N + 468, \tag{531}$$

$$P_{367} = 134N^6 + 402N^5 - 373N^4 - 1416N^3 + 1121N^2 + 1896N + 252, \tag{532}$$

$$P_{368} = 146N^6 + 603N^5 - 733N^4 - 2823N^3 + 2234N^2 + 3471N + 486, \tag{533}$$

$$P_{369} = 158N^6 + 178N^5 + 85N^4 + 28N^3 - 4043N^2 - 1126N - 3576, \tag{534}$$

$$P_{370} = 182N^6 - 1416N^5 + 1896N^4 + 9632N^3 - 15849N^2 - 8379N - 19494, \tag{535}$$

$$P_{371} = 208N^6 + 190N^5 + 711N^4 + 456N^3 - 7635N^2 - 2802N - 7488, \tag{536}$$

$$P_{372} = 248N^6 + 2595N^5 - 1572N^4 - 13771N^3 + 6576N^2 + 7356N + 12672, \tag{537}$$

$$P_{373} = 257N^6 + 705N^5 - 913N^4 - 2385N^3 + 3410N^2 + 3642N - 324, \tag{538}$$

$$P_{374} = 385N^6 + 99N^5 - 3089N^4 - 1239N^3 + 9376N^2 + 4644N - 2736, \tag{539}$$

$$P_{375} = 388N^6 + 6360N^5 - 6891N^4 - 36278N^3 + 34155N^2 + 24738N + 48240, \tag{540}$$

$$P_{376} = 429N^6 + 1849N^5 + 285N^4 - 7533N^3 - 7626N^2 + 716N - 3576, \tag{541}$$

$$P_{377} = 449N^6 + 705N^5 - 2395N^4 - 4669N^3 - 198N^2 + 3860N + 1896, \tag{542}$$

$$P_{378} = 521N^6 + 1497N^5 - 1177N^4 - 4233N^3 + 1826N^2 + 3114N + 2844, \tag{543}$$

$$P_{379} = 676N^6 + 5853N^5 - 2718N^4 - 32261N^3 + 8853N^2 + 19179N + 23166, \tag{544}$$

$$P_{380} = 913N^6 - 3282N^5 + 8202N^4 + 26842N^3 - 64629N^2 - 30582N - 73008, \tag{545}$$

$$P_{381} = 1046N^6 + 15756N^5 - 11349N^4 - 94906N^3 + 60393N^2 + 67464N + 102060, \tag{546}$$

$$P_{382} = 1360N^6 + 4524N^5 - 4169N^4 - 19314N^3 - 7739N^2 + 7914N + 5832, \tag{547}$$

$$P_{383} = 1472N^6 + 2121N^5 + 2868N^4 - 2785N^3 - 43668N^2 - 11448N - 39960, \tag{548}$$

$$P_{384} = 1540N^6 + 4053N^5 - 150N^4 - 13805N^3 - 28359N^2 - 3609N - 20682, \tag{549}$$

$$P_{385} = 1540N^6 + 9345N^5 - 792N^4 - 42161N^3 - 13683N^2 + 8649N + 13122, \tag{550}$$

$$P_{386} = 1545N^6 + 5807N^5 + 5289N^4 + 833N^3 + 2254N^2 + 3776N + 1232, \tag{551}$$

$$P_{387} = 2882N^6 + 12624N^5 - 5307N^4 - 59242N^3 - 19839N^2 + 22626N + 8424, \tag{552}$$

$$P_{388} = 4357N^6 + 20253N^5 + 23997N^4 + 10171N^3 + 666N^2 - 2700N - 1944, \tag{553}$$

$$P_{389} = 4502N^6 + 9708N^5 + 465N^4 - 27358N^3 - 93105N^2 - 17352N - 73548, \tag{554}$$

$$P_{390} = 4816N^6 + 5952N^5 + 12135N^4 - 1466N^3 - 156051N^2 - 48114N - 147312, \tag{555}$$

$$P_{391} = 6457N^6 + 43941N^5 - 12141N^4 - 206669N^3 - 9264N^2 + 63756N + 108864, \tag{556}$$

$$P_{392} = 7241N^6 + 50345N^5 + 124855N^4 + 122983N^3 + 29388N^2 - 3516N + 4032, \tag{557}$$

$$P_{393} = 7531N^6 + 26499N^5 + 27861N^4 + 8401N^3 + 336N^2 + 504N - 540, \tag{558}$$

$$P_{394} = 9187N^6 + 58477N^5 + 134975N^4 + 133571N^3 + 52890N^2 + 27036N + 24408, \tag{559}$$

$$P_{395} = -3221N^7 - 26286N^6 - 57930N^5 - 47240N^4 - 11865N^3 + 942N^2 - 72N - 1080, \tag{560}$$

$$P_{396} = 23N^7 + 88N^6 + 42N^5 - 144N^4 + 271N^3 + 824N^2 + 408N - 72, \tag{561}$$

$$P_{397} = 25N^7 - 379N^6 - 1119N^5 + 2215N^4 + 4787N^3 - 4173N^2 - 5436N - 4860, \tag{562}$$

$$P_{398} = 79N^7 + 169N^6 - 407N^5 - 1353N^4 - 912N^3 + 88N^2 + 1536N + 144, \tag{563}$$

$$P_{399} = 81N^7 - 1463N^6 - 5889N^5 - 333N^4 + 16148N^3 + 10448N^2 + 7072N + 3216, \tag{564}$$

$$P_{400} = 83N^7 + 487N^6 + 582N^5 - 1789N^4 - 4016N^3 - 339N^2 - 180N - 1836, \tag{565}$$

$$P_{401} = 115N^7 + 309N^6 + 141N^5 + 79N^4 + 1552N^3 + 1180N^2 - 48N - 288, \tag{566}$$

$$P_{402} = 173N^7 + 32N^6 - 2263N^5 - 2810N^4 + 3582N^3 + 4014N^2 + 2120N - 120, \tag{567}$$

$$P_{403} = 179N^7 + 700N^6 + 352N^5 - 1242N^4 - 407N^3 + 1830N^2 + 1572N + 216, \tag{568}$$

$$P_{404} = 379N^7 + 3227N^6 + 2631N^5 - 20843N^4 - 22294N^3 + 6252N^2 + 21744N + 18576, \tag{569}$$

$$P_{405} = 403N^7 + 3797N^6 + 6527N^5 - 9957N^4 - 29446N^3 - 8996N^2 + 4936N + 6240, \tag{570}$$

$$P_{406} = 417N^7 + 1362N^6 - 523N^5 - 4588N^4 - 4220N^3 - 2720N^2 + 864N + 2880, \tag{571}$$

$$P_{407} = 713N^7 + 1029N^6 - 2740N^5 - 1095N^4 + 11969N^3 + 6480N^2 - 2772N - 432, \tag{572}$$

$$P_{408} = 721N^7 + 2716N^6 - 38N^5 - 7892N^4 + 4489N^3 + 21520N^2 + 10308N - 1296, \tag{573}$$

$$P_{409} = 781N^7 - 2755N^6 - 22167N^5 - 45617N^4 + 12110N^3 + 59256N^2 + 100728N + 67824, \tag{574}$$

$$P_{410} = 1669N^7 + 7264N^6 + 3778N^5 - 16112N^4 - 2843N^3 + 29584N^2 + 22020N + 3024, \tag{575}$$

$$P_{411} = 1873N^7 + 8797N^6 + 5775N^5 - 38361N^4 - 70804N^3 - 27520N^2 + 20832N - 3888, \tag{576}$$

$$P_{412} = 1964N^7 + 11260N^6 + 8946N^5 - 44536N^4 - 72725N^3 - 15489N^2 + 21528N$$

$$+12204, \tag{577}$$

$$P_{413} = 2251N^7 + 11006N^6 + 6514N^5 - 39436N^4 - 64685N^3 - 23230N^2 + 1596N$$
$$-1656, \tag{578}$$

$$P_{414} = 2284N^7 + 12899N^6 + 12631N^5 - 41611N^4 - 89702N^3 - 32317N^2 + 10896N$$
$$-2700, \tag{579}$$

$$P_{415} = 2686N^7 - 128N^6 - 28605N^5 + 15784N^4 + 49639N^3 + 2700N^2 - 7740N$$
$$+4752, \tag{580}$$

$$P_{416} = 3098N^7 + 17416N^6 + 16611N^5 - 60352N^4 - 122465N^3 - 52368N^2 + 12312N$$
$$-7128, \tag{581}$$

$$P_{417} = 3691N^7 + 18675N^6 + 34038N^5 + 26401N^4 + 6405N^3 - 618N^2 - 36N - 540, \tag{582}$$

$$P_{418} = 4591N^7 + 32765N^6 + 46752N^5 - 90491N^4 - 251779N^3 - 89754N^2 + 43020N$$
$$+10584, \tag{583}$$

$$P_{419} = 4645N^7 + 15287N^6 - 15678N^5 - 107039N^4 - 63475N^3 + 26208N^2 + 51948N$$
$$+28080, \tag{584}$$

$$P_{420} = 16828N^7 + 802N^6 - 194889N^5 + 102862N^4 + 378583N^3 + 53370N^2 - 56556N$$
$$+26136, \tag{585}$$

$$P_{421} = 32287N^7 + 193848N^6 + 395214N^5 + 329536N^4 + 98223N^3 + 5076N^2 + 3996N$$
$$-324, \tag{586}$$

$$P_{422} = -116957N^8 - 894616N^7 - 2631828N^6 - 3627014N^5 - 2222155N^4 - 543882N^3$$
$$-286164N^2 - 177768N + 23328, \tag{587}$$

$$P_{423} = -28023N^8 - 203826N^7 - 566428N^6 - 738922N^5 - 430121N^4 - 91484N^3$$
$$-69852N^2 - 97776N - 34560, \tag{588}$$

$$P_{424} = 35N^8 + 136N^7 + 72N^6 - 222N^5 + 117N^4 + 702N^3 + 472N^2 + 32N + 96, \tag{589}$$

$$P_{425} = 296N^8 + 920N^7 - 1687N^6 - 6931N^5 - 679N^4 + 9827N^3 + 5634N^2 - 252$$
$$N - 216, \tag{590}$$

$$P_{426} = 805N^8 + 2632N^7 - 2678N^6 - 12656N^5 + 661N^4 + 16144N^3 + 5856N^2$$
$$-1044N + 648, \tag{591}$$

$$P_{427} = 1357N^8 + 5260N^7 + 1288N^6 - 12818N^5 - 3761N^4 + 16198N^3 + 16164N^2$$
$$+5976N + 864, \tag{592}$$

$$P_{428} = 1511N^8 + 9442N^7 + 22584N^6 + 26891N^5 + 16961N^4 + 5241N^3 + 38N^2 - 804N$$
$$-648, \tag{593}$$

$$P_{429} = 17529N^8 + 60627N^7 - 8987N^6 - 211343N^5 - 245062N^4 - 225100N^3 + 6936N^2$$
$$+155088N + 23328, \tag{594}$$

$$P_{430} = 28551N^8 + 63048N^7 + 57098N^6 + 23004N^5 + 3519N^4 + 3044N^3 - 9864N^2$$
$$-21888N - 9936, \tag{595}$$

$$P_{431} = 50689N^8 + 231286N^7 + 374706N^6 + 252688N^5 + 58645N^4 + 3486N^3$$
$$+22752N^2 + 19980N + 8100, \tag{596}$$

$$P_{432} = -97249N^9 - 674988N^8 - 1131968N^7 + 1296434N^6 + 5370597N^5 + 4078570N^4$$
$$-68076N^3 - 537576N^2 + 352224N + 114048, \tag{597}$$

$$P_{433} = -71503N^9 - 495343N^8 - 1283176N^7 - 1561498N^6 - 898003N^5 - 199315N^4$$

$$-12042N^3 - 12852N^2 - 4536N - 1620, \tag{598}$$

$$P_{434} = -47295N^9 - 50958N^8 + 608888N^7 + 213650N^6 - 1767525N^5 - 1823284N^4$$
$$+42212N^3 + 300888N^2 - 153216N - 85536, \tag{599}$$

$$P_{435} = -5021N^9 - 13792N^8 + 14010N^7 + 70516N^6 + 68039N^5 + 94644N^4 + 83628N^3$$
$$-75096N^2 - 108576N - 25056, \tag{600}$$

$$P_{436} = 10N^9 + 14N^8 - 108N^7 - 168N^6 + 213N^5 + 438N^4 + 29N^3 - 824N^2 - 180N$$
$$+288, \tag{601}$$

$$P_{437} = 14N^9 + 20N^8 - 133N^7 - 173N^6 + 407N^5 + 397N^4 - 384N^3 - 964N^2 - 192N$$
$$+288, \tag{602}$$

$$P_{438} = 25N^9 - 121N^8 - 775N^7 - 546N^6 + 1772N^5 - 1113N^4 - 5854N^3 - 2060N^2$$
$$+1112N + 384, \tag{603}$$

$$P_{439} = 79N^9 + 61N^8 - 785N^7 - 936N^6 + 1418N^5 - 2005N^4 - 8840N^3 - 7848N^2$$
$$-3104N - 1104, \tag{604}$$

$$P_{440} = 307N^9 + 3168N^8 + 6674N^7 - 2144N^6 - 18819N^5 - 6472N^4 + 12218N^3 - 3564N^2$$
$$-15128N - 6288, \tag{605}$$

$$P_{441} = 370N^9 + 536N^8 - 4027N^7 - 4267N^6 + 14417N^5 + 9647N^4 - 14144N^3$$
$$-16788N^2 - 3600N + 2592, \tag{606}$$

$$P_{442} = 430N^9 + 728N^8 - 3699N^7 - 6279N^6 + 7179N^5 + 13731N^4 - 616N^3 - 21338N^2$$
$$-5472N + 6552, \tag{607}$$

$$P_{443} = 575N^9 + 7672N^8 + 19410N^7 - 7588N^6 - 68521N^5 - 45276N^4 + 23872N^3$$
$$+21648N^2 - 8624N - 3840, \tag{608}$$

$$P_{444} = 1288N^9 + 33279N^8 + 109499N^7 - 4538N^6 - 391455N^5 - 190195N^4 + 311202N^3$$
$$+104832N^2 - 74520N - 16848, \tag{609}$$

$$P_{445} = 1963N^9 + 30012N^8 + 88574N^7 - 19280N^6 - 343611N^5 - 220246N^4 + 153144N^3$$
$$+49212N^2 - 44496N - 6480, \tag{610}$$

$$P_{446} = 2133N^9 + 810N^8 - 35584N^7 + 4418N^6 + 120351N^5 + 97028N^4 - 21724N^3$$
$$-15336N^2 + 17856N + 7776, \tag{611}$$

$$P_{447} = 3761N^9 - 14568N^8 - 104870N^7 - 23608N^6 + 335763N^5 + 109918N^4$$
$$-430176N^3 - 423972N^2 - 170640N + 9072, \tag{612}$$

$$P_{448} = 4258N^9 + 34062N^8 + 65735N^7 - 85433N^6 - 339093N^5 - 148117N^4 + 85182N^3$$
$$-99882N^2 - 153360N - 48600, \tag{613}$$

$$P_{449} = 5894N^9 - 12921N^8 - 126065N^7 - 48880N^6 + 374049N^5 + 55783N^4$$
$$-668856N^3 - 635868N^2 - 254448N - 20736, \tag{614}$$

$$P_{450} = 43949N^9 + 268194N^8 + 376282N^7 - 637870N^6 - 2123703N^5 - 1675742N^4$$
$$-209052N^3 + 96174N^2 - 61992N - 8424, \tag{615}$$

$$P_{451} = -5563N^{10} - 24261N^9 - 16394N^8 + 58642N^7 + 172417N^6 + 124231N^5$$
$$-76780N^4 - 55652N^3 + 153696N^2 + 131616N + 34560, \tag{616}$$

$$P_{452} = -4253N^{10} - 18719N^9 - 3594N^8 + 69026N^7 + 89183N^6 + 6165N^5 - 37744N^4$$
$$+8424N^3 + 50664N^2 + 35696N + 8160, \tag{617}$$

$$P_{453} = 24N^{10} - 230N^9 - 267N^8 + 2755N^7 - 219N^6 - 13309N^5 + 8190N^4 + 29248N^3$$

$$-144N^2 - 28400N - 17376, \tag{618}$$

$$P_{454} = 261N^{10} + 324N^9 - 1405N^8 + 651N^7 - 1688N^6 - 14839N^5 + 15318N^4 + 41614N^3$$
$$+ 29772N^2 - 27720N - 28944, \tag{619}$$

$$P_{455} = 561N^{10} + 2115N^9 + 794N^8 - 3146N^7 - 15755N^6 - 9537N^5 + 39368N^4$$
$$+ 15976N^3 - 84440N^2 - 83120N - 24864, \tag{620}$$

$$P_{456} = 6439N^{10} + 27179N^9 - 8632N^8 - 154834N^7 - 149641N^6 + 93575N^5 + 183334N^4$$
$$+ 75768N^3 - 2448N^2 - 12636N - 9720, \tag{621}$$

$$P_{457} = 7868N^{10} + 57302N^9 + 2431N^8 - 508191N^7 - 330541N^6 + 1419481N^5$$
$$+ 541702N^4 - 1330188N^3 + 96696N^2 + 1283040N + 663552, \tag{622}$$

$$P_{458} = -5729259N^{11} - 8768673N^{10} + 46945558N^9 + 46575050N^8 - 44721647N^7$$
$$- 98312261N^6 - 77570092N^5 - 6099524N^4 + 7097376N^3 - 18265104N^2$$
$$+ 8239104N + 14427072, \tag{623}$$

$$P_{459} = 27N^{11} + 69N^{10} - 176N^9 - 554N^8 + 209N^7 + 1231N^6 + 396N^5 - 1730N^4$$
$$- 2104N^3 - 792N^2 + 352N + 192, \tag{624}$$

$$P_{460} = 33N^{11} + 90N^{10} - 827N^9 - 2100N^8 + 2393N^7 + 7786N^6 + 2037N^5 - 7676N^4$$
$$- 8880N^3 - 936N^2 + 1712N + 1248, \tag{625}$$

$$P_{461} = 1103N^{11} + 2937N^{10} - 7526N^9 - 24942N^8 + 3681N^7 + 52899N^6 + 29794N^5$$
$$- 59478N^4 - 72556N^3 - 4968N^2 + 16272N + 1728, \tag{626}$$

$$P_{462} = 1937N^{11} + 4659N^{10} - 17094N^9 - 30326N^8 + 45693N^7 + 123927N^6 + 59528N^5$$
$$- 29940N^4 - 13440N^3 + 26832N^2 + 17088N + 2880, \tag{627}$$

$$P_{463} = 6009N^{11} + 89967N^{10} + 289824N^9 - 63490N^8 - 1473715N^7 - 1627513N^6$$
$$+ 315866N^5 + 800300N^4 + 227984N^3 + 702960N^2 + 787680N + 229824, \tag{628}$$

$$P_{464} = 8183N^{11} + 18327N^{10} - 109486N^9 - 251134N^8 + 289655N^7 + 850715N^6$$
$$+ 154440N^5 - 665500N^4 - 552096N^3 - 42048N^2 + 84672N + 62208, \tag{629}$$

$$P_{465} = 80453N^{11} + 688424N^{10} + 1757044N^9 + 44376N^8 - 6544585N^7 - 10461842N^6$$
$$- 6460412N^5 - 1603590N^4 - 41364N^3 + 220968N^2 + 31104N - 69984, \tag{630}$$

$$P_{466} = 428649N^{11} + 601299N^{10} - 3536240N^9 - 3241402N^8 + 3843517N^7 + 7383499N^6$$
$$+ 4538354N^5 + 20668N^4 + 528168N^3 + 1514448N^2 - 1263168N - 1311552, \tag{631}$$

$$P_{467} = 463143N^{11} + 3002511N^{10} + 5392892N^9 - 3891846N^8 - 24947513N^7$$
$$- 31706485N^6 - 15500034N^5 - 1653956N^4 - 1227864N^3 - 2638368N^2$$
$$- 1159488N - 124416, \tag{632}$$

$$P_{468} = 241N^{12} + 521N^{11} - 1523N^{10} - 3268N^9 + 659N^8 - 1815N^7 + 2291N^6 + 25482N^5$$
$$+ 45116N^4 + 19656N^3 - 27296N^2 - 32512N - 9984, \tag{633}$$

$$P_{469} = 10629N^{12} + 22185N^{11} - 70738N^{10} - 118858N^9 + 91525N^8 - 54183N^7 - 58576N^6$$
$$+ 818608N^5 + 1687828N^4 + 888516N^3 - 727992N^2 - 580176N - 28512, \tag{634}$$

$$P_{470} = 27765N^{12} + 58437N^{11} - 182597N^{10} - 325952N^9 + 200843N^8 - 157371N^7$$
$$- 55295N^6 + 2325230N^5 + 4593788N^4 + 2307744N^3 - 2192976N^2 - 2038176N$$
$$- 326592, \tag{635}$$

$$P_{471} = -3069N^{13} + 31221N^{12} + 144399N^{11} - 301293N^{10} - 795373N^9 + 250503N^8$$
$$+ 1886057N^7 + 1623213N^6 - 63486N^5 - 894316N^4 - 31760N^3 + 890256N^2$$

$$
\begin{aligned}
&+677088N + 157248, \tag{636}\\
P_{472} =\ & 494694N^{13} + 1075005N^{12} - 4137924N^{11} - 6752619N^{10} + 5087252N^9 \\
&+ 13650627N^8 + 2421476N^7 - 9000225N^6 - 3343434N^5 + 7931228N^4 \\
&+ 4200112N^3 - 6598896N^2 - 6774048N - 1729728, \tag{637}\\
P_{473} =\ & 599375N^{13} + 5709355N^{12} + 18035106N^{11} + 12196222N^{10} - 51833983N^9 \\
&- 129886893N^8 - 124003084N^7 - 53668604N^6 - 7785942N^5 + 2544156N^4 \\
&+ 6838020N^3 + 7118928N^2 + 4129056N + 1049760, \tag{638}\\
P_{474} =\ & -458487N^{14} - 3433746N^{13} - 4940481N^{12} + 20878856N^{11} + 64591263N^{10} \\
&+ 10248134N^9 - 139889479N^8 - 158068804N^7 - 33854512N^6 + 17126664N^5 \\
&- 5950960N^4 + 25711872N^3 + 46793088N^2 + 27948672N + 5971968, \tag{639}\\
P_{475} =\ & 3003N^{14} + 22594N^{13} + 47885N^{12} - 148240N^{11} - 659235N^{10} - 247430N^9 \\
&+ 1482531N^8 + 1874548N^7 + 339992N^6 - 373632N^5 - 131264N^4 - 1500992N^3 \\
&- 2232704N^2 - 1344000N - 301824, \tag{640}\\
P_{476} =\ & -7255N^{15} - 25524N^{14} + 35986N^{13} + 183186N^{12} + 88824N^{11} - 295334N^{10} \\
&- 321342N^9 + 60030N^8 + 10607N^7 - 441646N^6 - 382164N^5 + 194648N^4 \\
&+ 529968N^3 + 385824N^2 + 130752N + 17280. \tag{641}
\end{aligned}
$$

Further non–singlet contributions are

$$
\begin{aligned}
C_{F_2,q}^{d_{abc},(3)} =\ & \frac{d_{abc}d^{abc}N_F}{N_C}\Bigg\{ \frac{64\big(-3 - N + N^2\big)}{(N-2)(3+N)} - \frac{128(N-1)S_3 P_{479}}{(N-2)N(1+N)(2+N)(3+N)} \\
& - \frac{256 S_4 P_{483}}{(N-2)N^2(1+N)^2(2+N)(3+N)} + \frac{512 S_{3,1} P_{483}}{(N-2)N^2(1+N)^2(2+N)(3+N)} \\
& - \frac{256 S_{-2,1} P_{484}}{(N-2)N(1+N)(2+N)(3+N)} + \Bigg[-\frac{128\big(-66 + 5N - 9N^2 + N^3\big)}{(N-2)N(1+N)(2+N)(3+N)} \\
& - \frac{256 S_3 P_{483}}{(N-2)N^2(1+N)^2(2+N)(3+N)} - \frac{1024(N-3)S_{-2,1}}{(N-2)(1+N)(3+N)} \Bigg] S_1 \\
& - \Bigg[\frac{256 S_1 P_{480}}{(N-2)N(1+N)^2(2+N)(3+N)} + \frac{64 P_{487}}{(N-2)N^2(1+N)^2(2+N)(3+N)} \Bigg] \\
& \times S_{-2} + \Bigg[\frac{128 P_{484}}{(N-2)N(1+N)(2+N)(3+N)} + \frac{512(N-3)S_1}{(N-2)(1+N)(3+N)} \Bigg] S_{-3} \\
& + \frac{64(N-1)S_{-2}^2}{1+N} + \frac{64\big(-18 + N + N^3\big)S_{-4}}{(N-2)(1+N)(3+N)} - \frac{1024(N-3)S_{-2,2}}{(N-2)(1+N)(3+N)} \\
& - \frac{512(N-1)S_{-2,3}}{N(1+N)} - \frac{1024(N-3)S_{-3,1}}{(N-2)(1+N)(3+N)} + \frac{512(N-1)S_{-4,1}}{N(1+N)} \\
& + \frac{2048(N-3)S_{-2,1,1}}{(N-2)(1+N)(3+N)} + \Bigg[\frac{32 P_{485}}{3(N-2)N(1+N)(2+N)(3+N)} \\
& + \frac{1024\big(-12 - 4N - 3N^2 + N^3\big)S_1}{(N-2)N^2(1+N)^2(2+N)(3+N)} + \frac{1024(N-1)S_{-2}}{N(1+N)} \Bigg] \zeta_3
\end{aligned}
$$

$$-\frac{1280(N-3)(N-2)}{3N(1+N)}\zeta_5\Bigg\}, \tag{642}$$

$$C_{F_L,q}^{d_{abc},(3)} = \frac{d_{abc}d^{abc}N_F}{N_C}\Bigg\{-\frac{128N}{(N-2)(3+N)} - \frac{128S_3P_{478}}{(N-2)(N-1)(1+N)(2+N)(3+N)}$$

$$+\frac{512S_{-2,1}P_{482}}{(N-2)(N-1)(1+N)(2+N)(3+N)} + \Bigg[\frac{4096S_{-2,1}}{(N-2)(1+N)(3+N)}$$

$$+\frac{1024\left(6+N+2N^2+2N^3+N^4\right)S_3}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$+\frac{1536\left(8+N+N^2\right)}{(N-2)(N-1)(1+N)(2+N)(3+N)}\Bigg]S_1$$

$$+\frac{1024\left(6+N+2N^2+2N^3+N^4\right)S_4}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$+\Bigg[\frac{1536S_1P_{481}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$+\frac{128P_{486}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\Bigg]S_{-2} - \frac{128S_{-2}^2}{1+N}$$

$$+\Bigg[-\frac{256P_{482}}{(N-2)(N-1)(1+N)(2+N)(3+N)} - \frac{2048S_1}{(N-2)(1+N)(3+N)}\Bigg]S_{-3}$$

$$-\frac{128\left(10+N+N^2\right)S_{-4}}{(N-2)(1+N)(3+N)} - \frac{2048\left(6+N+2N^2+2N^3+N^4\right)S_{3,1}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$

$$+\frac{4096S_{-2,2}}{(N-2)(1+N)(3+N)} - \frac{512S_{-2,3}}{1+N} + \frac{4096S_{-3,1}}{(N-2)(1+N)(3+N)} + \frac{512S_{-4,1}}{1+N}$$

$$-\frac{8192S_{-2,1,1}}{(N-2)(1+N)(3+N)} + \Bigg[-\frac{64P_{477}}{(N-2)(N-1)(1+N)(2+N)(3+N)}$$

$$+\frac{1024S_{-2}}{1+N} - \frac{6144\left(2+N+N^2\right)S_1}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\Bigg]\zeta_3 + \frac{2560}{1+N}\zeta_5\Bigg\}, \tag{643}$$

with

$$P_{477} = N^4 + 2N^3 - 43N^2 - 44N + 36, \tag{644}$$
$$P_{478} = N^4 + 2N^3 + 5N^2 + 4N + 36, \tag{645}$$
$$P_{479} = N^4 + 6N^3 + 6N^2 - 8N - 24, \tag{646}$$
$$P_{480} = 2N^4 - 4N^3 - 17N^2 - 23N - 6, \tag{647}$$
$$P_{481} = 2N^4 + 4N^3 + 3N^2 + N + 2, \tag{648}$$
$$P_{482} = 4N^4 + 8N^3 - 31N^2 - 35N + 18, \tag{649}$$
$$P_{483} = N^5 - 5N^3 - 12N^2 - 8N - 24, \tag{650}$$
$$P_{484} = 2N^5 - 21N^3 + 21N^2 + 46N - 24, \tag{651}$$
$$P_{485} = 7N^5 + 22N^4 - 25N^3 + 128N^2 + 60N - 432, \tag{652}$$
$$P_{486} = 7N^6 + 21N^5 - 26N^4 - 87N^3 - 47N^2 - 12, \tag{653}$$
$$P_{487} = 5N^7 + 11N^6 - 31N^5 - 49N^4 + 28N^3 + 68N^2 + 40N + 48. \tag{654}$$

The pure singlet contributions are given by

$$C_{F_2,q}^{\mathrm{PS},(3)} =$$

$$C_F \left\{ T_F^2 N_F \left[-\frac{512 S_{-2} P_{495}}{9(N-2)(N-1)N^2(1+N)(2+N)^2(3+N)} \right. \right.$$

$$+\frac{128\zeta_3 P_{492}}{9(N-1)N^2(1+N)^2(2+N)} + \frac{16 S_1^2 P_{516}}{27(N-1)N^3(1+N)^3(2+N)^2}$$

$$-\frac{32 P_{547}}{243(N-2)(N-1)N^5(1+N)^5(2+N)^4(3+N)}$$

$$+\frac{16 S_2 P_{522}}{27(N-1)N^3(1+N)^3(2+N)^2} + \left(-\frac{32 P_{537}}{81(N-1)N^4(1+N)^4(2+N)^3} \right.$$

$$\left. -\frac{32\left(2+N+N^2\right)^2 S_2}{9(N-1)N^2(1+N)^2(2+N)} \right) S_1 + \frac{64\left(2+N+N^2\right)^2 S_1^3}{27(N-1)N^2(1+N)^2(2+N)}$$

$$\left. -\frac{160\left(2+N+N^2\right)^2 S_3}{27(N-1)N^2(1+N)^2(2+N)} + \frac{1024 S_{-3}}{3(N-1)N(1+N)(2+N)} \right]$$

$$+C_A T_F N_F \left[-\frac{48\left(2+N+N^2\right)^2}{(N-1)N^2(1+N)^2(2+N)}\zeta_4 - \frac{16 S_{-2,2} P_{488}}{(N-1)N^2(1+N)^2(2+N)} \right.$$

$$+\frac{64 S_{-3,1} P_{493}}{3(N-1)N^2(1+N)^2(2+N)} + \frac{64 S_{-2,1,1} P_{494}}{3(N-1)N^2(1+N)^2(2+N)}$$

$$-\frac{64 S_{3,1} P_{497}}{(N-2)N^2(1+N)^2(2+N)(3+N)} - \frac{8 S_{-4} P_{501}}{3(N-1)N^2(1+N)^2(2+N)}$$

$$-\frac{4 S_4 P_{508}}{3(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)} - \frac{8\left(2+N+N^2\right)S_1^3 P_{510}}{27(N-1)^2 N^3(1+N)^3(2+N)^2}$$

$$-\frac{16 S_{2,1} P_{519}}{3(N-1)N^3(1+N)^3(2+N)^2} - \frac{16 S_{-2,1} P_{520}}{3(N-1)N^3(1+N)^3(2+N)^2}$$

$$+\frac{8 S_3 P_{538}}{27(N-2)(N-1)^2 N^3(1+N)^3(2+N)^2(3+N)} - \frac{4 S_2 P_{542}}{27(N-1)^2 N^4(1+N)^4(2+N)^3}$$

$$+\frac{8 P_{550}}{243(N-2)(N-1)^2 N^6(1+N)^6(2+N)^5(3+N)} + \left(-\frac{16 S_{-2,1} P_{499}}{3(N-1)N^2(1+N)^2(2+N)} \right.$$

$$+\frac{8 S_3 P_{513}}{9(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)} + \frac{8 S_2 P_{526}}{9(N-1)^2 N^3(1+N)^3(2+N)^2}$$

$$\left. +\frac{4 P_{548}}{81(N-2)(N-1)^2 N^5(1+N)^5(2+N)^4(3+N)} \right) S_1$$

$$+\left(\frac{4 P_{541}}{27(N-1)^2 N^4(1+N)^4(2+N)^3} - \frac{80\left(2+N+N^2\right)^2 S_2}{3(N-1)N^2(1+N)^2(2+N)} \right) S_1^2$$

$$+\frac{28\left(2+N+N^2\right)^2 S_1^4}{9(N-1)N^2(1+N)^2(2+N)} + \frac{16\left(2+N+N^2\right)\left(5+2N+2N^2\right)S_2^2}{3(N-1)N^2(1+N)^2(2+N)}$$

$$+\left(-\frac{16 S_1 P_{531}}{3(N-2)(N-1)^2 N^3(1+N)^3(2+N)^2(3+N)} \right.$$

$$-\frac{8P_{543}}{9(N-2)(N-1)^2N^4(1+N)^4(2+N)^3(3+N)}$$

$$-\frac{16\big(-22+N+N^2\big)\big(2+N+N^2\big)S_1^2}{3(N-1)N^2(1+N)^2(2+N)}+\frac{32\big(-8+N+N^2\big)\big(2+N+N^2\big)S_2}{3(N-1)N^2(1+N)^2(2+N)}\bigg)S_{-2}$$

$$-\frac{16\big(2+N+N^2\big)\big(2+3N+3N^2\big)S_{-2}^2}{3(N-1)N^2(1+N)^2(2+N)}+\bigg(\frac{8S_1P_{503}}{3(N-1)N^2(1+N)^2(2+N)}$$

$$+\frac{8P_{527}}{3(N-1)^2N^3(1+N)^3(2+N)^2}\bigg)S_{-3}+\frac{32\big(2+N+N^2\big)^2S_{2,1,1}}{3(N-1)N^2(1+N)^2(2+N)}$$

$$+\bigg(-\frac{16S_1P_{509}}{3(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}$$

$$-\frac{16P_{534}}{9(N-2)(N-1)^2N^3(1+N)^3(2+N)^2(3+N)}\bigg)\zeta_3\bigg]\bigg\}$$

$$+C_F^2T_FN_F\bigg\{\frac{48\big(2+N+N^2\big)^2}{(N-1)N^2(1+N)^2(2+N)}\zeta_4-\frac{256S_{-2,1}P_{496}}{(N-1)N^3(1+N)^3(2+N)^2}$$

$$-\frac{64S_{3,1}P_{506}}{(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}$$

$$+\frac{8S_4P_{511}}{3(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}$$

$$+\frac{64S_{2,1}P_{518}}{3(N-1)N^3(1+N)^3(2+N)^2}+\frac{16S_1^3P_{521}}{9(N-1)N^3(1+N)^3(2+N)^2}$$

$$-\frac{8S_3P_{530}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^2(3+N)}-\frac{4S_2P_{532}}{3(N-1)N^4(1+N)^4(2+N)^3}$$

$$-\frac{4P_{549}}{3(N-2)^2(N-1)N^6(1+N)^6(2+N)^4(3+N)}$$

$$+\bigg(\frac{16S_3P_{515}}{9(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}-\frac{8S_2P_{523}}{3(N-1)N^3(1+N)^3(2+N)^2}$$

$$+\frac{8P_{546}}{3(N-2)(N-1)N^5(1+N)^5(2+N)^4(3+N)}-\frac{256S_{-2,1}}{N^2(1+N)^2}\bigg)S_1$$

$$+\bigg(\frac{4P_{535}}{3(N-1)N^4(1+N)^4(2+N)^3}-\frac{112\big(2+N+N^2\big)^2S_2}{3(N-1)N^2(1+N)^2(2+N)}\bigg)S_1^2$$

$$+\frac{44\big(2+N+N^2\big)^2S_1^4}{9(N-1)N^2(1+N)^2(2+N)}+\frac{68\big(2+N+N^2\big)^2S_2^2}{3(N-1)N^2(1+N)^2(2+N)}$$

$$+\bigg(-\frac{32S_1P_{524}}{(N-2)(N-1)N^3(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{32P_{539}}{(N-2)^2(N-1)N^4(1+N)^4(2+N)^2(3+N)}-\frac{128\big(4+N+N^2\big)S_1^2}{(N-1)N^2(1+N)^2(2+N)}$$

$$+\frac{128\big(4+N+N^2\big)S_2}{(N-1)N^2(1+N)^2(2+N)}\bigg)S_{-2}+\bigg(\frac{128\big(6+N+N^2\big)S_1}{(N-1)N^2(1+N)^2(2+N)}$$

$$
\begin{aligned}
&+\frac{64P_{528}}{(N-2)(N-1)N^3(1+N)^3(2+N)^2(3+N)}\Bigg)S_{-3}+\frac{128\big(6+5N+5N^2\big)S_{-4}}{(N-1)N^2(1+N)^2(2+N)}\\
&-\frac{256\big(2+3N+3N^2\big)S_{-2,2}}{(N-1)N^2(1+N)^2(2+N)}-\frac{1024\big(1+N+N^2\big)S_{-3,1}}{(N-1)N^2(1+N)^2(2+N)}\\
&-\frac{32\big(2+N+N^2\big)^2 S_{2,1,1}}{3(N-1)N^2(1+N)^2(2+N)}+\frac{1024S_{-2,1,1}}{(N-1)N(1+N)(2+N)}\\
&+\Bigg(-\frac{128S_1 P_{507}}{3(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}\\
&+\frac{32P_{529}}{3(N-2)(N-1)N^3(1+N)^3(2+N)^2(3+N)}\Bigg)\zeta_3\Bigg\}
\end{aligned}
\tag{655}
$$

and

$$
\begin{aligned}
&C_{F_L,q}^{\mathrm{PS},(3)}=\\
&C_F\Bigg\{T_F^2 N_F^2\Bigg[\frac{128P_{536}}{27(N-2)(N-1)N^3(1+N)^4(2+N)^3(3+N)}\\
&+\frac{128\big(16+27N+13N^2+8N^3\big)S_1}{9(N-1)N(1+N)^3(2+N)}-\frac{64\big(2+N+N^2\big)S_1^2}{3(N-1)N(1+N)^2(2+N)}\\
&-\frac{64\big(2+N+N^2\big)S_2}{3(N-1)N(1+N)^2(2+N)}-\frac{2048S_{-2}}{3(N-2)(N-1)(1+N)(2+N)(3+N)}\Bigg]\\
&+C_A T_F N_F\Bigg[-\frac{32S_3 P_{502}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}-\frac{16S_1^2 P_{512}}{3(N-1)^2 N^2(1+N)^3(2+N)^2}\\
&+\frac{16S_2 P_{514}}{3(N-1)^2 N^2(1+N)^3(2+N)^2}-\frac{32P_{545}}{27(N-2)(N-1)^2 N^4(1+N)^5(2+N)^4(3+N)}\\
&+\Bigg(-\frac{32P_{540}}{9(N-2)(N-1)^2 N^3(1+N)^4(2+N)^3(3+N)}+\frac{160\big(2+N+N^2\big)S_2}{(N-1)N(1+N)^2(2+N)}\\
&+\frac{256\big(2+N+N^2\big)S_3}{(N-2)(N-1)(1+N)(2+N)(3+N)}\Bigg)S_1-\frac{160\big(2+N+N^2\big)S_1^3}{3(N-1)N(1+N)^2(2+N)}\\
&+\frac{256\big(2+N+N^2\big)S_4}{(N-2)(N-1)(1+N)(2+N)(3+N)}+\Bigg(\frac{128S_1 P_{490}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\\
&+\frac{64P_{525}}{3(N-2)(N-1)^2 N^2(1+N)^3(2+N)^2(3+N)}\Bigg)S_{-2}-\frac{576\big(2+N+N^2\big)S_{-3}}{(N-1)N(1+N)^2(2+N)}\\
&-\frac{512\big(2+N+N^2\big)S_{3,1}}{(N-2)(N-1)(1+N)(2+N)(3+N)}+\frac{384\big(2+N+N^2\big)S_{-2,1}}{(N-1)N(1+N)^2(2+N)}\\
&+\Bigg(\frac{192\big(18+5N+5N^2\big)}{(N-2)N(1+N)^2(3+N)}-\frac{512\big(2+N+N^2\big)S_1}{(N-2)(N-1)(1+N)(2+N)(3+N)}\Bigg)\zeta_3\Bigg]\Bigg\}\\
&+C_F^2 T_F N_F\Bigg\{\frac{256S_{-3}P_{491}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}+\frac{16S_2 P_{504}}{(N-1)N^2(1+N)^3(2+N)^2}
\end{aligned}
$$

$$+\frac{64S_3P_{500}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)} - \frac{16S_1^2P_{505}}{(N-1)N^2(1+N)^3(2+N)^2}$$

$$+\frac{32P_{544}}{(N-2)^2(N-1)N^4(1+N)^5(2+N)^3(3+N)} + \left(\frac{96(2+N+N^2)S_2}{(N-1)N(1+N)^2(2+N)}\right.$$

$$-\frac{32P_{533}}{(N-2)(N-1)N^3(1+N)^4(2+N)^3(3+N)} - \frac{512(2+N+N^2)S_3}{(N-2)(N-1)(1+N)(2+N)(3+N)}\right)$$

$$\times S_1 - \frac{32(2+N+N^2)S_1^3}{3(N-1)N(1+N)^2(2+N)} - \frac{512(2+N+N^2)S_4}{(N-2)(N-1)(1+N)(2+N)(3+N)}$$

$$+\left(-\frac{128S_1P_{489}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\right.$$

$$+\frac{256P_{517}}{(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)}\right)S_{-2}$$

$$+\frac{1024(2+N+N^2)S_{3,1}}{(N-2)(N-1)(1+N)(2+N)(3+N)} - \frac{1024S_{-2,1}}{(N-1)N(1+N)^2(2+N)}$$

$$+\left(-\frac{128P_{498}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\right.$$

$$+\left.\frac{1024(2+N+N^2)S_1}{(N-2)(N-1)(1+N)(2+N)(3+N)}\right)\zeta_3\Bigg\} \tag{656}$$

with

$$P_{488} = N^4 + 2N^3 - 71N^2 - 72N - 20, \tag{657}$$
$$P_{489} = N^4 + 2N^3 - 15N^2 - 16N + 44, \tag{658}$$
$$P_{490} = N^4 + 2N^3 - 13N^2 - 14N + 16, \tag{659}$$
$$P_{491} = N^4 + 2N^3 + 3N^2 + 2N - 16, \tag{660}$$
$$P_{492} = N^4 + 2N^3 + 41N^2 + 40N + 4, \tag{661}$$
$$P_{493} = N^4 + 2N^3 + 65N^2 + 64N + 28, \tag{662}$$
$$P_{494} = 5N^4 + 10N^3 - 83N^2 - 88N - 4, \tag{663}$$
$$P_{495} = 5N^4 + 15N^3 - 38N^2 - 36N + 72, \tag{664}$$
$$P_{496} = 7N^4 + 16N^3 + 13N^2 + 12N + 4, \tag{665}$$
$$P_{497} = 8N^4 + 13N^3 + 15N^2 - 20N - 60, \tag{666}$$
$$P_{498} = 11N^4 + 22N^3 + 39N^2 + 28N - 84, \tag{667}$$
$$P_{499} = 25N^4 + 50N^3 - 87N^2 - 112N + 60, \tag{668}$$
$$P_{500} = 41N^4 + 82N^3 + 117N^2 + 76N - 204, \tag{669}$$
$$P_{501} = 47N^4 + 94N^3 + 487N^2 + 440N + 308, \tag{670}$$
$$P_{502} = 49N^4 + 98N^3 + 105N^2 + 56N - 372, \tag{671}$$
$$P_{503} = 63N^4 + 126N^3 - N^2 - 64N + 4, \tag{672}$$
$$P_{504} = N^5 - 6N^4 - 19N^3 - 52N^2 - 76N - 32, \tag{673}$$
$$P_{505} = 11N^5 + 30N^4 + 47N^3 + 36N^2 - 36N - 32, \tag{674}$$
$$P_{506} = N^6 - 13N^5 - 7N^4 - 3N^3 + 34N^2 + 44N - 120, \tag{675}$$

$$P_{507} = 4N^6 - 12N^5 - 11N^4 - 18N^3 - 43N^2 - 20N + 84, \tag{676}$$

$$P_{508} = 5N^6 - 177N^5 + 65N^4 + 297N^3 + 190N^2 + 140N - 3720, \tag{677}$$

$$P_{509} = 7N^6 + 117N^5 + 67N^4 + 3N^3 - 382N^2 - 428N - 600, \tag{678}$$

$$P_{510} = 7N^6 + 246N^5 + 558N^4 + 532N^3 + 377N^2 - 340N - 228, \tag{679}$$

$$P_{511} = 19N^6 - 135N^5 + 67N^4 + 231N^3 - 178N^2 - 188N - 3336, \tag{680}$$

$$P_{512} = 19N^6 + 9N^5 - 111N^4 - 329N^3 - 364N^2 + 32N + 24, \tag{681}$$

$$P_{513} = 37N^6 + 399N^5 + 301N^4 + 129N^3 - 2326N^2 - 2516N + 264, \tag{682}$$

$$P_{514} = 41N^6 + 87N^5 + 3N^4 - 175N^3 - 284N^2 - 80N - 24, \tag{683}$$

$$P_{515} = 43N^6 - 159N^5 - 173N^4 - 273N^3 + 278N^2 + 580N - 1896, \tag{684}$$

$$P_{516} = N^7 - 37N^6 - 248N^5 - 799N^4 - 1183N^3 - 970N^2 - 580N - 168, \tag{685}$$

$$P_{517} = N^7 + 2N^6 - 10N^5 + 37N^3 - 2N^2 + 4N - 40, \tag{686}$$

$$P_{518} = N^7 + 3N^6 + 2N^5 + 33N^4 + 73N^3 + 52N^2 + 36N + 16, \tag{687}$$

$$P_{519} = 3N^7 + 7N^6 + 7N^5 + 125N^4 + 274N^3 + 248N^2 + 200N + 80, \tag{688}$$

$$P_{520} = 15N^7 - 26N^6 - 187N^5 - 464N^4 - 734N^3 - 644N^2 - 504N - 176, \tag{689}$$

$$P_{521} = 23N^7 + 94N^6 + 147N^5 - 28N^4 - 288N^3 - 348N^2 - 384N - 176, \tag{690}$$

$$P_{522} = 49N^7 + 185N^6 + 340N^5 + 287N^4 + 65N^3 + 62N^2 - 196N - 168, \tag{691}$$

$$P_{523} = 51N^7 + 175N^6 + 139N^5 - 435N^4 - 1022N^3 - 1060N^2 - 952N - 384, \tag{692}$$

$$P_{524} = N^8 + 9N^7 - 9N^6 - 165N^5 - 148N^4 + 408N^3 + 464N^2 + 432N + 576, \tag{693}$$

$$P_{525} = 6N^8 - 6N^7 - 17N^6 + 141N^5 + 61N^4 + 297N^3 + 58N^2 - 900N - 216, \tag{694}$$

$$P_{526} = 11N^8 + 221N^7 + 754N^6 + 1826N^5 + 2301N^4 + 897N^3 - 150N^2 - 1180N$$
$$-648, \tag{695}$$

$$P_{527} = 37N^8 - 119N^7 - 883N^6 - 2063N^5 - 986N^4 + 1994N^3 + 2252N^2 + 1576N$$
$$+496, \tag{696}$$

$$P_{528} = N^9 + 10N^8 + 24N^7 + 16N^6 - 35N^5 - 214N^4 - 246N^3 + 100N^2 + 168N$$
$$+144, \tag{697}$$

$$P_{529} = 9N^9 - 54N^8 - 154N^7 - 140N^6 - 1159N^5 - 1550N^4 + 1192N^3 + 2896N^2$$
$$+3664N + 2208, \tag{698}$$

$$P_{530} = 37N^9 - 744N^8 - 2698N^7 - 1652N^6 + 1309N^5 + 2820N^4 + 4416N^3$$
$$+560N^2 + 560N - 768, \tag{699}$$

$$P_{531} = 3N^{10} - 15N^9 - 121N^8 + 106N^7 + 993N^6 + 693N^5 - 1643N^4 - 2376N^3$$
$$-2088N^2 - 688N + 528, \tag{700}$$

$$P_{532} = 6N^{10} + 53N^9 + 192N^8 - 30N^7 + 160N^6 + 4697N^5 + 12778N^4 + 17152N^3$$
$$+14848N^2 + 7920N + 1888, \tag{701}$$

$$P_{533} = 28N^{10} + 205N^9 + 641N^8 + 1139N^7 + 1035N^6 + 92N^5 - 748N^4 - 1656N^3$$
$$-2944N^2 - 2464N - 768, \tag{702}$$

$$P_{534} = 67N^{10} - 142N^9 - 27N^8 + 1572N^7 - 5543N^6 - 12210N^5 + 5535N^4 + 16444N^3$$
$$+20992N^2 + 3696N - 5040, \tag{703}$$

$$P_{535} = 114N^{10} + 681N^9 + 1398N^8 - 264N^7 - 4306N^6 - 2417N^5 + 7466N^4 + 14832N^3$$
$$+15008N^2 + 9040N + 2400, \tag{704}$$

$$P_{536} = 127N^{10} + 842N^9 + 1753N^8 - 236N^7 - 8468N^6 - 17624N^5 - 14130N^4$$

$$-2296N^3 + 288N^2 - 1440N - 864, \tag{705}$$

$$
\begin{aligned}
P_{537} = {} & 176N^{10} + 973N^9 + 1824N^8 - 948N^7 - 10192N^6 - 19173N^5 - 20424N^4 \\
& -16036N^3 - 7816N^2 - 1248N + 288,
\end{aligned} \tag{706}
$$

$$
\begin{aligned}
P_{538} = {} & 220N^{10} - 673N^9 - 3885N^8 - 2682N^7 + 1450N^6 + 10779N^5 + 19215N^4 \\
& -3632N^3 + 712N^2 - 2928N - 2448,
\end{aligned} \tag{707}
$$

$$
\begin{aligned}
P_{539} = {} & 9N^{11} + 23N^{10} - 100N^9 - 182N^8 + 333N^7 + 519N^6 + 446N^5 + 520N^4 - \\
& 1040N^3 - 2288N^2 - 2784N - 1152,
\end{aligned} \tag{708}
$$

$$
\begin{aligned}
P_{540} = {} & 64N^{11} + 343N^{10} - 190N^9 - 3834N^8 - 7845N^7 - 4410N^6 + 6491N^5 + 4865N^4 \\
& -9500N^3 - 11760N^2 - 9216N - 3024,
\end{aligned} \tag{709}
$$

$$
\begin{aligned}
P_{541} = {} & 82N^{11} - 107N^{10} + 1405N^9 + 19278N^8 + 66823N^7 + 123752N^6 + 143766N^5 \\
& +112177N^4 + 57120N^3 + 5000N^2 - 7584N - 1584,
\end{aligned} \tag{710}
$$

$$
\begin{aligned}
P_{542} = {} & 824N^{11} + 3749N^{10} + 6953N^9 + 11052N^8 + 22103N^7 + 50950N^6 + 83556N^5 \\
& +75245N^4 + 44472N^3 + 22840N^2 + 7584N + 720,
\end{aligned} \tag{711}
$$

$$
\begin{aligned}
P_{543} = {} & 39N^{13} + 66N^{12} - 1159N^{11} - 6196N^{10} - 3405N^9 + 38262N^8 + 50307N^7 \\
& -84768N^6 - 167366N^5 + 10252N^4 + 198720N^3 + 200928N^2 + 92064N \\
& +17856,
\end{aligned} \tag{712}
$$

$$
\begin{aligned}
P_{544} = {} & 43N^{13} + 253N^{12} + 286N^{11} - 985N^{10} - 2923N^9 - 1277N^8 + 5318N^7 \\
& +9049N^6 + 4744N^5 + 1348N^4 + 5408N^3 + 8624N^2 + 4736N + 960,
\end{aligned} \tag{713}
$$

$$
\begin{aligned}
P_{545} = {} & 1844N^{14} + 16184N^{13} + 50096N^{12} + 39487N^{11} - 152593N^{10} - 478032N^9 \\
& -483740N^8 + 106345N^7 + 667861N^6 + 593144N^5 + 330028N^4 + 334800N^3 \\
& +331056N^2 + 152064N + 25920,
\end{aligned} \tag{714}
$$

$$
\begin{aligned}
P_{546} = {} & 78N^{15} + 542N^{14} + 845N^{13} - 3182N^{12} - 15821N^{11} - 29382N^{10} - 37143N^9 \\
& -60252N^8 - 74387N^7 + 58234N^6 + 375052N^5 + 647480N^4 + 655920N^3 + \\
& 432480N^2 + 173632N + 31872,
\end{aligned} \tag{715}
$$

$$
\begin{aligned}
P_{547} = {} & 1733N^{15} + 14466N^{14} + 39452N^{13} + 4062N^{12} - 191762N^{11} - 320533N^{10} \\
& +130420N^9 + 775488N^8 + 348241N^7 - 756255N^6 - 802132N^5 + 107380N^4 \\
& +801072N^3 + 878256N^2 + 502848N + 119232,
\end{aligned} \tag{716}
$$

$$
\begin{aligned}
P_{548} = {} & 4373N^{16} + 39739N^{15} + 116771N^{14} - 491N^{13} - 811481N^{12} - 1974485N^{11} \\
& -1526011N^{10} + 1448375N^9 + 6113428N^8 + 13620182N^7 + 22903928N^6 \\
& +27507368N^5 + 24009536N^4 + 15346080N^3 + 7254144N^2 + 2401920N \\
& +414720,
\end{aligned} \tag{717}
$$

$$
\begin{aligned}
P_{549} = {} & 424N^{18} + 3206N^{17} + 5830N^{16} - 13976N^{15} - 63725N^{14} - 46169N^{13} \\
& +105366N^{12} + 112942N^{11} - 193953N^{10} - 59123N^9 + 1048298N^8 + 1685392N^7 \\
& +242464N^6 - 2245408N^5 - 3487424N^4 - 2893056N^3 - 1520896N^2 - \\
& 480000N - 69120,
\end{aligned} \tag{718}
$$

$$
\begin{aligned}
P_{550} = {} & 24754N^{19} + 255605N^{18} + 933618N^{17} + 783522N^{16} - 4107236N^{15} \\
& -12029850N^{14} - 4276384N^{13} + 29789918N^{12} + 46452820N^{11} - 4027635N^{10} \\
& -75545302N^9 - 98368740N^8 - 114975366N^7 - 183461620N^6 - 250266040N^5 \\
& -242010000N^4 - 166111776N^3 - 78065856N^2 - 22156416N - 2799360.
\end{aligned} \tag{719}
$$

Finally, the gluonic contributions read

$$C_{F_2,g}^{(3)} =$$

$$C_F \Bigg\{ T_F^2 N_F^2 \Bigg[-\frac{256 S_{2,1} P_{553}}{9N(1+N)^2(2+N)(3+N)} + \frac{16 S_1^3 P_{608}}{81(N-1)N^3(1+N)^3(2+N)^2}$$

$$+ \frac{32 S_3 P_{627}}{81(N-1)N^3(1+N)^3(2+N)^2(3+N)} - \frac{4 S_2 P_{648}}{27(N-1)N^4(1+N)^4(2+N)^3(3+N)}$$

$$+ \frac{P_{663}}{243(N-2)^2(N-1)N^6(1+N)^6(2+N)^5(3+N)}$$

$$- \frac{64\zeta_3 P_{630}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^2(3+N)}$$

$$+ \Bigg(-\frac{16 S_2 P_{625}}{27(N-1)N^3(1+N)^3(2+N)^2(3+N)}$$

$$- \frac{8 P_{659}}{243(N-2)(N-1)N^5(1+N)^5(2+N)^4(3+N)} + \frac{32\big(-74+35N+35N^2\big)S_3}{27N(1+N)(2+N)}$$

$$- \frac{128(N-1)S_{2,1}}{3N(1+N)} + \frac{512 S_{-2,1}}{3N(1+N)(2+N)} + \frac{64\big(46+5N+5N^2\big)\zeta_3}{9N(1+N)(2+N)} \Bigg) S_1$$

$$+ \Bigg(\frac{4 P_{641}}{81(N-1)N^4(1+N)^4(2+N)^3} - \frac{8\big(62+7N+7N^2\big)S_2}{9N(1+N)(2+N)} \Bigg) S_1^2 + \frac{68\big(2+N+N^2\big)S_1^4}{27N(1+N)(2+N)}$$

$$+ \frac{4\big(226+17N+17N^2\big)S_2^2}{9N(1+N)(2+N)} - \frac{56\big(-14+17N+17N^2\big)S_4}{9N(1+N)(2+N)}$$

$$+ \Bigg(-\frac{64 S_1 P_{594}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$- \frac{32 P_{636}}{9(N-2)^2(N-1)N^3(1+N)^2(2+N)^3(3+N)} - \frac{512 S_1^2}{3N(1+N)(2+N)}$$

$$+ \frac{512 S_2}{3N(1+N)(2+N)} \Bigg) S_{-2} + \frac{256 S_{-2}^2}{3N(1+N)(2+N)} + \Bigg(\frac{256 S_1}{N(1+N)(2+N)}$$

$$+ \frac{128 P_{593}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)} \Bigg) S_{-3} + \frac{1024 S_{-4}}{3N(1+N)(2+N)}$$

$$+ \frac{64\big(10+3N+3N^2\big)S_{3,1}}{3N(1+N)(2+N)} + \frac{512(-3+5N)S_{-2,1}}{9N^2(1+N)(2+N)} - \frac{512 S_{-2,2}}{3N(1+N)(2+N)}$$

$$- \frac{1024 S_{-3,1}}{3N(1+N)(2+N)} + \frac{64\big(-10+N+N^2\big)S_{2,1,1}}{3N(1+N)(2+N)} \Bigg]$$

$$+ C_A T_F N_F \Bigg[-\frac{96\zeta_2^2 P_{563}}{5(N-1)N^2(1+N)^2(2+N)^2} - \frac{16 S_{2,1,1} P_{564}}{3(N-1)N^2(1+N)^2(2+N)^2}$$

$$- \frac{2 S_1^4 P_{580}}{27(N-1)N^2(1+N)^2(2+N)^2} - \frac{2 S_2^2 P_{582}}{9(N-1)N^2(1+N)^2(2+N)^2}$$

$$- \frac{32 S_{-2,1,1} P_{595}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{16S_{-2,2}P_{596}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{16S_{-3,1}P_{597}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{16S_{3,1}P_{601}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{4S_4P_{616}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}+\frac{8S_{2,1}P_{632}}{9(N-1)N^3(1+N)^3(2+N)^3(3+N)}$$

$$-\frac{16S_{-2,1}P_{638}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}$$

$$-\frac{8\zeta_3P_{640}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}$$

$$-\frac{8S_3P_{642}}{81(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}$$

$$+\frac{P_{664}}{972(N-2)^2(N-1)N^6(1+N)^6(2+N)^5(3+N)}+\left(\frac{8S_{2,1}P_{555}}{3N^2(1+N)^2(2+N)^2}\right.$$

$$-\frac{32S_{-2,1}P_{584}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}-\frac{8S_3P_{612}}{27(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{8\zeta_3P_{618}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}+\frac{4S_2P_{635}}{27(N-1)N^3(1+N)^3(2+N)^3(3+N)}$$

$$+\frac{2P_{661}}{243(N-2)^2(N-1)N^5(1+N)^5(2+N)^5(3+N)}-\frac{80(2+N+N^2)S_2^2}{N(1+N)(2+N)}$$

$$+\frac{8(382+47N+47N^2)S_4}{3N(1+N)(2+N)}+\frac{64(-130+7N+7N^2)S_{3,1}}{3N(1+N)(2+N)}+\frac{128(54+5N+5N^2)S_{-2,2}}{N(1+N)(2+N)}$$

$$+\frac{16(1378+113N+113N^2)S_{-3,1}}{3N(1+N)(2+N)}-\frac{32(1214+127N+127N^2)S_{-2,1,1}}{3N(1+N)(2+N)}$$

$$\left.-\frac{80(2+N+N^2)S_{2,1,1}}{N(1+N)(2+N)}-\frac{96(2+N+N^2)\zeta_2^2}{5N(1+N)(2+N)}\right)S_1+\left(\frac{4S_2P_{581}}{9(N-1)N^2(1+N)^2(2+N)^2}\right.$$

$$+\frac{P_{654}}{81(N-2)(N-1)N^4(1+N)^4(2+N)^4(3+N)}-\frac{8(-310+13N+13N^2)S_3}{3N(1+N)(2+N)}$$

$$\left.-\frac{32(2+N+N^2)S_{2,1}}{N(1+N)(2+N)}+\frac{32(286+35N+35N^2)S_{-2,1}}{3N(1+N)(2+N)}+\frac{32(-166+19N+19N^2)\zeta_3}{3N(1+N)(2+N)}\right)$$

$$\times S_1^2+\left(-\frac{4P_{629}}{81(N-1)N^3(1+N)^3(2+N)^3}+\frac{136(2+N+N^2)S_2}{3N(1+N)(2+N)}\right)S_1^3$$

$$+\left(\frac{P_{656}}{27(N-2)(N-1)N^4(1+N)^4(2+N)^4(3+N)}+\frac{8(-898+7N+7N^2)S_3}{3N(1+N)(2+N)}\right.$$

$$\left.-\frac{128(72+7N+7N^2)S_{-2,1}}{N(1+N)(2+N)}-\frac{96(2+3N+3N^2)\zeta_3}{N(1+N)(2+N)}\right)S_2-\frac{256(11+N+N^2)S_5}{3N(1+N)(2+N)}$$

$$+\left(-\frac{8S_2P_{603}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}\right.$$

$$+\frac{8S_1^2 P_{606}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{8P_{655}}{9(N-2)^2(N-1)N^4(1+N)^4(2+N)^3(3+N)}$$

$$+\left(-\frac{16P_{644}}{9(N-2)^2(N-1)N^3(1+N)^3(2+N)^3(3+N)}-\frac{48\big(18+N+N^2\big)S_2}{N(1+N)(2+N)}\right)S_1$$

$$+\frac{16\big(178+17N+17N^2\big)S_1^3}{9N(1+N)(2+N)}-\frac{64\big(-2+35N+35N^2\big)S_3}{9N(1+N)(2+N)}+\frac{64\big(146+13N+13N^2\big)S_{2,1}}{N(1+N)(2+N)}$$

$$-\frac{128\big(2+N+N^2\big)S_{-2,1}}{3N(1+N)(2+N)}-\frac{1024\big(4-2N+N^2\big)\zeta_3}{N(1+N)(2+N)}\right)S_{-2}+\left(\frac{256\big(11+N+N^2\big)S_1}{3N(1+N)(2+N)}\right.$$

$$+\frac{32P_{586}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}\right)S_{-2}^2$$

$$+\left(-\frac{16S_1 P_{600}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}-\frac{16\big(490+41N+41N^2\big)S_1^2}{3N(1+N)(2+N)}\right.$$

$$+\frac{8P_{639}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}+\frac{128\big(107+10N+10N^2\big)S_2}{3N(1+N)(2+N)}$$

$$-\frac{32\big(-22+5N+5N^2\big)S_{-2}}{N(1+N)(2+N)}\right)S_{-3}+\left(\frac{8P_{602}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}\right.$$

$$-\frac{16\big(198+19N+19N^2\big)S_1}{N(1+N)(2+N)}\right)S_{-4}-\frac{16\big(-562+31N+31N^2\big)S_{-5}}{3N(1+N)(2+N)}+\frac{64\big(38+N+N^2\big)S_{2,3}}{N(1+N)(2+N)}$$

$$+\frac{160\big(86+7N+7N^2\big)S_{2,-3}}{3N(1+N)(2+N)}-\frac{64\big(134+13N+13N^2\big)S_{4,1}}{3N(1+N)(2+N)}-\frac{256\big(50+19N+N^2\big)S_{-2,3}}{3N(1+N)(2+N)}$$

$$+\frac{256\big(7+7N+N^2\big)S_{-4,1}}{N(1+N)(2+N)}-\frac{64\big(434+37N+37N^2\big)S_{2,1,-2}}{3N(1+N)(2+N)}-\frac{16\big(2+N+N^2\big)S_{2,2,1}}{3N(1+N)(2+N)}$$

$$-\frac{96\big(-46+N+N^2\big)S_{3,1,1}}{N(1+N)(2+N)}-\frac{64\big(118+11N+11N^2\big)S_{-2,1,-2}}{3N(1+N)(2+N)}$$

$$-\frac{2560\big(11+N+N^2\big)S_{-2,2,1}}{3N(1+N)(2+N)}-\frac{160\big(58+5N+5N^2\big)S_{-3,1,1}}{N(1+N)(2+N)}+\frac{160\big(2+N+N^2\big)S_{2,1,1,1}}{N(1+N)(2+N)}$$

$$+\frac{320\big(178+17N+17N^2\big)S_{-2,1,1,1}}{3N(1+N)(2+N)}-\frac{80\big(118-95N+N^2\big)\zeta_5}{N(1+N)(2+N)}\Bigg]\Bigg\}$$

$$+C_A T_F^2 N_F^2\Bigg\{\frac{32S_1^3 P_{556}}{81(N-1)N(1+N)^2(2+N)^2}+\frac{128S_{2,1}P_{557}}{27N^2(1+N)^2(2+N)^2}$$

$$-\frac{64S_3 P_{559}}{81(N-1)N(1+N)^2(2+N)^2}-\frac{16S_2 P_{611}}{81(N-1)N^2(1+N)^3(2+N)^3}$$

$$+\frac{8P_{660}}{243(N-2)(N-1)N^5(1+N)^5(2+N)^5(3+N)}+\frac{32\zeta_3 P_{577}}{9(N-1)N^2(1+N)^2(2+N)^2}$$

$$+\left(-\frac{32S_2 P_{578}}{27(N-1)N^2(1+N)^2(2+N)^2}+\frac{8P_{650}}{243(N-1)N^4(1+N)^4(2+N)^4}\right.$$

$$+\frac{32\big(110+19N+19N^2\big)S_3}{27N(1+N)(2+N)}+\frac{256\big(-1+N+N^2\big)S_{2,1}}{9N(1+N)(2+N)}-\frac{256S_{-2,1}}{3N(1+N)(2+N)}$$

$$-\frac{32\big(62+13N+13N^2\big)\zeta_3}{9N(1+N)(2+N)}\bigg)S_1+\bigg(\frac{16P_{615}}{81(N-1)N^2(1+N)^3(2+N)^3}$$

$$-\frac{8\big(14+19N+19N^2\big)S_2}{9N(1+N)(2+N)}\bigg)S_1^2+\frac{28\big(2+N+N^2\big)S_1^4}{27N(1+N)(2+N)}+\frac{4\big(-34+31N+31N^2\big)S_2^2}{9N(1+N)(2+N)}$$

$$+\frac{56\big(-10+7N+7N^2\big)S_4}{9N(1+N)(2+N)}+\bigg(-\frac{64S_1P_{568}}{9(N-1)N^2(1+N)^2(2+N)^2}$$

$$-\frac{32P_{631}}{81(N-2)(N-1)N^3(1+N)^2(2+N)^3(3+N)}-\frac{32(N-2)(3+N)S_1^2}{3N(1+N)(2+N)}$$

$$+\frac{32(N-2)(3+N)S_2}{3N(1+N)(2+N)}\bigg)S_{-2}-\frac{128S_{-2}^2}{3N(1+N)(2+N)}+\bigg(\frac{128P_{573}}{27(N-1)N^2(1+N)^2(2+N)^2}$$

$$+\frac{128\big(-1+N+N^2\big)S_1}{3N(1+N)(2+N)}\bigg)S_{-3}-\frac{64\big(38+7N+7N^2\big)S_{-4}}{9N(1+N)(2+N)}-\frac{128\big(13+5N+5N^2\big)S_{3,1}}{9N(1+N)(2+N)}$$

$$-\frac{256(-3+5N)S_{-2,1}}{9N^2(1+N)(2+N)}+\frac{256S_{-2,2}}{3N(1+N)(2+N)}+\frac{512S_{-3,1}}{3N(1+N)(2+N)}$$

$$-\frac{64\big(-16+N+N^2\big)S_{2,1,1}}{9N(1+N)(2+N)}\bigg\}$$

$$+C_A^2T_FN_F\bigg\{\frac{48(N-2)(3+N)\zeta_2^2P_{554}}{5(N-1)N^2(1+N)^2(2+N)^2}+\frac{S_2^2P_{561}}{9(N-1)N^2(1+N)^2(2+N)^2}$$

$$+\frac{S_1^4P_{562}}{27(N-1)N^2(1+N)^2(2+N)^2}+\frac{16S_{2,1,1}P_{569}}{9(N-1)N^2(1+N)^2(2+N)^2}$$

$$+\frac{8S_{3,1}P_{589}}{9(N-2)N^2(1+N)^2(2+N)^2(3+N)}-\frac{32S_{-2,1,1}P_{604}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{16S_{-2,2}P_{605}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{16S_{-3,1}P_{607}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{2S_4P_{617}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}-\frac{8S_{2,1}P_{628}}{27(N-1)N^3(1+N)^3(2+N)^3}$$

$$-\frac{32S_{-2,1}P_{637}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}$$

$$+\frac{4\zeta_3P_{643}}{9(N-2)(N-1)^2N^3(1+N)^3(2+N)^3(3+N)}$$

$$+\frac{8S_3P_{649}}{81(N-2)(N-1)^2N^3(1+N)^3(2+N)^3(3+N)}$$

$$-\frac{4P_{665}}{243(N-2)(N-1)^2N^6(1+N)^6(2+N)^6(3+N)}+\bigg(-\frac{16S_{2,1}P_{558}}{9N^2(1+N)^2(2+N)^2}$$

$$+\frac{16S_{-2,1}P_{598}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}-\frac{32\big(-70+N+N^2\big)S_{3,1}}{3N(1+N)(2+N)}$$

$$
+\frac{8\zeta_3 P_{613}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)} - \frac{48\big(2+N+N^2\big)S_4}{N(1+N)(2+N)}
$$

$$
-\frac{8S_3 P_{614}}{27(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)} + \frac{8S_2 P_{634}}{27(N-1)^2 N^3(1+N)^3(2+N)^3}
$$

$$
-\frac{4P_{662}}{243(N-2)(N-1)^2 N^5(1+N)^5(2+N)^5(3+N)} - \frac{124\big(2+N+N^2\big)S_2^2}{3N(1+N)(2+N)}
$$

$$
-\frac{32\big(226+5N+5N^2\big)S_{-2,2}}{3N(1+N)(2+N)} - \frac{80\big(106+5N+5N^2\big)S_{-3,1}}{3N(1+N)(2+N)} - \frac{16\big(2+N+N^2\big)S_{2,1,1}}{3N(1+N)(2+N)}
$$

$$
-\frac{32\big(-374+5N+5N^2\big)S_{-2,1,1}}{3N(1+N)(2+N)} + \frac{96\big(2+N+N^2\big)\zeta_2^2}{5N(1+N)(2+N)}\Bigg)S_1
$$

$$
+\Bigg(\frac{2S_2 P_{583}}{9(N-1)N^2(1+N)^2(2+N)^2} - \frac{2P_{653}}{81(N-1)^2 N^4(1+N)^4(2+N)^4}
$$

$$
-\frac{8\big(154+29N+29N^2\big)S_3}{3N(1+N)(2+N)} + \frac{32\big(2+N+N^2\big)S_{2,1}}{3N(1+N)(2+N)} + \frac{32\big(-22+3N+3N^2\big)S_{-2,1}}{N(1+N)(2+N)}
$$

$$
+\frac{16\big(190+17N+17N^2\big)\zeta_3}{3N(1+N)(2+N)}\Bigg)S_1^2 + \Bigg(-\frac{8P_{633}}{81(N-1)^2 N^3(1+N)^3(2+N)^3}
$$

$$
+\frac{128\big(2+N+N^2\big)S_2}{9N(1+N)(2+N)}\Bigg)S_1^3 - \frac{4\big(2+N+N^2\big)S_1^5}{3N(1+N)(2+N)} + \Bigg(\frac{2P_{652}}{81(N-1)^2 N^4(1+N)^4(2+N)^4}
$$

$$
-\frac{8\big(-658+31N+31N^2\big)S_3}{9N(1+N)(2+N)} + \frac{128\big(27+2N+2N^2\big)S_{-2,1}}{N(1+N)(2+N)} + \frac{48(N-1)\zeta_3}{N(1+N)}\Bigg)S_2
$$

$$
+\frac{32\big(26+7N+7N^2\big)S_5}{3N(1+N)(2+N)} + \Bigg[\frac{8S_1^2 P_{575}}{3(N-1)N^2(1+N)^2(2+N)^2} - \frac{8S_2 P_{576}}{3(N-1)N^2(1+N)^2(2+N)^2}
$$

$$
+\frac{8P_{658}}{81(N-2)(N-1)^2 N^4(1+N)^4(2+N)^4(3+N)}
$$

$$
+\Bigg(\frac{16P_{646}}{9(N-2)(N-1)^2 N^3(1+N)^3(2+N)^3(3+N)} - \frac{16\big(-50+11N+11N^2\big)S_2}{3N(1+N)(2+N)}\Bigg)S_1
$$

$$
-\frac{16\big(26+N+N^2\big)S_1^3}{3N(1+N)(2+N)} + \frac{128\big(5+4N+4N^2\big)S_3}{3N(1+N)(2+N)} - \frac{1024\big(11+N+N^2\big)S_{2,1}}{3N(1+N)(2+N)}
$$

$$
+\frac{128\big(2+N+N^2\big)S_{-2,1}}{3N(1+N)(2+N)} + \frac{128\big(7-4N+2N^2\big)\zeta_3}{N(1+N)(2+N)}\Bigg]S_{-2}
$$

$$
+\Bigg(-\frac{8P_{599}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)} - \frac{16\big(18+N+N^2\big)S_1}{N(1+N)(2+N)}\Bigg)S_{-2}^2
$$

$$
+\Bigg(-\frac{8S_1 P_{610}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)} - \frac{16\big(-122+29N+29N^2\big)S_1^2}{3N(1+N)(2+N)}
$$

$$
-\frac{16P_{647}}{27(N-2)(N-1)^2 N^3(1+N)^3(2+N)^3(3+N)} - \frac{32\big(64+9N+9N^2\big)S_2}{N(1+N)(2+N)}
$$

$$
+\frac{32\big(-10+13N+13N^2\big)S_{-2}}{3N(1+N)(2+N)}\Bigg)S_{-3} + \Bigg(-\frac{8P_{609}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}
$$

$$+\frac{128\big(13+2N+2N^2\big)S_1}{N(1+N)(2+N)}\bigg)S_{-4}+\frac{16\big(-238+N+N^2\big)S_{-5}}{3N(1+N)(2+N)}-\frac{48\big(14+N+N^2\big)S_{2,3}}{N(1+N)(2+N)}$$

$$+\frac{128\big(-34+N+N^2\big)S_{2,-3}}{3N(1+N)(2+N)}+\frac{16\big(146+19N+19N^2\big)S_{4,1}}{3N(1+N)(2+N)}-\frac{128\big(-38-7N+2N^2\big)S_{-2,3}}{3N(1+N)(2+N)}$$

$$-\frac{32\big(142+71N+35N^2\big)S_{-4,1}}{3N(1+N)(2+N)}+\frac{1024\big(11+N+N^2\big)S_{2,1,-2}}{3N(1+N)(2+N)}+\frac{176\big(2+N+N^2\big)S_{2,2,1}}{3N(1+N)(2+N)}$$

$$-\frac{1152S_{3,1,1}}{N(1+N)(2+N)}-\frac{64\big(-22+7N+7N^2\big)S_{-2,1,-2}}{3N(1+N)(2+N)}+\frac{64\big(178+17N+17N^2\big)S_{-2,2,1}}{3N(1+N)(2+N)}$$

$$+\frac{32\big(130+17N+17N^2\big)S_{-3,1,1}}{N(1+N)(2+N)}-\frac{64\big(322+17N+17N^2\big)S_{-2,1,1,1}}{3N(1+N)(2+N)}$$

$$+\frac{160\big(14-11N+N^2\big)\zeta_5}{N(1+N)(2+N)}\bigg\}$$

$$+C_F^2 T_F N_F\bigg\{\frac{S_1^4 P_{551}}{9N^2(1+N)^2(2+N)}-\frac{64S_{-2,2}P_{566}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$-\frac{32S_{-3,1}P_{570}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{64S_{-2,1,1}P_{571}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$+\frac{S_2^2 P_{552}}{3N^2(1+N)^2(2+N)}+\frac{32S_{3,1}P_{585}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}-\frac{16S_{2,1}P_{587}}{3N^3(1+N)^3(2+N)^2}$$

$$-\frac{2S_4 P_{590}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}+\frac{32S_{-2,1}P_{621}}{3(N-2)N^3(1+N)^3(2+N)^2(3+N)}$$

$$+\frac{4S_3 P_{626}}{9(N-2)N^3(1+N)^3(2+N)^2(3+N)}+\frac{P_{657}}{12(N-2)^2N^6(1+N)^6(2+N)(3+N)}$$

$$+\frac{16\zeta_3 P_{624}}{3(N-2)N^3(1+N)^3(2+N)^2(3+N)}+\bigg(-\frac{64S_{-2,1}P_{567}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$+\frac{16\zeta_3 P_{591}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}-\frac{8S_3 P_{592}}{9(N-2)N^2(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{2S_2 P_{588}}{3N^3(1+N)^3(2+N)^2}-\frac{2P_{651}}{3(N-2)N^5(1+N)^5(2+N)^2(3+N)}-\frac{188\big(2+N+N^2\big)S_2^2}{3N(1+N)(2+N)}$$

$$+\frac{32\big(-20+12N+18N^2+3N^3\big)S_{2,1}}{3N(1+N)(2+N)^2}-\frac{32\big(-262+13N+13N^2\big)S_{3,1}}{3N(1+N)(2+N)}$$

$$-\frac{8\big(766+95N+95N^2\big)S_4}{3N(1+N)(2+N)}+\frac{128\big(-14+N+N^2\big)S_{-2,2}}{N(1+N)(2+N)}+\frac{64\big(-70+13N+13N^2\big)S_{-3,1}}{3N(1+N)(2+N)}$$

$$+\frac{256\big(2+N+N^2\big)S_{2,1,1}}{3N(1+N)(2+N)}+\frac{128\big(98+N+N^2\big)S_{-2,1,1}}{3N(1+N)(2+N)}\bigg)S_1+\bigg(\frac{2S_2 P_{560}}{3N^2(1+N)^2(2+N)^2}$$

$$+\frac{P_{619}}{3N^4(1+N)^4(2+N)^2}-\frac{48\big(18+N+N^2\big)S_3}{N(1+N)(2+N)}+\frac{64\big(2+N+N^2\big)S_{2,1}}{3N(1+N)(2+N)}$$

$$-\frac{256\big(14+N+N^2\big)S_{-2,1}}{3N(1+N)(2+N)}-\frac{128\big(-14+5N+5N^2\big)\zeta_3}{3N(1+N)(2+N)}\bigg)S_1^2+\bigg(-\frac{2P_{579}}{9N^3(1+N)^3(2+N)}$$

$$+\frac{472\left(2+N+N^2\right)S_2}{9N(1+N)(2+N)}\Bigg)S_1^3-\frac{20\left(2+N+N^2\right)S_1^5}{3N(1+N)(2+N)}+\Bigg(\frac{P_{623}}{3N^4(1+N)^4(2+N)^2}$$

$$+\frac{16\left(1462+83N+83N^2\right)S_3}{9N(1+N)(2+N)}+\frac{256\left(38+N+N^2\right)S_{-2,1}}{3N(1+N)(2+N)}+\frac{192\left(2+N+N^2\right)\zeta_3}{N(1+N)(2+N)}\Bigg)S_2$$

$$+\frac{64\left(82+17N+17N^2\right)S_5}{3N(1+N)(2+N)}+\Bigg(\frac{8P_{645}}{3(N-2)^2N^4(1+N)^4(2+N)^2(3+N)}$$

$$+\Bigg(\frac{32P_{620}}{3(N-2)N^3(1+N)^3(2+N)^2(3+N)}-\frac{640\left(2+N+N^2\right)S_2}{3N(1+N)(2+N)}\Bigg)S_1$$

$$+\frac{64\left(-27+30N+22N^2+6N^3\right)S_1^2}{3N^2(1+N)^2(2+N)}-\frac{64\left(-7+3N-N^2+2N^3\right)S_2}{N^2(1+N)^2(2+N)}$$

$$+\frac{640\left(2+N+N^2\right)S_1^3}{9N(1+N)(2+N)}+\frac{512\left(5+7N+7N^2\right)S_3}{9N(1+N)(2+N)}-\frac{3072S_{2,1}}{N(1+N)(2+N)}$$

$$-\frac{128\left(2+N+N^2\right)S_{-2,1}}{3N(1+N)(2+N)}+\frac{512\left(9-4N+2N^2\right)\zeta_3}{N(1+N)(2+N)}\Bigg)S_{-2}+\Bigg(-\frac{256\left(8+N+N^2\right)S_1}{3N(1+N)(2+N)}$$

$$-\frac{32P_{574}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}\Bigg)S_{-2}^2+\Bigg(\frac{32S_1P_{565}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$-\frac{16P_{622}}{3(N-2)N^3(1+N)^3(2+N)^2(3+N)}-\frac{64\left(-2+11N+11N^2\right)S_1^2}{3N(1+N)(2+N)}$$

$$-\frac{64\left(74+N+N^2\right)S_2}{3N(1+N)(2+N)}+\frac{512\left(1+N+N^2\right)S_{-2}}{N(1+N)(2+N)}\Bigg)S_{-3}+\Bigg(\frac{512S_1}{N(1+N)(2+N)}$$

$$-\frac{32P_{572}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}\Bigg)S_{-4}+\frac{32\left(-10+67N+67N^2\right)S_{-5}}{3N(1+N)(2+N)}$$

$$-\frac{128\left(20+N+N^2\right)S_{2,3}}{N(1+N)(2+N)}+\frac{192(N-2)(3+N)S_{2,-3}}{N(1+N)(2+N)}+\frac{32\left(98+13N+13N^2\right)S_{4,1}}{N(1+N)(2+N)}$$

$$-\frac{64\left(2-47N+25N^2\right)S_{-2,3}}{3N(1+N)(2+N)}-\frac{32\left(20+13N+13N^2+12N^3\right)S_{2,1,1}}{3N^2(1+N)^2(2+N)}$$

$$-\frac{64\left(-2+27N+3N^2\right)S_{-4,1}}{N(1+N)(2+N)}-\frac{128\left(-70+N+N^2\right)S_{2,1,-2}}{3N(1+N)(2+N)}-\frac{160\left(2+N+N^2\right)S_{2,2,1}}{3N(1+N)(2+N)}$$

$$-\frac{4608S_{3,1,1}}{N(1+N)(2+N)}-\frac{128\left(-22+N+N^2\right)S_{-2,1,-2}}{3N(1+N)(2+N)}-\frac{64\left(-142+N+N^2\right)S_{-3,1,1}}{3N(1+N)(2+N)}$$

$$+\frac{3072S_{-2,2,1}}{N(1+N)(2+N)}-\frac{160\left(2+N+N^2\right)S_{2,1,1,1}}{N(1+N)(2+N)}-\frac{128\left(146+N+N^2\right)S_{-2,1,1,1}}{3N(1+N)(2+N)}$$

$$+\frac{48(N-1)\left(-2+3N+3N^2\right)\zeta_2^2}{5N^2(1+N)^2}-\frac{160\left(-66+49N+N^2\right)\zeta_5}{N(1+N)(2+N)}\Bigg\}. \tag{720}$$

with

$$P_{551}\quad=\quad-249N^4-346N^3-65N^2+336N+452, \tag{721}$$
$$P_{552}\quad=\quad-117N^4+14N^3-109N^2-256N+244, \tag{722}$$
$$P_{553}\quad=\quad2N^4+N^3+6N^2+34N+21, \tag{723}$$

$$P_{554} = 3N^4 + 6N^3 + 7N^2 + 4N + 4, \tag{724}$$

$$P_{555} = 17N^5 + 14N^4 - 45N^3 - 74N^2 - 68N - 24, \tag{725}$$

$$P_{556} = 19N^5 + 12N^4 - 48N^3 - 29N^2 - 112N - 58, \tag{726}$$

$$P_{557} = 19N^5 + 40N^4 + 37N^3 + 44N^2 - 32N - 36, \tag{727}$$

$$P_{558} = 44N^5 + 149N^4 + 221N^3 + 160N^2 - 10N - 36, \tag{728}$$

$$P_{559} = 53N^5 + 33N^4 + 12N^3 + 74N^2 - 68N + 112, \tag{729}$$

$$P_{560} = 177N^5 + 500N^4 + 317N^3 - 134N^2 - 692N - 712, \tag{730}$$

$$P_{561} = -305N^6 - 531N^5 + 1305N^4 + 2359N^3 + 2600N^2 + 1676N + 1248, \tag{731}$$

$$P_{562} = -293N^6 + 201N^5 + 1473N^4 + 595N^3 + 1628N^2 + 380N - 240, \tag{732}$$

$$P_{563} = 3N^6 + 9N^5 - 5N^4 - 25N^3 - 14N^2 - 16, \tag{733}$$

$$P_{564} = 5N^6 - 21N^5 - 147N^4 - 181N^3 + 100N^2 + 220N + 120, \tag{734}$$

$$P_{565} = 6N^6 + 24N^5 - 37N^4 - 22N^3 - 93N^2 + 210N + 144, \tag{735}$$

$$P_{566} = 6N^6 + 36N^5 - 65N^4 - 146N^3 + 81N^2 + 228N + 468, \tag{736}$$

$$P_{567} = 6N^6 + 48N^5 - 31N^4 - 154N^3 - 51N^2 - 66N + 720, \tag{737}$$

$$P_{568} = 8N^6 + 15N^5 - 14N^4 - 17N^3 + 60N^2 + 44N - 24, \tag{738}$$

$$P_{569} = 11N^6 - 3N^5 - 132N^4 - 193N^3 + 187N^2 + 298N + 120, \tag{739}$$

$$P_{570} = 12N^6 + 60N^5 - 139N^4 - 238N^3 + 159N^2 + 618N + 720, \tag{740}$$

$$P_{571} = 12N^6 + 84N^5 - 121N^4 - 346N^3 + 165N^2 + 294N + 1152, \tag{741}$$

$$P_{572} = 15N^6 + 83N^5 + 81N^4 - 315N^3 - 482N^2 - 370N - 12, \tag{742}$$

$$P_{573} = 19N^6 + 27N^5 + 12N^4 + 52N^3 + 41N^2 + 47N + 18, \tag{743}$$

$$P_{574} = 21N^6 + 105N^5 - 9N^4 - 425N^3 - 368N^2 + 84N + 432, \tag{744}$$

$$P_{575} = 23N^6 + 13N^5 - 3N^4 + 299N^3 + 484N^2 - 72N + 24, \tag{745}$$

$$P_{576} = 35N^6 - 3N^5 - 177N^4 - 65N^3 + 298N^2 + 328N + 160, \tag{746}$$

$$P_{577} = 42N^6 + 81N^5 - 61N^4 - 53N^3 + 383N^2 + 228N - 92, \tag{747}$$

$$P_{578} = 67N^6 + 102N^5 - 12N^4 - 35N^3 - 304N^2 - 106N + 72, \tag{748}$$

$$P_{579} = 127N^6 + 375N^5 + 165N^4 + 737N^3 + 1668N^2 + 1960N + 872, \tag{749}$$

$$P_{580} = 206N^6 + 522N^5 + 777N^4 - 556N^3 - 3197N^2 - 884N - 180, \tag{750}$$

$$P_{581} = 349N^6 + 1029N^5 + 588N^4 - 1499N^3 - 3295N^2 - 616N + 420, \tag{751}$$

$$P_{582} = 368N^6 + 1404N^5 + 1419N^4 - 190N^3 - 3353N^2 - 2636N - 324, \tag{752}$$

$$P_{583} = 497N^6 - 9N^5 - 1449N^4 - 1027N^3 - 4304N^2 - 1388N - 96, \tag{753}$$

$$P_{584} = 3N^7 - 260N^6 - 461N^5 + 1640N^4 + 2358N^3 - 1832N^2 - 5040N - 7056, \tag{754}$$

$$P_{585} = 12N^7 + 74N^6 - 315N^5 - 1024N^4 + 829N^3 + 4244N^2 + 2996N + 5568, \tag{755}$$

$$P_{586} = 15N^7 + 155N^6 + 292N^5 - 489N^4 - 1527N^3 - 594N^2 + 916N + 1128, \tag{756}$$

$$P_{587} = 16N^7 + 36N^6 - 89N^5 - 342N^4 - 395N^3 - 246N^2 - 124N - 24, \tag{757}$$

$$P_{588} = 23N^7 - 83N^6 - 933N^5 - 1405N^4 + 670N^3 + 2664N^2 + 2616N + 1072, \tag{758}$$

$$P_{589} = 85N^7 + 701N^6 + 305N^5 - 1809N^4 - 1686N^3 + 412N^2 + 8520N + 27360, \tag{759}$$

$$P_{590} = 93N^7 + 801N^6 - 2057N^5 - 9261N^4 + 4880N^3 + 31656N^2 + 23760N + 57456, \tag{760}$$

$$P_{591} = 147N^7 + 483N^6 - 1091N^5 - 3551N^4 + 844N^3 + 9776N^2 + 7552N + 6384, \tag{761}$$

$$P_{592} = 213N^7 + 985N^6 - 2533N^5 - 10357N^4 + 2972N^3 + 33504N^2 + 27296N$$
$$+ 33936, \tag{762}$$

$$P_{593} = 3N^8 + 24N^7 - 16N^6 - 204N^5 + 71N^4 + 588N^3 - 58N^2 - 336N - 648, \tag{763}$$

$$P_{594} = 3N^8 + 24N^7 + 34N^6 - 84N^5 - 269N^4 - 12N^3 + 592N^2 + 144N - 1008, \tag{764}$$

$$P_{595} = 9N^8 + 512N^7 + 186N^6 - 4406N^5 - 1547N^4 + 9986N^3 + 9116N^2 + 1936N - 13488, \tag{765}$$

$$P_{596} = 9N^8 + 512N^7 + 244N^6 - 4234N^5 - 2009N^4 + 8834N^3 + 9964N^2 + 3096N - 11808, \tag{766}$$

$$P_{597} = 9N^8 + 512N^7 + 302N^6 - 4062N^5 - 2471N^4 + 7682N^3 + 10812N^2 + 4256N - 10128, \tag{767}$$

$$P_{598} = 20N^8 - 34N^7 - 295N^6 + 167N^5 + 885N^4 + 243N^3 - 726N^2 - 1452N + 936, \tag{768}$$

$$P_{599} = 27N^8 + 174N^7 + 76N^6 - 902N^5 - 931N^4 + 1008N^3 + 1380N^2 + 32N - 1632, \tag{769}$$

$$P_{600} = 30N^8 + 493N^7 + 213N^6 - 3539N^5 - 1915N^4 + 5886N^3 + 6880N^2 + 3280N - 6720, \tag{770}$$

$$P_{601} = 33N^8 + 194N^7 - 666N^6 - 1212N^5 + 2993N^4 + 5026N^3 - 440N^2 + 6888N - 15120, \tag{771}$$

$$P_{602} = 39N^8 + 4N^7 + 188N^6 + 886N^5 - 895N^4 - 2378N^3 - 6316N^2 - 5352N + 4608, \tag{772}$$

$$P_{603} = 39N^8 + 200N^7 - 1006N^5 - 1299N^4 + 166N^3 + 3348N^2 + 1816N + 1344, \tag{773}$$

$$P_{604} = 60N^8 - 96N^7 - 793N^6 + 835N^5 + 2809N^4 - 603N^3 - 4488N^2 - 4252N + 2688, \tag{774}$$

$$P_{605} = 66N^8 - 106N^7 - 877N^6 + 933N^5 + 3225N^4 - 427N^3 - 5330N^2 - 5284N + 2040, \tag{775}$$

$$P_{606} = 69N^8 + 176N^7 - 576N^6 - 1906N^5 - 521N^4 + 6258N^3 + 7340N^2 - 4888N - 1344, \tag{776}$$

$$P_{607} = 72N^8 - 116N^7 - 961N^6 + 1031N^5 + 3641N^4 - 251N^3 - 6172N^2 - 6316N + 1392, \tag{777}$$

$$P_{608} = 124N^8 + 481N^7 + 609N^6 + 181N^5 - 759N^4 - 1226N^3 - 730N^2 - 192N - 216, \tag{778}$$

$$P_{609} = 143N^8 - 454N^7 - 2349N^6 + 3611N^5 + 12288N^4 + 4331N^3 - 27470N^2 - 32364N - 7272, \tag{779}$$

$$P_{610} = 176N^8 - 10N^7 - 2127N^6 - 153N^5 + 5809N^4 + 3419N^3 - 2982N^2 - 8524N + 1320, \tag{780}$$

$$P_{611} = 200N^8 + 1179N^7 + 952N^6 - 2962N^5 - 4454N^4 - 1081N^3 + 530N^2 + 236N + 216, \tag{781}$$

$$P_{612} = 205N^8 + 646N^7 + 7902N^6 + 6508N^5 - 42471N^4 - 77522N^3 + 3764N^2 - 16824N + 137376, \tag{782}$$

$$P_{613} = 233N^8 + 392N^7 - 3288N^6 - 4054N^5 + 3891N^4 + 9254N^3 + 28540N^2 + 22920N - 10656, \tag{783}$$

$$P_{614} = 263N^8 + 602N^7 - 4467N^6 - 6067N^5 + 9090N^4 + 17633N^3 + 33166N^2 + 27756N - 35352, \tag{784}$$

$$P_{615} = 274N^8 + 654N^7 - 544N^6 - 1793N^5 - 1288N^4 - 1853N^3 - 782N^2$$
$$-68N + 216, \tag{785}$$

$$P_{616} = 434N^8 + 1964N^7 - 5853N^6 - 14263N^5 + 25449N^4 + 54611N^3$$
$$-15746N^2 + 19140N - 129672, \tag{786}$$

$$P_{617} = 485N^8 + 1172N^7 - 6414N^6 - 8680N^5 + 23145N^4 + 38828N^3 - 5048N^2$$
$$-6192N - 96048, \tag{787}$$

$$P_{618} = 641N^8 + 1412N^7 - 8862N^6 - 11728N^5 + 24861N^4 + 49412N^3 + 9640N^2$$
$$+8784N - 51120, \tag{788}$$

$$P_{619} = -205N^9 - 474N^8 + 744N^7 + 3298N^6 + 5881N^5 + 10528N^4 + 17388N^3$$
$$+20760N^2 + 13184N + 3232, \tag{789}$$

$$P_{620} = 13N^9 + 88N^8 + 136N^7 - 122N^6 - 177N^5 - 738N^4 - 2500N^3 - 2060N^2$$
$$-1984N - 1776, \tag{790}$$

$$P_{621} = 35N^9 - 40N^8 - 760N^7 - 28N^6 + 3417N^5 + 2604N^4 - 1580N^3 - 2032N^2$$
$$+928N + 768, \tag{791}$$

$$P_{622} = 71N^9 + 84N^8 - 660N^7 + 36N^6 + 3113N^5 + 1232N^4 - 4772N^3 - 8240N^2$$
$$-3968N - 2112, \tag{792}$$

$$P_{623} = 93N^9 + 1002N^8 + 5144N^7 + 12790N^6 + 13999N^5 + 2456N^4 - 9140N^3$$
$$-11528N^2 - 6880N - 1632, \tag{793}$$

$$P_{624} = 114N^9 + 1125N^8 + 2886N^7 - 2796N^6 - 16498N^5 - 15241N^4 + 830N^3$$
$$+6052N^2 + 6248N + 4992, \tag{794}$$

$$P_{625} = 140N^9 + 1139N^8 + 2790N^7 + 1892N^6 - 1836N^5 - 4153N^4 - 3614N^3$$
$$-1902N^2 - 720N - 648, \tag{795}$$

$$P_{626} = 293N^9 + 2208N^8 + 754N^7 - 7064N^6 + 1245N^5 + 38896N^4 + 54988N^3$$
$$+40360N^2 + 512N - 6816, \tag{796}$$

$$P_{627} = 388N^9 + 2233N^8 + 4500N^7 + 4324N^6 + 2124N^5 - 3167N^4 - 8164N^3$$
$$-6414N^2 - 2088N - 648, \tag{797}$$

$$P_{628} = 1067N^9 + 4674N^8 + 6232N^7 + 2504N^6 - 2267N^5 - 3250N^4 + 6992N^3$$
$$+17600N^2 + 13680N + 3744, \tag{798}$$

$$P_{629} = 1721N^9 + 10404N^8 + 25600N^7 + 22820N^6 - 13643N^5 - 11908N^4$$
$$+45842N^3 + 52772N^2 + 23496N + 4464, \tag{799}$$

$$P_{630} = 30N^{10} + 177N^9 - 26N^8 - 932N^7 - 534N^6 + 589N^5 - 282N^4 - 314N^3$$
$$+2852N^2 + 2520N + 144, \tag{800}$$

$$P_{631} = 191N^{10} + 663N^9 - 914N^8 - 3481N^7 + 3013N^6 + 7856N^5 - 9352N^4$$
$$-14488N^3 + 5280N^2 + 6048N - 5184, \tag{801}$$

$$P_{632} = 383N^{10} + 2370N^9 + 5056N^8 + 4994N^7 + 5029N^6 + 5516N^5 + 2420N^4$$
$$+5360N^3 + 13992N^2 + 13344N + 3744, \tag{802}$$

$$P_{633} = 764N^{10} + 436N^9 - 4817N^8 + 1183N^7 + 16949N^6 + 12811N^5$$
$$+9684N^4 + 4650N^3 + 2836N^2 - 2160N - 864, \tag{803}$$

$$P_{634} = 1427N^{10} + 2932N^9 - 2816N^8 + 1429N^7 + 18422N^6 + 15883N^5$$
$$+15795N^4 + 6756N^3 - 6620N^2 - 12456N - 4464, \tag{804}$$

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$$P_{635} = 1471N^{10} + 17463N^9 + 74204N^8 + 138364N^7 + 86603N^6 - 47561N^5$$
$$-29066N^4 + 91534N^3 + 81060N^2 + 9720N - 5616, \tag{805}$$

$$P_{636} = 19N^{11} + 134N^{10} - 120N^9 - 1514N^8 + 237N^7 + 5580N^6 - 1912N^5$$
$$-7208N^4 + 4848N^3 + 6464N^2 - 1920N - 17280, \tag{806}$$

$$P_{637} = 42N^{11} - 21N^{10} - 902N^9 - 1743N^8 - 123N^7 + 3687N^6 + 5601N^5$$
$$+3531N^4 + 6320N^3 + 14040N^2 + 13488N + 5040, \tag{807}$$

$$P_{638} = 90N^{11} + 399N^{10} - 448N^9 + 2160N^8 + 17304N^7 + 9873N^6 - 29826N^5$$
$$-27432N^4 - 1208N^3 - 7968N^2 - 36096N - 16704, \tag{808}$$

$$P_{639} = 240N^{11} + 633N^{10} - 1246N^9 + 5868N^8 + 31038N^7 - 5913N^6 - 104064N^5$$
$$-90228N^4 + 28888N^3 + 118368N^2 + 64800N + 27648, \tag{809}$$

$$P_{640} = 339N^{11} + 4164N^{10} + 17783N^9 - 2526N^8 - 133535N^7 - 158392N^6$$
$$+173869N^5 + 322418N^4 - 27256N^3 - 251632N^2 - 232368N - 113760, \tag{810}$$

$$P_{641} = 1565N^{11} + 11465N^{10} + 30742N^9 + 35334N^8 + 10185N^7 - 2331N^6$$
$$+34756N^5 + 92380N^4 + 111968N^3 + 73104N^2 + 41472N + 15552, \tag{811}$$

$$P_{642} = 4724N^{11} + 32993N^{10} + 49126N^9 - 60426N^8 - 161406N^7 - 29073N^6$$
$$+68536N^5 + 87850N^4 + 173588N^3 - 14280N^2 + 92016N + 95040, \tag{812}$$

$$P_{643} = 39N^{12} + 1656N^{11} + 7771N^{10} - 9028N^9 - 78031N^8 - 36088N^7 + 227761N^6$$
$$+194300N^5 - 257892N^4 - 310104N^3 - 149632N^2 + 33696N + 71424, \tag{813}$$

$$P_{644} = 63N^{12} - 381N^{11} - 2219N^{10} + 1918N^9 + 14631N^8 - 6291N^7 - 35967N^6$$
$$+62346N^5 + 83132N^4 - 143680N^3 - 160656N^2 - 10464N - 6336, \tag{814}$$

$$P_{645} = 153N^{12} + 49N^{11} - 1696N^{10} + 126N^9 + 5857N^8 - 3075N^7 - 10674N^6$$
$$+19412N^5 + 39320N^4 - 14912N^3 - 43712N^2 - 41344N - 19200, \tag{815}$$

$$P_{646} = 226N^{12} + 579N^{11} - 1701N^{10} - 2617N^9 + 8109N^8 + 1923N^7$$
$$-33361N^6 - 11565N^5 + 57183N^4 + 42280N^3 - 7200N^2 - 6768N - 5616, \tag{816}$$

$$P_{647} = 430N^{12} + 2883N^{11} + 2713N^{10} - 8201N^9 - 3396N^8 + 3522N^7 - 44182N^6$$
$$-83352N^5 - 43465N^4 + 53360N^3 + 113496N^2 + 99072N + 31536, \tag{817}$$

$$P_{648} = 651N^{12} + 4796N^{11} + 12675N^{10} + 13520N^9 + 6837N^8 + 19740N^7 + 57489N^6$$
$$+94024N^5 + 124188N^4 + 127120N^3 + 85008N^2 + 46656N$$
$$+15552, \tag{818}$$

$$P_{649} = 1973N^{12} + 8607N^{11} - 2581N^{10} - 29530N^9 - 4464N^8 - 5301N^7 - 29117N^6$$
$$+129132N^5 + 193261N^4 - 30332N^3 + 31872N^2 - 54864N - 63504, \tag{819}$$

$$P_{650} = 5399N^{12} + 21717N^{11} - 8182N^{10} - 141466N^9 - 174693N^8 + 104127N^7$$
$$+435784N^6 + 484686N^5 + 327700N^4 + 206368N^3 + 101088N^2 + 31968N$$
$$+5184, \tag{820}$$

$$P_{651} = 254N^{13} + 807N^{12} - 2683N^{11} - 10266N^{10} - 3135N^9 + 52217N^8 + 158159N^7$$
$$+189724N^6 + 56845N^5 - 102970N^4 - 172072N^3 - 143952N^2 - 66544N$$
$$-12576, \tag{821}$$

$$P_{652} = 5459N^{13} + 43255N^{12} + 82772N^{11} - 53316N^{10} - 306641N^9 - 409197N^8$$
$$-673052N^7 - 1415068N^6 - 1869746N^5 - 1266354N^4 - 356752N^3 + 99792N^2$$
$$+139104N + 38880, \tag{822}$$

$$P_{653} = 13789N^{13} + 50057N^{12} - 25496N^{11} - 233268N^{10} - 154315N^9 - 125559N^8$$
$$-1154968N^7 - 2581772N^6 - 2738026N^5 - 1636170N^4 - 717392N^3$$
$$-119664N^2 + 52704N + 7776, \tag{823}$$

$$P_{654} = -12043N^{14} - 178942N^{13} - 923869N^{12} - 1714868N^{11} + 1262555N^{10}$$
$$+9673170N^9 + 13425517N^8 + 4916752N^7 - 5024960N^6 - 9988720N^5$$
$$-15826400N^4 - 20182656N^3 - 16420608N^2 - 7236864N - 1306368, \tag{824}$$

$$P_{655} = 205N^{14} - 1070N^{13} - 6997N^{12} + 9444N^{11} + 57515N^{10} - 20678N^9$$
$$-188539N^8 + 10016N^7 + 350968N^6 - 25824N^5 - 496784N^4 + 19520N^3$$
$$+470400N^2 + 418560N + 156672, \tag{825}$$

$$P_{656} = 9429N^{14} + 70894N^{13} + 159471N^{12} - 47508N^{11} - 712173N^{10} - 1057954N^9$$
$$-675639N^8 - 551176N^7 - 224640N^6 + 1703952N^5 + 3636768N^4$$
$$+4242688N^3 + 3770112N^2 + 1951488N + 380160, \tag{826}$$

$$P_{657} = 617N^{15} + 1213N^{14} + 3647N^{13} - 5135N^{12} - 62711N^{11} - 59557N^{10}$$
$$+245733N^9 + 472003N^8 - 3502N^7 - 620780N^6 - 330504N^5 + 457568N^4$$
$$+783264N^3 + 552896N^2 + 210048N + 34560, \tag{827}$$

$$P_{658} = 3805N^{15} + 25467N^{14} + 31782N^{13} - 102029N^{12} - 229612N^{11} - 7783N^{10}$$
$$-46240N^9 - 603459N^8 + 610299N^7 + 3183968N^6 + 1748734N^5 - 3156692N^4$$
$$-5185536N^3 - 3830112N^2 - 1561248N - 212544, \tag{828}$$

$$P_{659} = 1538N^{16} + 6349N^{15} - 4604N^{14} - 4419N^{13} + 247590N^{12} + 595697N^{11}$$
$$-155918N^{10} - 1529965N^9 - 660880N^8 + 802290N^7 - 3242454N^6$$
$$-12852560N^5 - 19050600N^4 - 16010208N^3 - 9510048N^2 - 4292352N$$
$$-1026432, \tag{829}$$

$$P_{660} = 4920N^{17} + 59829N^{16} + 268187N^{15} + 460995N^{14} - 275823N^{13} - 2426947N^{12}$$
$$-3950033N^{11} - 2689715N^{10} - 806789N^9 - 436062N^8 + 2361354N^7$$
$$+9732268N^6 + 15162440N^5 + 13427504N^4 + 7410144N^3 + 2426688N^2$$
$$+328320N - 20736, \tag{830}$$

$$P_{661} = 32968N^{18} + 200687N^{17} - 76724N^{16} - 2085743N^{15} + 226312N^{14}$$
$$+17762935N^{13} + 21073412N^{12} - 46005765N^{11} - 129344416N^{10} - 84661090N^9$$
$$+89668640N^8 + 206886704N^7 + 116843424N^6 - 124546688N^5 - 362045568N^4$$
$$-451275264N^3 - 326343168N^2 - 127858176N - 20404224, \tag{831}$$

$$P_{662} = 34763N^{18} + 253932N^{17} + 235627N^{16} - 2269948N^{15} - 5497382N^{14}$$
$$+4841882N^{13} + 30639918N^{12} + 32530946N^{11} - 9999541N^{10} - 49961642N^9$$
$$-76271245N^8 - 149423714N^7 - 245858156N^6 - 261281936N^5$$
$$-186926496N^4 - 92761632N^3 - 31078080N^2 - 7713792N - 1306368, \tag{832}$$

$$P_{663} = -10701N^{20} - 117180N^{19} - 383698N^{18} + 362182N^{17} + 4985840N^{16}$$
$$+7991006N^{15} - 14095594N^{14} - 52618222N^{13} - 4462979N^{12} + 154482966N^{11}$$
$$+143027116N^{10} - 208752656N^9 - 456973712N^8 - 131024384N^7 + 630587904N^6$$
$$+1328877824N^5 + 1500652800N^4 + 1142401536N^3 + 629462016N^2$$
$$+238215168N + 44789760, \tag{833}$$

$$P_{664} = 295317N^{20} + 3132540N^{19} + 8925794N^{18} - 15503126N^{17} - 128158720N^{16}$$

$$
\begin{aligned}
&\quad -159520558N^{15} + 378838970N^{14} + 1195612190N^{13} + 526578043N^{12} \\
&\quad -1946889990N^{11} - 3106593740N^{10} - 1641770960N^9 - 862068320N^8 \\
&\quad -1429854848N^7 + 580862976N^6 + 5174659328N^5 + 7525373184N^4 \\
&\quad +5839105536N^3 + 2748681216N^2 + 747823104N + 89579520, \quad\quad (834)
\end{aligned}
$$

$$
\begin{aligned}
P_{665} =~ & 23379N^{21} + 334890N^{20} + 1845809N^{19} + 4416076N^{18} + 910717N^{17} \\
& -19493138N^{16} - 43527868N^{15} - 31459862N^{14} + 7207567N^{13} - 2971600N^{12} \\
& -47214885N^{11} + 27754946N^{10} + 294517297N^9 + 665343056N^8 + 1059128256N^7 \\
& +1391632944N^6 + 1471240544N^5 + 1188614208N^4 + 711979776N^3 \\
& +304328448N^2 + 83960064N + 11197440 \quad\quad\quad (835)
\end{aligned}
$$

and

$$
\begin{aligned}
C_{F_L,g}^{(3)} =~ & C_F \Bigg\{ T_F^2 N_F \Bigg[\frac{64 S_1^2 P_{685}}{3(N-1)N^2(1+N)^3(2+N)^2} + \frac{64 S_2 P_{698}}{3(N-1)N^2(1+N)^3(2+N)^2(3+N)} \\
& - \frac{32 P_{721}}{9(N-2)^2(N-1)N^4(1+N)^5(2+N)^4(3+N)} + \Bigg(-\frac{128(5+3N)S_2}{3(1+N)(2+N)(3+N)} \\
& + \frac{32 P_{716}}{9(N-2)(N-1)N^3(1+N)^4(2+N)^3(3+N)} \Bigg) S_1 \\
& - \frac{128(5+3N)S_3}{3(1+N)(2+N)(3+N)} + \Bigg(-\frac{128 P_{699}}{9(N-2)^2(N-1)N(1+N)^2(2+N)^2(3+N)} \\
& - \frac{256(N-1)S_1}{3(N-2)(1+N)(3+N)} \Bigg) S_{-2} + \frac{512(N-1)S_{-3}}{3(N-2)(1+N)(3+N)} \\
& + \frac{256(5+3N)S_{2,1}}{3(1+N)(2+N)(3+N)} - \frac{256\left(-2-3N+N^2\right)\zeta_3}{(N-2)(1+N)(2+N)(3+N)} \Bigg] + C_A T_F N_F \Bigg[\\
& - \frac{128 S_{-4} P_{667}}{(N-2)N(1+N)^2(2+N)(3+N)} + \frac{512 S_{-3,1} P_{669}}{(N-2)N(1+N)^2(2+N)(3+N)} \\
& + \frac{128 S_{-2,2} P_{672}}{(N-2)N(1+N)^2(2+N)(3+N)} - \frac{256 S_{3,1} P_{683}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)} \\
& + \frac{64 S_4 P_{688}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)} \\
& + \frac{256 S_{-2,1} P_{697}}{(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)} \\
& - \frac{32 S_2 P_{703}}{3(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)} \\
& - \frac{32 S_3 P_{704}}{3(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)} \\
& - \frac{64 \zeta_3 P_{705}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)} \\
& - \frac{32 S_1^2 P_{706}}{3(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)} \\
& - \frac{8 P_{722}}{9(N-2)^2(N-1)N^4(1+N)^5(2+N)^4(3+N)}
\end{aligned}
$$

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$$+\left(-\frac{64S_3P_{678}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}\right.$$

$$+\frac{64\zeta_3P_{692}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{8P_{719}}{9(N-2)^2(N-1)N^3(1+N)^4(2+N)^3(3+N)}$$

$$+\frac{128S_{-2,1}P_{673}}{(N-2)N(1+N)^2(2+N)(3+N)}+\frac{16\left(90+293N+380N^2+117N^3\right)S_2}{3N(1+N)^2(2+N)(3+N)}\bigg)S_1$$

$$-\frac{16\left(2+13N+13N^2\right)S_1^3}{3N(1+N)^2(2+N)}+\left(-\frac{256S_1^2P_{666}}{(N-2)N(1+N)^2(2+N)(3+N)}\right.$$

$$+\frac{256S_1P_{707}}{3(N-2)^2(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{32P_{717}}{9(N-2)^2(N-1)N^3(1+N)^4(2+N)^2(3+N)}+\frac{3072\zeta_3}{(1+N)(2+N)}$$

$$-\frac{256\left(6+N+N^2\right)S_2}{(N-2)N(1+N)^2(2+N)(3+N)}\bigg)S_{-2}+\frac{64\left(-54+13N+13N^2\right)S_{-2}^2}{(N-2)(1+N)(2+N)(3+N)}$$

$$+\left(-\frac{64P_{709}}{3(N-2)(N-1)N^2(1+N)^3(2+N)^2(3+N)}\right.$$

$$-\frac{64S_1P_{674}}{(N-2)N(1+N)^2(2+N)(3+N)}\bigg)S_{-3}-\frac{128\left(9+35N+50N^2+18N^3\right)S_{2,1}}{3N(1+N)^2(2+N)(3+N)}$$

$$-\frac{1536S_{-2,3}}{(1+N)(2+N)}+\frac{1536S_{-4,1}}{(1+N)(2+N)}-\frac{4608(N-1)S_{-2,1,1}}{(N-2)(1+N)(3+N)}$$

$$+\frac{7680\zeta_5}{(1+N)(2+N)}\bigg]\bigg\}$$

$$+C_AT_F^2N_F\bigg\{\frac{128S_{-2}P_{691}}{9(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{32P_{718}}{27(N-2)(N-1)N^3(1+N)^4(2+N)^4(3+N)}-\frac{64\zeta_3}{(N+1)(N+2)}$$

$$+\left(-\frac{64P_{677}}{27(N-1)N(1+N)^3(2+N)^2}+\frac{128S_2}{(1+N)(2+N)}+\frac{256S_{-2}}{3(1+N)(2+N)}\right)S_1$$

$$-\frac{128\left(-3-8N-6N^2+11N^3\right)S_1^2}{9(N-1)N(1+N)^2(2+N)}+\frac{128\left(-6-19N+11N^2+5N^3\right)S_2}{9(N-1)N(1+N)(2+N)^2}$$

$$-\frac{128S_1^3}{9(1+N)(2+N)}+\frac{512S_3}{9(1+N)(2+N)}-\frac{512S_{-3}}{3(1+N)(2+N)}-\frac{512S_{2,1}}{3(1+N)(2+N)}\bigg\}$$

$$+C_F^2T_FN_F\bigg\{\frac{4096S_{3,1}P_{668}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}+\frac{8S_1^2P_{676}}{N^2(1+N)^3(2+N)}$$

$$-\frac{128S_{-4}P_{670}}{(N-2)N(1+N)^2(2+N)(3+N)}-\frac{128S_4P_{687}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{8S_2P_{671}}{N^2(1+N)^3(2+N)}+\frac{64S_3P_{700}}{3(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)}$$

$$-\frac{128S_{-2,1}P_{701}}{(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)}$$

$$+\frac{128\zeta_3P_{702}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{8P_{714}}{(N-2)^2N^4(1+N)^5(2+N)(3+N)}$$

$$+\left(\frac{8P_{712}}{(N-2)(N-1)N^3(1+N)^4(2+N)^2(3+N)}\right.$$

$$+\frac{128S_3P_{679}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{128\zeta_3P_{690}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}+\frac{48\big(2+N+N^2\big)S_2}{N(1+N)^2(2+N)}$$

$$-\frac{768(N-1)\big(2+N+N^2\big)S_{-2,1}}{(N-2)N(1+N)^2(3+N)}\bigg)S_1-\frac{80\big(2+N+N^2\big)S_1^3}{3N(1+N)^2(2+N)}$$

$$+\left(\frac{128S_1P_{696}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}+\frac{256\big(2+N+N^2\big)S_1^2}{N(1+N)^2(2+N)}\right.$$

$$-\frac{32P_{715}}{(N-2)^2(N-1)N^3(1+N)^4(2+N)^2(3+N)}-\frac{256\big(2+N+N^2\big)S_2}{N(1+N)^2(2+N)}$$

$$-\frac{3072\zeta_3}{(1+N)(2+N)}\bigg)S_{-2}-\frac{512\big(-4+N+N^2\big)S_{-2}^2}{(N-2)(1+N)(2+N)(3+N)}$$

$$+\left(\frac{64P_{708}}{(N-2)(N-1)N^2(1+N)^3(2+N)^2(3+N)}\right.$$

$$-\frac{128\big(-18+N+N^2\big)\big(2+N+N^2\big)S_1}{(N-2)N(1+N)^2(2+N)(3+N)}\bigg)S_{-3}+\frac{64\big(2+N+N^2\big)S_{2,1}}{N(1+N)^2(2+N)}$$

$$-\frac{256\big(2+N+N^2\big)\big(6+N+N^2\big)S_{-2,2}}{(N-2)N(1+N)^2(2+N)(3+N)}-\frac{3072\big(2+N+N^2\big)S_{-3,1}}{(N-2)N(1+N)^2(2+N)(3+N)}$$

$$+\frac{1536S_{-2,3}}{(1+N)(2+N)}-\frac{1536S_{-4,1}}{(1+N)(2+N)}+\frac{1024\big(2+N+N^2\big)S_{-2,1,1}}{(N-2)(1+N)(2+N)(3+N)}$$

$$-\frac{7680\zeta_5}{(1+N)(2+N)}\bigg\}$$

$$+C_A^2T_FN_F\left\{\frac{64S_4P_{680}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}\right.$$

$$+\frac{256S_{3,1}P_{682}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{128S_{-2,1}P_{689}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{16\zeta_3P_{693}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$
-\frac{32 S_3 P_{694}}{9(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)} - \frac{32 S_2 P_{710}}{9(N-1)^2 N^2(1+N)^3(2+N)^3}
$$

$$
+\frac{16 P_{723}}{27(N-2)(N-1)^2 N^4(1+N)^5(2+N)^5(3+N)}
$$

$$
+\left[-\frac{32 S_2 P_{675}}{(N-1)N(1+N)^2(2+N)^2} + \frac{128 S_3 P_{681}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}\right.
$$

$$
-\frac{128\zeta_3 P_{686}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)} - \frac{512\big(-14+3N+3N^2\big)S_{-2,1}}{(N-2)(1+N)(2+N)(3+N)}
$$

$$
\left.+\frac{32 P_{720}}{27(N-2)(N-1)^2 N^3(1+N)^4(2+N)^4(3+N)}\right]S_1 + \left[-\frac{256 S_2}{(1+N)(2+N)}\right.
$$

$$
\left.+\frac{32 P_{711}}{9(N-1)^2 N^2(1+N)^3(2+N)^3}\right]S_1^2 + \frac{32\big(-36-47N-72N^2+83N^3\big)S_1^3}{9(N-1)N(1+N)^2(2+N)}
$$

$$
+\frac{32 S_1^4}{(1+N)(2+N)} + \frac{32 S_2^2}{(1+N)(2+N)} + \left[-\frac{128 S_1^2}{(1+N)(2+N)}\right.
$$

$$
-\frac{64 S_1 P_{695}}{3(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)} + \frac{256 S_2}{(1+N)(2+N)}
$$

$$
\left.-\frac{32 P_{713}}{9(N-2)(N-1)^2 N^2(1+N)^3(2+N)^3(3+N)} - \frac{768\zeta_3}{(1+N)(2+N)}\right]S_{-2}
$$

$$
-\frac{32\big(-14+5N+5N^2\big)S_{-2}^2}{(N-2)(1+N)(2+N)(3+N)} + \left[\frac{512\big(-16+3N+3N^2\big)S_1}{(N-2)(1+N)(2+N)(3+N)}\right.
$$

$$
\left.-\frac{64 P_{684}}{3(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}\right]S_{-3} + \frac{1408 S_{2,1}}{3(1+N)(2+N)}
$$

$$
+\frac{32\big(-122+23N+23N^2\big)S_{-4}}{(N-2)(1+N)(2+N)(3+N)} - \frac{128\big(-62+13N+13N^2\big)S_{-2,2}}{(N-2)(1+N)(2+N)(3+N)}
$$

$$
+\frac{384 S_{-2,3}}{(1+N)(2+N)} - \frac{256\big(-34+7N+7N^2\big)S_{-3,1}}{(N-2)(1+N)(2+N)(3+N)} - \frac{384 S_{-4,1}}{(1+N)(2+N)}
$$

$$
\left.+\frac{1024\big(-14+3N+3N^2\big)S_{-2,1,1}}{(N-2)(1+N)(2+N)(3+N)} - \frac{1920\zeta_5}{(1+N)(2+N)}\right\}. \tag{836}
$$

with

$$
\begin{align}
P_{666} &= N^4 + 2N^3 - 6N^2 - 7N - 6, \tag{837}\\
P_{667} &= N^4 + 2N^3 + 16N^2 + 15N + 18, \tag{838}\\
P_{668} &= 2N^4 + 4N^3 + N^2 - N + 12, \tag{839}\\
P_{669} &= 4N^4 + 8N^3 - 3N^2 - 7N + 6, \tag{840}\\
P_{670} &= 5N^4 + 10N^3 - 27N^2 - 32N - 36, \tag{841}\\
P_{671} &= 15N^4 + 48N^3 + 71N^2 + 58N + 8, \tag{842}\\
P_{672} &= 17N^4 + 34N^3 - 15N^2 - 32N + 12, \tag{843}\\
P_{673} &= 21N^4 + 42N^3 - 31N^2 - 52N - 12, \tag{844}\\
P_{674} &= 25N^4 + 50N^3 - 27N^2 - 52N + 36, \tag{845}
\end{align}
$$

$$P_{675} = 35N^4 + 46N^3 - 79N^2 - 66N - 32, \tag{846}$$

$$P_{676} = 35N^4 + 84N^3 + 99N^2 + 78N + 16, \tag{847}$$

$$P_{677} = 302N^5 + 597N^4 - 286N^3 - 1017N^2 - 844N - 120, \tag{848}$$

$$P_{678} = N^6 + 3N^5 - 87N^4 - 179N^3 - 46N^2 + 44N - 312, \tag{849}$$

$$P_{679} = N^6 + 3N^5 - 35N^4 - 75N^3 - 26N^2 + 12N - 168, \tag{850}$$

$$P_{680} = N^6 + 3N^5 - 33N^4 - 71N^3 - 4N^2 + 32N - 72, \tag{851}$$

$$P_{681} = N^6 + 3N^5 - 19N^4 - 43N^3 + 22N - 36, \tag{852}$$

$$P_{682} = N^6 + 3N^5 + 9N^4 + 13N^3 + 8N^2 + 2N + 36, \tag{853}$$

$$P_{683} = N^6 + 3N^5 + 37N^4 + 69N^3 + 22N^2 - 12N + 168, \tag{854}$$

$$P_{684} = 2N^6 - 6N^5 + 143N^4 + 396N^3 - 403N^2 - 720N - 612, \tag{855}$$

$$P_{685} = 2N^6 + 6N^5 + 7N^4 + 4N^3 + 9N^2 + 8N + 12, \tag{856}$$

$$P_{686} = 3N^6 + 9N^5 - 35N^4 - 85N^3 - 4N^2 + 40N - 72, \tag{857}$$

$$P_{687} = 3N^6 + 9N^5 + 23N^4 + 31N^3 - 14N^2 - 28N + 264, \tag{858}$$

$$P_{688} = 5N^6 + 15N^5 + 65N^4 + 105N^3 + 14N^2 - 36N + 408, \tag{859}$$

$$P_{689} = 8N^6 + 20N^5 - 13N^4 - 26N^3 - 9N^2 - 32N - 60, \tag{860}$$

$$P_{690} = 11N^6 + 33N^5 - 85N^4 - 225N^3 - 130N^2 - 12N - 168, \tag{861}$$

$$P_{691} = 16N^6 + 48N^5 - 83N^4 - 246N^3 + 13N^2 + 144N - 36, \tag{862}$$

$$P_{692} = 17N^6 + 51N^5 - 211N^4 - 507N^3 - 154N^2 + 108N - 456, \tag{863}$$

$$P_{693} = 33N^6 + 99N^5 - 421N^4 - 1007N^3 + 68N^2 + 588N - 1152, \tag{864}$$

$$P_{694} = 35N^6 + 33N^5 - 976N^4 - 1407N^3 + 131N^2 + 132N - 3132, \tag{865}$$

$$P_{695} = 59N^6 + 129N^5 - 427N^4 - 669N^3 + 794N^2 + 678N - 612, \tag{866}$$

$$P_{696} = N^7 - N^6 - 11N^5 + 9N^4 - 66N^3 - 68N^2 + 168N - 96, \tag{867}$$

$$P_{697} = 2N^7 + 7N^6 - 48N^5 - 160N^4 - 120N^3 + 29N^2 + 138N + 72, \tag{868}$$

$$P_{698} = 6N^7 + 24N^6 + 19N^5 - 37N^4 - 73N^3 - 59N^2 - 36N - 36, \tag{869}$$

$$P_{699} = 13N^7 + 13N^6 - 103N^5 - 113N^4 + 74N^3 + 172N^2 + 520N - 864, \tag{870}$$

$$P_{700} = 17N^7 + 56N^6 + 216N^5 + 554N^4 + 891N^3 + 854N^2 + 748N + 120, \tag{871}$$

$$P_{701} = 21N^7 + 80N^6 - 74N^5 - 468N^4 - 343N^3 + 164N^2 + 316N + 48, \tag{872}$$

$$P_{702} = 23N^7 + 75N^6 - 152N^5 - 473N^4 - 109N^3 + 136N^2 - 156N + 144, \tag{873}$$

$$P_{703} = 27N^7 + 42N^6 - 68N^5 - 51N^4 + 109N^3 + 15N^2 - 554N - 384, \tag{874}$$

$$P_{704} = 31N^7 + 156N^6 + 722N^5 + 1524N^4 + 1595N^3 + 960N^2 + 1828N + 1248, \tag{875}$$

$$P_{705} = 32N^7 + 146N^6 - 269N^5 - 972N^4 - 149N^3 + 416N^2 - 612N + 144, \tag{876}$$

$$P_{706} = 77N^7 + 278N^6 - 260N^5 - 1501N^4 - 725N^3 + 1145N^2 + 1466N + 384, \tag{877}$$

$$P_{707} = 14N^8 + 11N^7 - 137N^6 - 85N^5 + 514N^4 - 109N^3 - 880N^2 + 828N - 144, \tag{878}$$

$$P_{708} = 29N^8 + 112N^7 - 42N^6 - 484N^5 - 463N^4 - 12N^3 + 300N^2 + 112N + 192, \tag{879}$$

$$P_{709} = 49N^8 + 190N^7 - 244N^6 - 1346N^5 - 1213N^4 + 4N^3 + 1096N^2 + 696N + 288, \tag{880}$$

$$P_{710} = 67N^8 + 334N^7 - 73N^6 - 1247N^5 - 406N^4 + 1645N^3 + 1708N^2 - 12N - 72, \tag{881}$$

$$P_{711} = 358N^8 + 1078N^7 - 397N^6 - 2729N^5 - 529N^4 + 2743N^3 + 2944N^2 - 12N$$
$$- 216, \tag{882}$$

$$P_{712} = 91N^{10} + 417N^9 - 128N^8 - 2686N^7 - 6937N^6 - 9783N^5 - 4234N^4 + 2324N^3$$
$$+ 424N^2 - 1440N - 576, \tag{883}$$

$$P_{713} = 281N^{10} + 1405N^9 - 105N^8 - 7986N^7 - 4629N^6 + 13197N^5 + 7549N^4$$
$$-15904N^3 - 2448N^2 + 16848N + 2160, \tag{884}$$

$$P_{714} = 18N^{11} + 68N^{10} + 179N^9 - 304N^8 - 1767N^7 - 2263N^6 - 230N^5 + 1039N^4$$
$$+76N^3 - 1424N^2 - 1040N - 240, \tag{885}$$

$$P_{715} = 59N^{11} + 161N^{10} - 392N^9 - 1310N^8 + 31N^7 + 2625N^6 + 1478N^5 - 1220N^4$$
$$-152N^3 - 1088N^2 + 64N + 768, \tag{886}$$

$$P_{716} = 93N^{11} + 543N^{10} + 650N^9 - 1782N^8 - 5427N^7 - 4905N^6 - 900N^5 + 4272N^4$$
$$+7024N^3 + 5616N^2 + 7488N + 3456, \tag{887}$$

$$P_{717} = 581N^{11} + 1653N^{10} - 3824N^9 - 13086N^8 + 1281N^7 + 26757N^6 + 12850N^5$$
$$-15036N^4 - 808N^3 - 5184N^2 - 6336N + 3456, \tag{888}$$

$$P_{718} = 51N^{12} + 423N^{11} - 314N^{10} - 9178N^9 - 21183N^8 + 4305N^7 + 82314N^6$$
$$+134826N^5 + 92180N^4 + 11968N^3 - 18048N^2 - 11232N - 1728, \tag{889}$$

$$P_{719} = 1347N^{12} + 5385N^{11} - 13244N^{10} - 70178N^9 - 33309N^8 + 167445N^7$$
$$+305262N^6 + 206148N^5 - 13096N^4 - 53200N^3 + 70560N^2 + 100800N$$
$$+31104, \tag{890}$$

$$P_{720} = 1820N^{13} + 12757N^{12} + 15253N^{11} - 73492N^{10} - 200208N^9 - 38667N^8$$
$$+354250N^7 + 351875N^6 - 74335N^5 - 203477N^4 + 26664N^3 + 91944N^2$$
$$+58320N + 19440, \tag{891}$$

$$P_{721} = 12N^{15} + 134N^{14} + 501N^{13} + 453N^{12} - 1684N^{11} - 5780N^{10} - 10781N^9$$
$$-18749N^8 - 8256N^7 + 28470N^6 + 10840N^5 - 52984N^4 - 74944N^3$$
$$-83808N^2 - 58752N - 17280, \tag{892}$$

$$P_{722} = 96N^{15} + 266N^{14} - 5889N^{13} - 31881N^{12} - 27580N^{11} + 147580N^{10} + 402667N^9$$
$$+282187N^8 - 247206N^7 - 501336N^6 - 287480N^5 - 222400N^4 - 437536N^3$$
$$-461952N^2 - 206208N - 34560, \tag{893}$$

$$P_{723} = 150N^{16} + 1614N^{15} - 1820N^{14} - 61808N^{13} - 210369N^{12} - 122317N^{11}$$
$$+723938N^{10} + 1759218N^9 + 1253237N^8 - 923191N^7 - 2589744N^6$$
$$-2736284N^5 - 2333312N^4 - 1942608N^3 - 1354752N^2 - 565056N - 103680. \tag{894}$$

The d_{abc} contribution read

$$C_{F_2,g}^{d_{abc},(3)} = \frac{d_{abc}d^{abc}N_F^2}{N_A}\Bigg\{$$

$$-\frac{256S_{-2,1}P_{725}}{45N(1+N)(2+N)^2} + \frac{128S_3P_{726}}{(N-2)N(1+N)(2+N)^2(3+N)}$$

$$-\frac{256S_4P_{734}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)} + \frac{512S_{3,1}P_{734}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{128(N-1)S_5P_{724}}{45N(1+N)(2+N)} + \frac{32P_{742}}{225(N-2)(N-1)^3N^3(1+N)^5(2+N)^5(3+N)^3}$$

$$+\Bigg(-\frac{64P_{740}}{225(N-2)(N-1)^2N^2(1+N)^4(2+N)^4(3+N)^3} - \frac{1024(N-4)(N-1)S_{3,1}}{N(1+N)(2+N)}$$

$$-\frac{128S_3P_{733}}{(N-2)N^2(1+N)^2(2+N)^2(3+N)} + \frac{512(N-4)(N-1)S_4}{N(1+N)(2+N)}$$

$$
+\frac{512\bigl(8-3N+3N^2\bigr)S_{-2,1}}{3N(1+N)(2+N)}\Biggr)S_1+\left(\frac{128\bigl(19-30N+3N^2\bigr)}{3N(1+N)(2+N)}+\frac{256(N-4)(N-1)S_3}{N(1+N)(2+N)}\right)
$$

$$
\times S_1^2-\frac{768(N-4)(N-1)S_2S_3}{N(1+N)(2+N)}+\Biggl(\frac{256S_1P_{736}}{45(N-2)N^2(1+N)^2(2+N)^2(3+N)}
$$

$$
+\frac{64P_{738}}{45(N-2)N^2(1+N)^3(2+N)^3(3+N)}-\frac{256\bigl(8-3N+3N^2\bigr)S_1^2}{3N(1+N)(2+N)}
$$

$$
+\frac{256}{45}(N-1)\bigl(21-2N+N^2\bigr)S_3+\frac{256}{45}(N-1)\bigl(21-2N+N^2\bigr)S_{-2,1}\Biggr)S_{-2}
$$

$$
-\frac{128(N-1)S_{-2}^2}{(1+N)(2+N)}+\left(\frac{128P_{725}}{45N(1+N)(2+N)^2}-\frac{256\bigl(8-3N+3N^2\bigr)S_1}{3N(1+N)(2+N)}\right)S_{-3}
$$

$$
-\frac{256\bigl(4-3N+3N^2\bigr)S_{-4}}{3N(1+N)(2+N)}+\frac{128}{45}(N-1)\bigl(21-2N+N^2\bigr)S_{-5}
$$

$$
+\frac{768(N-4)(N-1)S_{2,3}}{N(1+N)(2+N)}-\frac{768(N-4)(N-1)S_{4,1}}{N(1+N)(2+N)}+\frac{256\bigl(8-3N+3N^2\bigr)S_{-2,2}}{3N(1+N)(2+N)}
$$

$$
-\frac{256}{45}(N-1)\bigl(21-2N+N^2\bigr)S_{-2,3}+\frac{256\bigl(8-3N+3N^2\bigr)S_{-3,1}}{3N(1+N)(2+N)}
$$

$$
+\frac{1536(N-4)(N-1)S_{3,1,1}}{N(1+N)(2+N)}-\frac{512\bigl(8-3N+3N^2\bigr)S_{-2,1,1}}{3N(1+N)(2+N)}
$$

$$
-\frac{256}{45}(N-1)\bigl(21-2N+N^2\bigr)S_{-2,1,-2}+\Biggl(-\frac{64P_{732}}{45(N-2)N(1+N)(2+N)^2(3+N)}
$$

$$
+\frac{512S_1P_{735}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}-\frac{512(N-4)(N-1)S_1^2}{N(1+N)(2+N)}
$$

$$
+\frac{256}{45}(N-1)\bigl(21-2N+N^2\bigr)S_{-2}\Biggr)\zeta_3+\frac{64(N-1)P_{724}}{9N(1+N)(2+N)}\zeta_5\Biggr\},\tag{895}
$$

$$
C_{F_L,g}^{d_{abc},(3)}=\frac{d_{abc}d^{abc}N_F^2}{N_A}\Biggl\{
$$

$$
\frac{256S_4P_{729}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}-\frac{512S_{3,1}P_{729}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}
$$

$$
-\frac{256P_{741}}{225(N-2)(N-1)^3N^3(1+N)^5(2+N)^5(3+N)^3}+\left(\frac{6144S_{3,1}}{(1+N)(2+N)}-\frac{3072S_4}{(1+N)(2+N)}\right.
$$

$$
+\frac{512S_3P_{728}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}-\frac{1024S_{-2,1}}{(1+N)(2+N)}
$$

$$
+\frac{256P_{739}}{225(N-2)(N-1)^2N^2(1+N)^4(2+N)^4(3+N)^3}\Biggr)S_1+\left(-\frac{1408}{(1+N)(2+N)}\right.
$$

$$
\left.-\frac{1536S_3}{(1+N)(2+N)}\right)S_1^2-\frac{1536\bigl(1+N+N^2\bigr)\bigl(-10+3N+3N^2\bigr)S_3}{(N-2)(N-1)(1+N)(2+N)^2(3+N)}+\frac{4608S_2S_3}{(1+N)(2+N)}
$$

$$
-\frac{128(N-4)(5+N)\bigl(18+N+N^2\bigr)S_5}{15(1+N)(2+N)}+\left(\frac{512S_1^2}{(1+N)(2+N)}-\frac{256}{15}(N-1)NS_3\right.
$$

$$-\frac{256 S_1 P_{731}}{15(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)} - \frac{256}{15}(N-1)N S_{-2,1}$$

$$-\frac{128 P_{737}}{15(N-2)(N-1)N(1+N)^3(2+N)^3(3+N)}\Bigg) S_{-2} + \frac{256 S_{-2}^2}{(1+N)(2+N)}$$

$$+\left(-\frac{128\left(10+N+N^2\right)\left(30+N+N^2\right)}{15(N-1)(1+N)(2+N)^2} + \frac{512 S_1}{(1+N)(2+N)}\right) S_{-3} + \frac{512 S_{-4}}{(1+N)(2+N)}$$

$$-\frac{128}{15}(N-1)N S_{-5} - \frac{4608 S_{2,3}}{(1+N)(2+N)} + \frac{256\left(10+N+N^2\right)\left(30+N+N^2\right) S_{-2,1}}{15(N-1)(1+N)(2+N)^2}$$

$$+\frac{4608 S_{4,1}}{(1+N)(2+N)} - \frac{512 S_{-2,2}}{(1+N)(2+N)} + \frac{256}{15}(N-1)N S_{-2,3} - \frac{512 S_{-3,1}}{(1+N)(2+N)}$$

$$-\frac{9216 S_{3,1,1}}{(1+N)(2+N)} + \frac{1024 S_{-2,1,1}}{(1+N)(2+N)} + \frac{256}{15}(N-1)N S_{-2,1,-2} + \left(\frac{3072 S_1^2}{(1+N)(2+N)}\right.$$

$$-\frac{256}{15}(N-1)N S_{-2} + \frac{128 P_{727}}{15(N-2)(N-1)(1+N)(2+N)^2(3+N)}$$

$$\left.-\frac{512 S_1 P_{730}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}\right) \zeta_3 - \frac{64(N-4)(5+N)\left(18+N+N^2\right)}{3(1+N)(2+N)}\zeta_5\Bigg\},$$

$$(896)$$

with the polynomials

$$
\begin{aligned}
P_{724} &= N^5 + N^4 + 17N^3 + 59N^2 - 138N + 720, & (897)\\
P_{725} &= N^5 + N^4 + 59N^3 + 59N^2 + 225N - 450, & (898)\\
P_{726} &= 5N^5 - 16N^4 - 58N^3 + 25N^2 + 32N + 60, & (899)\\
P_{727} &= N^6 + 3N^5 + 1162N^4 + 2319N^3 - 1661N^2 - 2820N - 4860, & (900)\\
P_{728} &= 11N^6 + 33N^5 - 15N^4 - 85N^3 - 50N^2 - 2N + 36, & (901)\\
P_{729} &= 21N^6 + 63N^5 - 25N^4 - 155N^3 - 104N^2 - 16N + 72, & (902)\\
P_{730} &= 23N^6 + 69N^5 - 35N^4 - 185N^3 - 96N^2 + 8N + 72, & (903)\\
P_{731} &= 45N^6 + 135N^5 + 215N^4 + 205N^3 - 1340N^2 - 1420N + 624, & (904)\\
P_{732} &= 2N^7 + 4N^6 + 873N^5 - 3466N^4 - 11255N^3 + 6582N^2 + 7020N + 16200, & (905)\\
P_{733} &= 5N^7 - 19N^6 - 105N^5 + 23N^4 + 240N^3 + 104N^2 - 8N - 96, & (906)\\
P_{734} &= 6N^7 - 33N^6 - 154N^5 + 41N^4 + 358N^3 + 202N^2 + 36N - 144, & (907)\\
P_{735} &= 9N^7 - 24N^6 - 161N^5 + 28N^4 + 362N^3 + 110N^2 - 60N - 144, & (908)\\
P_{736} &= 45N^7 + 90N^6 + 50N^5 - 130N^4 - 1895N^3 - 740N^2 + 984N - 1944, & (909)\\
P_{737} &= N^{10} + 7N^9 - 58N^8 - 438N^7 - 1017N^6 - 987N^5 - 130N^4 + 674N^3 \\
&\quad + 3940N^2 + 9960N + 6480, & (910)\\
P_{738} &= 2N^{11} + 12N^{10} - 315N^9 - 2240N^8 - 4508N^7 + 2140N^6 + 22169N^5 \\
&\quad + 36208N^4 + 30892N^3 + 10680N^2 - 23040N - 21600, & (911)\\
P_{739} &= 1065N^{14} + 14910N^{13} + 91950N^{12} + 334830N^{11} + 786240N^{10} + 1070370N^9 \\
&\quad + 148290N^8 - 2319270N^7 - 3721785N^6 - 1334280N^5 + 2039760N^4 \\
&\quad + 2233440N^3 + 654480N^2, & (912)\\
P_{740} &= 220N^{15} - 960N^{14} - 28415N^{13} - 151530N^{12} - 369410N^{11} - 556890N^{10}
\end{aligned}
$$

$$-716080N^9 - 97590N^8 + 2725310N^7 + 4299810N^6 - 627745N^5 - 5315880N^4$$
$$-2040120N^3 + 1823040N^2 + 1056240N, \tag{913}$$

$$P_{741} = 405N^{18} + 6480N^{17} + 43335N^{16} + 152280N^{15} + 272970N^{14} + 103680N^{13}$$
$$-541890N^{12} - 1017360N^{11} - 393255N^{10} + 810000N^9 + 1042875N^8$$
$$+217080N^7 - 366120N^6 - 272160N^5 - 58320N^4, \tag{914}$$

$$P_{742} = 555N^{19} + 6195N^{18} + 17955N^{17} - 54135N^{16} - 471780N^{15} - 1092330N^{14}$$
$$-398730N^{13} + 2590050N^{12} + 4158675N^{11} - 126225N^{10} - 5426505N^9$$
$$-3556395N^8 + 1998870N^7 + 2874360N^6 + 341280N^5 - 641520N^4$$
$$-220320N^3. \tag{915}$$

Despite the fact that the above equations contain terms in which the power of N is larger in the numerator than the denominator, the whole expressions behave maximally $\propto \ln(N)/N$ for large values of N for the Wilson coefficient of F_2 and $\propto \ln(N)/N^2$ for that of F_L.

6 The three–loop Wilson coefficients for the structure function $xF_3(x, Q^2)$

The Wilson coefficient for the structure function $xF_3(x, Q^2)$ has already received the renormalizations of the coupling and those of the axial-vector coupling, as well the collinear singularities have been removed, to arrive at $\mathbb{C}^{(3)}_{\hat{F}_3, q}$, which still requires the finite renormalization to restore the Ward identities performed by

$$\mathbb{C}^{(3)}_{F_3, q} = Z_5 \mathbb{C}^{(3)}_{\hat{F}_3, q}, \tag{916}$$

with, [17],

$$Z_5 = 1 + a_s C_F \left[-4 - 10\varepsilon + \varepsilon^2 \left(-22 + 2\zeta_2 \right) \right] + a_s^2 \left[-\frac{107}{9} C_A C_F + 22 C_F^2 + \frac{4}{9} C_F T_F N_F \right.$$
$$\left. +\varepsilon \left(\frac{662}{27} C_F T_F N_F + C_F^2 (132 - 48\zeta_3) + C_A C_F \left(-\frac{7229}{54} + 48\zeta_3 \right) \right) \right]$$
$$+a_s^3 \left[\frac{208}{81} C_F T_F^2 N_F^2 + C_A C_F^2 \left(\frac{5834}{27} - 160\zeta_3 \right) + 2C_F^2 T_F N_F \left(-\frac{62}{27} - \frac{32\zeta_3}{3} \right) \right.$$
$$\left. +2C_A C_F T_F N_F \left(\frac{356}{81} + \frac{32\zeta_3}{3} \right) + C_A^2 C_F \left(-\frac{2147}{27} + 56\zeta_3 \right) + C_F^3 \left(-\frac{370}{3} + 96\zeta_3 \right) \right]. \tag{917}$$

One obtains

$$\mathbb{C}^{(3)}_{F_3, q} = C^{\mathrm{NS},(3)}_{F_3, q} + C^{d_{abc},(3)}_{F_3, q} \tag{918}$$

with the contributions

$$C^{\mathrm{NS},(3)}_{F_3} = C_F^2 \left\{ C_A \left[-\frac{20\zeta_5 P_{754}}{3N(1+N)} + \frac{8S_{-2,1,-2} P_{758}}{3N(1+N)} + \frac{16S_{-2,3} P_{760}}{9N(1+N)} + \frac{36\zeta_2^2 P_{764}}{5N^2(1+N)^2} \right. \right.$$

$$-\frac{4S_5 P_{766}}{9N(1+N)} - \frac{4S_{2,1,1}P_{776}}{3N^2(1+N)^2} + \frac{S_2^2 P_{788}}{27N^2(1+N)^2} - \frac{4S_{-3,1}P_{815}}{9(-1+N)N^2(1+N)^2(2+N)}$$

$$+\frac{8S_{-2,2}P_{821}}{27(-1+N)N^2(1+N)^2(2+N)} + \frac{4S_4 P_{837}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$+\frac{4S_{3,1}P_{838}}{27(-1+N)N^2(1+N)^2(2+N)} - \frac{8S_{-2,1,1}P_{839}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$+\frac{\zeta_3 P_{850}}{3(-1+N)N^3(1+N)^3(2+N)} + \frac{P_{884}}{648(-1+N)^2 N^6(1+N)^6(2+N)}$$

$$+\left(\frac{2\zeta_3 P_{828}}{3(-1+N)N^2(1+N)^2(2+N)} - \frac{2S_3 P_{820}}{27(-1+N)N^2(1+N)^2(2+N)}\right.$$

$$+\frac{8S_{-2,1}P_{836}}{27(-1+N)N^2(1+N)^2(2+N)} + \frac{P_{878}}{648(-1+N)^2 N^5(1+N)^5(2+N)}$$

$$+\frac{4S_{2,1}P_{782}}{9N^2(1+N)^2} + \left(\frac{P_{843}}{162N^3(1+N)^3(2+N)} - \frac{2960}{3}S_3 + \frac{160}{3}S_{2,1} - \frac{15104}{3}S_{-2,1}\right.$$

$$\left.+480\zeta_3\right)S_2 + \frac{8\left(15+236N+236N^2\right)S_2^2}{9N(1+N)} + \frac{8\left(-99+734N+734N^2\right)S_4}{9N(1+N)}$$

$$+256S_5 + \frac{2080}{3}S_{2,3} + 3920S_{2,-3} - \frac{8\left(-1212+869N+869N^2\right)S_{3,1}}{9N(1+N)} + 240S_{4,1}$$

$$-\frac{8\left(1662+61N+61N^2\right)S_{-2,2}}{9N(1+N)} - 3840S_{-2,3} - \frac{32\left(107+50N+50N^2\right)S_{-3,1}}{3N(1+N)}$$

$$+64S_{-2,-3} - \frac{4000}{3}S_{-4,1} - \frac{40\left(6+13N+13N^2\right)S_{2,1,1}}{3N(1+N)} - \frac{11552}{3}S_{2,1,-2} - \frac{32}{3}S_{2,2,1}$$

$$+1984S_{3,1,1} - \frac{32\left(-1020+239N+239N^2\right)S_{-2,1,1}}{9N(1+N)} - \frac{11744}{3}S_{-2,1,-2} - \frac{12224}{3}S_{-2,2,1}$$

$$-\frac{6784}{3}S_{-3,1,1} - 128S_{2,1,1,1} + 11776S_{-2,1,1,1} + 560\zeta_5\right)S_1 + \left(-\frac{4S_2 P_{786}}{9N^2(1+N)^2}\right.$$

$$+\frac{P_{868}}{36(-1+N)N^4(1+N)^4(2+N)} - \frac{40}{3}S_2^2 + \frac{4\left(-744+733N+733N^2\right)S_3}{9N(1+N)} - 88S_4$$

$$+\frac{16\left(30+29N+29N^2\right)S_{2,1}}{9N(1+N)} - \frac{2464}{3}S_{3,1} + \frac{208\left(-54+19N+19N^2\right)S_{-2,1}}{9N(1+N)}$$

$$+\frac{7216}{3}S_{-2,2} + \frac{6304}{3}S_{-3,1} + 80S_{2,1,1} - \frac{16256}{3}S_{-2,1,1} + \frac{4\left(94+35N+35N^2\right)\zeta_3}{N(1+N)}\right)S_1^2$$

$$+\left(\frac{P_{842}}{54N^3(1+N)^3} - \frac{4\left(108+491N+491N^2\right)S_2}{27N(1+N)} + \frac{1648}{9}S_3 - \frac{320}{9}S_{2,1} + 1120S_{-2,1}\right.$$

$$\left.-96\zeta_3\right)S_1^3 + \left(\frac{-110+433N+433N^2}{9N(1+N)} + 8S_2\right)S_1^4 + \frac{44}{9}S_1^5 + \left(-\frac{616}{3}S_4 + 880S_{-3,1}\right.$$

$$+\frac{P_{873}}{324(-1+N)N^4(1+N)^4(2+N)} - \frac{16\left(-1143+1159N+1159N^2\right)S_3}{27N(1+N)}$$

$$-\frac{3760}{3}S_{-2,2} - \frac{20\big(4+9N+9N^2\big)S_{2,1}}{3N(1+N)} + \frac{3136}{3}S_{3,1} - \frac{8\big(-446+79N+79N^2\big)S_{-2,1}}{3N(1+N)}$$

$$-\frac{544}{3}S_{2,1,1} + \frac{12736}{3}S_{-2,1,1} - \frac{12\big(18+77N+77N^2\big)\zeta_3}{N(1+N)}\bigg)S_2 + \frac{64}{3}S_2^3$$

$$+\bigg(\frac{P_{849}}{81(-1+N)N^3(1+N)^3(2+N)} - \frac{1072}{9}S_{2,1} + 2656S_{-2,1} + 256\zeta_3\bigg)S_3$$

$$+\frac{3392}{9}S_3^2 + 224S_6 + \bigg(-\frac{16\zeta_3 P_{753}}{3N(1+N)} + \frac{4S_2 P_{832}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{8S_{-2,1}P_{759}}{3N(1+N)} - \frac{8S_3 P_{768}}{27N(1+N)} - \frac{2P_{876}}{81(-1+N)^2 N^4(1+N)^4(2+N)}$$

$$+\bigg(-\frac{4P_{867}}{81(-1+N)^2 N^3(1+N)^3(2+N)} + \frac{8\big(24+185N+185N^2\big)S_2}{3N(1+N)} + 608S_3$$

$$+\frac{11264}{3}S_{2,1} + \frac{1312}{3}S_{-2,1} + 1248\zeta_3\bigg)S_1 + \bigg(-\frac{4P_{823}}{9(-1+N)N^2(1+N)^2(2+N)}$$

$$-112S_2\bigg)S_1^2 - \frac{16\big(81+139N+139N^2\big)S_1^3}{27N(1+N)} + 32S_1^4 - \frac{32\big(150+71N+71N^2\big)S_{2,1}}{9N(1+N)}$$

$$+\frac{128}{3}S_2^2 + 2224S_4 - 3136S_{3,1} - 1728S_{-2,2} - 2048S_{-3,1} + 4480S_{-2,1,1} + \frac{288}{5}\zeta_2^2\bigg)S_{-2}$$

$$+\bigg(\frac{2P_{818}}{3(-1+N)N^2(1+N)^2(2+N)} + \frac{4\big(-222+161N+161N^2\big)S_1}{3N(1+N)} + 392S_1^2$$

$$-168S_2\bigg)S_{-2}^2 - \frac{32}{3}S_{-2}^3 + \bigg(\frac{4P_{859}}{81(-1+N)N^3(1+N)^3(2+N)} + \bigg(\frac{2176}{3}S_2$$

$$+\frac{4P_{822}}{27(-1+N)N^2(1+N)^2(2+N)}\bigg)S_1 + \frac{8\big(1044+335N+335N^2\big)S_1^2}{9N(1+N)} - \frac{2416}{3}S_1^3$$

$$-\frac{4\big(-678+773N+773N^2\big)S_2}{9N(1+N)} - 2720S_3 + \bigg(\frac{8\big(-212+N+N^2\big)}{3N(1+N)} + \frac{4064S_1}{3}\bigg)S_{-2}$$

$$-80S_{2,1} + 3584S_{-2,1} - 64\zeta_3\bigg)S_{-3} - 1184S_{-3}^2 + \bigg(\frac{2P_{840}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{4\big(-126+31N+31N^2\big)S_1}{9N(1+N)} - 456S_1^2 + \frac{1400}{3}S_2 + 960S_{-2}\bigg)S_{-4} + \bigg(-\frac{4P_{765}}{9N(1+N)}$$

$$+\frac{3920S_1}{3}\bigg)S_{-5} - \frac{544}{3}S_{-6} + \bigg(\frac{4P_{846}}{9N^3(1+N)^3(2+N)} - 288\zeta_3\bigg)S_{2,1} + 40S_{2,1}^2$$

$$+\frac{16\big(-303+431N+431N^2\big)S_{2,3}}{9N(1+N)} - \frac{4\big(2898+487N+487N^2\big)S_{2,-3}}{9N(1+N)}$$

$$-\frac{4\big(-162+1807N+1807N^2\big)S_{4,1}}{9N(1+N)} + 552S_{4,2} - 2080S_{4,-2} + \frac{5888}{3}S_{5,1}$$

$$+\left(\frac{8P_{855}}{81(-1+N)N^3(1+N)^3(2+N)}+96S_{2,1}-640\zeta_3\right)S_{-2,1}-2688S_{-2,1}^2$$

$$+\frac{16(-1+N)(2+N)S_{-2,-3}}{N(1+N)}+3456S_{-3,3}-\frac{8\left(-484+189N+189N^2\right)S_{-4,1}}{3N(1+N)}$$

$$-2384S_{-4,2}-1216S_{-4,-2}+288S_{-5,1}+\frac{8\left(654+293N+293N^2\right)S_{2,1,-2}}{9N(1+N)}$$

$$-\frac{4\left(-4+65N+65N^2\right)S_{2,2,1}}{3N(1+N)}-\frac{1760}{3}S_{2,3,1}+\frac{32\left(-387+361N+361N^2\right)S_{3,1,1}}{9N(1+N)}$$

$$-2160S_{2,-3,1}+2784S_{3,1,-2}+48S_{3,2,1}+\frac{320}{3}S_{4,1,1}+\frac{16\left(390+133N+133N^2\right)S_{-2,2,1}}{9N(1+N)}$$

$$+704S_{-2,2,2}+64S_{-2,2,-2}+2816S_{-2,3,1}+\frac{16\left(-40+127N+127N^2\right)S_{-3,1,1}}{3N(1+N)}$$

$$+960S_{-3,1,-2}-112S_{-3,2,1}-704S_{-3,-2,1}+5504S_{-4,1,1}+\frac{32\left(6+13N+13N^2\right)S_{2,1,1,1}}{3N(1+N)}$$

$$+\frac{208}{3}S_{2,2,1,1}-1792S_{2,-2,1,1}-\frac{6368}{3}S_{3,1,1,1}+\frac{160\left(-180+23N+23N^2\right)S_{-2,1,1,1}}{9N(1+N)}$$

$$-288S_{-2,1,1,2}+192S_{-2,2,1,1}-1408S_{-2,-2,1,1}-896S_{-3,1,1,1}+\frac{640}{3}S_{2,1,1,1,1}$$

$$-9728S_{-2,1,1,1,1}\Bigg]$$

$$+T_F N_F\Bigg[-\frac{128S_{-2,1}P_{770}}{81N^2(1+N)^2}-\frac{16S_{2,1}P_{789}}{9N^2(1+N)^2(2+N)}+\frac{4S_3P_{792}}{81N^2(1+N)^2(2+N)}$$

$$+\frac{16\zeta_3P_{817}}{9(-1+N)N^2(1+N)^2(2+N)}+\frac{P_{879}}{162(-1+N)^2N^5(1+N)^5(2+N)}$$

$$+\left(\frac{2S_2P_{793}}{81N^2(1+N)^2(2+N)}+\frac{P_{871}}{162(-1+N)N^4(1+N)^4(2+N)}-\frac{752}{9}S_2^2\right.$$

$$-\frac{8\left(-210+61N+61N^2\right)S_3}{27N(1+N)}-\frac{1664}{9}S_4-\frac{16\left(-10+13N+13N^2\right)S_{2,1}}{9N(1+N)}+\frac{1088}{9}S_{3,1}$$

$$-\frac{512\left(-3+10N+10N^2\right)S_{-2,1}}{27N(1+N)}+\frac{256}{9}S_{-2,2}+\frac{512}{3}S_{-3,1}+\frac{320}{3}S_{2,1,1}+\frac{2048}{9}S_{-2,1,1}$$

$$+\frac{64}{5}\zeta_2^2-\frac{16\left(42+73N+73N^2\right)\zeta_3}{9N(1+N)}\right)S_1+\Bigg(\frac{16\left(-23+82N+82N^2\right)S_2}{9N(1+N)}$$

$$+\frac{P_{805}}{9N^3(1+N)^3}-\frac{464}{9}S_3-\frac{352}{9}S_{2,1}-\frac{1280}{9}S_{-2,1}-32\zeta_3\Bigg)S_1^2+\left(-\frac{2P_{783}}{27N^2(1+N)^2}\right.$$

$$+\frac{832S_2}{27}\Bigg)S_1^3-\frac{20\left(-2+7N+7N^2\right)S_1^4}{9N(1+N)}-\frac{16}{9}S_1^5+\Bigg(\frac{P_{844}}{81N^3(1+N)^3(2+N)}$$

$$+\frac{4976}{27}S_3+32S_{2,1}+\frac{256}{3}S_{-2,1}+\frac{352}{3}\zeta_3\Bigg)S_2-\frac{4\left(-210+167N+167N^2\right)S_2^2}{27N(1+N)}$$

$$-\frac{16\left(-3+170N+170N^2\right)S_4}{27N(1+N)}+\frac{2368}{9}S_5$$

$$+\left(\frac{32P_{866}}{81(-1+N)^2N^3(1+N)^3(2+N)}+\left(\frac{128P_{806}}{81(-1+N)N^2(1+N)^2(2+N)}\right.\right.$$

$$\left.-\frac{512}{3}S_2\right)S_1+\frac{256\left(-1+5N+5N^2\right)S_1^2}{9N(1+N)}$$

$$+\frac{1280}{27}S_1^3-\frac{128\left(-9+20N+20N^2\right)S_2}{27N(1+N)}+\frac{3712}{27}S_3+\frac{512}{9}S_{2,1}+\left.\frac{896}{3}\zeta_3\right)S_{-2}$$

$$+\left(-\frac{32\left(-2+13N+13N^2\right)}{3N(1+N)}-\frac{256}{3}S_1\right)S_{-2}^2-\left(\frac{64P_{795}}{81(-1+N)N^2(1+N)^2(2+N)}\right.$$

$$+\frac{512\left(-3+5N+5N^2\right)S_1}{27N(1+N)}+\frac{896}{9}S_1^2-\frac{640}{9}S_2-\left.\frac{512}{3}S_{-2}\right)S_{-3}+\left(-\frac{256}{9}S_1\right.$$

$$\left.-\frac{32\left(42+197N+197N^2\right)}{27N(1+N)}\right)S_{-4}-\frac{16\left(48+431N+431N^2\right)S_{3,1}}{27N(1+N)}+\frac{3008}{9}S_{-5}$$

$$-\frac{1984}{9}S_{2,3}+\frac{512}{9}S_{2,-3}+\frac{3008}{9}S_{4,1}+\frac{128\left(-3+10N+10N^2\right)S_{-2,2}}{27N(1+N)}-\frac{896}{9}S_{-2,3}$$

$$+\frac{128\left(-3+10N+10N^2\right)S_{-3,1}}{9N(1+N)}+\frac{16\left(-4+5N+5N^2\right)S_{2,1,1}}{3N(1+N)}-\frac{512}{9}S_{2,1,-2}$$

$$+\frac{64}{3}S_{2,2,1}-\frac{2944}{9}S_{3,1,1}+\frac{256\left(-3+10N+10N^2\right)S_{-2,1,1}}{27N(1+N)}+\frac{512}{3}S_{-2,1,-2}$$

$$\left.\left.-\frac{512}{9}S_{-2,2,1}-\frac{512}{3}S_{-3,1,1}-\frac{256}{3}S_{2,1,1,1}-\frac{1024}{9}S_{-2,1,1,1}-\frac{16\left(2+3N+3N^2\right)\zeta_2^2}{5N(1+N)}\right]\right\}$$

$$+C_F\left\{C_AT_FN_F\left[-\frac{8S_{3,1}P_{773}}{27(-1+N)N(1+N)(2+N)}+\frac{4S_4P_{787}}{27(-1+N)N(1+N)(2+N)}\right.\right.$$

$$+\frac{64S_{-2,1}P_{770}}{81N^2(1+N)^2}+\frac{8S_{2,1}P_{791}}{27N^2(1+N)^2(2+N)}-\frac{8S_3P_{830}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{4\zeta_3P_{834}}{27(-1+N)N^2(1+N)^2(2+N)}+\frac{P_{881}}{729(-1+N)^2N^5(1+N)^5(2+N)}$$

$$+\left(-\frac{32S_3P_{772}}{27(-1+N)N(1+N)(2+N)}+\frac{16\zeta_3P_{778}}{27(-1+N)N(1+N)(2+N)}\right.$$

$$+\frac{8S_2P_{790}}{27N^2(1+N)^2(2+N)}-\frac{4P_{872}}{729(-1+N)N^4(1+N)^4(2+N)}+\frac{224}{9}S_2^2+\frac{1312}{9}S_4$$

$$+\frac{16\left(-13+40N+40N^2\right)S_{2,1}}{9N(1+N)}-96S_{3,1}+\frac{256\left(-3+10N+10N^2\right)S_{-2,1}}{27N(1+N)}$$

$$\left.-\frac{128}{9}S_{-2,2}-\frac{256}{3}S_{-3,1}-\frac{160}{3}S_{2,1,1}-\frac{1024}{9}S_{-2,1,1}-\frac{64}{5}\zeta_2^2\right)S_1+\left(-\frac{2P_{841}}{81N^3(1+N)^3}\right.$$

$$+\frac{8\left(11+2N+2N^2\right)S_2}{9N(1+N)}-\frac{16}{9}S_3+\frac{304}{9}S_{2,1}+\frac{640}{9}S_{-2,1}+\left.32\zeta_3\right)S_1^2+\left(-\frac{224}{27}S_2\right.$$

$$-\frac{16\left(-33+194N+194N^2\right)}{81N(1+N)}\bigg)S_1^3 - \frac{88}{27}S_1^4 + \left(\frac{2P_{848}}{81N^3(1+N)^3(2+N)} - \frac{1360}{27}S_3\right.$$

$$\left.-16S_{2,1} - \frac{128}{3}S_{-2,1} - \frac{352}{3}\zeta_3\right)S_2 - \frac{4\left(16+109N+109N^2\right)S_2^2}{9N(1+N)} - \frac{1312}{9}S_5$$

$$+\left(-\frac{16P_{866}}{81(-1+N)^2N^3(1+N)^3(2+N)} + \left(-\frac{64P_{806}}{81(-1+N)N^2(1+N)^2(2+N)}\right.\right.$$

$$\left.+\frac{256}{3}S_2\right)S_1 - \frac{128\left(-1+5N+5N^2\right)S_1^2}{9N(1+N)} - \frac{640}{27}S_1^3 + \frac{64\left(-9+20N+20N^2\right)S_2}{27N(1+N)}$$

$$-\frac{1856}{27}S_3 - \frac{256}{9}S_{2,1} - \frac{448}{3}\zeta_3\right)S_{-2} + \left(\frac{16\left(-2+13N+13N^2\right)}{3N(1+N)} + \frac{128}{3}S_1\right)S_{-2}^2$$

$$+\left(\frac{32P_{795}}{81(-1+N)N^2(1+N)^2(2+N)} + \frac{256\left(-3+5N+5N^2\right)S_1}{27N(1+N)} + \frac{448}{9}S_1^2 - \frac{320}{9}S_2\right.$$

$$\left.-\frac{256}{3}S_{-2}\right)S_{-3} + \left(\frac{16\left(42+197N+197N^2\right)}{27N(1+N)} + \frac{128}{9}S_1\right)S_{-4} - \frac{1504}{9}S_{-5} + \frac{320}{3}S_{2,3}$$

$$-\frac{256}{9}S_{2,-3} - \frac{64\left(-3+10N+10N^2\right)S_{-2,2}}{27N(1+N)} - \frac{64\left(-3+10N+10N^2\right)S_{-3,1}}{9N(1+N)}$$

$$+\frac{448}{9}S_{-2,3} - \frac{1120}{9}S_{4,1} - \frac{32\left(-3+8N+8N^2\right)S_{2,1,1}}{9N(1+N)} + \frac{256}{9}S_{2,1,-2} - \frac{64}{3}S_{2,2,1}$$

$$+\frac{1696}{9}S_{3,1,1} - \frac{128\left(-3+10N+10N^2\right)S_{-2,1,1}}{27N(1+N)} - \frac{256}{3}S_{-2,1,-2} + \frac{256}{9}S_{-2,2,1}$$

$$+\frac{256}{3}S_{-3,1,1} + \frac{64}{9}S_{2,1,1,1} + \frac{512}{9}S_{-2,1,1,1} + \frac{16\left(2+3N+3N^2\right)\zeta_2^2}{5N(1+N)}\bigg]$$

$$+T_F^2N_F^2\bigg[-\frac{8S_2P_{784}}{81N^2(1+N)^2} - \frac{2P_{861}}{729N^4(1+N)^4} + \left(-\frac{16\left(-6+29N+29N^2\right)S_2}{27N(1+N)}\right.$$

$$\left.+\frac{8P_{835}}{729N^3(1+N)^3} + \frac{128}{27}S_3 + \frac{128}{27}\zeta_3\right)S_1 + \left(\frac{8P_{775}}{81N^2(1+N)^2} - \frac{32S_2}{9}\right)S_1^2$$

$$+\frac{16\left(-6+29N+29N^2\right)S_1^3}{81N(1+N)} + \frac{16}{27}S_1^4 + \frac{80}{9}S_2^2 + \frac{32\left(-6+247N+247N^2\right)S_3}{81N(1+N)}$$

$$-\frac{992}{27}S_4 - \frac{1216}{27}S_{2,1} + \frac{256}{9}S_{3,1} - \frac{128}{9}S_{2,1,1} - \frac{32\left(2+3N+3N^2\right)\zeta_3}{27N(1+N)}\bigg]$$

$$+C_A^2\bigg[-\frac{4S_{-2,1,-2}P_{745}}{3N(1+N)} - \frac{4S_{-2,3}P_{761}}{9N(1+N)} + \frac{2S_5P_{763}}{9N(1+N)} - \frac{12\zeta_2^2P_{764}}{5N^2(1+N)^2} + \frac{S_2^2P_{779}}{9N^2(1+N)^2}$$

$$-\frac{8S_{-2,2}P_{811}}{27(-1+N)N^2(1+N)^2(2+N)} + \frac{4S_{-3,1}P_{813}}{9(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{4S_{3,1}P_{814}}{27(-1+N)N^2(1+N)^2(2+N)} + \frac{8S_{-2,1,1}P_{824}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{2S_4P_{831}}{9(-1+N)N^2(1+N)^2(2+N)} + \frac{2\zeta_3P_{860}}{27(-1+N)N^3(1+N)^3(2+N)}$$

$$+\frac{P_{882}}{5832(-1+N)^2 N^6(1+N)^6(2+N)} + \left(-\frac{8S_{-2,1}P_{825}}{27(-1+N)N^2(1+N)^2(2+N)} \right.$$

$$+\frac{4S_3 P_{829}}{27(-1+N)N^2(1+N)^2(2+N)} - \frac{8\zeta_3 P_{833}}{27(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{4S_{2,1}P_{781}}{9N^2(1+N)^2} + \frac{P_{877}}{729(-1+N)N^5(1+N)^5(2+N)} + \left(\frac{P_{845}}{27N^3(1+N)^3(2+N)} \right.$$

$$+\frac{1184}{3}S_3 - 32S_{2,1} + 1376S_{-2,1} - 64\zeta_3 \Bigg) S_2 - \frac{4\big(-6+145N+145N^2\big)S_2^2}{9N(1+N)}$$

$$-\frac{4\big(-162+893N+893N^2\big)S_4}{9N(1+N)} - 1040S_{2,-3} + \frac{16\big(-73+48N+48N^2\big)S_{3,1}}{3N(1+N)}$$

$$+\frac{448}{3}S_5 - \frac{1120}{3}S_{2,3} - \frac{176}{3}S_{4,1} + \frac{32\big(105+11N+11N^2\big)S_{-2,2}}{9N(1+N)} + \frac{2912}{3}S_{-2,3}$$

$$-\frac{64}{3}S_{-2,-3} + \frac{64\big(12+11N+11N^2\big)S_{-3,1}}{3N(1+N)} + \frac{1184}{3}S_{-4,1} + \frac{8\big(12+55N+55N^2\big)S_{2,1,1}}{3N(1+N)}$$

$$+992S_{2,1,-2} + 64S_{2,2,1} - \frac{2816}{3}S_{3,1,1} + \frac{128\big(-69+22N+22N^2\big)S_{-2,1,1}}{9N(1+N)}$$

$$+\frac{3424}{3}S_{-2,1,-2} + \frac{3200}{3}S_{-2,2,1} + \frac{1600}{3}S_{-3,1,1} + 128S_{2,1,1,1} - 3200S_{-2,1,1,1} + 240\zeta_5 \Bigg) S_1$$

$$+\left(\frac{2S_2 P_{774}}{9N^2(1+N)^2} + \frac{P_{863}}{162N^4(1+N)^4} - \frac{8}{3}S_2^2 + \frac{4\big(240+11N+11N^2\big)S_3}{9N(1+N)} - 72S_4 \right.$$

$$-\frac{4\big(12+209N+209N^2\big)S_{2,1}}{9N(1+N)} + \frac{1168}{3}S_{3,1} - \frac{352\big(-9+5N+5N^2\big)S_{-2,1}}{9N(1+N)}$$

$$-\frac{1888}{3}S_{-2,2} - 512S_{-3,1} - 32S_{2,1,1} + \frac{4480}{3}S_{-2,1,1} - \frac{16\big(4+11N+11N^2\big)\zeta_3}{N(1+N)} \Bigg) S_1^2$$

$$+\left(\frac{-726+4649N+4649N^2}{81N(1+N)} + \frac{616}{27}S_2 - \frac{640}{9}S_3 + \frac{32}{9}S_{2,1} - 320S_{-2,1} \right) S_1^3 + \frac{121}{27}S_1^4$$

$$+\left(\frac{P_{864}}{54N^4(1+N)^4(2+N)} + \frac{296\big(-18+13N+13N^2\big)S_3}{27N(1+N)} + \frac{424}{3}S_4 \right.$$

$$+\frac{16\big(1+3N+3N^2\big)S_{2,1}}{N(1+N)} - \frac{1168}{3}S_{3,1} + \frac{16\big(-57+22N+22N^2\big)S_{-2,1}}{3N(1+N)} + 352S_{-2,2}$$

$$-\frac{1040}{3}S_{-3,1} + \frac{272}{3}S_{2,1,1} - 1088S_{-2,1,1} + \frac{32\big(1+18N+18N^2\big)\zeta_3}{N(1+N)} \Bigg) S_2 - \frac{56}{9}S_2^3$$

$$+\left(\frac{2P_{862}}{81(-1+N)N^3(1+N)^3(2+N)} + \frac{880}{9}S_{2,1} - \frac{2080}{3}S_{-2,1} - 128\zeta_3 \right) S_3 - \frac{1280}{9}S_3^2$$

$$+\frac{80}{9}S_6 + \left(\frac{4S_{-2,1}P_{752}}{3N(1+N)} + \frac{4\zeta_3 P_{756}}{3N(1+N)} + \frac{4S_3 P_{767}}{27N(1+N)} - \frac{4S_2 P_{785}}{27N^2(1+N)^2} \right.$$

$$
+\frac{2P_{875}}{81(-1+N)^2N^4(1+N)^4(2+N)}+\left(\frac{4P_{858}}{81(-1+N)N^3(1+N)^3(2+N)}\right.
$$

$$
-\frac{32\big(-3+22N+22N^2\big)S_2}{3N(1+N)}-\frac{320}{3}S_3-960S_{2,1}-160S_{-2,1}-320\zeta_3\Bigg)S_1
$$

$$
+\left(\frac{4P_{780}}{9N^2(1+N)^2}-32S_2\right)S_1^2+\frac{1760}{27}S_1^3-\frac{16}{3}S_2^2+\frac{32\big(27+22N+22N^2\big)S_{2,1}}{9N(1+N)}
$$

$$
-\frac{1648}{3}S_4+768S_{3,1}+\frac{1280}{3}S_{-2,2}+\frac{1600}{3}S_{-3,1}-\frac{3584}{3}S_{-2,1,1}-\frac{96}{5}\zeta_2^2\Bigg)S_{-2}
$$

$$
+\left(-\frac{8P_{800}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{8\big(-38+47N+47N^2\big)S_1}{3N(1+N)}-\frac{400}{3}S_1^2\right.
$$

$$
+\frac{208}{3}S_2\Bigg)S_{-2}^2+\frac{32}{9}S_{-2}^3+\left(-\frac{2P_{857}}{81(-1+N)N^3(1+N)^3(2+N)}\right.
$$

$$
+\left(-\frac{4P_{819}}{27(-1+N)N^2(1+N)^2(2+N)}-144S_2\right)S_1-\frac{16\big(117+77N+77N^2\big)S_1^2}{9N(1+N)}
$$

$$
+\frac{544}{3}S_1^3+\frac{40\big(-27+22N+22N^2\big)S_2}{9N(1+N)}+\frac{2080}{3}S_3+\left(\frac{8\big(40+13N+13N^2\big)}{3N(1+N)}\right.
$$

$$
-\frac{832}{3}S_1\Bigg)S_{-2}+\frac{80}{3}S_{2,1}-\frac{3136}{3}S_{-2,1}+32\zeta_3\Bigg)S_{-3}+\frac{1072}{3}S_{-3}^2+\left(\frac{320}{3}S_1^2\right.
$$

$$
-\frac{4P_{826}}{27(-1+N)N^2(1+N)^2(2+N)}-\frac{8\big(12+53N+53N^2\big)S_1}{9N(1+N)}-\frac{320}{3}S_2-\frac{736}{3}S_{-2}\Bigg)
$$

$$
\times S_{-4}+\left(\frac{2P_{762}}{9N(1+N)}-208S_1\right)S_{-5}+\frac{544}{9}S_{-6}+\left(-\frac{4P_{847}}{27N^3(1+N)^3(2+N)}\right.
$$

$$
+96\zeta_3\Bigg)S_{2,1}-\frac{16\big(-35+52N+52N^2\big)S_{2,3}}{3N(1+N)}+\frac{8\big(369+106N+106N^2\big)S_{2,-3}}{9N(1+N)}
$$

$$
-\frac{56}{3}S_{2,1}^2+\frac{8\big(33+367N+367N^2\big)S_{4,1}}{9N(1+N)}-\frac{952}{3}S_{4,2}+\frac{1600}{3}S_{4,-2}-\frac{2144}{3}S_{5,1}
$$

$$
+\left(-\frac{4P_{856}}{81(-1+N)N^3(1+N)^3(2+N)}-32S_{2,1}+192\zeta_3\right)S_{-2,1}+\frac{32S_{-2,-3}}{3N(1+N)}
$$

$$
+\frac{2240}{3}S_{-2,1}^2-\frac{2848}{3}S_{-3,3}+\frac{8\big(-140+57N+57N^2\big)S_{-4,1}}{3N(1+N)}+\frac{2096}{3}S_{-4,2}
$$

$$
-224S_{-5,1}+\frac{4\big(-84+607N+562N^2\big)S_{2,1,1}}{9N(1+N)}-\frac{16\big(63+44N+44N^2\big)S_{2,1,-2}}{9N(1+N)}
$$

$$
+\frac{1216}{3}S_{-4,-2}+\frac{4\big(-24+29N+29N^2\big)S_{2,2,1}}{3N(1+N)}+\frac{640}{3}S_{2,3,1}+\frac{2096}{3}S_{2,-3,1}
$$

$$
-\frac{16\big(-264+287N+287N^2\big)S_{3,1,1}}{9N(1+N)}-\frac{1952}{3}S_{3,1,-2}-\frac{208}{3}S_{3,2,1}+\frac{320}{3}S_{4,1,1}
$$

$$-\frac{64\big(21+11N+11N^2\big)S_{-2,2,1}}{9N(1+N)}-\frac{832}{3}S_{-2,2,2}-\frac{64}{3}S_{-2,2,-2}-\frac{1984}{3}S_{-2,3,1}$$

$$-\frac{352\big(-1+2N+2N^2\big)S_{-3,1,1}}{3N(1+N)}-320S_{-3,1,-2}+\frac{112}{3}S_{-3,2,1}+\frac{704}{3}S_{-3,-2,1}$$

$$-\frac{4480}{3}S_{-4,1,1}-\frac{16\big(36+11N+11N^2\big)S_{2,1,1,1}}{9N(1+N)}-\frac{272}{3}S_{2,2,1,1}+384S_{2,-2,1,1}$$

$$+\frac{2720}{3}S_{3,1,1,1}-\frac{64\big(-117+22N+22N^2\big)S_{-2,1,1,1}}{9N(1+N)}+96S_{-2,1,1,2}-64S_{-2,2,1,1}$$

$$+\frac{1408}{3}S_{-2,-2,1,1}+384S_{-3,1,1,1}-\frac{640}{3}S_{2,1,1,1,1}+2560S_{-2,1,1,1,1}$$

$$+\frac{5(-5+N)(6+N)\big(8+N+N^2\big)\zeta_5}{3N(1+N)}\bigg]\bigg\}$$

$$+C_F^3\bigg\{\frac{S_2^2P_{743}}{3N^2(1+N)^2}-\frac{16S_{-2,1,-2}P_{746}}{3N(1+N)}-\frac{16S_{-2,3}P_{748}}{3N(1+N)}+\frac{8S_5P_{755}}{3N(1+N)}+\frac{20\zeta_5P_{757}}{3N(1+N)}$$

$$-\frac{24\zeta_2^2P_{764}}{5N^2(1+N)^2}+\frac{8S_{2,1,1}P_{769}}{N^2(1+N)^2}-\frac{8S_{-3,1}P_{798}}{3(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{16S_{-2,2}P_{801}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{8S_{3,1}P_{810}}{3(-1+N)N^2(1+N)^2(2+N)}$$

$$+\frac{16S_{-2,1,1}P_{812}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{2S_4P_{816}}{3(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{4\zeta_3P_{853}}{3(-1+N)N^3(1+N)^3(2+N)}+\frac{P_{883}}{24(-1+N)^2N^6(1+N)^6(2+N)}$$

$$+\bigg(\frac{2S_3P_{804}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{16S_{-2,1}P_{809}}{3(-1+N)N^2(1+N)^2(2+N)}$$

$$-\frac{4S_{2,1}P_{771}}{3N^2(1+N)^2}-\frac{8\zeta_3P_{803}}{(N-1)N^2(1+N)^2(2+N)}+\frac{P_{880}}{24(N-1)^2N^5(1+N)^5(2+N)}$$

$$+\bigg(\frac{P_{827}}{6N^3(1+N)^3}+\frac{976}{3}S_3-\frac{64}{3}S_{2,1}+\frac{13696}{3}S_{-2,1}-416\zeta_3\bigg)S_2$$

$$+\frac{2\big(-54+25N+25N^2\big)S_2^2}{N(1+N)}-\frac{4\big(-2+85N+85N^2\big)S_4}{N(1+N)}-\frac{2560}{3}S_5-64S_{2,3}$$

$$-3680S_{2,-3}+\frac{8\big(-106+109N+109N^2\big)S_{3,1}}{N(1+N)}-\frac{16\big(-274+9N+9N^2\big)S_{-2,2}}{3N(1+N)}$$

$$-\frac{1696}{3}S_{4,1}+\frac{11392}{3}S_{-2,3}-\frac{128}{3}S_{-2,-3}+\frac{64\big(59+6N+6N^2\big)S_{-3,1}}{3N(1+N)}+1088S_{-4,1}$$

$$-\frac{24\big(-2+3N+3N^2\big)S_{2,1,1}}{N(1+N)}+\frac{11200}{3}S_{2,1,-2}+\frac{64\big(-52+7N+7N^2\big)S_{-2,1,1}}{N(1+N)}$$

$$-\frac{1984}{3}S_{3,1,1}+3264S_{-2,1,-2}+\frac{11648}{3}S_{-2,2,1}+\frac{7168}{3}S_{-3,1,1}-10752S_{-2,1,1,1}-1120\zeta_5$$

$$-\frac{160}{3}S_{2,2,1}\bigg)S_1+\bigg(\frac{2S_2P_{777}}{3N^2(1+N)^2}+\frac{P_{870}}{12(N-1)N^4(1+N)^4(2+N)}+108S_2^2$$

$$-\frac{4\big(-46+75N+75N^2\big)S_3}{N(1+N)}+344S_4+\frac{24\big(-2+3N+3N^2\big)S_{2,1}}{N(1+N)}+336S_{3,1}$$

$$-\frac{32\big(-34+3N+3N^2\big)S_{-2,1}}{N(1+N)}-\frac{6880}{3}S_{-2,2}-\frac{6464}{3}S_{-3,1}-48S_{2,1,1}+4864S_{-2,1,1}$$

$$-\frac{352\zeta_3}{N(1+N)}\bigg)S_1^2+\bigg(\frac{P_{794}}{6N^3(1+N)^3}-\frac{12\big(-6+7N+7N^2\big)S_2}{N(1+N)}-48S_3+32S_{2,1}$$

$$-960S_{-2,1}+96\zeta_3\bigg)S_1^3+\bigg(\frac{P_{744}}{N^2(1+N)^2}-36S_2\bigg)S_1^4+\frac{2\big(-2+3N+3N^2\big)S_1^5}{N(1+N)}$$

$$+\frac{4}{3}S_1^6+\bigg(\frac{P_{869}}{12(-1+N)N^4(1+N)^4(2+N)}+\frac{4\big(-398+513N+513N^2\big)S_3}{3N(1+N)}+40S_4$$

$$-\frac{8\big(-4+15N+15N^2\big)S_{2,1}}{3N(1+N)}-432S_{3,1}-\frac{16\big(218+9N+9N^2\big)S_{-2,1}}{3N(1+N)}+\frac{3296}{3}S_{-2,2}$$

$$-\frac{1120}{3}S_{-3,1}+\frac{272}{3}S_{2,1,1}-\frac{12416}{3}S_{-2,1,1}+\frac{32\big(5+12N+12N^2\big)\zeta_3}{N(1+N)}\bigg)S_2-\frac{148}{9}S_2^3$$

$$+\bigg(\frac{P_{854}}{3(-1+N)N^3(1+N)^3(2+N)}+\frac{64}{3}S_{2,1}-\frac{7616}{3}S_{-2,1}-\frac{320}{3}\zeta_3\bigg)S_3-\frac{512}{3}S_3^2$$

$$-\frac{1664}{9}S_6+\bigg(\frac{16\zeta_3P_{747}}{3N(1+N)}+\frac{16S_3P_{749}}{3N(1+N)}-\frac{8S_2P_{802}}{3(-1+N)N^2(1+N)^2(2+N)}$$

$$+\frac{16S_{-2,1}P_{750}}{3N(1+N)}+\frac{4P_{874}}{3(-1+N)^2N^4(1+N)^4(2+N)}-\bigg(\frac{2368}{3}S_3+\frac{11008}{3}S_{2,1}$$

$$+\frac{8P_{865}}{3(N-1)^2N^3(1+N)^3(2+N)}+\frac{16\big(16+3N+3N^2\big)S_2}{N(1+N)}+\frac{704}{3}S_{-2,1}+1216\zeta_3\bigg)S_1$$

$$+\bigg(\frac{8P_{807}}{3(-1+N)N^2(1+N)^2(2+N)}+352S_2\bigg)S_1^2-\frac{96\big(-1+N+N^2\big)S_1^3}{N(1+N)}-64S_1^4$$

$$-64S_2^2-\frac{6752}{3}S_4+\frac{64\big(32+9N+9N^2\big)S_{2,1}}{3N(1+N)}+3200S_{3,1}+\frac{5248}{3}S_{-2,2}+\frac{5888}{3}S_{-3,1}$$

$$-\frac{12544}{3}S_{-2,1,1}-\frac{192}{5}\zeta_2^2\bigg)S_{-2}+\bigg(-\frac{752}{3}S_1^2-\frac{4P_{799}}{3(-1+N)N^2(1+N)^2(2+N)}$$

$$+\frac{176}{3}S_2+\frac{8\big(70+27N+27N^2\big)S_1}{3N(1+N)}\bigg)S_{-2}^2+\frac{64}{9}S_{-2}^3+\bigg(\bigg(-\frac{2624}{3}S_2$$

$$+\frac{8P_{797}}{3(-1+N)N^2(1+N)^2(2+N)}\bigg)S_1-\frac{8P_{852}}{3(-1+N)N^3(1+N)^3(2+N)}$$

$$-\frac{16\big(64+3N+3N^2\big)S_1^2}{N(1+N)}+\frac{2656}{3}S_1^3+\frac{8\big(-46+111N+111N^2\big)S_2}{3N(1+N)}+\frac{8000}{3}S_3$$

$$+\bigg(-\frac{16\big(-44+9N+9N^2\big)}{N(1+N)}-1600S_1\bigg)S_{-2}+\frac{160}{3}S_{2,1}-\frac{8960}{3}S_{-2,1}\bigg)S_{-3}$$

$$+\frac{2816}{3}S_{-3}^2 + \left(-\frac{4P_{796}}{3(-1+N)N^2(1+N)^2(2+N)} + \frac{8\left(-26+81N+81N^2\right)S_1}{3N(1+N)} \right.$$

$$+\frac{1456}{3}S_1^2 - \frac{1520}{3}S_2 - \frac{2816}{3}S_{-2} \Big)S_{-4} + \left(\frac{8P_{751}}{3N(1+N)} - \frac{5344S_1}{3} \right)S_{-5} + \frac{1088}{9}S_{-6}$$

$$+\left(-\frac{4P_{808}}{3N^3(1+N)^3} + 192\zeta_3 \right)S_{2,1} - \frac{64}{3}S_{2,1}^2 - \frac{32\left(-13+17N+17N^2\right)S_{2,3}}{N(1+N)}$$

$$+\frac{8\left(158+7N+7N^2\right)S_{2,-3}}{N(1+N)} + \frac{8\left(-38+135N+135N^2\right)S_{4,1}}{3N(1+N)} - \frac{1088}{3}S_{4,2}$$

$$+\frac{6080}{3}S_{4,-2} - 1120S_{5,1} + \left(\frac{16P_{851}}{3(-1+N)N^3(1+N)^3(2+N)} - 64S_{2,1} + 512\zeta_3 \right)$$

$$\times S_{-2,1} - \frac{32\left(-2+3N+3N^2\right)S_{-2,-3}}{3N(1+N)} + \frac{16\left(-68+25N+25N^2\right)S_{-4,1}}{N(1+N)}$$

$$-\frac{9344}{3}S_{-3,3} + \frac{5920}{3}S_{-4,2} + \frac{2432}{3}S_{-4,-2} - \frac{16\left(134+39N+39N^2\right)S_{2,1,-2}}{3N(1+N)}$$

$$+320S_{-5,1} + \frac{80S_{2,2,1}}{3N(1+N)} + \frac{352}{3}S_{2,3,1} - \frac{16\left(-206+195N+195N^2\right)S_{3,1,1}}{3N(1+N)}$$

$$+\frac{4576}{3}S_{2,-3,1} - \frac{8896}{3}S_{3,1,-2} + \frac{64}{3}S_{3,2,1} + \frac{128}{3}S_{4,1,1} - \frac{32\left(74+15N+15N^2\right)S_{-2,2,1}}{3N(1+N)}$$

$$-\frac{896}{3}S_{-2,2,2} - \frac{128}{3}S_{-2,2,-2} - \frac{8960}{3}S_{-2,3,1} - \frac{32\left(4+39N+39N^2\right)S_{-3,1,1}}{3N(1+N)}$$

$$-640S_{-3,1,-2} + \frac{224}{3}S_{-3,2,1} + \frac{1408}{3}S_{-3,-2,1} - \frac{15104}{3}S_{-4,1,1} + \frac{64}{3}S_{2,2,1,1}$$

$$+2048S_{2,-2,1,1} + 704S_{3,1,1,1} - \frac{192\left(-16+N+N^2\right)S_{-2,1,1,1}}{N(1+N)} + 192S_{-2,1,1,2}$$

$$-128S_{-2,2,1,1} + \frac{2816}{3}S_{-2,-2,1,1} + 256S_{-3,1,1,1} + 9216S_{-2,1,1,1,1} + \frac{7168}{3}S_{-2,1}^2 \bigg\} \qquad (919)$$

with the polynomials

$$P_{743} = -617N^4 - 1180N^3 - 485N^2 + 90N + 52, \qquad (920)$$

$$P_{744} = -9N^4 - 12N^3 - 9N^2 + 22N + 20, \qquad (921)$$

$$P_{745} = N^4 + 2N^3 - 293N^2 - 294N + 356, \qquad (922)$$

$$P_{746} = N^4 + 2N^3 - 120N^2 - 121N + 258, \qquad (923)$$

$$P_{747} = N^4 + 2N^3 + 15N^2 + 14N + 60, \qquad (924)$$

$$P_{748} = N^4 + 2N^3 + 15N^2 + 14N + 194, \qquad (925)$$

$$P_{749} = N^4 + 2N^3 + 24N^2 + 23N + 68, \qquad (926)$$

$$P_{750} = N^4 + 2N^3 + 42N^2 + 41N + 10, \qquad (927)$$

$$P_{751} = N^4 + 2N^3 + 90N^2 + 89N + 166, \qquad (928)$$

$$P_{752} = N^4 + 2N^3 + 111N^2 + 110N + 36, \qquad (929)$$

$$P_{753} = N^4 + 2N^3 + 115N^2 + 114N + 54, \qquad (930)$$

$$P_{754} = N^4 + 2N^3 + 195N^2 + 194N - 156, \qquad (931)$$

$$P_{755} = N^4 + 2N^3 + 204N^2 + 203N + 40, \tag{932}$$

$$P_{756} = N^4 + 2N^3 + 215N^2 + 214N + 48, \tag{933}$$

$$P_{757} = N^4 + 2N^3 + 303N^2 + 302N - 144, \tag{934}$$

$$P_{758} = 2N^4 + 4N^3 - 413N^2 - 415N + 614, \tag{935}$$

$$P_{759} = 2N^4 + 4N^3 + 153N^2 + 151N + 46, \tag{936}$$

$$P_{760} = 3N^4 + 6N^3 + 100N^2 + 97N + 495, \tag{937}$$

$$P_{761} = 3N^4 + 6N^3 + 155N^2 + 152N + 408, \tag{938}$$

$$P_{762} = 3N^4 + 6N^3 + 1321N^2 + 1318N - 252, \tag{939}$$

$$P_{763} = 3N^4 + 6N^3 + 1435N^2 + 1432N - 480, \tag{940}$$

$$P_{764} = 5N^4 + 10N^3 + 9N^2 + 4N + 4, \tag{941}$$

$$P_{765} = 6N^4 + 12N^3 + 1591N^2 + 1585N + 246, \tag{942}$$

$$P_{766} = 6N^4 + 12N^3 + 1871N^2 + 1865N - 216, \tag{943}$$

$$P_{767} = 9N^4 + 18N^3 + 925N^2 + 916N + 144, \tag{944}$$

$$P_{768} = 18N^4 + 36N^3 + 1141N^2 + 1123N + 756, \tag{945}$$

$$P_{769} = 29N^4 + 51N^3 + 16N^2 - 20N - 6, \tag{946}$$

$$P_{770} = 83N^4 + 166N^3 + 239N^2 + 192N + 63, \tag{947}$$

$$P_{771} = 85N^4 + 134N^3 + 57N^2 - 80N - 52, \tag{948}$$

$$P_{772} = 97N^4 + 194N^3 - 121N^2 - 218N - 6, \tag{949}$$

$$P_{773} = 131N^4 + 262N^3 - 23N^2 - 154N + 216, \tag{950}$$

$$P_{774} = 147N^4 + 294N^3 + 38N^2 - 121N + 6, \tag{951}$$

$$P_{775} = 235N^4 + 524N^3 + 211N^2 + 30N + 72, \tag{952}$$

$$P_{776} = 277N^4 + 536N^3 + 200N^2 - 143N - 36, \tag{953}$$

$$P_{777} = 291N^4 + 488N^3 + 217N^2 - 304N - 256, \tag{954}$$

$$P_{778} = 361N^4 + 722N^3 - 433N^2 - 794N - 72, \tag{955}$$

$$P_{779} = 423N^4 + 792N^3 + 413N^2 + 44N - 90, \tag{956}$$

$$P_{780} = 536N^4 + 1075N^3 + 601N^2 + 62N + 162, \tag{957}$$

$$P_{781} = 536N^4 + 1087N^3 + 414N^2 - 149N + 6, \tag{958}$$

$$P_{782} = 575N^4 + 1084N^3 + 420N^2 - 365N - 150, \tag{959}$$

$$P_{783} = 683N^4 + 1510N^3 + 479N^2 + 68N + 292, \tag{960}$$

$$P_{784} = 1055N^4 + 2164N^3 + 1031N^2 + 30N + 72, \tag{961}$$

$$P_{785} = 1108N^4 + 2261N^3 + 1099N^2 - 126N + 288, \tag{962}$$

$$P_{786} = 1124N^4 + 2224N^3 + 847N^2 - 331N - 57, \tag{963}$$

$$P_{787} = 1687N^4 + 3374N^3 - 1657N^2 - 3344N + 372, \tag{964}$$

$$P_{788} = 4513N^4 + 9026N^3 + 2815N^2 - 1698N + 612, \tag{965}$$

$$P_{789} = 112N^5 + 313N^4 + 188N^3 - 73N^2 + 8N + 52, \tag{966}$$

$$P_{790} = 328N^5 + 1513N^4 + 2150N^3 + 959N^2 - 102N - 120, \tag{967}$$

$$P_{791} = 1103N^5 + 4010N^4 + 4309N^3 + 1096N^2 - 60N + 240, \tag{968}$$

$$P_{792} = 7241N^5 + 26102N^4 + 28297N^3 + 8428N^2 - 540N - 384, \tag{969}$$

$$P_{793} = 9187N^5 + 33508N^4 + 33155N^3 + 2138N^2 + 2844N + 8136, \tag{970}$$

$$P_{794} = -279N^6 - 705N^5 - 663N^4 - 159N^3 - 402N^2 - 816N - 304, \tag{971}$$

$$P_{795} = 25N^6 + 75N^5 - 161N^4 - 303N^3 + 901N^2 + 705N + 54, \tag{972}$$

$$P_{796} = 33N^6 + 9N^5 - 669N^4 - 1433N^3 - 1180N^2 + 368N - 8, \tag{973}$$

$$P_{797} = 34N^6 + 114N^5 + 695N^4 + 1472N^3 + 1051N^2 - 430N - 632, \tag{974}$$

$$P_{798} = 36N^6 + 138N^5 + 969N^4 + 2040N^3 + 1355N^2 - 682N - 400, \tag{975}$$

$$P_{799} = 37N^6 + 49N^5 + 161N^4 + 491N^3 + 242N^2 - 188N - 216, \tag{976}$$

$$P_{800} = 49N^6 + 148N^5 + 110N^4 - 4N^3 - 24N^2 - 61N - 74, \tag{977}$$

$$P_{801} = 61N^6 + 190N^5 + 530N^4 + 926N^3 + 617N^2 - 320N - 276, \tag{978}$$

$$P_{802} = 74N^6 + 206N^5 + 5N^4 - 328N^3 - 203N^2 + 94N + 8, \tag{979}$$

$$P_{803} = 79N^6 + 215N^5 + 73N^4 - 143N^3 - 52N^2 + 36N - 96, \tag{980}$$

$$P_{804} = 81N^6 + 103N^5 - 651N^4 - 1335N^3 - 570N^2 + 748N + 568, \tag{981}$$

$$P_{805} = 83N^6 - 293N^5 + 15N^4 - 23N^3 + 170N^2 + 1040N + 704, \tag{982}$$

$$P_{806} = 83N^6 + 249N^5 + 164N^4 - 33N^3 + 14N^2 - 63N - 90, \tag{983}$$

$$P_{807} = 96N^6 + 250N^5 + 49N^4 - 300N^3 - 333N^2 + 78N + 16, \tag{984}$$

$$P_{808} = 100N^6 + 349N^5 + 301N^4 + 31N^3 + 79N^2 + 128N + 76, \tag{985}$$

$$P_{809} = 158N^6 + 454N^5 + 651N^4 + 780N^3 + 471N^2 - 290N - 496, \tag{986}$$

$$P_{810} = 173N^6 + 466N^5 - 273N^4 - 1264N^3 - 472N^2 + 462N + 380, \tag{987}$$

$$P_{811} = 182N^6 + 528N^5 + 2678N^4 + 5724N^3 + 4385N^2 - 1383N - 1746, \tag{988}$$

$$P_{812} = 208N^6 + 622N^5 + 1151N^4 + 1664N^3 + 1113N^2 - 598N - 704, \tag{989}$$

$$P_{813} = 248N^6 + 771N^5 - 1216N^4 - 4599N^3 - 3616N^2 + 828N + 672, \tag{990}$$

$$P_{814} = 379N^6 + 1065N^5 - 3293N^4 - 9309N^3 - 5402N^2 + 2304N + 3240, \tag{991}$$

$$P_{815} = 388N^6 + 1128N^5 - 5339N^4 - 15318N^3 - 11297N^2 + 3702N + 2544, \tag{992}$$

$$P_{816} = 403N^6 + 1191N^5 + 633N^4 - 599N^3 - 440N^2 - 116N - 16, \tag{993}$$

$$P_{817} = 417N^6 + 900N^5 - 253N^4 - 1298N^3 - 458N^2 + 20N - 192, \tag{994}$$

$$P_{818} = 429N^6 + 1233N^5 + 1041N^4 + 459N^3 + 50N^2 - 676N - 808, \tag{995}$$

$$P_{819} = 676N^6 + 2109N^5 - 2858N^4 - 10617N^3 - 5747N^2 + 2811N + 3258, \tag{996}$$

$$P_{820} = 781N^6 + 1227N^5 - 5909N^4 - 14895N^3 - 12548N^2 + 5496N + 8568, \tag{997}$$

$$P_{821} = 913N^6 + 2766N^5 + 10126N^4 + 19782N^3 + 14323N^2 - 5646N - 5976, \tag{998}$$

$$P_{822} = 1046N^6 + 3192N^5 - 11971N^4 - 34482N^3 - 20953N^2 + 9492N + 12204, \tag{999}$$

$$P_{823} = 1360N^6 + 3972N^5 + 1355N^4 - 3874N^3 - 2955N^2 + 310N - 600, \tag{1000}$$

$$P_{824} = 1472N^6 + 4425N^5 + 7064N^4 + 9099N^3 + 6692N^2 - 3048N - 4968, \tag{1001}$$

$$P_{825} = 1540N^6 + 4629N^5 + 4822N^4 + 3087N^3 + 1945N^2 - 1785N - 3870, \tag{1002}$$

$$P_{826} = 1540N^6 + 4701N^5 + 1444N^4 - 5937N^3 - 7331N^2 - 1671N - 522, \tag{1003}$$

$$P_{827} = 1545N^6 + 4511N^5 + 3537N^4 + 737N^3 + 1222N^2 + 2384N + 784, \tag{1004}$$

$$P_{828} = 1873N^6 + 5331N^5 + 2385N^4 - 3299N^3 - 2470N^2 + 44N - 1368, \tag{1005}$$

$$P_{829} = 1964N^6 + 5856N^5 + 68N^4 - 10116N^3 - 6055N^2 + 1461N + 1314, \tag{1006}$$

$$P_{830} = 2251N^6 + 6552N^5 + 1826N^4 - 6688N^3 - 3669N^2 + 292N - 132, \tag{1007}$$

$$P_{831} = 2284N^6 + 6951N^5 + 3143N^4 - 5233N^3 - 2970N^2 - 53N - 450, \tag{1008}$$

$$P_{832} = 2882N^6 + 8592N^5 + 2333N^4 - 10050N^3 - 5899N^2 + 1926N - 1080, \tag{1009}$$

$$P_{833} = 3098N^6 + 9240N^5 + 3107N^4 - 8898N^3 - 5665N^2 - 18N - 1188, \tag{1010}$$

$$P_{834} = 3153N^6 + 7047N^5 - 1489N^4 - 9635N^3 - 5120N^2 - 148N - 720, \tag{1011}$$

$$P_{835} = 4357N^6 + 16149N^5 + 16977N^4 + 9091N^3 + 3798N^2 - 1404N - 1944, \tag{1012}$$

$$P_{836} = 4502N^6 + 13344N^5 + 15503N^4 + 13194N^3 + 8129N^2 - 6180N - 12204, \tag{1013}$$

$$P_{837} = 4591N^6 + 13935N^5 + 6934N^4 - 8871N^3 - 5873N^2 - 1464N - 612, \tag{1014}$$

$$P_{838} = 4645N^6 + 13341N^5 - 4124N^4 - 31599N^3 - 16541N^2 + 8430N + 8568, \tag{1015}$$

$$P_{839} = 4816N^6 + 14448N^5 + 24487N^4 + 33174N^3 + 23401N^2 - 11478N - 16272, \tag{1016}$$

$$P_{840} = 6457N^6 + 18885N^5 - 245N^4 - 36645N^3 - 39944N^2 - 3372N - 2160, \tag{1017}$$

$$P_{841} = 7531N^6 + 23619N^5 + 23253N^4 + 7825N^3 + 2064N^2 + 1080N - 252, \tag{1018}$$

$$P_{842} = 8425N^6 + 26643N^5 + 24471N^4 + 6857N^3 + 2520N^2 + 1484N - 864, \tag{1019}$$

$$P_{843} = -116957N^7 - 556705N^6 - 889137N^5 - 480947N^4 - 48370N^3 - 93780N^2$$
$$-76536N + 2592, \tag{1020}$$

$$P_{844} = -28023N^7 - 129045N^6 - 212881N^5 - 149635N^4 - 36908N^3 - 4628N^2$$
$$-17376N - 9216, \tag{1021}$$

$$P_{845} = -3221N^7 - 19897N^6 - 43683N^5 - 43139N^4 - 15736N^3 + 3828N^2 + 3720N$$
$$+720, \tag{1022}$$

$$P_{846} = 1511N^7 + 6361N^6 + 8811N^5 + 3500N^4 - 61N^3 + 1842N^2 + 1772N + 552, \tag{1023}$$

$$P_{847} = 3691N^7 + 16568N^6 + 25248N^5 + 13189N^4 + 236N^3 + 1188N^2 + 1860N$$
$$+360, \tag{1024}$$

$$P_{848} = 32287N^7 + 161273N^6 + 288291N^5 + 220675N^4 + 64886N^3 + 4416N^2$$
$$+1188N - 792, \tag{1025}$$

$$P_{849} = -97249N^8 - 360322N^7 - 305212N^6 + 226586N^5 + 292481N^4 - 76528N^3$$
$$-50508N^2 + 100320N + 63072, \tag{1026}$$

$$P_{850} = -5021N^8 - 15158N^7 - 6690N^6 + 17488N^5 + 14927N^4 - 3058N^3 + 1392N^2$$
$$+6920N + 4560, \tag{1027}$$

$$P_{851} = 25N^8 + 105N^7 + 101N^6 + 52N^5 + 1332N^4 + 2547N^3 + 890N^2 - 932N$$
$$-664, \tag{1028}$$

$$P_{852} = 79N^8 + 323N^7 + 315N^6 + 22N^5 + 1546N^4 + 3031N^3 + 1088N^2 - 1076N$$
$$-720, \tag{1029}$$

$$P_{853} = 307N^8 + 1174N^7 + 1142N^6 - 608N^5 - 2451N^4 - 2370N^3 - 250N^2$$
$$+888N + 824, \tag{1030}$$

$$P_{854} = 575N^8 + 2226N^7 + 1974N^6 - 1440N^5 + 15N^4 + 5006N^3 + 1172N^2 - 3320N$$
$$-1984, \tag{1031}$$

$$P_{855} = 1288N^8 + 4747N^7 + 10549N^6 + 7126N^5 - 99269N^4 - 195599N^3 - 68574N^2$$
$$+70380N + 51624, \tag{1032}$$

$$P_{856} = 1963N^8 + 7582N^7 + 13276N^6 + 8530N^5 - 63305N^4 - 126830N^3 - 44544N^2$$
$$+45216N + 33696, \tag{1033}$$

$$P_{857} = 3761N^8 + 15422N^7 + 6284N^6 - 22930N^5 + 78905N^4 + 182846N^3 + 61872N^2$$
$$-51408N - 36288, \tag{1034}$$

$$P_{858} = 4258N^8 + 16762N^7 + 18841N^6 - 1469N^5 - 28625N^4 - 39305N^3 - 12672N^2$$
$$+15102N + 12636, \tag{1035}$$

$$P_{859} = 5894N^8 + 24143N^7 + 14789N^6 - 22336N^5 + 120647N^4 + 264683N^3$$

$$+91248N^2 - 80460N - 55728, \tag{1036}$$

$$P_{860} = 17529N^8 + 56958N^7 + 34840N^6 - 52470N^5 - 72773N^4 - 21696N^3 - 4220N^2$$
$$-10440N - 7344, \tag{1037}$$

$$P_{861} = 28551N^8 + 92280N^7 + 118370N^6 + 46548N^5 - 11961N^4 + 10748N^3$$
$$+12600N^2 - 14112N - 9936, \tag{1038}$$

$$P_{862} = 43949N^8 + 169784N^7 + 155918N^6 - 99922N^5 - 174643N^4 - 20848N^3$$
$$+15276N^2 - 15534N - 7884, \tag{1039}$$

$$P_{863} = 50689N^8 + 208318N^7 + 323478N^6 + 236812N^5 + 86401N^4 + 16554N^3$$
$$-2304N^2 - 2772N - 540, \tag{1040}$$

$$P_{864} = -71503N^9 - 425688N^8 - 980826N^7 - 1101180N^6 - 615171N^5$$
$$-147456N^4 - 4292N^3 + 4428N^2 + 2940N + 72, \tag{1041}$$

$$P_{865} = 24N^9 + 82N^8 - N^7 - 166N^6 + 590N^5 + 262N^4 - 1321N^3 - 1098N^2$$
$$+52N + 424, \tag{1042}$$

$$P_{866} = 261N^9 + 783N^8 - 127N^7 - 1760N^6 - 493N^5 - 1289N^4 - 1879N^3$$
$$-758N^2 + 114N - 36, \tag{1043}$$

$$P_{867} = 7868N^9 + 22794N^8 + 4185N^7 - 36138N^6 - 70242N^5 - 28434N^4$$
$$+88933N^3 + 85194N^2 - 6336N - 36720, \tag{1044}$$

$$P_{868} = -5563N^{10} - 22141N^9 - 36762N^8 - 34822N^7 + 5681N^6 + 48719N^5$$
$$+42980N^4 + 28996N^3 + 19056N^2 - 7936N - 10560, \tag{1045}$$

$$P_{869} = -4253N^{10} - 19223N^9 - 23706N^8 + 10042N^7 + 39207N^6 + 16757N^5$$
$$-6072N^4 + 1800N^3 + 6168N^2 - 528N - 1760, \tag{1046}$$

$$P_{870} = 561N^{10} + 2523N^9 + 4538N^8 + 4510N^7 + 1005N^6 - 5025N^5 - 8992N^4$$
$$-7960N^3 - 8936N^2 - 1968N + 1312, \tag{1047}$$

$$P_{871} = 6009N^{10} + 21117N^9 + 38586N^8 + 68306N^7 + 61705N^6 + 28925N^5$$
$$-2468N^4 - 31900N^3 + 38584N^2 + 73824N + 29088, \tag{1048}$$

$$P_{872} = 80453N^{10} + 440578N^9 + 771262N^8 + 284116N^7 - 538565N^6 - 656852N^5$$
$$-328702N^4 - 44730N^3 + 83592N^2 + 19008N - 16848, \tag{1049}$$

$$P_{873} = 463143N^{10} + 2173209N^9 + 3165914N^8 + 374654N^7 - 3058933N^6$$
$$-2659411N^5 - 640204N^4 - 29428N^3 - 117840N^2 + 26496N + 53568, \tag{1050}$$

$$P_{874} = 245N^{11} + 944N^{10} + 786N^9 - 975N^8 - 1959N^7 - 1998N^6 - 1756N^5$$
$$-695N^4 + 748N^3 + 228N^2 - 112N - 64, \tag{1051}$$

$$P_{875} = 10683N^{11} + 42732N^{10} + 25931N^9 - 78468N^8 - 83703N^7 - 19116N^6$$
$$-61407N^5 - 68436N^4 - 10052N^3 + 14856N^2 - 468N - 648, \tag{1052}$$

$$P_{876} = 27981N^{11} + 110952N^{10} + 73084N^9 - 183261N^8 - 220299N^7$$
$$-92178N^6 - 170226N^5 - 155637N^4 + 92N^3 + 35868N^2 - 3960N - 3024, \tag{1053}$$

$$P_{877} = 599375N^{12} + 3815193N^{11} + 8947106N^{10} + 8383052N^9 - 1037899N^8$$
$$-9404623N^7 - 8442036N^6 - 2234074N^5 + 1430550N^4 + 814536N^3$$
$$-384156N^2 - 321408N - 50544, \tag{1054}$$

$$P_{878} = -458487N^{13} - 2189691N^{12} - 3476151N^{11} - 706147N^{10}$$
$$+5181991N^9 + 6607155N^8 - 353013N^7 - 6891201N^6 - 4645260N^5$$

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$$+669484N^4 + 2725928N^3 + 920640N^2 - 844704N - 521856, \tag{1055}$$

$$\begin{aligned}
P_{879} &= -3069N^{13} + 2556N^{12} + 79368N^{11} + 221328N^{10} + 43770N^9 - 547648N^8 \\
&\quad -435404N^7 - 15640N^6 - 194549N^5 - 563692N^4 - 68324N^3 + 170136N^2 \\
&\quad +10848N - 26784, \tag{1056}
\end{aligned}$$

$$\begin{aligned}
P_{880} &= 3003N^{13} + 13255N^{12} + 27059N^{11} + 32543N^{10} + 397N^9 - 54447N^8 - 26095N^7 \\
&\quad +141989N^6 + 192852N^5 + 62644N^4 - 92576N^3 - 56336N^2 + 29952N \\
&\quad +20672, \tag{1057}
\end{aligned}$$

$$\begin{aligned}
P_{881} &= 428649N^{13} + 1845381N^{12} + 2026189N^{11} - 1935731N^{10} - 5520669N^9 \\
&\quad -1634037N^8 + 3712287N^7 + 2628591N^6 + 266216N^5 + 1213892N^4 \\
&\quad +461088N^3 - 542160N^2 - 82944N + 119232, \tag{1058}
\end{aligned}$$

$$\begin{aligned}
P_{882} &= -5729259N^{15} - 30802914N^{14} - 53071856N^{13} - 478888N^{12} \\
&\quad +104531506N^{11} + 96490716N^{10} - 31073520N^9 - 90914040N^8 - 63564367N^7 \\
&\quad -54366650N^6 - 31332904N^5 + 18604224N^4 + 17762832N^3 - 2637792N^2 \\
&\quad -4240512N - 559872, \tag{1059}
\end{aligned}$$

$$\begin{aligned}
P_{883} &= -7255N^{15} - 40527N^{14} - 54850N^{13} + 77690N^{12} + 228204N^{11} + 45864N^{10} \\
&\quad -240778N^9 - 240422N^8 - 203989N^7 - 270809N^6 - 72500N^5 + 144732 \\
&\quad N^4 + 110144N^3 - 17904N^2 - 37248N - 10176, \tag{1060}
\end{aligned}$$

$$\begin{aligned}
P_{884} &= 494694N^{15} + 2631933N^{14} + 3703662N^{13} - 3856542N^{12} - 13210842N^{11} \\
&\quad -3078616N^{10} + 14028102N^9 + 13282506N^8 + 9865380N^7 + 14654523N^6 \\
&\quad +7299204N^5 - 6176300N^4 - 4977096N^3 + 807456N^2 + 1390176N \\
&\quad +300672. \tag{1061}
\end{aligned}$$

and

$$\begin{aligned}
C_{F_3}^{d_{abc}} &= \frac{d_{abc}d^{abc}N_F}{N_c}\Bigg\{ -\frac{64S_{-2,1}P_{887}}{3(N-1)N^3(1+N)^3(2+N)} - \frac{32S_3P_{888}}{3(N-1)N^3(1+N)^3(2+N)} \\
&\quad +\frac{64S_{2,1}P_{885}}{3N^3(1+N)^3} + \frac{16S_2P_{890}}{3(N-1)N^4(1+N)^4(2+N)} - \frac{80S_1^2P_{891}}{3(N-1)N^4(1+N)^4(2+N)} \\
&\quad +\frac{64P_{894}}{3(N-1)^2N^6(1+N)^6(2+N)^2} - \Bigg[\frac{16P_{895}}{3(N-1)^2N^5(1+N)^5(2+N)^2} \\
&\quad +\frac{32(2+N+N^2)(18+5N+5N^2)S_3}{3(N-1)N^2(1+N)^2(2+N)} + \frac{64(2+N+N^2)(17N^2+17N-10)S_{-2,1}}{3(N-1)N^2(1+N)^2(2+N)} \\
&\quad +\frac{32S_2P_{885}}{3N^3(1+N)^3}\Bigg]S_1 - \frac{16(2+N+N^2)S_2^2}{3N^2(1+N)^2} + \frac{16(2+N+N^2)(-14+3N+3N^2)S_4}{(N-1)N^2(1+N)^2(2+N)} \\
&\quad +\Bigg[-\frac{64S_1P_{892}}{3(N-1)^2N^3(1+N)^3(2+N)^2} - \frac{32P_{893}}{3(N-1)^2N^4(1+N)^4(2+N)^2} \\
&\quad -\frac{320(2+N+N^2)^2S_1^2}{3(N-1)N^2(1+N)^2(2+N)} + \frac{256(2+N+N^2)S_2}{3(N-1)N^2(1+N)^2(2+N)}\Bigg]S_{-2} \\
&\quad -\frac{64(2+N+N^2)S_{-2}^2}{3N^2(1+N)^2} + \Bigg[\frac{32(2+N+N^2)(18+35N+35N^2)S_1}{3(N-1)N^2(1+N)^2(2+N)}
\end{aligned}$$

$$
\left. + \frac{32P_{889}}{3(N-1)N^3(1+N)^3(2+N)} \right] S_{-3} + \frac{32\left(2+N+N^2\right)\left(6+N+N^2\right)S_{-4}}{(N-1)N^2(1+N)^2(2+N)}
$$

$$
+ \frac{256\left(2+N+N^2\right)S_{3,1}}{(N-1)N^2(1+N)^2(2+N)} - \frac{64\left(2+N+N^2\right)\left(2+3N+3N^2\right)S_{-2,2}}{(N-1)N^2(1+N)^2(2+N)}
$$

$$
- \frac{128\left(2+N+N^2\right)^2 S_{-3,1}}{(N-1)N^2(1+N)^2(2+N)} + \frac{512\left(2+N+N^2\right)S_{-2,1,1}}{(N-1)N(1+N)(2+N)}
$$

$$
+ \left[\frac{64P_{886}}{(N-1)N^3(1+N)^3(2+N)} + \frac{256\left(2+N+N^2\right)\left(4+N+N^2\right)S_1}{(N-1)N^2(1+N)^2(2+N)} \right] \zeta_3 \right\}, \quad (1062)
$$

with the polynomials

$$
\begin{align}
P_{885} &= N^4 + 2N^3 - 5N^2 - 6N - 4, \tag{1063} \\
P_{886} &= 3N^6 + 9N^5 - 9N^4 - 33N^3 - 74N^2 - 56N - 32, \tag{1064} \\
P_{887} &= 7N^6 + 12N^5 - 73N^4 - 170N^3 - 104N^2 + 56N + 80, \tag{1065} \\
P_{888} &= 11N^6 + 30N^5 - 26N^4 - 106N^3 - 113N^2 - 24N + 4, \tag{1066} \\
P_{889} &= 13N^6 + 28N^5 - 139N^4 - 326N^3 - 288N^2 - 40N + 48, \tag{1067} \\
P_{890} &= N^8 + 6N^7 + 20N^6 + 28N^5 + 32N^4 + 44N^3 + 73N^2 + 40N + 12, \tag{1068} \\
P_{891} &= 3N^8 + 12N^7 + 16N^6 + 6N^5 + 30N^4 + 64N^3 + 73N^2 + 40N + 12, \tag{1069} \\
P_{892} &= 8N^8 + 24N^7 - 51N^6 - 281N^5 - 459N^4 - 383N^3 - 190N^2 + 76N + 104, \tag{1070} \\
P_{893} &= 9N^{10} + 37N^9 + 10N^8 + 2N^7 + 491N^6 + 1291N^5 + 1566N^4 + 926N^3 \\
&\quad + 300N^2 - 8N - 16, \tag{1071} \\
P_{894} &= 4N^{12} + 21N^{11} + 20N^{10} - 158N^9 - 683N^8 - 1644N^7 - 2947N^6 \\
&\quad - 3115N^5 - 1578N^4 + 92N^3 + 484N^2 + 240N + 48, \tag{1072} \\
P_{895} &= 43N^{12} + 232N^{11} + 319N^{10} - 336N^9 - 1485N^8 - 2782N^7 - 5229N^6 \\
&\quad - 7186N^5 - 5296N^4 - 304N^3 + 1896N^2 + 1344N + 352. \tag{1073}
\end{align}
$$

7 The three–loop Wilson coefficients for the structure function $g_1(x, Q^2)$

The non–singlet Wilson coefficient for the polarized structure function $g_1(x, Q^2)$ has the representation

$$
\begin{align}
\Delta\mathbb{C}^{(3)}_{g_1,q} &= \Delta C^{\text{NS},(3)}_{g_1,q} + \Delta C^{d_{abc},(3)}_{g_1,q} + \Delta C^{\text{PS},(3)}_{g_1,q}, \tag{1074} \\
\Delta\mathbb{C}^{(3)}_{g_1,g} &= \Delta C^{(3)}_{g_1,g}, \tag{1075}
\end{align}
$$

with

$$
\Delta C^{\text{NS},(3),\text{M}}_{g_1,q} = C^{\text{NS},(3)}_{F_3,q} \tag{1076}
$$

in the $\overline{\text{MS}}$ scheme similar to the observation at one– and two–loop order in Eqs. (134, 135) and (390, 400). $\Delta C^{\text{NS},(3),\text{L}}_{g_1,q}$ is given by

$$
\Delta C^{\text{NS},(3),\text{M}}_{g_1,q}(N, a_s) = Z_5^{-1}(N, a_s)\,\Delta C^{\text{NS},(3),\text{L}}_{g_1,q}(N, a_s) \tag{1077}
$$

The Z-factor $Z_5^{-1}(N, a_s)$ providing the finite renormalization is calculated in Appendix A.

The function $\Delta C_{g_1,q}^{d_{abc},(3)}$ is given by

$$
\Delta C_{g_1,q}^{d_{abc},(3)} = \frac{d_{abc}d^{abc}}{N_C}N_F\Bigg\{\frac{64P_{896}}{(N-1)N(1+N)(2+N)} - \frac{512P_{897}}{(N-1)N(1+N)(2+N)}S_{-2,1}
$$

$$
-\frac{256P_{898}}{(N-1)N^2(1+N)^2(2+N)}S_4 + \frac{512P_{898}}{(N-1)N^2(1+N)^2(2+N)}S_{3,1}
$$

$$
+\frac{32P_{900}}{3(N-1)N(1+N)(2+N)}\zeta_3 + \Bigg(-\frac{128\big(-16+5N+5N^2\big)}{(N-1)N(1+N)(2+N)}
$$

$$
-\frac{256P_{898}}{(N-1)N^2(1+N)^2(2+N)}S_3 - \frac{1024}{(N-1)(2+N)}S_{-2,1}
$$

$$
-\frac{1024\big(2+N+N^2\big)}{(N-1)N^2(1+N)^2(2+N)}\zeta_3\Bigg)S_1 - \frac{128\big(2-N+2N^3+N^4\big)}{(N-1)N(1+N)(2+N)}S_3
$$

$$
+\Bigg(-\frac{64P_{899}}{(N-1)N(1+N)(2+N)} - \frac{256\big(-1+2N+2N^2\big)}{(N-1)N(1+N)(2+N)}S_1
$$

$$
+\frac{1024}{N(1+N)}\zeta_3\Bigg)S_{-2} + 64S_{-2}^2 + \Bigg(\frac{256P_{897}}{(N-1)N(1+N)(2+N)}
$$

$$
+\frac{512}{(N-1)(2+N)}S_1\Bigg)S_{-3} + \frac{64\big(6+N+N^2\big)}{(N-1)(2+N)}S_{-4} - \frac{1024}{(N-1)(2+N)}S_{-2,2}
$$

$$
-\frac{512}{N(1+N)}S_{-2,3} - \frac{1024}{(N-1)(2+N)}S_{-3,1} + \frac{512}{N(1+N)}S_{-4,1}
$$

$$
+\frac{2048}{(N-1)(2+N)}S_{-2,1,1} - \frac{1280(N-2)(3+N)}{3N(1+N)}\zeta_5\Bigg\}, \tag{1078}
$$

with

$$
\begin{aligned}
P_{896} &= N^4 + 2N^3 - 7N^2 - 8N + 30, & (1079)\\
P_{897} &= N^4 + 2N^3 - 4N^2 - 5N + 2, & (1080)\\
P_{898} &= N^4 + 2N^3 - N^2 - 2N - 4, & (1081)\\
P_{899} &= 5N^4 + 10N^3 - 19N^2 - 24N + 10, & (1082)\\
P_{900} &= 7N^4 + 14N^3 - 19N^2 - 26N + 216. & (1083)
\end{aligned}
$$

The Wilson coefficient $\Delta C_{g_1,q}^{d_{abc},(3)}$ obeys

$$
\Delta C_{g_1,q}^{d_{abc},(3)}(N=1) = 0 \tag{1084}
$$

and does not contribute to the Bjorken sum rule [46,93]. Four-index d_{abcd} structures do, however, contribute in four–loop order. Let us also present this term in z–space. One obtains

$$
\Delta C_{g_1,q}^{d_{abc},(3),\delta(1-z)}(z) = \frac{d_{abc}d^{abc}N_F}{N_C}\Bigg[64 + 160\zeta_2 + \frac{224}{3}\zeta_3 - \frac{32}{5}\zeta_2^2 - \frac{1280}{3}\zeta_5\Bigg]\delta(1-z) \tag{1085}
$$

$$
\Delta C_{g_1,q}^{d_{abc},(3),+}(z) = 0 \tag{1086}
$$

$$
\begin{aligned}
\Delta C_{g_1,q}^{d_{abc},(3),reg}(z) &= \frac{d_{abc}d^{abc}N_F}{N_C}\Bigg\{ -768(1-z) + \frac{1}{1+z}\Bigg[\frac{1}{1-z}\Bigg(32\big(2+z+3z^2-3z^3+z^4\big)\mathrm{H}_0^2 \\
&\quad -\frac{64}{3}z^4\mathrm{H}_0^3 + \frac{128}{3}\big(-6-11z+3z^2+11z^3+6z^4\big)\mathrm{H}_0\zeta_2 \Bigg) + 64\big(-3 \\
&\quad +3z+11z^2\big)\mathrm{H}_0 - 128\big(1+6z+z^2\big)\mathrm{H}_{0,1} + 64\big(-5+5z+3z^2 \\
&\quad +z^3\big)\zeta_2 \Bigg] + (1-z)\Bigg[-896\mathrm{H}_1 + \frac{64}{3}\mathrm{H}_0^2\mathrm{H}_1 + \frac{64\big(1+7z+z^2\big)}{3z}\mathrm{H}_0^2\mathrm{H}_1^2 \\
&\quad -\frac{128}{3}\mathrm{H}_0\mathrm{H}_{0,1} - \frac{256\big(1+7z+z^2\big)}{3z}\mathrm{H}_0\mathrm{H}_1\mathrm{H}_{0,1} + \frac{128\big(1+7z+z^2\big)}{3z}\mathrm{H}_{0,1}^2 \\
&\quad +\frac{256\big(1+7z+z^2\big)}{3z}\mathrm{H}_0\mathrm{H}_{0,1,1} - \frac{256\big(1+7z+z^2\big)}{3z}\mathrm{H}_{0,0,1,1} + \frac{512}{3}z\mathrm{H}_{0,-1,0,1} \\
&\quad +\frac{256\big(1+7z+z^2\big)}{3z}\mathrm{H}_0\mathrm{H}_1\zeta_2 - \frac{256\big(1+7z+z^2\big)}{3z}\mathrm{H}_{0,1}\zeta_2 - \frac{512}{3}z\mathrm{H}_{0,-1}\zeta_2 \\
&\quad -\frac{512\big(1+4z+z^2\big)}{3z}\mathrm{H}_1\zeta_3 \Bigg] + (1+z)\Bigg[\frac{(1-z)^2}{z}\big(64\mathrm{H}_{-1}^2\mathrm{H}_0 - 128\mathrm{H}_{-1}\mathrm{H}_{0,-1} \\
&\quad +128\mathrm{H}_{0,-1,-1}\big) + \frac{512(1-z)}{3z}\big(\mathrm{H}_{0,-1,0,1} - \mathrm{H}_{0,-1}\zeta_2\big) + \frac{64\big(1-8z+z^2\big)}{z} \\
&\quad \times\mathrm{H}_{-1}\mathrm{H}_0 - \frac{64\big(3+4z+3z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_0^2 + \frac{128\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}^2\mathrm{H}_0^2 \\
&\quad +256\mathrm{H}_{-1}\mathrm{H}_0^2\mathrm{H}_{0,1} - \frac{128\big(3+14z+3z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,1} + \frac{512\big(1-z+z^2\big)}{3z} \\
&\quad \times\mathrm{H}_{-1}^2\mathrm{H}_{0,1} - 256\mathrm{H}_0\mathrm{H}_{0,1}^2 - \frac{64\big(1-8z+z^2\big)}{z}\mathrm{H}_{0,-1} - \frac{512\big(1-z+z^2\big)}{3z} \\
&\quad \mathrm{H}_{-1}\mathrm{H}_0\mathrm{H}_{0,-1} + \frac{128}{3}(7+3z)\mathrm{H}_{0,0,1} - 1024\mathrm{H}_{-1}\mathrm{H}_0\mathrm{H}_{0,0,1} - \frac{512\big(1-z+z^2\big)}{3z} \\
&\quad \times\mathrm{H}_{-1}\mathrm{H}_{0,0,1} + \frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,0,-1} + \frac{128\big(3+14z+3z^2\big)}{3z}\mathrm{H}_{0,1,-1} \\
&\quad +256\big(\mathrm{H}_{0,1,1} - \mathrm{H}_{0,1,-1}\big)\mathrm{H}_0^2 - \frac{1024\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,1,-1} - 256\mathrm{H}_{0,-1,1}\mathrm{H}_0^2 \\
&\quad +\frac{128\big(3+14z+3z^2\big)}{3z}\mathrm{H}_{0,-1,1} - \frac{1024\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,-1,1} \\
&\quad +\frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_0\mathrm{H}_{0,-1,-1} + 1024\mathrm{H}_{-1}\mathrm{H}_{0,0,0,1} + \frac{512\big(1-z+z^2\big)}{3z} \\
&\quad \times\mathrm{H}_{0,0,1,-1} + 1024\mathrm{H}_0\mathrm{H}_{0,0,1,-1} + \frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_{0,0,-1,1} + 1024\mathrm{H}_0\mathrm{H}_{0,0,-1,1} \\
&\quad -\frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_{0,0,-1,-1} + \frac{1024\big(1-z+z^2\big)}{3z}\mathrm{H}_{0,1,-1,-1} + 512\mathrm{H}_0 \\
&\quad \times\mathrm{H}_{0,-1,0,1} + \frac{1024\big(1-z+z^2\big)}{3z}\mathrm{H}_{0,-1,1,-1} + \frac{1024\big(1-z+z^2\big)}{3z}\mathrm{H}_{0,-1,-1,1} \\
&\quad +1536\mathrm{H}_{0,0,0,1,1} - 1024\mathrm{H}_{0,0,0,1,-1} - 1024\mathrm{H}_{0,0,0,-1,1} + 1024\mathrm{H}_{0,0,1,0,1} \\
&\quad -512\mathrm{H}_{0,0,-1,0,1} + \frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_0\zeta_2 + 256\mathrm{H}_{-1}\mathrm{H}_0^2\zeta_2
\end{aligned}
$$

$$+\frac{64\left(9+22z+9z^2\right)}{3z}\mathrm{H}_{-1}\zeta_2 - \frac{512\left(1-z+z^2\right)}{3z}\mathrm{H}_{-1}^2\zeta_2 + 512\mathrm{H}_0\mathrm{H}_{0,1}\zeta_2$$

$$-512\mathrm{H}_0\mathrm{H}_{0,-1}\zeta_2 - 1024\mathrm{H}_{0,0,1}\zeta_2 + 512\mathrm{H}_{0,0,-1}\zeta_2 - \frac{2048}{5}\mathrm{H}_{-1}\zeta_2^2$$

$$-512\mathrm{H}_{-1}\mathrm{H}_0\zeta_3 + \frac{256\left(1-z+z^2\right)}{z}\mathrm{H}_{-1}\zeta_3 - 512\mathrm{H}_{0,1}\zeta_3 + 512\mathrm{H}_{0,-1}\zeta_3 \Bigg]$$

$$-\frac{128}{3}z(6+z)\mathrm{H}_0^2\mathrm{H}_{0,1} - \frac{128\left(-3-4z-6z^2+10z^3\right)}{3(1-z)z}\mathrm{H}_0\mathrm{H}_{0,-1}$$

$$-\frac{256}{3}z^2\mathrm{H}_0^2\mathrm{H}_{0,-1} + \frac{512}{3}z(6+z)\mathrm{H}_0\mathrm{H}_{0,0,1} + \frac{128\left(-3-z-13z^2+3z^3\right)}{3z}$$

$$\times\mathrm{H}_{0,0,-1} + \frac{1024}{3}z^2\mathrm{H}_0\mathrm{H}_{0,0,-1} - 1024z\mathrm{H}_{0,0,0,1} - 512z^2\mathrm{H}_{0,0,0,-1}$$

$$-128z(2+z)\mathrm{H}_0^2\zeta_2 + \frac{64(1-z)(1+z)^2}{z}\mathrm{H}_1\zeta_2 + \frac{512}{5}z(4+z)\zeta_2^2$$

$$+\left(-\frac{64}{3}\left(79-25z+15z^2\right)+256\zeta_2\right)\zeta_3 - \frac{256}{3}(-6+z)z\mathrm{H}_0\zeta_3 + 2560\zeta_5$$

$$-\left(\frac{256\mathrm{H}_{0,0,-1}-192\zeta_3}{1-z}\right)\Bigg\}.$$

$$(1087)$$

Illustrating three–loop heavy flavor effects on the polarized non–singlet structure function $g_1^{\mathrm{NS}}(x,Q^2)$ in Ref. [53] it was assumed that there is no contribution by $\Delta C_{g_1,q}^{d_{abc},(3)}$, despite this function is only known now. This is correct for the three massless quark flavors dealt with there, but changes if four or five massless flavors are considered at very high virtualities or energies. One may generally assume some suppression due to (1084), but there is a finite contribution. Note that $\Delta C_{g_1,q}^{d_{abc},(3)}$ is independent of the scheme employed for the treatment of γ_5, since its contribution is finite.

The other Wilson coefficients are calculated in the Larin scheme. The Wilson coefficient $\Delta C_{g_1,q}^{\mathrm{PS},(3),\mathrm{L}}$ reads

$$\Delta C_{g_1,q}^{\mathrm{PS},(3),\mathrm{L}} =$$

$$C_F\Bigg\{T_F^2N_F^2\Bigg[\frac{128\zeta_3 P_{902}}{9(N-1)N^2(1+N)^2(2+N)} - \frac{32P_{936}}{243(N-1)^2N^5(1+N)^5(2+N)}$$

$$+\left(-\frac{32(2+N)P_{919}}{81N^4(1+N)^4} - \frac{32(N-1)(2+N)S_2}{9N^2(1+N)^2}\right)S_1 + \frac{64(N-1)(2+N)S_1^3}{27N^2(1+N)^2}$$

$$+\frac{16(2+N)\left(21+17N-6N^2+16N^3\right)S_1^2}{27N^3(1+N)^3} + \frac{16(2+N)\left(21+5N-24N^2+46N^3\right)}{27N^3(1+N)^3}$$

$$\times S_2 - \frac{160(N-1)(2+N)S_3}{27N^2(1+N)^2} + \frac{512\left(-5-12N+5N^2\right)S_{-2}}{9(N-1)^2N(1+N)^2(2+N)}$$

$$-\frac{1024S_{-3}}{3(N-1)N(1+N)(2+N)}\Bigg]$$

$$+C_AT_FN_F\Bigg[\frac{64S_{-3,1}P_{901}}{3(N-1)N^2(1+N)^2(2+N)} - \frac{20S_4P_{903}}{3(N-1)N^2(1+N)^2(2+N)}$$

$$-\frac{16S_{-2,2}P_{905}}{(N-1)N^2(1+N)^2(2+N)} + \frac{16S_{2,1}P_{906}}{3N^3(1+N)^3} + \frac{64S_{-2,1,1}P_{907}}{3(N-1)N^2(1+N)^2(2+N)}$$

$$+\frac{8S_1^3P_{910}}{27N^3(1+N)^3} - \frac{8S_{-4}P_{915}}{3(N-1)N^2(1+N)^2(2+N)} - \frac{16S_{-2,1}P_{923}}{3(N-1)N^3(1+N)^3(2+N)}$$

$$+\frac{8S_3P_{927}}{27(N-1)N^3(1+N)^3(2+N)} + \frac{8P_{938}}{243(N-1)^2N^6(1+N)^6(2+N)}$$

$$-\left(\frac{8S_2P_{913}}{9N^3(1+N)^3} + \frac{16S_{-2,1}P_{911}}{3(N-1)N^2(1+N)^2(2+N)} - \frac{8S_3P_{912}}{9(N-1)N^2(1+N)^2(2+N)}\right.$$

$$\left.-\frac{8P_{934}}{81(N-1)N^5(1+N)^5(2+N)}\right)S_1 + \left(\frac{4P_{926}}{27N^4(1+N)^4} - \frac{80(N-1)(2+N)S_2}{3N^2(1+N)^2}\right)$$

$$\times S_1^2 + \frac{28(N-1)(2+N)S_1^4}{9N^2(1+N)^2} - \frac{16\left(5-2N-2N^2\right)S_2^2}{3N^2(1+N)^2} - \left(\frac{16\left(22+N+N^2\right)S_1^2}{3N^2(1+N)^2}\right.$$

$$+\frac{16S_1P_{922}}{3(N-1)N^3(1+N)^3(2+N)} + \frac{16P_{933}}{9(N-1)^2N^4(1+N)^4(2+N)}$$

$$\left.-\frac{32\left(8+N+N^2\right)S_2}{3N^2(1+N)^2}\right)S_{-2} - \frac{16\left(-2+3N+3N^2\right)S_{-2}^2}{3N^2(1+N)^2} - \frac{4S_2P_{928}}{27N^4(1+N)^4}$$

$$+\left(\frac{8P_{925}}{3(N-1)N^3(1+N)^3(2+N)} + \frac{8S_1P_{916}}{3(N-1)N^2(1+N)^2(2+N)}\right)S_{-3}$$

$$+\frac{320\left(2+N+N^2\right)S_{3,1}}{(N-1)N^2(1+N)^2(2+N)} + \frac{32(N-1)(2+N)S_{2,1,1}}{3N^2(1+N)^2} - \frac{96(N-1)(2+N)\zeta_2^2}{5N^2(1+N)^2}$$

$$+\left(-\frac{16S_1P_{908}}{3(N-1)N^2(1+N)^2(2+N)} - \frac{16P_{924}}{9(N-1)N^3(1+N)^3(2+N)}\right)\zeta_3\Bigg]\Bigg\}$$

$$+C_F^2T_FN_F\Bigg\{-\frac{64S_{3,1}P_{904}}{(N-1)N^2(1+N)^2(2+N)} + \frac{8S_4P_{909}}{3(N-1)N^2(1+N)^2(2+N)}$$

$$-\frac{8S_3P_{920}}{9(N-1)N^3(1+N)^3(2+N)} - \frac{4S_2P_{931}}{3(N-1)N^4(1+N)^4(2+N)}$$

$$-\frac{8P_{937}}{3(N-1)^2N^6(1+N)^6(2+N)} + \left(\frac{16S_3P_{914}}{9(N-1)N^2(1+N)^2(2+N)}\right.$$

$$-\frac{8S_2P_{917}}{3N^3(1+N)^3} + \frac{8P_{935}}{3(N-1)^2N^5(1+N)^5(2+N)}$$

$$\left.+\frac{256\left(2+N+N^2\right)S_{-2,1}}{(N-1)N^2(1+N)^2(2+N)}\right)S_1$$

$$+\left(\frac{4P_{932}}{3(N-1)N^4(1+N)^4(2+N)} - \frac{112(N-1)(2+N)S_2}{3N^2(1+N)^2}\right)S_1^2$$

$$+\frac{16(2+N)\left(22+2N-15N^2+27N^3\right)S_1^3}{9N^3(1+N)^3} + \frac{68(N-1)(2+N)S_2^2}{3N^2(1+N)^2}$$

$$+\left(-\frac{32P_{929}}{(N-1)^2N^3(1+N)^3(2+N)} - \frac{32S_1P_{930}}{(N-1)^2N^3(1+N)^3(2+N)}\right.$$

$$+\frac{128\big(-4+N+N^2\big)S_1^2}{(N-1)N^2(1+N)^2(2+N)}-\frac{128\big(-4+N+N^2\big)S_2}{(N-1)N^2(1+N)^2(2+N)}\bigg)S_{-2}$$

$$+\bigg(\frac{64P_{918}}{(N-1)N^3(1+N)^2(2+N)}-\frac{128(N-2)(3+N)S_1}{(N-1)N^2(1+N)^2(2+N)}\bigg)S_{-3}$$

$$-\frac{128\big(-6+5N+5N^2\big)S_{-4}}{(N-1)N^2(1+N)^2(2+N)}-\frac{64\big(4+3N-3N^2+2N^3\big)S_{2,1}}{3N^3(1+N)^3}$$

$$-\frac{256\big(6-N-2N^2+N^3\big)S_{-2,1}}{(N-1)N^3(1+N)^2(2+N)}+\frac{256\big(-2+3N+3N^2\big)S_{-2,2}}{(N-1)N^2(1+N)^2(2+N)}$$

$$+\frac{1024\big(-1+N+N^2\big)S_{-3,1}}{(N-1)N^2(1+N)^2(2+N)}-\frac{32(N-1)(2+N)S_{2,1,1}}{3N^2(1+N)^2}$$

$$+\frac{96(N-1)(2+N)\zeta_2^2}{5N^2(1+N)^2}-\frac{1024S_{-2,1,1}}{(N-1)N(1+N)(2+N)}+\frac{44(N-1)(2+N)S_1^4}{9N^2(1+N)^2}$$

$$+\bigg(\frac{32P_{921}}{3(N-1)N^3(1+N)^3(2+N)}-\frac{128\big(7+4N+4N^2\big)S_1}{3N^2(1+N)^2}\bigg)\zeta_3\bigg\},\tag{1088}$$

with the polynomials

$$
\begin{aligned}
P_{901} &= N^4+2N^3-63N^2-64N+28, & (1089)\\
P_{902} &= N^4+2N^3-39N^2-40N+4, & (1090)\\
P_{903} &= N^4+2N^3-15N^2-16N+124, & (1091)\\
P_{904} &= N^4+2N^3+5N^2+4N+20, & (1092)\\
P_{905} &= N^4+2N^3+73N^2+72N-20, & (1093)\\
P_{906} &= N^4+10N^3-5N^2+18N+12, & (1094)\\
P_{907} &= 5N^4+10N^3+93N^2+88N-4, & (1095)\\
P_{908} &= 7N^4+14N^3-81N^2-88N+100, & (1096)\\
P_{909} &= 19N^4+38N^3-105N^2-124N+556, & (1097)\\
P_{910} &= 23N^4+N^3-107N^2+167N+222, & (1098)\\
P_{911} &= 25N^4+50N^3+137N^2+112N+60, & (1099)\\
P_{912} &= 37N^4+74N^3-375N^2-412N-44, & (1100)\\
P_{913} &= 37N^4+77N^3-97N^2+151N+246, & (1101)\\
P_{914} &= 43N^4+86N^3+87N^2+44N+316, & (1102)\\
P_{915} &= 47N^4+94N^3-393N^2-440N+308, & (1103)\\
P_{916} &= 63N^4+126N^3+127N^2+64N+4, & (1104)\\
P_{917} &= 65N^4+86N^3-49N^2+66N+96, & (1105)\\
P_{918} &= N^5+2N^4+7N^3-6N+12, & (1106)\\
P_{919} &= 161N^5+23N^4+55N^3+55N^2-12N-18, & (1107)\\
P_{920} &= 3N^6+27N^5+157N^4+461N^3+1212N^2+380N+64, & (1108)\\
P_{921} &= 3N^6+33N^5+71N^4+115N^3+294N^2+4N-88, & (1109)\\
P_{922} &= 10N^6+24N^5+7N^4-30N^3-135N^2-64N+92, & (1110)\\
P_{923} &= 21N^6+34N^5-191N^4-248N^3-40N^2-184N-160, & (1111)\\
P_{924} &= 58N^6+237N^5-811N^4-1741N^3+873N^2+652N-852, & (1112)
\end{aligned}
$$

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$$P_{925} = 63N^6 + 138N^5 + 23N^4 + 228N^3 + 308N^2 - 456N - 240, \tag{1113}$$

$$P_{926} = 79N^6 + 78N^5 - 1526N^4 - 1632N^3 - 794N^2 - 3351N - 1710, \tag{1114}$$

$$P_{927} = 199N^6 + 588N^5 + 20N^4 - 208N^3 + 3177N^2 + 1612N - 1068, \tag{1115}$$

$$P_{928} = 953N^6 + 2298N^5 - 664N^4 - 2490N^3 + 980N^2 - 1455N - 1422, \tag{1116}$$

$$P_{929} = N^7 + N^6 - 10N^5 - 14N^4 - 35N^3 + 9N^2 + 20N - 4, \tag{1117}$$

$$P_{930} = N^7 + 2N^6 + 20N^5 + 10N^4 - 25N^3 - 92N^2 - 12N + 32, \tag{1118}$$

$$P_{931} = 56N^8 + 205N^7 + 154N^6 + 40N^5 - 84N^4 - 977N^3 - 942N^2 + 308N + 472, \tag{1119}$$

$$P_{932} = 156N^8 + 501N^7 + 164N^6 - 374N^5 - 242N^4 - 1139N^3 - 958N^2 + 524N \\ + 600, \tag{1120}$$

$$P_{933} = 45N^9 + 57N^8 + 386N^7 + 232N^6 - 1896N^5 - 2008N^4 + 937N^3 + 63N^2 \\ - 228N - 84, \tag{1121}$$

$$P_{934} = 2701N^{10} + 11348N^9 + 6606N^8 - 13281N^7 + 8301N^6 + 20247N^5 + 980N^4 \\ + 32572N^3 + 10626N^2 - 17460N - 10800, \tag{1122}$$

$$P_{935} = 96N^{11} + 302N^{10} - 111N^9 - 890N^8 - 689N^7 + 706N^6 + 2477N^5 + 2544N^4 \\ - 497N^3 - 1854N^2 + 324N + 664, \tag{1123}$$

$$P_{936} = 2042N^{11} + 6095N^{10} - 2575N^9 - 15042N^8 - 2412N^7 + 15972N^6 + 22057N^5 \\ + 18358N^4 + 2452N^3 - 8427N^2 + 468N + 2484, \tag{1124}$$

$$P_{937} = 260N^{13} + 1033N^{12} + 579N^{11} - 2050N^{10} - 2768N^9 - 582N^8 + 146N^7 - 680N^6 \\ + 752N^5 + 1237N^4 - 765N^3 - 950N^2 + 356N + 360, \tag{1125}$$

$$P_{938} = 30520N^{13} + 123245N^{12} + 50588N^{11} - 274955N^{10} - 241017N^9 + 129972N^8 \\ + 331898N^7 + 404113N^6 + 59461N^5 - 137557N^4 - 25818N^3 + 116766N^2 \\ + 1080N - 29160. \tag{1126}$$

Finally, the Wilson coefficient $\Delta C^{(3)}_{g_1,g}$ is given by

$$\Delta C^{(3)}_{g_1,g} = \Delta C^{(3),a}_{g_1,g} + \Delta C^{(3),d_{abc}}_{g_1,g}, \tag{1127}$$

with

$$\Delta C^{(3),a}_{g_1,g} = C_F \Bigg\{ T_F^2 N_F^2 \Bigg[-\frac{256 P_{941}}{9N(1+N)^2(2+N)^2} S_{2,1} - \frac{64 P_{1002}}{9(N-1)N^3(1+N)^3(2+N)} \zeta_3 \\ + \frac{16(N-1)P_{946}}{81N^3(1+N)^3} S_1^3 + \frac{32 P_{1007}}{81N^3(1+N)^3(2+N)^2} S_3 - \frac{4 P_{1029}}{27N^4(1+N)^4(2+N)^2} \\ \times S_2 + \frac{P_{1045}}{243(N-1)^2 N^6(1+N)^6(2+N)^2} + \Bigg(-\frac{16 P_{1003}}{27N^3(1+N)^3(2+N)^2} \\ \times S_2 - \frac{8 P_{1039}}{243(N-1)N^5(1+N)^5(2+N)^2} + \frac{32(74 + 35N + 35N^2)}{27N(1+N)(2+N)} S_3 \\ - \frac{128(2 + N + N^2)}{3N(1+N)(2+N)} S_{2,1} - \frac{512}{3N(1+N)(2+N)} S_{-2,1} \\ + \frac{64(-46 + 5N + 5N^2)}{9N(1+N)(2+N)} \zeta_3 \Bigg) S_1 + \frac{68(N-1)}{27N(1+N)} S_1^4 + \Bigg(\frac{4 P_{1025}}{81N^4(1+N)^4(2+N)} \\ - \frac{8(-62 + 7N + 7N^2)}{9N(1+N)(2+N)} S_2 \Bigg) S_1^2 + \frac{4(-226 + 17N + 17N^2)}{9N(1+N)(2+N)} S_2^2$$

$$-\frac{56\left(14+17N+17N^2\right)}{9N(1+N)(2+N)}S_4+\left(-\frac{64P_{960}}{9(N-1)N^2(1+N)^2(2+N)^2}S_1\right.$$

$$-\frac{32P_{1028}}{9(N-1)^2N^3(1+N)^3(2+N)^2}+\frac{512}{3N(1+N)(2+N)}S_1^2$$

$$\left.-\frac{512}{3N(1+N)(2+N)}S_2\right)S_{-2}+\left(\frac{128P_{961}}{9(N-1)N^2(1+N)^2(2+N)^2}\right.$$

$$\left.-\frac{256}{N(1+N)(2+N)}S_1\right)S_{-3}-\frac{256S_{-2}^2}{3N(1+N)(2+N)}-\frac{1024}{3N(1+N)(2+N)}S_{-4}$$

$$+\frac{64\left(-10+3N+3N^2\right)}{3N(1+N)(2+N)}S_{3,1}-\frac{512(-3+5N)}{9N^2(1+N)(2+N)}S_{-2,1}$$

$$+\frac{512}{3N(1+N)(2+N)}S_{-2,2}+\frac{1024}{3N(1+N)(2+N)}S_{-3,1}+\frac{64\left(10+N+N^2\right)}{3N(1+N)(2+N)}$$

$$\left.\times S_{2,1,1}\right]+C_A T_F N_F\left[-\frac{32P_{965}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1,1}\right.$$

$$-\frac{2P_{949}}{9N^2(1+N)^2(2+N)}S_2^2+\frac{16P_{966}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,2}$$

$$+\frac{16P_{967}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-3,1}-\frac{16P_{978}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}$$

$$+\frac{4P_{997}}{9(N-1)N^2(1+N)^2(2+N)^2}S_4+\frac{8P_{1006}}{9N^3(1+N)^3(2+N)^2}S_{2,1}$$

$$-\frac{16P_{1017}}{9(N-1)N^3(1+N)^3(2+N)^2}S_{-2,1}-\frac{8P_{1023}}{9(N-1)N^3(1+N)^3(2+N)^2}\zeta_3$$

$$-\frac{8P_{1027}}{81(N-1)N^3(1+N)^3(2+N)^2}S_3+\frac{P_{1047}}{972(N-1)^2N^6(1+N)^6(2+N)^3}$$

$$+\left(\frac{8P_{944}}{3N^2(1+N)^2(2+N)}S_{2,1}-\frac{32P_{959}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1}\right.$$

$$-\frac{8P_{992}}{27(N-1)N^2(1+N)^2(2+N)^2}S_3-\frac{8P_{999}}{9(N-1)N^2(1+N)^2(2+N)^2}\zeta_3$$

$$+\frac{4P_{1010}}{27N^3(1+N)^3(2+N)^2}S_2+\frac{2P_{1043}}{243(N-1)^2N^5(1+N)^5(2+N)^3}$$

$$-\frac{80(N-1)}{N(1+N)}S_2^2+\frac{8\left(-382+47N+47N^2\right)}{3N(1+N)(2+N)}S_4+\frac{64\left(130+7N+7N^2\right)}{3N(1+N)(2+N)}S_{3,1}$$

$$+\frac{128\left(-54+5N+5N^2\right)}{N(1+N)(2+N)}S_{-2,2}+\frac{16\left(-1378+113N+113N^2\right)}{3N(1+N)(2+N)}S_{-3,1}$$

$$-\frac{80(N-1)}{N(1+N)}S_{2,1,1}-\frac{32\left(-1214+127N+127N^2\right)}{3N(1+N)(2+N)}S_{-2,1,1}-\frac{96(N-1)\zeta_2^2}{5N(1+N)}\right)S_1$$

$$+\left(\frac{4P_{948}}{9N^2(1+N)^2(2+N)}S_2+\frac{P_{1030}}{81(N-1)N^4(1+N)^4(2+N)^2}\right.$$

$$-\frac{8\left(310+13N+13N^2\right)}{3N(1+N)(2+N)}S_3-\frac{32(N-1)}{N(1+N)}S_{2,1}+\frac{32\left(-286+35N+35N^2\right)}{3N(1+N)(2+N)}$$

$$\times S_{-2,1} + \frac{32\left(166 + 19N + 19N^2\right)}{3N(1+N)(2+N)}\zeta_3\right)S_1^2 + \left(-\frac{4P_{1001}}{81N^3(1+N)^3(2+N)}\right.$$

$$+\frac{136(N-1)}{3N(1+N)}S_2\right)S_1^3 + \left(\frac{P_{1035}}{27(N-1)N^4(1+N)^4(2+N)^2} + \frac{8\left(898 + 7N + 7N^2\right)}{3N(1+N)(2+N)}\right.$$

$$\times S_3 - \frac{128\left(-72 + 7N + 7N^2\right)}{N(1+N)(2+N)}S_{-2,1} - \frac{96\left(-2 + 3N + 3N^2\right)}{N(1+N)(2+N)}\zeta_3\right)S_2$$

$$-\frac{2\left(369 - 23N - 96N^2 + 206N^3\right)}{27N^2(1+N)^2}S_1^4 + \frac{256\left(11 - N - N^2\right)}{3N(1+N)(2+N)}S_5 + \left(-\frac{8P_{981}}{3(N-1)N^2(1+N)^2(2+N)}\right.$$

$$+\frac{8P_{985}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2 - \frac{8P_{1037}}{9(N-1)^2N^4(1+N)^4(2+N)^3}$$

$$+\left(-\frac{16P_{1031}}{9(N-1)^2N^3(1+N)^3(2+N)^3} - \frac{48\left(-18 + N + N^2\right)}{N(1+N)(2+N)}S_2\right)S_1$$

$$-\frac{16\left(178 - 17N - 17N^2\right)}{9N(1+N)(2+N)}S_1^3 - \frac{64\left(2 + 35N + 35N^2\right)}{9N(1+N)(2+N)}$$

$$\times S_3 + \frac{64\left(-146 + 13N + 13N^2\right)}{N(1+N)(2+N)}S_{2,1} - \frac{1024\left(-4 + N + N^2\right)}{N(1+N)(2+N)}\zeta_3$$

$$-\frac{128(N-1)}{3N(1+N)}S_{-2,1}\right)S_{-2} + \left(\frac{32P_{971}}{3(N-1)N^2(1+N)^2(2+N)^2} + \frac{256\left(-11 + N + N^2\right)}{3N(1+N)(2+N)}\right.$$

$$\times S_1\right)S_{-2}^2 + \left(-\frac{16P_{977}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1 + \frac{8P_{1022}}{9(N-1)N^3(1+N)^3(2+N)^2}\right.$$

$$-\frac{16\left(-490 + 41N + 41N^2\right)}{3N(1+N)(2+N)}S_1^2 + \frac{128\left(-107 + 10N + 10N^2\right)}{3N(1+N)(2+N)}S_2$$

$$-\frac{32\left(22 + 5N + 5N^2\right)}{N(1+N)(2+N)}S_{-2}\right)S_{-3} + \left(\frac{8P_{980}}{3(N-1)N^2(1+N)^2(2+N)^2}\right.$$

$$-\frac{16\left(-198 + 19N + 19N^2\right)}{N(1+N)(2+N)}S_1\right)S_{-4} - \frac{16\left(562 + 31N + 31N^2\right)}{3N(1+N)(2+N)}S_{-5}$$

$$+\frac{64\left(-38 + N + N^2\right)}{N(1+N)(2+N)}S_{2,3} - \frac{64\left(-134 + 13N + 13N^2\right)}{3N(1+N)(2+N)}S_{4,1}$$

$$+\frac{160\left(-86 + 7N + 7N^2\right)}{3N(1+N)(2+N)}S_{2,-3} - \frac{256\left(-50 + N + N^2\right)}{3N(1+N)(2+N)}S_{-2,3}$$

$$+\frac{256\left(-7 + N + N^2\right)}{N(1+N)(2+N)}S_{-4,1} - \frac{16\left(128 + 67N - 26N^2 + 5N^3\right)}{3N(1+N)^2(2+N)}S_{2,1,1}$$

$$-\frac{64\left(-434 + 37N + 37N^2\right)}{3N(1+N)(2+N)}S_{2,1,-2} - \frac{96\left(46 + N + N^2\right)}{N(1+N)(2+N)}S_{3,1,1}$$

$$-\frac{16(N-1)}{3N(1+N)}S_{2,2,1} - \frac{64\left(-118 + 11N + 11N^2\right)}{3N(1+N)(2+N)}S_{-2,1,-2} - \frac{2560\left(-11 + N + N^2\right)}{3N(1+N)(2+N)}$$

$$\times S_{-2,2,1} - \frac{160\left(-58 + 5N + 5N^2\right)}{N(1+N)(2+N)}S_{-3,1,1} + \frac{320\left(-178 + 17N + 17N^2\right)}{3N(1+N)(2+N)}S_{-2,1,1,1}$$

$$+\frac{160(N-1)}{N(1+N)}S_{2,1,1,1}+\frac{96(N-1)\big(5-3N-3N^2\big)}{5N^2(1+N)^2}\zeta_2^2-\frac{80\big(-118+N+N^2\big)}{N(1+N)(2+N)}$$

$$\times\zeta_5\bigg]\bigg]+C_AT_F^2N_F^2\bigg\{\frac{128P_{954}}{27N^2(1+N)^2(2+N)^2}S_{2,1}+\frac{32P_{955}}{9(N-1)N^2(1+N)^2(2+N)}\zeta_3$$

$$-\frac{64P_{956}}{81N^2(1+N)^2(2+N)^2}S_3+\frac{8P_{1041}}{243(N-1)^2N^5(1+N)^5(2+N)^2}$$

$$-\frac{16P_{1004}}{81N^3(1+N)^3(2+N)^2}S_2+\bigg[\frac{8P_{1033}}{243(N-1)N^4(1+N)^4(2+N)^2}$$

$$-\frac{32P_{957}}{27N^2(1+N)^2(2+N)^2}S_2+\frac{32\big(-110+19N+19N^2\big)}{27N(1+N)(2+N)}S_3+\frac{256\big(1+N+N^2\big)}{9N(1+N)(2+N)}$$

$$\times S_{2,1}+\frac{256}{3N(1+N)(2+N)}S_{-2,1}-\frac{32\big(-62+13N+13N^2\big)}{9N(1+N)(2+N)}\zeta_3\bigg]S_1$$

$$+\bigg[\frac{16P_{996}}{81N^3(1+N)^3(2+N)}-\frac{8\big(-14+19N+19N^2\big)}{9N(1+N)(2+N)}S_2\bigg]S_1^2+\frac{28(N-1)}{27N(1+N)}S_1^4$$

$$+\frac{32\big(27-22N-9N^2+19N^3\big)}{81N^2(1+N)^2}S_1^3+\frac{4\big(34+31N+31N^2\big)}{9N(1+N)(2+N)}S_2^2+\frac{56\big(10+7N+7N^2\big)}{9N(1+N)(2+N)}$$

$$\times S_4+\bigg[-\frac{64P_{964}}{9(N-1)N^2(1+N)^2(2+N)^2}S_1-\frac{32P_{1021}}{81(N-1)^2N^2(1+N)^3(2+N)^2}$$

$$-\frac{32\big(6+N+N^2\big)}{3N(1+N)(2+N)}S_1^2+\frac{32\big(6+N+N^2\big)}{3N(1+N)(2+N)}S_2\bigg]S_{-2}+\frac{128}{3N(1+N)(2+N)}S_{-2}^2$$

$$+\bigg[\frac{128P_{972}}{27(N-1)N^2(1+N)^2(2+N)^2}+\frac{128\big(1+N+N^2\big)}{3N(1+N)(2+N)}S_1\bigg]S_{-3}$$

$$-\frac{64\big(-38+7N+7N^2\big)}{9N(1+N)(2+N)}S_{-4}-\frac{128\big(-13+5N+5N^2\big)}{9N(1+N)(2+N)}S_{3,1}+\frac{256(-3+5N)}{9N^2(1+N)(2+N)}$$

$$\times S_{-2,1}-\frac{256}{3N(1+N)(2+N)}S_{-2,2}-\frac{512}{3N(1+N)(2+N)}S_{-3,1}$$

$$-\frac{64\big(16+N+N^2\big)}{9N(1+N)(2+N)}S_{2,1,1}\bigg\}+C_A^2T_FN_F\bigg\{\frac{P_{939}}{9N^2(1+N)^2(2+N)}S_2^2$$

$$+\frac{16P_{943}}{9N^2(1+N)^2(2+N)}S_{2,1,1}-\frac{32P_{983}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1,1}$$

$$+\frac{16P_{984}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,2}+\frac{16P_{986}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-3,1}$$

$$+\frac{8P_{987}}{9(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}-\frac{2P_{998}}{9(N-1)N^2(1+N)^2(2+N)^2}S_4$$

$$-\frac{8P_{1008}}{27N^3(1+N)^3(2+N)^2}S_{2,1}+\frac{4P_{1014}}{9(N-1)N^3(1+N)^3(2+N)^2}\zeta_3$$

$$-\frac{32P_{1015}}{9(N-1)N^3(1+N)^3(2+N)^2}S_{-2,1}+\frac{8P_{1026}}{81(N-1)N^3(1+N)^3(2+N)^2}S_3$$

$$-\frac{4P_{1046}}{243(N-1)^2N^6(1+N)^6(2+N)^3} + \Bigg[-\frac{16P_{945}}{9N^2(1+N)^2(2+N)}S_{2,1}$$

$$+\frac{16P_{982}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1} + \frac{8P_{994}}{9(N-1)N^2(1+N)^2(2+N)^2}\zeta_3$$

$$-\frac{8P_{995}}{27(N-1)N^2(1+N)^2(2+N)^2}S_3 + \frac{8P_{1009}}{27N^3(1+N)^3(2+N)^2}S_2$$

$$-\frac{4P_{1044}}{243(N-1)^2N^5(1+N)^5(2+N)^3} - \frac{124(N-1)}{3N(1+N)}S_2^2 - \frac{48(N-1)}{N(1+N)}S_4$$

$$-\frac{32\big(70+N+N^2\big)}{3N(1+N)(2+N)}S_{3,1} - \frac{80\big(-106+5N+5N^2\big)}{3N(1+N)(2+N)}S_{-3,1} - \frac{16(N-1)}{3N(1+N)}S_{2,1,1}$$

$$-\frac{32\big(-226+5N+5N^2\big)}{3N(1+N)(2+N)}S_{-2,2} - \frac{32\big(374+5N+5N^2\big)}{3N(1+N)(2+N)}S_{-2,1,1} + \frac{96(N-1)}{5N(1+N)}\zeta_2^2\Bigg]$$

$$\times S_1 + \Bigg[\frac{2P_{950}}{9N^2(1+N)^2(2+N)}S_2 - \frac{2P_{1036}}{81(N-1)N^4(1+N)^4(2+N)^2}$$

$$-\frac{8\big(-154+29N+29N^2\big)}{3N(1+N)(2+N)}S_3 + \frac{32(N-1)}{3N(1+N)}S_{2,1} + \frac{32\big(22+3N+3N^2\big)}{N(1+N)(2+N)}S_{-2,1}$$

$$+\frac{16\big(-190+17N+17N^2\big)}{3N(1+N)(2+N)}\zeta_3\Bigg]S_1^2 + \Bigg[-\frac{8P_{1000}}{81N^3(1+N)^3(2+N)} + \frac{128(N-1)}{9N(1+N)}S_2\Bigg]$$

$$\times S_1^3 + \frac{\big(-528+221N+216N^2-293N^3\big)}{27N^2(1+N)^2}S_1^4 - \frac{4(N-1)}{3N(1+N)}S_1^5$$

$$+\Bigg[\frac{2P_{1034}}{81(N-1)N^4(1+N)^4(2+N)^2} - \frac{8\big(658+31N+31N^2\big)}{9N(1+N)(2+N)}S_3 + \frac{48\big(2+N+N^2\big)}{N(1+N)(2+N)}$$

$$\times\zeta_3 + \frac{128\big(-27+2N+2N^2\big)}{N(1+N)(2+N)}S_{-2,1}\Bigg]S_2 + \frac{32\big(-26+7N+7N^2\big)}{3N(1+N)(2+N)}S_5$$

$$+\Bigg[\frac{8P_{975}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2 - \frac{8P_{979}}{3(N-1)N^2(1+N)^2(2+N)^2}S_2$$

$$+\frac{8P_{1040}}{81(N-1)^2N^4(1+N)^4(2+N)^3} + \Bigg(\frac{16P_{1032}}{9(N-1)^2N^3(1+N)^3(2+N)^3}$$

$$-\frac{16\big(50+11N+11N^2\big)}{3N(1+N)(2+N)}S_2\Bigg)S_1 - \frac{16\big(-26+N+N^2\big)}{3N(1+N)(2+N)}S_1^3 + \frac{128\big(-5+4N+4N^2\big)}{3N(1+N)(2+N)}$$

$$\times S_3 + \frac{1024\big(11-N-N^2\big)}{3N(1+N)(2+N)}S_{2,1} + \frac{128(N-1)}{3N(1+N)}S_{-2,1} - \frac{128\big(7-2N-2N^2\big)}{N(1+N)(2+N)}$$

$$\times\zeta_3\Bigg]S_{-2} + \Bigg[-\frac{8P_{976}}{3(N-1)N^2(1+N)^2(2+N)^2} - \frac{16\big(-18+N+N^2\big)}{N(1+N)(2+N)}S_1\Bigg]S_{-2}^2$$

$$+\Bigg[-\frac{8P_{991}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1 - \frac{16P_{1024}}{27(N-1)N^3(1+N)^3(2+N)^2}$$

$$-\frac{16\big(122+29N+29N^2\big)}{3N(1+N)(2+N)}S_1^2 - \frac{32\big(-64+9N+9N^2\big)}{N(1+N)(2+N)}S_2$$

$$+\frac{32\left(10+13N+13N^2\right)}{3N(1+N)(2+N)}S_{-2}\Bigg]S_{-3}+\Bigg[-\frac{8P_{989}}{9(N-1)N^2(1+N)^2(2+N)^2}$$

$$+\frac{128\left(-13+2N+2N^2\right)}{N(1+N)(2+N)}S_1\Bigg]S_{-4}+\frac{16\left(238+N+N^2\right)}{3N(1+N)(2+N)}S_{-5}$$

$$-\frac{48\left(-14+N+N^2\right)}{N(1+N)(2+N)}S_{2,3}+\frac{128\left(34+N+N^2\right)}{3N(1+N)(2+N)}S_{2,-3}-\frac{256\left(19+N+N^2\right)}{3N(1+N)(2+N)}S_{-2,3}$$

$$+\frac{16\left(-146+19N+19N^2\right)}{3N(1+N)(2+N)}S_{4,1}-\frac{32\left(-142+35N+35N^2\right)}{3N(1+N)(2+N)}S_{-4,1}$$

$$+\frac{1024\left(-11+N+N^2\right)}{3N(1+N)(2+N)}S_{2,1,-2}+\frac{176(N-1)}{3N(1+N)}S_{2,2,1}+\frac{1152}{N(1+N)(2+N)}S_{3,1,1}$$

$$-\frac{64\left(22+7N+7N^2\right)}{3N(1+N)(2+N)}S_{-2,1,-2}+\frac{64\left(-178+17N+17N^2\right)}{3N(1+N)(2+N)}S_{-2,2,1}$$

$$+\frac{32\left(-130+17N+17N^2\right)}{N(1+N)(2+N)}S_{-3,1,1}-\frac{64\left(-322+17N+17N^2\right)}{3N(1+N)(2+N)}S_{-2,1,1,1}$$

$$+\frac{48(N-1)\left(-8+3N+3N^2\right)}{5N^2(1+N)^2}\zeta_2^2+\frac{160\left(-14+N+N^2\right)}{N(1+N)(2+N)}\zeta_5\Bigg\}$$

$$+C_F^2T_FN_F\Bigg\{\frac{P_{940}}{3N^2(1+N)^2(2+N)}S_2^2+\frac{64P_{952}}{3(N-1)N(1+N)^2(2+N)^2}S_{-2,1,1}$$

$$-\frac{16P_{953}}{3N^3(1+N)^2(2+N)}S_{2,1}-\frac{64P_{962}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,2}$$

$$+\frac{32P_{968}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}-\frac{32P_{969}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-3,1}$$

$$-\frac{2P_{988}}{3(N-1)N^2(1+N)^2(2+N)^2}S_4+\frac{4P_{1005}}{9(N-1)N^3(1+N)^2(2+N)^2}S_3$$

$$+\frac{32P_{1013}}{3(N-1)N^3(1+N)^3(2+N)^2}S_{-2,1}+\frac{16P_{1019}}{3(N-1)N^3(1+N)^3(2+N)^2}\zeta_3$$

$$+\frac{P_{1042}}{12(N-1)N^6(1+N)^6(2+N)^2}+\Bigg[\frac{32P_{942}}{3N^2(1+N)^2(2+N)}S_{2,1}$$

$$-\frac{64P_{951}}{3(N-1)N(1+N)^2(2+N)^2}S_{-2,1}+\frac{16P_{990}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3$$

$$+\frac{2P_{974}}{3N^3(1+N)^3(2+N)}S_2-\frac{8P_{993}}{9(N-1)N^2(1+N)^2(2+N)^2}S_3$$

$$-\frac{2P_{1038}}{3(N-1)N^5(1+N)^5(2+N)^2}-\frac{8\left(-766+95N+95N^2\right)}{3N(1+N)(2+N)}S_4-\frac{188(N-1)}{3N(1+N)}$$

$$\times S_2^2-\frac{32\left(262+13N+13N^2\right)}{3N(1+N)(2+N)}S_{3,1}+\frac{128\left(14+N+N^2\right)}{N(1+N)(2+N)}S_{-2,2}$$

$$+\frac{64\left(70+13N+13N^2\right)}{3N(1+N)(2+N)}S_{-3,1}+\frac{256(N-1)}{3N(1+N)}S_{2,1,1}+\frac{128\left(-98+N+N^2\right)}{3N(1+N)(2+N)}S_{-2,1,1}$$

$$\Bigg]S_1+\Bigg[\frac{2P_{947}}{3N^2(1+N)^2(2+N)}S_2+\frac{48\left(18-N-N^2\right)}{N(1+N)(2+N)}S_3$$

$$+\frac{P_{1011}}{3N^4(1+N)^4(2+N)} + \frac{64(N-1)}{3N(1+N)}S_{2,1} - \frac{256\left(-14+N+N^2\right)}{3N(1+N)(2+N)}S_{-2,1}$$

$$-\frac{128\left(14+5N+5N^2\right)}{3N(1+N)(2+N)}\zeta_3\Bigg]S_1^2 + \left[-\frac{2P_{958}}{9N^3(1+N)^3} + \frac{472(N-1)}{9N(1+N)}S_2\right]S_1^3$$

$$-\frac{20(N-1)}{3N(1+N)}S_1^5 - \frac{(N-1)\left(-226+97N+249N^2\right)}{9N^2(1+N)^2}S_1^4 + \Bigg[\frac{P_{1018}}{3N^4(1+N)^4(2+N)}$$

$$+\frac{16\left(-1462+83N+83N^2\right)}{9N(1+N)(2+N)}S_3 + \frac{256\left(-38+N+N^2\right)}{3N(1+N)(2+N)}S_{-2,1} + \frac{192(N-1)}{N(1+N)}\zeta_3\Bigg]S_2$$

$$+\frac{64\left(-82+17N+17N^2\right)}{3N(1+N)(2+N)}S_5 + \Bigg[\frac{8P_{1020}}{3(N-1)N^2(1+N)^4(2+N)^2}$$

$$+\left(\frac{32P_{1012}}{3(N-1)N^3(1+N)^3(2+N)^2} - \frac{640(N-1)}{3N(1+N)}S_2\right)S_1 + \frac{640(N-1)}{9N(1+N)}S_1^3$$

$$-\frac{64\left(-3+10N+26N^2+6N^3\right)}{3N^2(1+N)^2(2+N)}S_1^2 + \frac{64\left(-5+N+13N^2+6N^3\right)}{3N^2(1+N)^2(2+N)}S_2$$

$$+\frac{512\left(-5+7N+7N^2\right)}{9N(1+N)(2+N)}S_3 + \frac{3072}{N(1+N)(2+N)}S_{2,1} - \frac{128(N-1)}{3N(1+N)}S_{-2,1}$$

$$+\frac{512\left(-9+2N+2N^2\right)}{N(1+N)(2+N)}\zeta_3\Bigg]S_{-2} + \Bigg[-\frac{32P_{973}}{3(N-1)N^2(1+N)^2(2+N)^2}$$

$$-\frac{256\left(-8+N+N^2\right)}{3N(1+N)(2+N)}S_1\Bigg]S_{-2}^2 + \Bigg[\frac{32P_{963}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1$$

$$-\frac{16P_{1016}}{3(N-1)N^3(1+N)^3(2+N)^2} - \frac{64\left(2+11N+11N^2\right)}{3N(1+N)(2+N)}S_1^2 - \frac{64\left(-74+N+N^2\right)}{3N(1+N)(2+N)}$$

$$\times S_2 + \frac{512\left(-1+N+N^2\right)}{N(1+N)(2+N)}S_{-2}\Bigg]S_{-3} + \Bigg[-\frac{32P_{970}}{3(N-1)N^2(1+N)^2(2+N)^2}$$

$$-\frac{512S_1}{N(1+N)(2+N)}\Bigg]S_{-4} + \frac{32\left(10+67N+67N^2\right)}{3N(1+N)(2+N)}S_{-5} - \frac{128(-4+N)(5+N)}{N(1+N)(2+N)}$$

$$\times S_{2,3} + \frac{192\left(6+N+N^2\right)}{N(1+N)(2+N)}S_{2,-3} + \frac{32\left(-98+13N+13N^2\right)}{N(1+N)(2+N)}S_{4,1}$$

$$-\frac{64\left(-2+25N+25N^2\right)}{3N(1+N)(2+N)}S_{-2,3} - \frac{64\left(2+3N+3N^2\right)}{N(1+N)(2+N)}S_{-4,1} - \frac{160(N-1)}{3N(1+N)}S_{2,2,1}$$

$$-\frac{32\left(-8-5N+19N^2+12N^3\right)}{3N^2(1+N)^2(2+N)}S_{2,1,1} - \frac{128\left(70+N+N^2\right)}{3N(1+N)(2+N)}S_{2,1,-2}$$

$$+\frac{4608}{N(1+N)(2+N)}S_{3,1,1} - \frac{128\left(22+N+N^2\right)}{3N(1+N)(2+N)}S_{-2,1,-2} - \frac{3072}{N(1+N)(2+N)}S_{-2,2,1}$$

$$-\frac{64\left(142+N+N^2\right)}{3N(1+N)(2+N)}S_{-3,1,1} - \frac{160(N-1)}{N(1+N)}S_{2,1,1,1} + \frac{128\left(146-N-N^2\right)}{3N(1+N)(2+N)}$$

$$\times S_{-2,1,1,1} + \frac{48(N-1)\left(-2+3N+3N^2\right)}{5N^2(1+N)^2}\zeta_2^2 - \frac{160\left(66+N+N^2\right)}{N(1+N)(2+N)}\zeta_5\Bigg\}$$

and

$$P_{939} = -305N^4 - 514N^3 - 19N^2 + 142N - 1416, \tag{1129}$$

$$P_{940} = -117N^4 + 14N^3 + 791N^2 + 676N - 116, \tag{1130}$$

$$P_{941} = 2N^4 + 8N^3 - 6N^2 - 35N - 20, \tag{1131}$$

$$P_{942} = 3N^4 + 15N^3 + 14N^2 + 2N + 8, \tag{1132}$$

$$P_{943} = 11N^4 - 14N^3 + 148N^2 + 191N - 48, \tag{1133}$$

$$P_{944} = 17N^4 - 20N^3 + 51N^2 + 112N - 36, \tag{1134}$$

$$P_{945} = 44N^4 + 61N^3 + 55N^2 + 74N - 6, \tag{1135}$$

$$P_{946} = 124N^4 + 233N^3 + 49N^2 - 60N + 54, \tag{1136}$$

$$P_{947} = 177N^4 + 146N^3 - 731N^2 - 492N + 324, \tag{1137}$$

$$P_{948} = 349N^4 + 680N^3 - 550N^2 - 245N + 1038, \tag{1138}$$

$$P_{949} = 368N^4 + 1036N^3 - 1145N^2 - 1741N + 954, \tag{1139}$$

$$P_{950} = 497N^4 + 646N^3 - 785N^2 + 482N + 1848, \tag{1140}$$

$$P_{951} = 6N^5 + 12N^4 + 65N^3 + 138N^2 + 225N + 130, \tag{1141}$$

$$P_{952} = 12N^5 + 36N^4 + 191N^3 + 318N^2 + 369N + 226, \tag{1142}$$

$$P_{953} = 16N^5 - 47N^3 + 19N^2 - 4N + 12, \tag{1143}$$

$$P_{954} = 19N^5 + 76N^4 + 45N^3 - 52N^2 + 20N + 36, \tag{1144}$$

$$P_{955} = 42N^5 + 15N^4 + 23N^3 - 141N^2 - 153N + 70, \tag{1145}$$

$$P_{956} = 53N^5 + 158N^4 + 9N^3 - 119N^2 + 34N - 108, \tag{1146}$$

$$P_{957} = 67N^5 + 277N^4 + 216N^3 - 145N^2 + 44N + 180, \tag{1147}$$

$$P_{958} = 127N^5 + N^4 - 421N^3 + 267N^2 - 114N - 436, \tag{1148}$$

$$P_{959} = 3N^6 - 14N^5 - 551N^4 - 838N^3 - 644N^2 - 572N - 264, \tag{1149}$$

$$P_{960} = 3N^6 + 9N^5 - N^4 + 7N^3 + 82N^2 + 20N + 24, \tag{1150}$$

$$P_{961} = 3N^6 + 9N^5 + 49N^4 + 77N^3 - 28N^2 - 50N + 84, \tag{1151}$$

$$P_{962} = 6N^6 + 24N^5 + 113N^4 + 158N^3 + 159N^2 + 112N + 4, \tag{1152}$$

$$P_{963} = 6N^6 + 36N^5 + 147N^4 + 158N^3 + 87N^2 + 78N + 64, \tag{1153}$$

$$P_{964} = 8N^6 + 27N^5 + 18N^4 - 25N^3 - 80N^2 - 44N + 24, \tag{1154}$$

$$P_{965} = 9N^6 + 71N^5 + 981N^4 + 1675N^3 + 1936N^2 + 1224N - 136, \tag{1155}$$

$$P_{966} = 9N^6 + 95N^5 + 955N^4 + 1477N^3 + 2132N^2 + 1620N - 528, \tag{1156}$$

$$P_{967} = 9N^6 + 119N^5 + 929N^4 + 1279N^3 + 2328N^2 + 2016N - 920, \tag{1157}$$

$$P_{968} = 12N^6 + 50N^5 + 565N^4 + 958N^3 + 433N^2 + 86N + 488, \tag{1158}$$

$$P_{969} = 12N^6 + 60N^5 + 261N^4 + 314N^3 + 267N^2 + 222N + 16, \tag{1159}$$

$$P_{970} = 15N^6 + 23N^5 - 101N^4 - 131N^3 - 32N^2 - 82N + 20, \tag{1160}$$

$$P_{971} = 15N^6 + 47N^5 + 107N^4 + 139N^3 + 108N^2 + 24N - 8, \tag{1161}$$

$$P_{972} = 19N^6 + 63N^5 - 28N^4 - 154N^3 - 63N^2 - 35N - 18, \tag{1162}$$

$$P_{973} = 21N^6 + 57N^5 + 29N^4 + 3N^3 + 126N^2 + 76N - 24, \tag{1163}$$

$$P_{974} = 23N^6 - 201N^5 - 955N^4 - 583N^3 + 828N^2 - 48N - 536, \tag{1164}$$

$$P_{975} = 23N^6 + 61N^5 + 93N^4 - 117N^3 - 692N^2 - 256N + 600, \tag{1165}$$

$$P_{976} = 27N^6 + 87N^5 + 179N^4 + 213N^3 + 94N^2 - 40N + 16, \tag{1166}$$

$$P_{977} = 30N^6 + 163N^5 + 788N^4 + 865N^3 + 1322N^2 + 1408N - 544, \tag{1167}$$

$$P_{978} = 33N^6 + 113N^5 + 815N^4 + 1319N^3 + 1140N^2 + 708N + 1632, \tag{1168}$$

$$P_{979} = 35N^6 + 93N^5 - 17N^4 - 97N^3 - 238N^2 - 320N + 256, \tag{1169}$$

$$P_{980} = 39N^6 + 13N^5 - 439N^4 - 421N^3 - 2156N^2 - 2412N + 1344, \tag{1170}$$

$$P_{981} = 39N^6 + 161N^5 + 45N^4 - 345N^3 + 600N^2 + 908N - 832, \tag{1171}$$

$$P_{982} = 60N^6 + 126N^5 - 491N^4 - 724N^3 - 271N^2 - 584N - 420, \tag{1172}$$

$$P_{983} = 60N^6 + 132N^5 - 367N^4 - 652N^3 - 711N^2 - 782N + 16, \tag{1173}$$

$$P_{984} = 66N^6 + 140N^5 - 337N^4 - 550N^3 - 835N^2 - 1024N + 236, \tag{1174}$$

$$P_{985} = 69N^6 + 203N^5 - 153N^4 + 21N^3 + 1536N^2 + 308N - 1408, \tag{1175}$$

$$P_{986} = 72N^6 + 148N^5 - 307N^4 - 448N^3 - 959N^2 - 1266N + 456, \tag{1176}$$

$$P_{987} = 85N^6 + 243N^5 + 443N^4 + 425N^3 + 2592N^2 + 2644N + 3072, \tag{1177}$$

$$P_{988} = 93N^6 + 39N^5 + 3465N^4 + 7681N^3 + 5118N^2 + 924N + 3416, \tag{1178}$$

$$P_{989} = 143N^6 + 231N^5 + 298N^4 + 1495N^3 - 4263N^2 - 6796N + 3708, \tag{1179}$$

$$P_{990} = 147N^6 + 453N^5 + 811N^4 + 755N^3 + 606N^2 + 500N + 760, \tag{1180}$$

$$P_{991} = 176N^6 + 390N^5 - 829N^4 - 1288N^3 - 429N^2 - 1208N - 268, \tag{1181}$$

$$P_{992} = 205N^6 + 225N^5 - 7345N^4 - 15529N^3 - 14712N^2 - 4748N - 9936, \tag{1182}$$

$$P_{993} = 213N^6 + 703N^5 + 3205N^4 + 4849N^3 + 3122N^2 + 1340N + 2120, \tag{1183}$$

$$P_{994} = 233N^6 + 591N^5 - 713N^4 - 683N^3 + 7632N^2 + 5372N - 4656, \tag{1184}$$

$$P_{995} = 263N^6 + 771N^5 - 848N^4 - 1349N^3 + 9819N^2 + 7532N - 1932, \tag{1185}$$

$$P_{996} = 274N^6 + 624N^5 - 278N^4 - 615N^3 + 625N^2 + 108N - 540, \tag{1186}$$

$$P_{997} = 434N^6 + 810N^5 + 5245N^4 + 10912N^3 + 3351N^2 - 2956N + 16764, \tag{1187}$$

$$P_{998} = 485N^6 + 975N^5 + 2077N^4 + 4777N^3 + 666N^2 - 3820N + 13848, \tag{1188}$$

$$P_{999} = 641N^6 + 1995N^5 + 3601N^4 + 5437N^3 + 11178N^2 + 6116N - 1320, \tag{1189}$$

$$P_{1000} = 764N^6 + 1896N^5 - 592N^4 - 1797N^3 + 413N^2 - 3762N - 3600, \tag{1190}$$

$$P_{1001} = 1721N^6 + 3837N^5 + 29N^4 + 909N^3 + 4838N^2 - 5394N - 5508, \tag{1191}$$

$$P_{1002} = 30N^7 + 51N^6 - 5N^5 - 173N^4 - 63N^3 - 22N^2 - 118N + 12, \tag{1192}$$

$$P_{1003} = 140N^7 + 643N^6 + 902N^5 + N^4 - 676N^3 + 94N^2 + 264N - 216, \tag{1193}$$

$$P_{1004} = 200N^7 + 1357N^6 + 2729N^5 + 1069N^4 - 481N^3 + 1174N^2 - 324N - 1080, \tag{1194}$$

$$P_{1005} = 293N^7 + 513N^6 + 2089N^5 + 4287N^4 + 11362N^3 + 9848N^2 + 3496N - 784, \tag{1195}$$

$$P_{1006} = 383N^7 + 1702N^6 + 689N^5 - 4466N^4 - 4516N^3 - 176N^2 - 312N - 336, \tag{1196}$$

$$P_{1007} = 388N^7 + 1781N^6 + 1660N^5 - 2725N^4 - 5366N^3 - 2458N^2 - 192N - 216, \tag{1197}$$

$$P_{1008} = 1067N^7 + 5191N^6 + 5663N^5 - 2933N^4 - 2176N^3 + 4672N^2 - 288N - 1008, \tag{1198}$$

$$P_{1009} = 1427N^7 + 6712N^6 + 6878N^5 - 5021N^4 - 4822N^3 + 280N^2 - 12132N - 8856, \tag{1199}$$

$$P_{1010} = 1471N^7 + 7367N^6 + 12949N^5 + 9107N^4 + 6508N^3 + 4334N^2 - 12144N - 10440, \tag{1200}$$

$$P_{1011} = -205N^8 - 904N^7 + 88N^6 + 3218N^5 + 2309N^4 + 254N^3 - 944N^2 - 2584N - 1424, \tag{1201}$$

$$P_{1012} = 13N^8 + 50N^7 + 104N^6 + 130N^5 + 127N^4 + 136N^3 + 100N^2 + 36N - 24, \tag{1202}$$

$$P_{1013} = 35N^8 + 142N^7 + 16N^6 - 424N^5 + 341N^4 + 1418N^3 + 704N^2 + 24N + 48, \tag{1203}$$

$$P_{1014} = 39N^8 + 390N^7 - 2192N^6 - 4350N^5 + 10577N^4 + 8384N^3 - 28776N^2 - 9800N + 13632, \tag{1204}$$

$$P_{1015} = 42N^8 + 141N^7 + 284N^6 + 351N^5 - 1843N^4 - 2925N^3 - 730N^2 - 1272N$$
$$-960, \tag{1205}$$

$$P_{1016} = 71N^8 + 290N^7 + 108N^6 - 624N^5 + 437N^4 + 1838N^3 + 384N^2$$
$$-248N + 48, \tag{1206}$$

$$P_{1017} = 90N^8 + 417N^7 - 155N^6 - 1779N^5 + 7189N^4 + 14142N^3 + 6376N^2 + 4776N$$
$$+3504, \tag{1207}$$

$$P_{1018} = 93N^8 + 1048N^7 + 2200N^6 + 1622N^5 + 1891N^4 + 890N^3 - 936N^2 + 376N$$
$$+624, \tag{1208}$$

$$P_{1019} = 114N^8 + 345N^7 - 222N^6 - 1074N^5 - 1960N^4 - 3335N^3 - 2280N^2 - 68N$$
$$+416, \tag{1209}$$

$$P_{1020} = 153N^8 + 725N^7 + 380N^6 - 2494N^5 - 3431N^4 - 399N^3 - 510N^2 - 1672N$$
$$-1008, \tag{1210}$$

$$P_{1021} = 191N^8 + 684N^7 - 217N^6 - 1949N^5 - 1246N^4 + 2305N^3 + 4728N^2 + 1120N$$
$$-432, \tag{1211}$$

$$P_{1022} = 240N^8 + 1089N^7 - 917N^6 - 6465N^5 + 6181N^4 + 19980N^3 + 8596N^2 + 3840N$$
$$+3168, \tag{1212}$$

$$P_{1023} = 339N^8 + 843N^7 - 2092N^6 - 2274N^5 + 1147N^4 - 14645N^3 - 26082N^2 + 116N$$
$$+6360, \tag{1213}$$

$$P_{1024} = 430N^8 + 1951N^7 - 340N^6 - 7385N^5 + 2891N^4 + 11059N^3 + 970N^2 + 8136N$$
$$+3888, \tag{1214}$$

$$P_{1025} = 1565N^8 + 5906N^7 + 3476N^6 - 7570N^5 - 4705N^4 + 152N^3 - 8328N^2 + 3888, \tag{1215}$$

$$P_{1026} = 1973N^8 + 5921N^7 + 1402N^6 - 1417N^5 + 14062N^4 + 21596N^3 + 41243N^2$$
$$+14832N - 14076, \tag{1216}$$

$$P_{1027} = 4724N^8 + 15473N^7 + 3469N^6 - 22813N^5 + 48721N^4 + 140300N^3 + 112538N^2$$
$$+15108N - 6480, \tag{1217}$$

$$P_{1028} = 19N^9 + 63N^8 + 192N^7 + 162N^6 - 513N^5 - 513N^4 + 230N^3 - 720N^2$$
$$-216N + 144, \tag{1218}$$

$$P_{1029} = 651N^9 + 3056N^8 + 1492N^7 - 12986N^6 - 24959N^5 - 15422N^4 - 6056N^3$$
$$-5680N^2 + 1296N + 2592, \tag{1219}$$

$$P_{1030} = -12043N^{10} - 52199N^9 - 68592N^8 - 22926N^7 + 115557N^6 + 458949N^5$$
$$+428806N^4 - 302176N^3 - 337584N^2 + 149040N + 140832, \tag{1220}$$

$$P_{1031} = 63N^{10} + 291N^9 + 581N^8 + 1646N^7 + 2335N^6 + 4289N^5 + 13213N^4 + 5150N^3$$
$$-12616N^2 + 552N + 5232, \tag{1221}$$

$$P_{1032} = 226N^{10} + 1061N^9 + 860N^8 - 1831N^7 - 2444N^6 + 2045N^5 + 8496N^4 + 6381N^3$$
$$-4438N^2 - 2268N + 2280, \tag{1222}$$

$$P_{1033} = 5399N^{10} + 21865N^9 + 4191N^8 - 66642N^7 - 36909N^6 + 68193N^5 + 32287N^4$$
$$+4016N^3 + 45360N^2 + 3888N - 19440, \tag{1223}$$

$$P_{1034} = 5459N^{10} + 35950N^9 + 52398N^8 - 48294N^7 - 52359N^6 + 143592N^5 + 37432N^4$$
$$-26596N^3 + 134946N^2 - 70632N - 87480, \tag{1224}$$

$$P_{1035} = 9429N^{10} + 33109N^9 - 26912N^8 - 199686N^7 - 184155N^6 + 5745N^5 + 39702N^4$$

$$+97328N^3 + 107792N^2 - 12816N - 35424, \tag{1225}$$

$$P_{1036} = 13789N^{10} + 60242N^9 + 31062N^8 - 128262N^7 - 48825N^6 + 174492N^5$$
$$+110408N^4 + 144844N^3 + 66474N^2 - 183816N - 115992, \tag{1226}$$

$$P_{1037} = 205N^{12} + 1122N^{11} - 2485N^{10} - 21156N^9 - 27453N^8 + 17874N^7 + 42893N^6$$
$$-5568N^5 - 25832N^4 + 2784N^3 + 10512N^2 + 576N - 384, \tag{1227}$$

$$P_{1038} = 254N^{12} + 1391N^{11} + 413N^{10} - 8542N^9 - 8007N^8 + 22639N^7 + 38167N^6$$
$$+10124N^5 - 9715N^4 - 4980N^3 + 2736N^2 + 3344N + 1328, \tag{1228}$$

$$P_{1039} = 1538N^{12} + 8703N^{11} + 20491N^{10} + 20280N^9 - 17340N^8 - 20277N^7$$
$$+105025N^6 + 77580N^5 - 83254N^4 + 66642N^3 + 131004N^2 - 18792N$$
$$-42768, \tag{1229}$$

$$P_{1040} = 3805N^{12} + 23880N^{11} + 31020N^{10} - 63109N^9 - 153501N^8 - 33090N^7 + 131168N^6$$
$$+144597N^5 - 48240N^4 - 172070N^3 + 49356N^2 + 44280N + 10800, \tag{1230}$$

$$P_{1041} = 4920N^{13} + 22305N^{12} + 218N^{11} - 95044N^{10} - 22165N^9 + 174177N^8 + 27618N^7$$
$$-156150N^6 - 99101N^5 - 98972N^4 - 72698N^3 + 67716N^2 + 20376N - 22032 \tag{1231}$$

$$P_{1042} = 617N^{14} + 4175N^{13} + 14349N^{12} + 30747N^{11} + 2959N^{10} - 140311N^9 - 226729N^8$$
$$-109247N^7 - 131196N^6 - 283508N^5 - 248768N^4 - 18848N^3 + 45408N^2$$
$$+6528N - 2944, \tag{1232}$$

$$P_{1043} = 32968N^{14} + 214291N^{13} + 230166N^{12} - 1212607N^{11} - 2507302N^{10} + 1955271N^9$$
$$+5076722N^8 - 7562245N^7 - 17306682N^6 - 3182438N^5 + 7389328N^4 + 339888N^3$$
$$-2572992N^2 + 515808N + 673920, \tag{1233}$$

$$P_{1044} = 34763N^{14} + 211157N^{13} + 229596N^{12} - 789770N^{11} - 1434236N^{10} + 1014978N^9$$
$$+2087956N^8 - 1947938N^7 - 3974499N^6 - 2470711N^5 + 66548N^4 + 2887812N^3$$
$$+1290168N^2 - 982368N - 702432, \tag{1234}$$

$$P_{1045} = -10701N^{15} - 42606N^{14} - 112294N^{13} - 235574N^{12} - 177548N^{11} - 133306N^{10}$$
$$-388110N^9 + 1317318N^8 + 3159337N^7 + 1521824N^6 + 1661012N^5 + 2875000N^4$$
$$-65712N^3 - 1912320N^2 + 39744N + 466560, \tag{1235}$$

$$P_{1046} = 23379N^{16} + 176637N^{15} + 336908N^{14} - 418342N^{13} - 1616541N^{12} + 367147N^{11}$$
$$+3663481N^{10} + 1241253N^9 - 2594354N^8 - 1895204N^7 + 1153767N^6$$
$$+3040589N^5 - 340384N^4 - 2307312N^3 - 1311840N^2 + 714096N + 513216, \tag{1236}$$

$$P_{1047} = 295317N^{16} + 2034216N^{15} + 4013522N^{14} - 2161486N^{13} - 8694264N^{12}$$
$$+30856978N^{11} + 88306930N^{10} + 17465238N^9 - 110878589N^8 - 41117522N^7$$
$$+85727340N^6 + 17006960N^5 - 58178704N^4 - 9306528N^3 + 13384512N^2$$
$$-1838592N - 3027456. \tag{1237}$$

$$\Delta C_{g_1,g}^{(3),d_{abc}} = \frac{d_{abc}d^{abc}N_F^2}{N_A} \left\{ -\frac{128(N-2)(3+N)P_{1048}}{45N(1+N)(2+N)}S_5 - \frac{64(N-2)(3+N)P_{1048}}{9N(1+N)(2+N)}\zeta_5 \right.$$
$$+\frac{128P_{1049}}{(N-1)N(1+N)(2+N)^2}S_3 + \frac{32P_{1052}}{45(N-1)N(1+N)(2+N)^2}$$
$$-\frac{256P_{1053}}{45N^2(1+N)^2(2+N)}S_{-2,1} - \frac{512P_{1055}}{3(N-1)N^2(1+N)^2(2+N)^2}S_4$$

$$+\frac{1024P_{1055}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}-\frac{64P_{1058}}{45(N-1)N^2(1+N)^2(2+N)^2}\zeta_3$$

$$+\left[-\frac{128\left(-4+N+N^2\right)P_{1050}}{(N-1)N^2(1+N)^2(2+N)^2}S_3-\frac{64P_{1051}}{45(N-1)N(1+N)(2+N)^2}\right.$$

$$+\frac{512P_{1056}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3+\frac{512\left(-4+N+N^2\right)}{N(1+N)(2+N)}S_4$$

$$\left.-\frac{1024\left(-4+N+N^2\right)S_{3,1}}{N(1+N)(2+N)}+\frac{512\left(-8+3N+3N^2\right)}{3N(1+N)(2+N)}S_{-2,1}\right]S_1$$

$$+\left[\frac{128\left(-19+3N+3N^2\right)}{3N(1+N)(2+N)}+\frac{256\left(-4+N+N^2\right)}{N(1+N)(2+N)}S_3-\frac{512\left(-4+N+N^2\right)}{N(1+N)(2+N)}\right.$$

$$\left.\times\zeta_3\right]S_1^2-\frac{768\left(-4+N+N^2\right)}{N(1+N)(2+N)}S_2S_3+\left[-\frac{64P_{1054}}{45(N-1)N(1+N)(2+N)^2}\right.$$

$$+\frac{256P_{1057}}{45(N-1)N^2(1+N)^2(2+N)^2}S_1-\frac{256\left(-8+3N+3N^2\right)}{3N(1+N)(2+N)}S_1^2$$

$$\left.-\frac{256}{45}(N-1)\left(-21+N+N^2\right)[S_3+S_{-2,1}+\zeta_3]\right]S_{-2}-\frac{128}{2+N}S_{-2}^2$$

$$+\left[\frac{128P_{1053}}{45N^2(1+N)^2(2+N)}-\frac{256\left(-8+3N+3N^2\right)}{3N(1+N)(2+N)}S_1\right]S_{-3}$$

$$-\frac{256\left(-4+3N+3N^2\right)}{3N(1+N)(2+N)}S_{-4}-\frac{128}{45}(N-1)\left(-21+N+N^2\right)S_{-5}$$

$$+\frac{768\left(-4+N+N^2\right)}{N(1+N)(2+N)}[S_{2,3}-S_{4,1}]+\frac{256\left(-8+3N+3N^2\right)}{3N(1+N)(2+N)}S_{-2,2}$$

$$+\frac{256}{45}(N-1)\left(-21+N+N^2\right)[S_{-2,3}+S_{-2,1,-2}]+\frac{256\left(-8+3N+3N^2\right)}{3N(1+N)(2+N)}S_{-3,1}$$

$$+\frac{1536\left(-4+N+N^2\right)}{N(1+N)(2+N)}S_{3,1,1}-\frac{512\left(-8+3N+3N^2\right)}{3N(1+N)(2+N)}S_{-2,1,1}\Bigg\} \qquad (1238)$$

and

$$P_{1048} = N^4+2N^3-16N^2-17N+120, \qquad (1239)$$

$$P_{1049} = 5N^4+10N^3-16N^2-21N+10, \qquad (1240)$$

$$P_{1050} = 5N^4+10N^3+3N^2-2N-4, \qquad (1241)$$

$$P_{1051} = 46N^4+92N^3+1049N^2+1003N-3342, \qquad (1242)$$

$$P_{1052} = 65N^4+130N^3-593N^2-658N+5232, \qquad (1243)$$

$$P_{1053} = N^6+3N^5+24N^4+43N^3-142N^2-163N-156, \qquad (1244)$$

$$P_{1054} = 2N^6+6N^5+315N^4+620N^3-1391N^2-1700N+60, \qquad (1245)$$

$$P_{1055} = 3N^6+9N^5-4N^4-23N^3-14N^2-N+12, \qquad (1246)$$

$$P_{1056} = 9N^6+27N^5-13N^4-71N^3-26N^2+14N+24, \qquad (1247)$$

$$P_{1057} = 45N^6+135N^5-140N^4-505N^3+35N^2+310N+156, \qquad (1248)$$

$$P_{1058} = 2N^8+8N^7+815N^6+2417N^5-1509N^4-7037N^3-1780N^2$$

$$+2140N + 624. \tag{1249}$$

The above Wilson coefficients complete the massless parts of the polarized single and two mass heavy flavor Wilson coefficients still missing in Refs. [49–53].

8   The small- and large x expansions of the Wilson coefficients

We will represent the small x and large x behaviour of the different Wilson coefficients first in Mellin N space, where the former corresponds to the expansion around $N = 0$ and $N = 1$, while the latter corresponds to the expansion in the limit $N \to \infty$. Finally, we also consider the large N_F limit.

The non–singlet Wilson coefficients receive their leading small x contributions from the expansion around $N = 0$, retaining the pole terms. The singlet ones also receive contributions from the pole terms at $N = 1$. This is due to the Mellin transforms

$$\frac{1}{(N-1)^k} = \frac{(-1)^{k-1}}{(k-1)!} \mathbf{M}\left[\frac{\ln^{k-1}(x)}{x}\right](N) \tag{1250}$$

$$\frac{1}{N^k} = \frac{(-1)^{k-1}}{(k-1)!} \mathbf{M}[\ln^{k-1}(x)](N). \tag{1251}$$

In the large x region we will retain all terms up to $\propto 1/N$ in the asymptotic expansion. Here, terms of the kind

$$L_a^l \equiv (\ln(N) + \gamma_E)^l \tag{1252}$$

contribute, which we will replace by

$$\ln(N) + \gamma_E = S_1(N) - \frac{1}{2N} + O\left(\frac{1}{N^2}\right), \tag{1253}$$

to derive in a more direct way the inverse Mellin transforms. One obtains asymptotically up to terms of $O(L_a^k/N^2)$, $k \geq 0$, and smaller

$$\mathbf{M}^{-1}[S_1(N)](x) \simeq -\frac{1}{1-x} + 1, \tag{1254}$$

$$\mathbf{M}^{-1}[S_1^2(N)](x) \simeq \frac{2\ln(1-x)}{1-x} - 2\ln(1-x) - \zeta_2\delta(1-x) + 1, \tag{1255}$$

$$\mathbf{M}^{-1}[S_1^3(N)](x) \simeq -\frac{3\ln^2(1-x)}{1-x} + \frac{3\zeta_2}{1-x} + 3\ln^2(1-x) - 3\ln(1-x) - 2\zeta_3\delta(1-x)$$
$$-3\zeta_2, \tag{1256}$$

$$\mathbf{M}^{-1}[S_1^4(N)](x) \simeq \frac{4\ln^3(1-x)}{1-x} - \frac{12\zeta_2\ln(1-x)}{1-x} + \frac{8\zeta_3}{1-x} - 4\ln^3(1-x) + 6\ln^2(1-x)$$
$$+12\zeta_2\ln(1-x) + \frac{3}{5}\zeta_2^2\delta(1-x) - 6\zeta_2 - 8\zeta_3, \tag{1257}$$

$$\mathbf{M}^{-1}[S_1^5(N)](x) \simeq -\frac{5\ln^4(1-x)}{1-x} + \frac{30\zeta_2\ln^2(1-x)}{1-x} - \frac{40\zeta_3\ln(1-x)}{1-x} - \frac{3\zeta_2^2}{1-x}$$
$$+5\ln^4(1-x) - 10\ln^3(1-x) - 30\zeta_2\ln^2(1-x) + (30\zeta_2 + 40\zeta_3)$$
$$\times \ln(1-x) + (20\zeta_2\zeta_3 - 24\zeta_5)\delta(1-x) + 3\zeta_2^2 - 20\zeta_3, \tag{1258}$$

$$\mathbf{M}^{-1}[S_1^6(N)](x) \simeq \frac{6\ln^5(1-x)}{1-x} - \frac{60\zeta_2 \ln^3(1-x)}{1-x} + \frac{120\zeta_3 \ln^2(1-x)}{1-x} + \frac{18\zeta_2^2 \ln(1-x)}{1-x}$$

$$-(120\zeta_2\zeta_3 - 144\zeta_5)\frac{1}{1-x} - 6\ln^5(1-x) + 15\ln^4(1-x)$$

$$+60\zeta_2 \ln^3(1-x) - (90\zeta_2 + 120\zeta_3)\ln^2(1-x) + (120\zeta_3 - 18\zeta_2^2)\ln(1-x)$$

$$+\left(40\zeta_3^2 - \frac{45}{7}\zeta_2^3\right)\delta(1-x) + 9\zeta_2^2 + 120\zeta_2\zeta_3 - 144\zeta_5, \tag{1259}$$

and

$$\mathbf{M}^{-1}\left[\frac{S_1(N)}{N}\right](x) \simeq -\ln(1-x), \tag{1260}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^2(N)}{N}\right](x) \simeq \ln^2(1-x) - \zeta_2, \tag{1261}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^3(N)}{N}\right](x) \simeq -\ln^3(1-x) + 3\zeta_2 \ln(1-x) - 2\zeta_3, \tag{1262}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^4(N)}{N}\right](x) \simeq \ln^4(1-x) - 6\zeta_2 \ln^2(1-x) + 8\zeta_3 \ln(1-x) + \frac{3}{5}\zeta_2^2, \tag{1263}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^5(N)}{N}\right](x) \simeq -\ln^5(1-x) + 10\zeta_2 \ln^3(1-x) - 20\zeta_3 \ln^2(1-x) - 3\zeta_2^2 \ln(1-x)$$

$$+20\zeta_2\zeta_3 - 24\zeta_5, \tag{1264}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^6(N)}{N}\right](x) \simeq \ln^6(1-x) - 15\zeta_2 \ln^4(1-x) + 40\zeta_3 \ln^3(1-x) + 9\zeta_2^2 \ln^2(1-x)$$

$$+(144\zeta_5 - 120\zeta_2\zeta_3)\ln(1-x) - \frac{45}{7}\zeta_2^3 + 40\zeta_3^2. \tag{1265}$$

We will now present the asymptotic behaviour of the Wilson coefficients in the small x and large x region. For the pure singlet and gluonic Wilson coefficients of the structure function $g_1(x, Q^2)$ we refer to the Larin scheme.[4]

8.1 The small x limit

In this section we will show the small x expansion of the Wilson coefficients calculated above. The non–singlet unpolarized Wilson coefficients have leading singularities at $N = 0$, $\propto \ln^k(x)$ and the unpolarized pure singlet and gluonic Wilson coefficients have their leading singularity at $N = 1$, i.e. terms as $\ln(x)/x$ and $1/x$. Here we also list the logarithmic contributions up to the constant terms. One obtains

$$C_{F_2,q}^{(1),\mathrm{NS}} \simeq C_F[3 - 2\ln(x)], \tag{1266}$$

$$C_{F_2,q}^{(2),\mathrm{NS}} \simeq C_F\left\{T_F N_F\left[\frac{10}{3}\ln^2(x) + 8\ln(x) + \frac{178}{27} - \frac{16}{3}\zeta_2\right] + C_A\left[-\frac{55}{6}\ln^2(x) - 32\ln(x)\right.\right.$$

$$\left.\left. -\frac{1693}{54} + \frac{44}{3}\zeta_2 - 12\zeta_3\right]\right\} + C_F^2\left\{-\frac{5}{3}\ln^3(x) + 9\ln^2(x) + (17 + 24\zeta_2)\ln(x) - 8\zeta_2\right.$$

---

[4]To one- and two-loop order it is known, that the gluonic Wilson coefficients are the same in the Larin and M-scheme [26].

$$+ \frac{3}{2} + 56\zeta_3 \Bigg\}, \tag{1267}$$

$$
\begin{aligned}
C_{F_2,q}^{(3),\text{NS}} \simeq\ & -\frac{1}{2}C_F^3 \ln^5(x) + \left[\frac{67}{12}C_F^3 + C_F^2\left(-\frac{1001}{108}C_A + \frac{91}{27}T_F N_F\right)\right]\ln^4(x) \\
& + \left[C_F\left(\frac{2024}{81}C_A T_F N_F - \frac{368}{81}T_F^2 N_F^2 + \left(-\frac{2783}{81} + 20\zeta_2\right)C_A^2\right) + C_F^2\left(\frac{10}{27}N_F T_F\right.\right. \\
& \left.\left. + \left(-\frac{835}{54} - 64\zeta_2\right)C_A\right) + C_F^3\left(5 + \frac{262}{3}\zeta_2\right)\right]\ln^3(x) + \left[C_F\left(-\frac{1984}{81}T_F^2 N_F^2\right.\right. \\
& + \left(\frac{14392}{81} - 16\zeta_2\right)C_A T_F N_F + \left(-\frac{23062}{81} + 84\zeta_2\right)C_A^2\right) + C_F^2\left(T_F N_F\left(-\frac{2630}{81}\right.\right. \\
& \left.\left. - \frac{532}{9}\zeta_2\right) + C_A\left(\frac{17315}{162} - \frac{265\zeta_2}{9} - 64\zeta_3\right)\right) + \left(-\frac{113}{6} + \frac{299}{3}\zeta_2\right. \\
& \left.\left. + \frac{646}{3}\zeta_3\right)C_F^3\right]\ln^2(x) + \left[C_F\left[T_F^2 N_F^2\left(-\frac{4816}{81} + \frac{64}{3}\zeta_2\right) + C_A^2\left(-\frac{78338}{81} + \frac{3058}{9}\zeta_2\right.\right.\right. \\
& \left.\left. + 32\zeta_3 - 24\zeta_2^2\right) + C_A T_F N_F\left(\frac{14152}{27} - \frac{1376}{9}\zeta_2 + \frac{256}{3}\zeta_3\right)\right] + C_F^2\left[T_F N_F\left(\right.\right. \\
& \left.\left. - \frac{2999}{81} - \frac{964}{9}\zeta_2 - 264\zeta_3\right) + C_A\left(\frac{106801}{324} + \frac{599}{9}\zeta_2 + \frac{418}{3}\zeta_3 + \frac{656}{15}\zeta_2^2\right)\right] \\
& \left. + C_F^3\left(\frac{1619}{12} + \frac{764}{3}\zeta_2 + 154\zeta_3 - \frac{556}{3}\zeta_2^2\right)\right]\ln(x) \\
& + \left[C_F^2\left(C_A\left(\frac{193961}{648} + \frac{6014}{81}\zeta_2 + \frac{13189}{27}\zeta_3 - \frac{3434}{45}\zeta_2^2 + 520\zeta_2\zeta_3 + \frac{1970}{3}\zeta_5\right)\right.\right. \\
& \left. + T_F N_F\left(-\frac{2881}{162} + \frac{944}{81}\zeta_2 - \frac{11804}{27}\zeta_3 - \frac{848}{45}\zeta_2^2\right)\right) + C_F\left(C_A^2\left(-\frac{1779023}{1458}\right.\right. \\
& + \frac{14917}{27}\zeta_2 - \frac{1960}{27}\zeta_3 - \frac{148}{3}\zeta_2^2 - \frac{436}{3}\zeta_2\zeta_3 - \frac{152}{3}\zeta_5\right) + T_F^2 N_F^2\left(-\frac{44680}{729}\right. \\
& \left.\left. + \frac{928\zeta_2}{27} + \frac{128\zeta_3}{27}\right) + C_A T_F N_F\left(\frac{448438}{729} - \frac{9040}{27}\zeta_2 + \frac{704}{15}\zeta_2^2 + \frac{2824}{27}\zeta_3\right)\right) \\
& \left. + C_F^3\left(\frac{5603}{24} + \frac{533}{3}\zeta_2 + \frac{1730}{3}\zeta_3 - \frac{68}{15}\zeta_2^2 - 904\zeta_2\zeta_3 - 872\zeta_5\right)\right], \tag{1268}
\end{aligned}
$$

$$C_{F_2,q}^{(3,d_{abc})} \simeq\ 0, \tag{1269}$$

$$
\begin{aligned}
C_{F_2,q}^{(2),\text{PS}} \simeq\ & C_F T_F N_F\left[\frac{1}{x}\left(\frac{688}{27} - \frac{32}{3}\zeta_2\right) + \frac{20}{3}\ln^3(x) - 2\ln^2(x) + \left(112 - 32\zeta_2\right)\ln(x)\right. \\
& \left. + 12 - 16\zeta_3\right], \tag{1270}
\end{aligned}
$$

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$$
\begin{aligned}
C_{F2,q}^{(3),\mathrm{PS}} \simeq\ & C_A C_F T_F N_F \left[-\frac{78976}{243} + \frac{832}{9}\zeta_2 - \frac{256}{9}\zeta_3 \right] \frac{\ln(x)}{x} + \left\{ C_F \left[T_F^2 N_F^2 \left(\frac{88448}{729} - \frac{128}{9}\zeta_2 \right. \right. \right. \\
& \left. + \frac{512}{27}\zeta_3 \right) + C_A T_F N_F \left(-\frac{1942568}{729} + \frac{30080}{81}\zeta_2 + \frac{1504}{9}\zeta_3 + \frac{7936}{45}\zeta_2^2 \right) \Bigg] \\
& + C_F^2 T_F N_F \left(\frac{2180}{27} - 32\zeta_2 - \frac{1600}{9}\zeta_3 + \frac{384}{5}\zeta_2^2 \right) \Bigg\} \frac{1}{x} + \left[-4 C_F C_A T_F N_F \right. \\
& + 4 C_F^2 T_F N_F \Bigg] \ln^5(x) + \left[-\frac{110}{9} C_F^2 T_F N_F + C_F \left(\frac{854}{27} C_A T_F N_F - \frac{184}{27} T_F^2 N_F^2 \right) \right] \\
& \times \ln^4(x) + \left[C_F \left(-\frac{3568}{81} T_F^2 N_F^2 + C_A T_F N_F \left(-\frac{16984}{81} + \frac{320}{9}\zeta_2 \right) \right) \right. \\
& + C_F^2 T_F N_F \left(\frac{572}{3} - \frac{1328}{9}\zeta_2 \right) \Bigg] \ln^3(x) + \left[C_F \left(T_F^2 N_F^2 \left(-\frac{22912}{81} + \frac{64}{9}\zeta_2 \right) \right. \right. \\
& \left. + C_A T_F N_F \left(\frac{101192}{81} - \frac{920}{9}\zeta_2 - \frac{1024}{3}\zeta_3 \right) \right) + C_F^2 T_F N_F \left(-\frac{218}{3} - \frac{512}{3}\zeta_2 \right. \\
& \left. - \frac{640}{3}\zeta_3 \right) \Bigg] \ln^2(x) + \left[C_F \left(C_A T_F N_F \left(-\frac{304304}{243} + \frac{30872}{27}\zeta_2 - \frac{640}{9}\zeta_3 - \frac{752}{5}\zeta_2^2 \right) \right. \right. \\
& + T_F^2 N_F^2 \left(-\frac{198560}{243} + \frac{3392}{27}\zeta_2 - \frac{64}{9}\zeta_3 \right) \Bigg) + C_F^2 T_F N_F \left(692 - 2200\zeta_2 - \frac{160}{3}\zeta_3 \right. \\
& \left. + \frac{2896}{15}\zeta_2^2 \right) \Bigg] \ln(x) + C_F \left[C_A T_F N_F \left(\frac{1102604}{243} - \frac{71896}{81}\zeta_2 - \frac{32864}{27}\zeta_3 + \frac{1856}{15}\zeta_2^2 \right. \right. \\
& \left. + 592\zeta_2\zeta_3 - 1240\zeta_5 \right) + T_F^2 N_F^2 \left(-\frac{225376}{243} + \frac{6176}{81}\zeta_2 + \frac{1312}{27}\zeta_3 + \frac{224}{15}\zeta_2^2 \right) \Bigg] \\
& + C_F^2 T_F N_F \left(-248 - \frac{4936}{3}\zeta_2 - 880\zeta_3 + \frac{736}{5}\zeta_2^2 + 288\zeta_2\zeta_3 + 352\zeta_5 \right), \quad\quad (1271)
\end{aligned}
$$

$$
C_{F2,g}^{(1)} \simeq -4 T_F N_F (\ln(x) + 1), \quad\quad (1272)
$$

$$
\begin{aligned}
C_{F2,g}^{(2)} \simeq\ & C_F T_F N_F \left[-\frac{10}{3}\ln^3(x) - 3\ln^2(x) - 32(1 - \zeta_2)\ln(x) - 86 + 32\zeta_2 + 64\zeta_3 \right] \\
& + C_A T_F N_F \left[\frac{1}{x}\left(\frac{688}{27} - \frac{32}{3}\zeta_2 \right) + \frac{20}{3}\ln^3(x) - 2\ln^2(x) + (116 - 16\zeta_2)\ln(x) + 30 \right. \\
& + 16\zeta_2 + 8\zeta_3 \Bigg], \quad\quad (1273)
\end{aligned}
$$

$$
\begin{aligned}
C_{F2,g}^{(3),a} \simeq\ & C_A^2 T_F N_F \left[-\frac{78976}{243} + \frac{832}{9}\zeta_2 - \frac{256}{9}\zeta_3 \right] \frac{\ln(x)}{x} + \left\{ C_F \left[C_A T_F N_F \left(\frac{2180}{27} - 32\zeta_2 \right. \right. \right. \\
& \left. - \frac{1600}{9}\zeta_3 + \frac{384}{5}\zeta_2^2 \right) + T_F^2 N_F^2 \left(\frac{181472}{729} - \frac{2048}{27}\zeta_2 + \frac{512}{27}\zeta_3 \right) \Bigg]
\end{aligned}
$$

$$
+C_A T_F^2 N_F^2 \left(-\frac{2288}{729} + \frac{640}{27}\zeta_2 + \frac{256}{27}\zeta_3 \right) + C_A^2 T_F N_F \left(-\frac{2004664}{729} + \frac{32192}{81}\zeta_2 \right.
$$

$$
\left. \left. + \frac{7936}{45}\zeta_2^2 + \frac{4384}{27}\zeta_3 \right) \right\} \frac{1}{x} + \left[-4C_A^2 T_F N_F - C_F^2 T_F N_F + C_F \left(2C_A T_F N_F \right. \right.
$$

$$
\left. \left. + 4T_F^2 N_F^2 \right) \right] \ln^5(x) + \left[\frac{359}{27}C_A^2 T_F N_F + \frac{11}{9}C_F^2 T_F N_F - \frac{4}{27}C_A T_F^2 N_F^2 \right.
$$

$$
\left. + C_F \left(-\frac{409}{27}C_A T_F N_F + \frac{818}{27}T_F^2 N_F^2 \right) \right] \ln^4(x) + \left[-\frac{1064}{81}C_A T_F^2 N_F^2 + C_F \left(\right. \right.
$$

$$
\left. C_A T_F N_F \left(\frac{4589}{81} - \frac{656}{9}\zeta_2 \right) + T_F^2 N_F^2 \left(\frac{35564}{81} - \frac{176}{3}\zeta_2 \right) \right) + C_A^2 T_F N_F \left(\right.
$$

$$
\left. -\frac{27338}{81} + \frac{56}{3}\zeta_2 \right) + C_F^2 T_F N_F \left(-\frac{178}{9} + \frac{524}{9}\zeta_2 \right) \Big] \ln^3(x) + \left[C_F \left(C_A T_F N_F \left(\right. \right. \right.
$$

$$
\left. \left. -\frac{20815}{81} + \frac{284}{3}\zeta_2 - \frac{320}{3}\zeta_3 \right) + T_F^2 N_F^2 \left(\frac{156500}{81} - \frac{872}{3}\zeta_2 - \frac{304}{3}\zeta_3 \right) \right)
$$

$$
+C_A T_F^2 N_F^2 \left(-\frac{1024}{81} - \frac{16}{3}\zeta_2 \right) + C_A^2 T_F N_F \left(\frac{28580}{81} - 32\zeta_2 - \frac{1136}{3}\zeta_3 \right)
$$

$$
\left. + C_F^2 T_F N_F \left(-88 + \frac{284}{3}\zeta_2 + 284\zeta_3 \right) \right] \ln^2(x) + \left[C_F \left(T_F^2 N_F^2 \left(\frac{1683904}{243} \right. \right. \right.
$$

$$
\left. -\frac{17632}{9}\zeta_2 - \frac{5072}{9}\zeta_3 + \frac{224}{5}\zeta_2^2 \right) + C_A T_F N_F \left(-\frac{147806}{243} - \frac{1516}{9}\zeta_2 + \frac{2632}{9}\zeta_3 \right.
$$

$$
\left. \left. -\frac{2176}{15}\zeta_2^2 \right) \right) + C_A^2 T_F N_F \left(-\frac{935806}{243} + \frac{10912}{9}\zeta_2 - \frac{2432}{15}\zeta_2^2 - \frac{832}{9}\zeta_3 \right)
$$

$$
+C_A T_F^2 N_F^2 \left(-\frac{53512}{243} + \frac{112}{9}\zeta_2 - \frac{256}{9}\zeta_3 \right) + C_F^2 T_F N_F \left(\frac{368}{3} + \frac{550}{3}\zeta_2 + 600\zeta_3 \right.
$$

$$
\left. -\frac{56}{15}\zeta_2^2 \right) \Big] \ln(x) + C_F \left[T_F^2 N_F^2 \left(\frac{2179673}{243} - \frac{226144}{81}\zeta_2 - \frac{41744}{27}\zeta_3 - \frac{3632}{45}\zeta_2^2 \right. \right.
$$

$$
\left. +128\zeta_2\zeta_3 - 96\zeta_5 \right) + C_A T_F N_F \left(-\frac{2310157}{972} + \frac{91424}{81}\zeta_2 + \frac{1840}{27}\zeta_3 - \frac{13064}{45}\zeta_2^2 \right.
$$

$$
\left. \left. -352\zeta_2\zeta_3 + \frac{1852}{3}\zeta_5 \right) \right] + C_A^2 T_F N_F \left[\frac{359827}{243} - \frac{4138}{9}\zeta_2 - \frac{7348}{9}\zeta_3 + \frac{908}{15}\zeta_2^2 \right.
$$

$$
\left. +\frac{1288}{3}\zeta_2\zeta_3 - \frac{5224}{3}\zeta_5 \right] + C_F^2 T_F N_F \left[\frac{1925}{4} - \frac{572}{3}\zeta_2 + \frac{4478}{3}\zeta_3 + \frac{2176}{15}\zeta_2^2 \right.
$$

$$
\left. -320\zeta_2\zeta_3 - 248\zeta_5 \right] + C_A T_F^2 N_F^2 \left[-\frac{52712}{243} - \frac{400}{9}\zeta_2 - \frac{1072}{9}\zeta_3 + \frac{224}{15}\zeta_2^2 \right], \quad (1274)
$$

$$
C_{F_2,g}^{(3),d_{abc}} \simeq 0, \quad\quad (1275)
$$

$$
\begin{align}
(1276) \\
C_{F_L,q}^{(1),\text{NS}} &\simeq 0, \tag{1277} \\
C_{F_L,q}^{(2),\text{NS}} &\simeq C_F\left[-C_A\frac{44}{3}+T_FN_F\frac{16}{3}\right]-4C_F^2\left[2\ln(x)-3\right], \tag{1278}
\end{align}
$$

$$
\begin{align}
C_{F_L,q}^{(3),\text{NS}} &\simeq -\frac{20}{3}C_F^3\ln^3(x)+\left[32C_F^3-66C_F^2C_A+24C_F^2T_FN_F\right]\ln^2(x)+\left[C_F\left(\frac{704}{9}C_AT_FN_F\right.\right. \\
&\quad \left.-\frac{128}{9}T_F^2N_F^2+\left(-\frac{968}{9}+120\zeta_2\right)C_A^2\right)+C_F^2\left(\frac{448}{9}T_FN_F-\left(\frac{1832}{9}+384\zeta_2\right)C_A\right) \\
&\quad \left.+\left(168+416\zeta_2\right)C_F^3\right]\ln(x)+C_F\left[-\frac{1216}{27}T_F^2N_F^2+\left(\frac{8368}{27}-32\zeta_2\right)C_AT_FN_F\right. \\
&\quad \left.+\left(-\frac{13060}{27}+244\zeta_2\right)C_A^2\right]+C_F^2\left(\left(-\frac{2288}{27}-\frac{64}{3}\zeta_2\right)T_FN_F+\left(\frac{5800}{27}-\frac{1696}{3}\zeta_2\right.\right. \\
&\quad \left.\left.-96\zeta_3\right)C_A\right)+\left(608\zeta_2+288\zeta_3\right)C_F^3, \tag{1279}
\end{align}
$$

$$
\begin{align}
(1280) \\
C_{F_L,q}^{(3),d_{abc}} &\simeq 0, \tag{1281} \\
C_{F_L,q}^{(2),\text{PS}} &\simeq 32C_FT_FN_F\left[-\frac{1}{9x}+\ln(x)\right], \tag{1282}
\end{align}
$$

$$
\begin{align}
C_{F_L,q}^{(3),\text{PS}} &\simeq C_F\left\{T_F^2N_F^2\left[\frac{1}{x}\left(\frac{13568}{81}-\frac{256}{9}\zeta_2\right)-\frac{128}{3}\ln^2(x)-\frac{2048}{9}\ln(x)-\frac{11264}{27}\right]\right. \\
&\quad +C_AT_FN_F\left[\frac{1}{x}\left(-\frac{28768}{27}+224\zeta_2+\frac{640}{3}\zeta_3\right)-\frac{160}{3}\ln^3(x)+\frac{496}{3}\ln^2(x)\right. \\
&\quad \left.+\left(-\frac{800}{9}+\frac{1}{x}\left(-\frac{4352}{27}+\frac{128}{3}\zeta_2\right)+224\zeta_2\right)\ln(x)+\frac{35200}{27}+64\zeta_2-704\zeta_3\right]\Bigg\} \\
&\quad +C_F^2T_FN_F\left[\frac{1}{x}\left(\frac{3584}{27}-\frac{64}{3}\zeta_2-\frac{256}{3}\zeta_3\right)-576\zeta_2+\left(160-512\zeta_2\right)\ln(x)\right. \\
&\quad \left.-16\ln^2(x)+\frac{160}{3}\ln^3(x)\right], \tag{1283}
\end{align}
$$

$$
\begin{align}
C_{F_L,g}^{(1)} &\simeq 0, \tag{1284} \\
C_{F_L,g}^{(2)} &\simeq 32C_AT_FN_F\left[-\frac{1}{9x}+\ln(x)\right]-16C_FT_FN_F(\ln(x)+1), \tag{1285}
\end{align}
$$

$$
\begin{align}
C_{F_L,g}^{(3),a} &\simeq C_A^2T_FN_F\left[-\frac{4352}{27}+\frac{128}{3}\zeta_2\right]\frac{\ln(x)}{x}+\left\{C_F\left[T_F^2N_F^2\left(\frac{7744}{81}-\frac{256}{9}\zeta_2\right)+C_AT_FN_F\left(\right.\right.\right. \\
&\quad \left.\left.\left.\frac{3584}{27}-\frac{64}{3}\zeta_2-\frac{256}{3}\zeta_3\right)\right]+C_AT_F^2N_F^2\left(\frac{3232}{27}-\frac{128}{9}\zeta_2\right)+C_A^2T_FN_F\left(-\frac{88192}{81}\right.\right.
\end{align}
$$

$$+ \frac{2080}{9}\zeta_2 + \frac{640}{3}\zeta_3 \Big) \Big\} \frac{1}{x} + \Big[-\frac{160}{3}C_A^2 T_F N_F - \frac{40}{3}C_F^2 T_F N_F + C_F \Big(\frac{80}{3}C_A T_F N_F$$

$$+ \frac{160}{3}T_F^2 N_F^2 \Big) \Big] \ln^3(x) + \Big[\frac{56}{3}C_A^2 T_F N_F - 20 C_F^2 T_F N_F + \frac{32}{3}C_A T_F^2 N_F^2$$

$$+ C_F \Big(-\frac{152}{3}C_A T_F N_F + \frac{928}{3}T_F^2 N_F^2 \Big) \Big] \ln^2(x) + \Big[-\frac{320}{9}C_A T_F^2 N_F^2 + C_F \Big($$

$$C_A T_F N_F \Big(-\frac{616}{9} - 288\zeta_2 \Big) + T_F^2 N_F^2 \Big(\frac{13664}{9} - 256\zeta_2 \Big) \Big) + C_F^2 T_F N_F \Big(32$$

$$+192\zeta_2 \Big) + C_A^2 T_F N_F \Big(-\frac{7280}{9} + 240\zeta_2 \Big) \Big] \ln(x) + C_F \Big[T_F^2 N_F^2 \Big(\frac{20864}{9}$$

$$-\frac{1664}{3}\zeta_2 - 128\zeta_3 \Big) + C_A T_F N_F \Big(-\frac{2776}{9} - \frac{512}{3}\zeta_2 + 64\zeta_3 \Big) \Big] + C_A^2 T_F N_F \Big($$

$$\frac{5168}{27} + 200\zeta_2 - 704\zeta_3 \Big) + C_F^2 T_F N_F \Big(-24 + 256\zeta_2 + 384\zeta_3 \Big)$$

$$-\frac{3520}{27}C_A T_F^2 N_F^2, \tag{1286}$$

$$C_{F_L,g}^{(3),d_{abc}} \simeq 0, \tag{1287}$$

$$C_{F_3,q}^{(1),\mathrm{NS}} \simeq C_F\big[1 - 2\log(x)\big], \tag{1288}$$

$$C_{F_3,q}^{(2),\mathrm{NS}} \simeq C_F \Big\{ C_A \Big[-2\ln^3(x) - \frac{103}{6}\ln^2(x) + \Big(-\frac{122}{3} + 8\zeta_2 \Big)\ln(x) - \frac{1327}{54} + \frac{80}{3}\zeta_2$$

$$+8\zeta_3 \Big] + T_F N_F \Big[\frac{10}{3}\ln^2(x) + \frac{40}{3}\ln(x) + \frac{262}{27} - \frac{16}{3}\zeta_2 \Big] \Big\} + C_F^2 \Big[\frac{7}{3}\ln^3(x)$$

$$+21\ln^2(x) + (21 + 8\zeta_2)\ln(x) - \frac{59}{2} - 20\zeta_2 + 16\zeta_3 \Big], \tag{1289}$$

$$C_{F_3,q}^{(3),\mathrm{NS}} \simeq C_F^2 \Big\{ T_F N_F \Big[-\frac{31}{9}\ln^4(x) - \frac{4358}{81}\ln^3(x) + \Big(-\frac{5374}{27} - \frac{124}{3}\zeta_2 \Big)\ln^2(x) + \Big(-\frac{271}{81}$$

$$-\frac{1436}{27}\zeta_2 - \frac{2056}{9}\zeta_3 \Big)\ln(x) + \frac{4123}{54} + \frac{8344}{81}\zeta_2 + \frac{752}{9}\zeta_2^2 - \frac{9884}{27}\zeta_3 \Big]$$

$$+C_A \Big[-\frac{29}{15}\ln^5(x) + \frac{43}{36}\ln^4(x) + \Big(\frac{24245}{162} + \frac{220}{3}\zeta_2 \Big)\ln^3(x) + \Big(\frac{13297}{54} + 401\zeta_2$$

$$+328\zeta_3 \Big)\ln^2(x) + \Big(-\frac{404119}{324} + \frac{16273}{27}\zeta_2 + \frac{9206}{9}\zeta_3 + \frac{3416}{15}\zeta_2^2 \Big)\ln(x) - \frac{496687}{216}$$

$$+\frac{54412}{81}\zeta_2 + \frac{1252}{45}\zeta_2^2 + \Big(-\frac{1865}{27} - 648\zeta_2 \Big)\zeta_3 + \frac{5306}{3}\zeta_5 \Big] \Big\}$$

$$+C_F \Big\{ T_F^2 N_F^2 \Big[-\frac{368}{81}\ln^3(x) - \frac{2848}{81}\ln^2(x) + \Big(-\frac{6736}{81} + \frac{64}{3}\zeta_2 \Big)\ln(x) - \frac{35896}{729}$$

$$
+\frac{1312}{27}\zeta_2 + \frac{128}{27}\zeta_3 \Bigg] + C_A T_F N_F \Bigg[\frac{92}{27}\ln^4(x) + \frac{1544}{27}\ln^3(x) + \left(\frac{8432}{27} - \frac{224}{9}\zeta_2 \right)
$$

$$
\times \ln^2(x) + \left(\frac{52168}{81} - \frac{6224}{27}\zeta_2 + \frac{608}{9}\zeta_3 \right)\ln(x) + \frac{433462}{729} - \frac{37480}{81}\zeta_2 + \frac{1288}{27}\zeta_3
$$

$$
-\frac{64}{15}\zeta_2^2 \Bigg] + C_A^2 \Bigg[\frac{2}{5}\ln^5(x) - \frac{166}{27}\ln^4(x) + \left(-\frac{9799}{81} - \frac{64}{9}\zeta_2 \right)\ln^3(x) + \Bigg(-\frac{38642}{81}
$$

$$
+\frac{226}{9}\zeta_2 - \frac{212}{3}\zeta_3 \Bigg)\ln^2(x) + \left(-\frac{53650}{81} + \frac{11260}{27}\zeta_2 - \frac{416}{5}\zeta_2^2 - \frac{1696}{9}\zeta_3 \right)\ln(x)
$$

$$
-\frac{1010495}{1458} + \frac{52214}{81}\zeta_2 - \frac{1258}{15}\zeta_2^2 + \left(\frac{9932}{27} + \frac{292}{3}\zeta_2 \right)\zeta_3 - \frac{896}{3}\zeta_5 \Bigg] \Bigg\}
$$

$$
+C_F^3 \Bigg[\frac{53}{30}\ln^5(x) + \frac{89}{12}\ln^4(x) + \left(-\frac{49}{3} - \frac{710}{9}\zeta_2 \right)\ln^3(x) + \Bigg(\frac{1085}{6} - \frac{1373}{3}\zeta_2
$$

$$
-286\zeta_3 \Bigg)\ln^2(x) + \left(\frac{5553}{4} - \frac{2308}{3}\zeta_2 - \frac{1978}{3}\zeta_3 - \frac{4748}{15}\zeta_2^2 \right)\ln(x) + \frac{71777}{24}
$$

$$
-\frac{2561}{3}\zeta_2 - \frac{2248}{15}\zeta_2^2 + \left(\frac{710}{3} + \frac{1384}{3}\zeta_2 \right)\zeta_3 - 2104\zeta_5 \Bigg], \tag{1290}
$$

$$
C_{F3,q}^{(3),d_{abc}} \;\simeq\; \frac{d_{abc}d^{abc}N_F}{N_C} \Bigg[-\frac{32}{15}\ln^5(x) + \left(-\frac{704}{9} + \frac{256}{3}\zeta_2 \right)\ln^3(x) + \Bigg(-1600 - \frac{2176}{5}\zeta_2^2
$$

$$
+\frac{1024}{3}\zeta_2 + \frac{1088}{3}\zeta_3 \Bigg)\ln(x) + \left(-1056 + 96\zeta_2 + \frac{1216}{3}\zeta_3 \right)\ln^2(x) - \frac{11456}{3}
$$

$$
+2560\zeta_2 - \frac{3104}{15}\zeta_2^2 + \left(\frac{3904}{3} - \frac{1856\zeta_2}{3} \right)\zeta_3 - \frac{3584}{3}\zeta_5 \Bigg], \tag{1291}
$$

$$
\Delta C_{g_1,q}^{(1),\mathrm{NS,L}} \;\simeq\; -C_F(7 + 2\ln(x)), \tag{1292}
$$

$$
\Delta C_{g_1,q}^{(2),\mathrm{NS,L}} \;\simeq\; C_F \Bigg\{ C_A \Bigg[-2\ln^3(x) - \frac{151}{6}\ln^2(x) + \left(-\frac{238}{3} + 8\zeta_2 \right)\ln(x) - \frac{6391}{54} + \frac{128}{3}\zeta_2 + 8\zeta_3 \Bigg]
$$

$$
+T_F N_F \Bigg[\frac{10}{3}\ln^2(x) + \frac{56}{3}\ln(x) + \frac{502}{27} - \frac{16}{3}\zeta_2 \Bigg] \Bigg\} + C_F^2 \Bigg[\frac{7}{3}\ln^3(x) + 21\ln^2(x)
$$

$$
+\left(21 + 8\zeta_2 \right)\ln(x) + \frac{117}{2} - 20\zeta_2 + 16\zeta_3 \Bigg], \tag{1293}
$$

$$
\tag{1294}
$$

$$
\Delta C_{g_1,q}^{(2),\mathrm{PS,L}} \;\simeq\; C_F T_F N_F \Bigg[\frac{20}{3}\ln^3(x) + 58\ln^2(x) + \left(172 - 32\zeta_2 \right)\ln(x) + 280 - 96\zeta_2 - 16\zeta_3 \Bigg], \tag{1295}
$$

$$
\Delta C_{g_1,q}^{(3),\mathrm{PS,L}} \;\simeq\; C_F \Bigg\{ T_F^2 N_F^2 \Bigg[-\frac{184}{27}\ln^4(x) - \frac{7312}{81}\ln^3(x) + \left(-\frac{37744}{81} + \frac{64}{9}\zeta_2 \right)\ln^2(x)
$$

$$+\left(-\frac{245504}{243}+\frac{1088}{27}\zeta_2-\frac{64}{9}\zeta_3\right)\ln(x)-\frac{292640}{243}+\frac{14528}{81}\zeta_2-\frac{32}{27}\zeta_3+\frac{224}{15}\zeta_2^2\bigg]$$

$$+C_FT_FN_F\bigg[4\ln^5(x)+\frac{632}{9}\ln^4(x)+\left(\frac{5156}{9}-\frac{1328}{9}\zeta_2\right)\ln^3(x)+\bigg(2708$$

$$-\frac{4096\zeta_2}{3}-\frac{256\zeta_3}{3}\bigg)\ln^2(x)+\left(\frac{23360}{3}-5064\zeta_2-224\zeta_3+\frac{3856}{15}\zeta_2^2\right)\ln(x)$$

$$+\frac{31712}{3}-6856\zeta_2+\left(-\frac{6160}{3}+32\zeta_2\right)\zeta_3+\frac{15008}{15}\zeta_2^2+352\zeta_5\bigg]$$

$$+C_AT_FN_F\bigg[4\ln^5(x)+\frac{2450}{27}\ln^4(x)+\left(\frac{67472}{81}-\frac{688}{9}\zeta_2\right)\ln^3(x)+\bigg(\frac{325580}{81}$$

$$-\frac{6992\zeta_2}{9}-352\zeta_3\bigg)\ln^2(x)+\left(\frac{2395192}{243}-\frac{74464}{27}\zeta_2-\frac{21136}{9}\zeta_3+\frac{3088}{15}\zeta_2^2\right)\ln(x)$$

$$+\frac{2166976}{243}-\frac{315832}{81}\zeta_2+\left(-\frac{93968}{27}+\frac{1808}{3}\zeta_2\right)\zeta_3+\frac{10312}{15}\zeta_2^2-264\zeta_5\bigg]\bigg\},\quad(1296)$$

$$\Delta C_{g_1,g}^{(1)}\ \simeq\ 4T_FN_F[3+\ln(x)],\quad(1297)$$

$$\Delta C_{g_1,g}^{(2)}\ \simeq\ C_FT_FN_F\bigg[\frac{10}{3}\ln^3(x)+41\ln^2(x)+\big(160-32\zeta_2\big)\ln(x)+134-88\zeta_2\bigg]$$

$$+C_AT_FN_F\bigg[\frac{28}{3}\ln^3(x)+90\ln^2(x)+\big(304-64\zeta_2\big)\ln(x)+484-168\zeta_2$$

$$-48\zeta_3\bigg],\quad(1298)$$

$$\Delta C_{g_1,g}^{(3)}\ \simeq\ T_FN_F\bigg\{\bigg[\frac{44}{5}C_A^2-\frac{23}{45}C_F^2+C_F\left(\frac{146}{45}C_A-4T_FN_F\right)\bigg]\ln^5(x)+\bigg[\frac{1627}{9}C_A^2-21C_F^2$$

$$-\frac{68}{9}C_AT_FN_F+C_F\left(\frac{2143}{27}C_A-\frac{2654}{27}T_FN_F\right)\bigg]\ln^4(x)+\bigg[-\frac{12056}{81}C_AT_FN_F$$

$$+C_F\left(C_A\left(\frac{61699}{81}-\frac{976}{9}\zeta_2\right)+T_FN_F\left(-\frac{79220}{81}+\frac{176}{3}\zeta_2\right)\right)+C_A^2\bigg(\frac{132454}{81}$$

$$-\frac{656}{3}\zeta_2\bigg)+C_F^2\left(-\frac{602}{3}-\frac{332}{9}\zeta_2\right)\bigg]\ln^3(x)+\bigg[C_F\left(T_FN_F\left(-\frac{422240}{81}+\frac{2600}{3}\zeta_2\right.\right.$$

$$\left.\left.+\frac{304}{3}\zeta_3\right)+C_A\left(\frac{221032}{81}-1188\zeta_2+\frac{824}{3}\zeta_3\right)\right)+C_AT_FN_F\left(-\frac{30296}{27}+80\zeta_2\right)$$

$$+C_A^2\left(\frac{236002}{27}-\frac{6844}{3}\zeta_2-\frac{2216}{3}\zeta_3\right)+C_F^2\left(-\frac{877}{3}-\frac{824}{3}\zeta_2-252\zeta_3\right)\bigg]\ln^2(x)$$

$$+\bigg[C_F\left(T_FN_F\left(-\frac{3711364}{243}+\frac{37264\zeta_2}{9}+\frac{10064\zeta_3}{9}-\frac{224\zeta_2^2}{5}\right)+C_A\left(\frac{728285}{243}\right.\right.$$

$$-\frac{46892}{9}\zeta_2 + \frac{2944}{15}\zeta_2^2 + \frac{27296}{9}\zeta_3\bigg)\bigg) + C_A^2\bigg(\frac{6175024}{243} - \frac{25564}{3}\zeta_2 + \frac{9968}{15}\zeta_2^2$$

$$-\frac{52064}{9}\zeta_3\bigg) + C_F^2\bigg(\frac{2755}{3} - \frac{1450}{3}\zeta_2 - \frac{5632}{3}\zeta_3 + \frac{5464}{15}\zeta_2^2\bigg)$$

$$+C_A T_F N_F\bigg(-\frac{970424}{243} + \frac{2144}{3}\zeta_2 + \frac{1408}{9}\zeta_3\bigg)\bigg]\ln(x) + C_F^2\bigg(\frac{39635}{12} - 1280\zeta_2$$

$$+\frac{2494}{3}\zeta_3 + \frac{9568}{15}\zeta_2^2 + \frac{640}{3}\zeta_2\zeta_3 - \frac{11032}{3}\zeta_5\bigg) + C_A^2\bigg(\frac{2218820}{81} - \frac{1035208}{81}\zeta_2$$

$$-\frac{83068}{9}\zeta_3 + \frac{86584}{45}\zeta_2^2 + 1544\zeta_2\zeta_3 - 664\zeta_5\bigg) + C_A T_F N_F\bigg(-\frac{495448}{81} + \frac{135320}{81}\zeta_2$$

$$+\frac{5696}{9}\zeta_3 - \frac{1376}{45}\zeta_2^2\bigg) + C_F\bigg[T_F N_F\bigg(-\frac{1456493}{81} + \frac{536656}{81}\zeta_2 + \frac{75680}{27}\zeta_3$$

$$-\frac{22288}{45}\zeta_2^2 - 128\zeta_2\zeta_3 + 96\zeta_5\bigg) + C_A\bigg(-\frac{367691}{324} - \frac{488996}{81}\zeta_2 - \frac{7432}{27}\zeta_3$$

$$+\frac{72272}{45}\zeta_2^2 - \frac{1936}{3}\zeta_2\zeta_3 + \frac{12844}{3}\zeta_5\bigg)\bigg]\bigg\}, \tag{1299}$$

$$\Delta C_{g_1,g}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F^2}{N_A}\bigg[-\frac{13952}{15} - \frac{64}{3}\zeta_2 + 640\zeta_3 + \frac{8192}{75}\zeta_2^2\bigg]. \tag{1300}$$

Note, that the expansion has to be performed in z–space, since the Mellin space expression is either applicable for even or odd moments. A small x expansion of the non–singlet Wilson coefficient of the structure function $g_1(x, Q^2)$ has been considered in [94] based on [95], however, in a different not fully specified scheme.

8.2 The large x limit

In the large x limit one obtains the following leading term behaviour for the Wilson coefficients calculated in the present paper. Here we dropped the terms $O(L_a^k/N^2), \quad k \geq 0$.

$$C_{F_2,q}^{(1),\mathrm{NS}} \simeq C_F\bigg[2S_1^2 + 3S_1 - 9 - 2\zeta_2 + \frac{9}{N}\bigg], \tag{1301}$$

$$C_{F_2,q}^{(2),\mathrm{NS}} \simeq C_F\bigg\{T_F N_F\bigg[-\frac{8}{9}S_1^3 - \frac{58}{9}S_1^2 - \bigg(\frac{494}{27} - \frac{8}{3}\zeta_2 + \frac{52}{3N}\bigg)S_1 + \frac{457}{18} + \frac{170}{9}\zeta_2 + \frac{8}{9}\zeta_3 - \frac{484}{9N}\bigg]$$

$$+C_A\bigg[\frac{22}{9}S_1^3 + \bigg(\frac{367}{18} - 4\zeta_2\bigg)S_1^2 + \bigg(\frac{3155}{54} - \frac{22}{3}\zeta_2 - 40\zeta_3 + \frac{161}{3N} - \frac{16}{N}\zeta_2\bigg)S_1 - \frac{5465}{72}$$

$$-\frac{1139}{18}\zeta_2 + \frac{464}{9}\zeta_3 + \frac{51}{5}\zeta_2^2 + \frac{1202}{9N} + \frac{8}{N}\zeta_2 - \frac{24}{N}\zeta_3\bigg]\bigg\} + C_F^2\bigg\{2S_1^4 + 6S_1^3$$

$$+\bigg(-\frac{27}{2} + \frac{30}{N} - 4\zeta_2\bigg)S_1^2 + \bigg(-\frac{51}{2} - 18\zeta_2 + 24\zeta_3 - \frac{11}{N} + \frac{32\zeta_2}{N}\bigg)S_1 + \frac{331}{8}$$

$$+\frac{111}{2}\zeta_2 - 66\zeta_3 + \frac{4}{5}\zeta_2^2 - \frac{95}{N} - \frac{38}{N}\zeta_2 + \frac{48}{N}\zeta_3\Bigg\}, \tag{1302}$$

$$
\begin{aligned}
C_{F_2,q}^{(3),\mathrm{NS}} \;\simeq\; & C_F\Bigg\{ T_F^2 N_F^2\Bigg[\frac{16}{27}S_1^4 + \frac{464}{81}S_1^3 + \left(\frac{1880}{81} - \frac{32}{9}\zeta_2 + \frac{208}{9N}\right)S_1^2 + \left(\frac{34856}{729} - \frac{464}{27}\zeta_2\right. \\
& + \frac{256}{27}\zeta_3 + \frac{4192}{27N}\Bigg)S_1 - \frac{19034}{243} - \frac{8440}{81}\zeta_2 + \frac{320}{81}\zeta_3 - \frac{1168}{135}\zeta_2^2 + \frac{24608}{81N} - \frac{208}{9N}\zeta_2\Bigg] \\
& + C_A T_F N_F\Bigg[-\frac{88}{27}S_1^4 + \left(-\frac{3104}{81} + \frac{32}{9}\zeta_2\right)S_1^3 + \left(-\frac{15062}{81} - \frac{1252}{9N} + \frac{112}{3}\zeta_2\right. \\
& + 16\zeta_3 + \frac{32}{N}\zeta_2\Bigg)S_1^2 + \left(-\frac{321812}{729} + \frac{10528}{81}\zeta_2 + \frac{3952}{27}\zeta_3 - \frac{256}{15}\zeta_2^2 - \frac{29908}{27N} + \frac{488}{3N}\zeta_2\right. \\
& + \frac{512}{3N}\zeta_3\Bigg)S_1 + \frac{142883}{243} + \frac{66662}{81}\zeta_2 - \frac{42836}{81}\zeta_3 + \frac{328}{135}\zeta_2^2 - \frac{128}{9}\zeta_2\zeta_3 + \frac{16}{3}\zeta_5 \\
& - \frac{55130}{27N} + \frac{2140\zeta_2}{27N} + \frac{2552\zeta_3}{9N} + \frac{96\zeta_2^2}{5N}\Bigg] + C_A^2\Bigg[\frac{121}{27}S_1^4 + \left(\frac{4649}{81} - \frac{88}{9}\zeta_2\right)S_1^3 \\
& + \left(\frac{50689}{162} - \frac{778}{9}\zeta_2 - 132\zeta_3 + \frac{88}{5}\zeta_2^2 + \frac{2011}{9N} - \frac{178}{3N}\zeta_2 - \frac{32}{N}\zeta_3\right)S_1^2 + \left(\frac{599375}{729}\right. \\
& - \frac{18179}{81}\zeta_2 - \frac{6688}{9}\zeta_3 + \frac{212}{15}\zeta_2^2 + \frac{176}{3}\zeta_2\zeta_3 + 232\zeta_5 + \frac{50410}{27N} - \frac{568}{N}\zeta_2 - \frac{904}{3N}\zeta_3 \\
& + \frac{128}{5N}\zeta_2^2\Bigg)S_1 - \frac{1909753}{1944} - \frac{78607}{54}\zeta_2 + \frac{115010}{81}\zeta_3 + \frac{13151}{135}\zeta_2^2 + \frac{3496}{9}\zeta_2\zeta_3 - \frac{416}{3}\zeta_5 \\
& - \frac{12016}{315}\zeta_2^3 - \frac{248}{3}\zeta_3^2 + \frac{534221}{162N} + \frac{4399}{27N}\zeta_2 + \frac{518}{15N}\zeta_2^2 - \frac{12844}{9N}\zeta_3 - \frac{112}{N}\zeta_2\zeta_3 \\
& - \frac{160}{N}\zeta_5\Bigg]\Bigg\} + C_F^2\Bigg\{ T_F N_F\Bigg[-\frac{16}{9}S_1^5 - \frac{140}{9}S_1^4 + \left(-\frac{1366}{27} + \frac{64\zeta_2}{9} - \frac{1504}{27N}\right)S_1^3 \\
& + \left(\frac{83}{9} + \frac{224}{3}\zeta_2 + \frac{16}{9}\zeta_3 - \frac{602}{3N} - \frac{64}{N}\zeta_2\right)S_1^2 + \left(\frac{2003}{54} + \frac{4354}{27}\zeta_2 - \frac{40}{9}\zeta_3 - \frac{16}{3}\zeta_2^2\right. \\
& + \frac{19690}{81N} - \frac{1456}{9N}\zeta_2 - \frac{640}{3N}\zeta_3\Bigg)S_1 - \frac{341}{18} - \frac{10733}{27}\zeta_2 + \frac{21532}{27}\zeta_3 - \frac{21604}{135}\zeta_2^2 \\
& - \frac{80}{3}\zeta_2\zeta_3 - \frac{1568}{9}\zeta_5 + \frac{47101}{54N} + \frac{10610\zeta_2}{27N} - \frac{16136\zeta_3}{27N} + \frac{192\zeta_2^2}{5N}\Bigg] + C_A\Bigg[\frac{44}{9}S_1^5 \\
& + \left(\frac{433}{9} - 8\zeta_2\right)S_1^4 + \left(\frac{8425}{54} - \frac{284}{9}\zeta_2 - 80\zeta_3 + \frac{4460}{27N} - \frac{32}{N}\zeta_2\right)S_1^3 + \left(-\frac{5563}{36}\right. \\
& - \frac{592}{3}\zeta_2 + \frac{142}{5}\zeta_2^2 + \frac{640}{9}\zeta_3 + \frac{3127}{6N} - \frac{8}{3N}\zeta_2 + \frac{80}{N}\zeta_3\Bigg)S_1^2 + \left(-\frac{16981}{24} - \frac{28495}{54}\zeta_2\right. \\
& + 752\zeta_3 + \frac{299}{3}\zeta_2^2 + 96\zeta_2\zeta_3 + 120\zeta_5 - \frac{152317}{162N} + \frac{7532}{9N}\zeta_2 - \frac{1496}{3N}\zeta_3 + \frac{672}{5N}\zeta_2^2\Bigg)S_1
\end{aligned}
$$

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$$+\frac{191545}{108}\zeta_2 - \frac{49346}{27}\zeta_3 + \frac{11419}{27}\zeta_2^2 - \frac{3896}{9}\zeta_5 - 828\zeta_2\zeta_3 - \frac{23098}{315}\zeta_2^3 + \frac{536}{3}\zeta_3^2$$

$$-\frac{751331}{216N} - \frac{88043}{54N}\zeta_2 + \frac{78052}{27N}\zeta_3 - \frac{1721}{5N}\zeta_2^2 + \frac{432}{N}\zeta_2\zeta_3 + \frac{1200}{N}\zeta_5 + \frac{9161}{12}\Bigg]\Bigg\}$$

$$+C_F^3\Bigg[\frac{4}{3}S_1^6 + 6S_1^5 + \left(-9 - 4\zeta_2 + \frac{42}{N}\right)S_1^4 + \left(-\frac{93}{2} - 36\zeta_2 + 48\zeta_3 + \frac{14}{N} + \frac{64}{N}\zeta_2\right)$$

$$\times S_1^3 + \left(\frac{187}{4} + 66\zeta_2 - 60\zeta_3 + \frac{8}{5}\zeta_2^2 - \frac{1237}{6N} - \frac{4}{N}\zeta_2 - \frac{32}{N}\zeta_3\right)S_1^2 + \left(\frac{1001}{8} + \frac{579}{2}\zeta_2\right.$$

$$-346\zeta_3 + 84\zeta_2^2 - 80\zeta_2\zeta_3 - 240\zeta_5 + \frac{119}{6N} + \frac{984}{N}\zeta_3 - \frac{998}{3N}\zeta_2 - \frac{1856}{5N}\zeta_2^2\Bigg)S_1 - \frac{7255}{24}$$

$$-\frac{6197}{12}\zeta_2 - 411\zeta_3 - \frac{1791}{5}\zeta_2^2 + 556\zeta_2\zeta_3 + 1384\zeta_5 + \frac{8144}{315}\zeta_2^3 - \frac{176}{3}\zeta_3^2 + \frac{16889}{24N}$$

$$+\frac{5029}{6N}\zeta_2 - \frac{650}{N}\zeta_3 + \frac{1780}{3N}\zeta_2^2 - \frac{416}{N}\zeta_2\zeta_3 - \frac{1760}{N}\zeta_5\Bigg], \tag{1303}$$

$$C_{F_2,q}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F}{N_C}\Bigg[64 + 160\zeta_2$$

$$+\frac{224}{3}\zeta_3 - \frac{32}{5}\zeta_2^2 - \frac{1280}{3}\zeta_5 - \frac{128}{N} - \frac{448}{N}\zeta_2 + \frac{64}{5N}\zeta_2^2 - \frac{704}{N}\zeta_3 + \frac{128}{N}\zeta_2\zeta_3$$

$$+\frac{1280}{N}\zeta_5\Bigg], \tag{1304}$$

$$C_{F_2,q}^{(2),\text{PS}} \simeq 0, \tag{1305}$$

$$C_{F_2,q}^{(3),\text{PS}} \simeq 0, \tag{1306}$$

$$C_{F_2,g}^{(1)} \simeq -4T_F N_F\frac{1}{N}[S_1 + 1], \tag{1307}$$

$$C_{F_2,g}^{(2)} \simeq \frac{1}{N}\Bigg\{C_A T_F N_F\left[-\frac{4}{3}S_1^3 - 8S_1^2 + \left(-28 + 12\zeta_2\right)S_1 - 16 - \frac{92}{3}\zeta_3\right]$$

$$+C_F T_F N_F\left[-\frac{20}{3}S_1^3 - 18S_1^2 + \left(4 + 12\zeta_2\right)S_1 - 4 + 10\zeta_2 + \frac{176}{3}\zeta_3\right]\Bigg\}, \tag{1308}$$

$$C_{F_2,g}^{(3),a} \simeq \frac{1}{N}\Bigg\{C_F\Bigg\{T_F^2 N_F^2\Bigg[\frac{68}{27}S_1^4 + \frac{1984}{81}S_1^3 + \left(\frac{6260}{81} - \frac{56}{9}\zeta_2\right)S_1^2 + \left(-\frac{12304}{243} - \frac{1952}{27}\zeta_2\right.$$

$$\left.-\frac{224}{27}\zeta_3\right)S_1 - \frac{1189}{27} - \frac{188}{3}\zeta_2 - \frac{16672}{81}\zeta_3 + \frac{1028}{45}\zeta_2^2\Bigg] + C_A T_F N_F\Bigg[-\frac{412}{27}S_1^4$$

$$+\left(-\frac{6884}{81} + \frac{272}{9}\zeta_2\right)S_1^3 + \left(-\frac{12043}{81} + \frac{568}{9}\zeta_2 + \frac{104}{3}\zeta_3\right)S_1^2 + \left(\frac{65936}{243} + \frac{7396}{27}\zeta_2\right.$$

$$\left.-\frac{10796}{27}\zeta_3 + \frac{976}{15}\zeta_2^2\right)S_1 + \frac{32813}{108} + \frac{1321}{3}\zeta_2 - \frac{11908}{81}\zeta_3 - \frac{1172}{9}\zeta_2^2 + \frac{352}{9}\zeta_2\zeta_3$$

$$+\frac{1496}{3}\zeta_5\Bigg]\Bigg\}+C_AT_F^2N_F^2\Bigg[\frac{28}{27}S_1^4+\frac{608}{81}S_1^3+\left(\frac{4384}{81}-\frac{104}{9}\zeta_2\right)S_1^2+\Bigg(\frac{43192}{243}$$

$$-\frac{1376}{27}\zeta_2+\frac{32}{27}\zeta_3\Bigg)S_1+\frac{13120}{81}-\frac{16}{9}\zeta_2+\frac{17824}{81}\zeta_3-\frac{4}{5}\zeta_2^2\Bigg]+C_F^2T_FN_F\Bigg[-\frac{20}{3}S_1^5$$

$$-\frac{83}{3}S_1^4+\left(-\frac{254}{9}+\frac{152}{9}\zeta_2\right)S_1^3+\left(-\frac{205}{3}+118\zeta_2+\frac{32}{3}\zeta_3\right)S_1^2+\Bigg(-\frac{508}{3}$$

$$-54\zeta_2+\frac{2072}{3}\zeta_3-\frac{2732}{15}\zeta_2^2\Bigg)S_1+\frac{617}{12}-173\zeta_2+\frac{5564}{9}\zeta_3+\frac{193}{5}\zeta_2^2-\frac{1040}{9}\zeta_2\zeta_3$$

$$-\frac{2384}{3}\zeta_5\Bigg]+C_A^2T_FN_F\Bigg[-\frac{4}{3}S_1^5-\frac{293}{27}S_1^4+\left(-\frac{6112}{81}+\frac{152}{9}\zeta_2\right)S_1^3+\Bigg(-\frac{27578}{81}$$

$$+\frac{718}{9}\zeta_2+\frac{272}{3}\zeta_3\Bigg)S_1^2+\left(-\frac{139052}{243}+\frac{5992}{27}\zeta_2+\frac{3368}{27}\zeta_3-\frac{136}{3}\zeta_2^2\right)S_1-\frac{31172}{81}$$

$$-\frac{478}{9}\zeta_2-\frac{10988}{81}\zeta_3+\frac{163}{5}\zeta_2^2-\frac{344}{9}\zeta_2\zeta_3-16\zeta_5\Bigg]\Bigg\}, \tag{1309}$$

$$C_{F2,g}^{(3),d_{abc}}\simeq\frac{d_{abc}d^{abc}N_F^2}{N_A}\frac{1}{N}\Bigg[\left(128+128\zeta_2-256\zeta_3\right)S_1^2+\Bigg(-64-128\zeta_2-\frac{1536}{5}\zeta_2^2$$

$$+768\zeta_3\Bigg)S_1+80+256\zeta_2-512\zeta_3+\frac{1728}{5}\zeta_2^2-640\zeta_5\Bigg], \tag{1310}$$

$$C_{FL,q}^{(1),\text{NS}}\simeq C_F\frac{4}{N}, \tag{1311}$$

$$C_{FL,q}^{(2),\text{NS}}\simeq\frac{1}{N}\Bigg\{C_F\Bigg[T_FN_F\left(-\frac{16S_1}{3}-\frac{152}{9}\right)+C_A\Bigg(\left(\frac{92}{3}-16\zeta_2\right)S_1+\frac{430}{9}+16\zeta_2$$

$$-24\zeta_3\Bigg)\Bigg]+C_F^2\Bigg[8S_1^2+\left(-36+32\zeta_2\right)S_1-34-40\zeta_2+48\zeta_3\Bigg]\Bigg\}, \tag{1312}$$

$$C_{FL,q}^{(3),\text{NS}}\simeq\frac{1}{N}\Bigg\{C_F\Bigg\{T_F^2N_F^2\Bigg[\frac{64}{9}S_1^2+\frac{1216}{27}S_1+\frac{6496}{81}-\frac{64}{9}\zeta_2\Bigg]+C_AT_FN_F\Bigg[\left(-\frac{640}{9}+32\zeta_2\right)$$

$$\times S_1^2+\left(-\frac{12880}{27}+\frac{640\zeta_2}{9}+\frac{512\zeta_3}{3}\right)S_1-\frac{42976}{81}-64\zeta_2+\frac{128}{3}\zeta_3+\frac{96}{5}\zeta_2^2\Bigg]$$

$$+C_A^2\Bigg[\left(\frac{1276}{9}-56\zeta_2-32\zeta_3\right)S_1^2+\left(\frac{25756}{27}-\frac{3008\zeta_2}{9}+\frac{128\zeta_2^2}{5}-\frac{880\zeta_3}{3}\right)S_1$$

$$+\frac{67312}{81}+\frac{3820}{9}\zeta_2-\frac{1240}{3}\zeta_3-112\zeta_2\zeta_3-160\zeta_5\Bigg]\Bigg\}+C_F^2\Bigg\{T_FN_F\Bigg[-\frac{128}{9}S_1^3$$

$$+\left(\frac{184}{9}-64\zeta_2\right)S_1^2+\left(\frac{9472}{27}-\frac{1088\zeta_2}{9}-\frac{640\zeta_3}{3}\right)S_1+\frac{2312}{9}\zeta_2-\frac{2656}{9}\zeta_3+158$$

$$+\frac{192}{5}\zeta_2^2\Bigg]+C_A\Bigg[\left(-\frac{32732}{27}+\frac{6640}{9}\zeta_2-\frac{472}{3}\zeta_3+\frac{672}{5}\zeta_2^2\right)S_1+\Bigg(-\frac{530}{9}$$

$$+80\zeta_2 + 80\zeta_3\Big)S_1^2 + \left(\frac{640}{9} - 32\zeta_2\right)S_1^3 - \frac{5255}{6} - \frac{11806}{9}\zeta_2 + \frac{10832}{9}\zeta_3$$

$$-436\zeta_2^2 + 432\zeta_2\zeta_3 + 1200\zeta_5\bigg]\bigg\} + C_F^3\bigg\{8S_1^4 + \left(-72 + 64\zeta_2\right)S_1^3 + \Big(-34 + 16\zeta_2$$

$$-32\zeta_3\Big)S_1^2 + \left(264 - 232\zeta_2 + 816\zeta_3 - \frac{1856}{5}\zeta_2^2\right)S_1 + \frac{1937}{6} + 506\zeta_2 - 200\zeta_3$$

$$+688\zeta_2^2 - 416\zeta_2\zeta_3 - 1760\zeta_5\bigg\}\bigg\}, \tag{1313}$$

$$C_{F_L,q}^{(3),d_{abc}} \;\simeq\; \frac{d_{abc}d^{abc}N_F}{N_C}\frac{1}{N}\left[-128 - 448\zeta_2 - 704\zeta_3 + \frac{64}{5}\zeta_2^2 + 128\zeta_2\zeta_3 + 1280\zeta_5\right], \tag{1314}$$

$$C_{F_L,q}^{(2),\mathrm{PS}} \;\simeq\; 0, \tag{1315}$$

$$C_{F_L,q}^{(3),\mathrm{PS}} \;\simeq\; 0, \tag{1316}$$

$$C_{F_L,g}^{(1)} \;\simeq\; 0, \tag{1317}$$

$$C_{F_L,g}^{(2)} \;\simeq\; 0, \tag{1318}$$

$$C_{F_L,g}^{(3),a} \;\simeq\; 0, \tag{1319}$$

$$C_{F_L,g}^{(3),d_{abc}} \;\simeq\; 0, \tag{1320}$$

$$C_{F_3,q}^{(1),\mathrm{NS}} \;\simeq\; C_F\left[2S_1^2 + 3S_1 - 9 - 2\zeta_2 + \frac{5}{N}\right], \tag{1321}$$

$$C_{F_3,q}^{(2),\mathrm{NS}} \;\simeq\; C_F\bigg\{T_F N_F\bigg[-\frac{8}{9}S_1^3 - \frac{58}{9}S_1^2 + \left(-\frac{494}{27} + \frac{8}{3}\zeta_2 - \frac{12}{N}\right)S_1 + \frac{457}{18} + \frac{170}{9}\zeta_2 + \frac{8}{9}\zeta_3$$

$$-\frac{332}{9N}\bigg] + C_A\bigg[\frac{22}{9}S_1^3 + \left(\frac{367}{18} - 4\zeta_2\right)S_1^2 + \left(\frac{3155}{54} + \frac{23}{N} - \frac{22\zeta_2}{3} - 40\zeta_3\right)S_1$$

$$-\frac{5465}{72} - \frac{1139}{18}\zeta_2 + \frac{464}{9}\zeta_3 + \frac{51}{5}\zeta_2^2 + \frac{772}{9N} - \frac{8\zeta_2}{N}\bigg]\bigg\} + C_F^2\bigg[2S_1^4 + 6S_1^3$$

$$+\left(-\frac{27}{2} - 4\zeta_2 + \frac{22}{N}\right)S_1^2 + \left(-\frac{51}{2} - 18\zeta_2 + 24\zeta_3 + \frac{25}{N}\right)S_1 + \frac{331}{8} + \frac{111}{2}\zeta_2$$

$$-66\zeta_3 + \frac{4}{5}\zeta_2^2 - \frac{61}{N} + \frac{2\zeta_2}{N}\bigg], \tag{1322}$$

$$C_{F_3,q}^{(3),\mathrm{NS}} \;\simeq\; C_F^2\bigg\{C_A\bigg[\frac{44}{9}S_1^5 + \left(\frac{433}{9} - 8\zeta_2\right)S_1^4 + \left(\frac{8425}{54} - \frac{284\zeta_2}{9} - 80\zeta_3 + \frac{2540}{27N}\right)S_1^3$$

$$+\left(-\frac{5563}{36} - \frac{592}{3}\zeta_2 + \frac{640}{9}\zeta_3 + \frac{142}{5}\zeta_2^2 + \frac{10441}{18N} - \frac{248}{3N}\zeta_2\right)S_1^2 + \Big(-\frac{16981}{24} - \frac{28495}{54}\zeta_2$$

$$+752\zeta_3 + \frac{299}{3}\zeta_2^2 + 96\zeta_2\zeta_3 + 120\zeta_5 + \frac{44075}{162N} + \frac{892}{9N}\zeta_2 - \frac{1024}{3N}\zeta_3\Big)S_1 + \frac{9161}{12}$$

$$+\frac{191545}{108}\zeta_2 - \frac{49418}{27}\zeta_3 + \frac{11419}{27}\zeta_2^2 - 828\zeta_2\zeta_3 - \frac{3896}{9}\zeta_5 - \frac{23098}{315}\zeta_2^3 + \frac{536}{3}\zeta_3^2$$

$$-\frac{562151}{216N} - \frac{17207}{54N}\zeta_2 + \frac{45556}{27N}\zeta_3 + \frac{459}{5N}\zeta_2^2\bigg] + T_F N_F\bigg[-\frac{16}{9}S_1^5 - \frac{140}{9}S_1^4$$

$$+\left(-\frac{1366}{27} - \frac{1120}{27N} + \frac{64}{9}\zeta_2\right)S_1^3 + \left(\frac{83}{9} - \frac{1990}{9N} + \frac{224}{3}\zeta_2 + \frac{16}{9}\zeta_3\right)S_1^2$$

$$+\left(\frac{2003}{54} - \frac{8726}{81N} + \frac{4354}{27}\zeta_2 - \frac{368}{9N}\zeta_2 - \frac{16}{3}\zeta_2^2 - \frac{40}{9}\zeta_3\right)S_1 - \frac{341}{18} - \frac{10733}{27}\zeta_2$$

$$+\frac{21532}{27}\zeta_3 - \frac{21604}{135}\zeta_2^2 - \frac{80}{3}\zeta_2\zeta_3 - \frac{1568}{9}\zeta_5 + \frac{38569}{54N} + \frac{3674}{27N}\zeta_2 - \frac{8168}{27N}\zeta_3\bigg]\bigg\}$$

$$+C_F\bigg\{T_F^2 N_F^2\bigg[\frac{16}{27}S_1^4 + \frac{464}{81}S_1^3 + \left(\frac{1880}{81} - \frac{32}{9}\zeta_2 + \frac{16}{N}\right)S_1^2 + \left(\frac{34856}{729} - \frac{464}{27}\zeta_2\right.$$

$$+\frac{256}{27}\zeta_3 + \frac{992}{9N}\bigg)S_1 - \frac{19034}{243} - \frac{8440}{81}\zeta_2 + \frac{320}{81}\zeta_3 - \frac{1168}{135}\zeta_2^2 + \frac{18112}{81N} - \frac{16}{N}\zeta_2\bigg]$$

$$+C_A^2\bigg[\frac{121}{27}S_1^4 + \left(\frac{4649}{81} - \frac{88}{9}\zeta_2\right)S_1^3 + \left(\frac{50689}{162} - \frac{778}{9}\zeta_2 - 132\zeta_3 + \frac{88}{5}\zeta_2^2\right.$$

$$+\frac{245}{3N} - \frac{10}{3N}\zeta_2\bigg)S_1^2 + \left(\frac{599375}{729} - \frac{18179}{81}\zeta_2 - \frac{6688}{9}\zeta_3 + \frac{212}{15}\zeta_2^2 + \frac{176}{3}\zeta_2\zeta_3\right.$$

$$+232\zeta_5 + \frac{8218}{9N} - \frac{2104}{9N}\zeta_2 - \frac{8}{N}\zeta_3\bigg)S_1 - \frac{1909753}{1944} - \frac{78607}{54}\zeta_2 + \frac{115064}{81}\zeta_3$$

$$+\frac{13151}{135}\zeta_2^2 + \frac{3496}{9}\zeta_2\zeta_3 - \frac{416}{3}\zeta_5 - \frac{12016}{315}\zeta_2^3 - \frac{248}{3}\zeta_3^2 + \frac{133199}{54N} - \frac{7061}{27N}\zeta_2$$

$$-\frac{9124}{9N}\zeta_3 + \frac{518}{15N}\zeta_2^2\bigg] + C_A T_F N_F\bigg[-\frac{88}{27}S_1^4 + \left(-\frac{3104}{81} + \frac{32\zeta_2}{9}\right)S_1^3 + \left(-\frac{15062}{81}\right.$$

$$-\frac{68}{N} + \frac{112\zeta_2}{3} + 16\zeta_3\bigg)S_1^2 + \left(-\frac{321812}{729} - \frac{1892}{3N} + \frac{10528}{81}\zeta_2\right.$$

$$+\frac{824}{9N}\zeta_2 - \frac{256}{15}\zeta_2^2 + \frac{3952}{27}\zeta_3\bigg)S_1 + \frac{142883}{243} - \frac{122414}{81N} + \frac{66662}{81}\zeta_2 - \frac{42836}{81}\zeta_3 + \frac{328}{135}\zeta_2^2$$

$$-\frac{128}{9}\zeta_2\zeta_3 + \frac{16}{3}\zeta_5 + \frac{3868\zeta_2}{27N} + \frac{2168\zeta_3}{9N}\bigg]\bigg\} + C_F^3\bigg\{\frac{4}{3}S_1^6 + 6S_1^5 + \left(-\frac{93}{2} - 36\zeta_2\right.$$

$$+48\zeta_3 + \frac{86}{N}\bigg)S_1^3 + \left(-9 - 4\zeta_2 + \frac{34}{N}\right)S_1^4 + \left(\frac{187}{4} + 66\zeta_2 - 60\zeta_3 + \frac{8}{5}\zeta_2^2\right.$$

$$-\frac{1033}{6N} - \frac{20}{N}\zeta_2\bigg)S_1^2 + \left(\frac{1001}{8} + \frac{579}{2}\zeta_2 - 346\zeta_3 + 84\zeta_2^2 - 80\zeta_2\zeta_3 - 240\zeta_5\right.$$

$$-\frac{1465}{6N} - \frac{302}{3N}\zeta_2 + \frac{168}{N}\zeta_3\bigg)S_1 - \frac{7255}{24} - \frac{6197}{12}\zeta_2 - \frac{1225}{3}\zeta_3 - \frac{1791}{5}\zeta_2^2$$

$$+556\zeta_2\zeta_3 + 1384\zeta_5 + \frac{8144}{315}\zeta_2^3 - \frac{176}{3}\zeta_3^2 + \frac{3047}{8N} + \frac{1993}{6N}\zeta_2 - \frac{450}{N}\zeta_3 - \frac{284}{3N}\zeta_2^2\Big\}, \quad (1323)$$

$$C_{F_3,q}^{(3),d_{abc}} \simeq 0, \quad (1324)$$

and

$$\Delta C_{g_1,q}^{(1),\mathrm{NS,L}} \simeq C_F\left[2S_1^2 + 3S_1 - 9 - 2\zeta_2 + \frac{5}{N}\right], \quad (1325)$$

$$\begin{aligned}
\Delta C_{g_1,q}^{(2),\mathrm{NS,L}} \simeq\ & C_F\Bigg\{ T_F N_F\left[-\frac{58}{9}S_1^2 - \frac{8}{9}S_1^3 + \left(-\frac{494}{27} + \frac{8\zeta_2}{3} - \frac{12}{N}\right)S_1 + \frac{457}{18} + \frac{170}{9}\zeta_2 + \frac{8}{9}\zeta_3 \right.\\
& \left. -\frac{332}{9N}\right] + C_A\left[\frac{22}{9}S_1^3 + \left(\frac{367}{18} - 4\zeta_2\right)S_1^2 + \left(\frac{3155}{54} + \frac{23}{N} - \frac{22\zeta_2}{3} - 40\zeta_3\right)S_1 \right.\\
& \left. -\frac{5465}{72} - \frac{1139}{18}\zeta_2 + \frac{464}{9}\zeta_3 + \frac{51}{5}\zeta_2^2 + \frac{772}{9N} - \frac{8}{N}\zeta_2\right]\Bigg\} + C_F^2\Bigg[2S_1^4 + 6S_1^3 \\
& + \left(-\frac{27}{2} + \frac{22}{N} - 4\zeta_2\right)S_1^2 + \left(-\frac{51}{2} + \frac{25}{N} - 18\zeta_2 + 24\zeta_3\right)S_1 + \frac{331}{8} + \frac{111}{2}\zeta_2 \\
& -66\zeta_3 + \frac{4}{5}\zeta_2^2 - \frac{61}{N} + \frac{2\zeta_2}{N}\Bigg],
\end{aligned} \quad (1326)$$

$$\quad (1327)$$

$$\Delta C_{g_1,q}^{(2),\mathrm{PS,L}} \simeq 0, \quad (1328)$$

$$\Delta C_{g_1,q}^{(3),\mathrm{PS,L}} \simeq 0, \quad (1329)$$

$$\Delta C_{g_1,g}^{(1)} \simeq -\frac{4}{N}T_F N_F\big(1 + S_1\big), \quad (1330)$$

$$\begin{aligned}
\Delta C_{g_1,g}^{(2)} \simeq\ & \frac{1}{N}\Bigg\{ C_A T_F N_F\left[-\frac{4}{3}S_1^3 - 8S_1^2 + \left(-28 + 12\zeta_2\right)S_1 - 16 - \frac{92}{3}\zeta_3\right]\\
& +C_F T_F N_F\left[-\frac{20}{3}S_1^3 - 18S_1^2 + \left(4 + 12\zeta_2\right)S_1 - 4 + 10\zeta_2 + \frac{176}{3}\zeta_3\right]\Bigg\},
\end{aligned} \quad (1331)$$

$$\begin{aligned}
\Delta C_{g_1,g}^{(3)} \simeq\ & \frac{1}{N}T_F N_F\Bigg\{ -\left[\frac{4}{3}C_A^2 + \frac{20}{3}C_F^2\right]S_1^5 + \left[-\frac{293}{27}C_A^2 - \frac{83}{3}C_F^2 + \frac{28}{27}C_A T_F N_F \right.\\
& \left. +C_F\left(-\frac{412}{27}C_A + \frac{68}{27}T_F N_F\right)\right]S_1^4 + \left[\frac{608}{81}C_A T_F N_F + C_F\left(\frac{1984}{81}T_F N_F + C_A\left(\right.\right.\right.\\
& \left.\left.\left. -\frac{6884}{81} + \frac{272}{9}\zeta_2\right)\right) + C_A^2\left(-\frac{6112}{81} + \frac{152}{9}\zeta_2\right) + C_F^2\left(-\frac{254}{9} + \frac{152}{9}\zeta_2\right)\right]S_1^3 \\
& + \left[C_F\left(T_F N_F\left(\frac{6260}{81} - \frac{56}{9}\zeta_2\right) + C_A\left(-\frac{12043}{81} + \frac{568}{9}\zeta_2 + \frac{104}{3}\zeta_3\right)\right) + C_A T_F N_F \right.\\
& \left. \times\left(\frac{4384}{81} - \frac{104}{9}\zeta_2\right) + C_F^2\left(-\frac{205}{3} + 118\zeta_2 + \frac{32}{3}\zeta_3\right) + C_A^2\left(-\frac{27578}{81} + \frac{718}{9}\zeta_2\right.\right.
\end{aligned}$$

$$+\frac{272}{3}\zeta_3\Big)\Big]S_1^2 + \Big[C_F\Big(C_A\Big(\frac{65936}{243}+\frac{7396}{27}\zeta_2-\frac{10796}{27}\zeta_3+\frac{976}{15}\zeta_2^2\Big)$$

$$+T_FN_F\Big(-\frac{12304}{243}-\frac{1952}{27}\zeta_2-\frac{224}{27}\zeta_3\Big)\Big)+C_AT_FN_F\Big(\frac{43192}{243}-\frac{1376}{27}\zeta_2$$

$$+\frac{32}{27}\zeta_3\Big)+C_A^2\Big(-\frac{139052}{243}+\frac{5992}{27}\zeta_2+\frac{3368}{27}\zeta_3-\frac{136}{3}\zeta_2^2\Big)+C_F^2\Big(-\frac{508}{3}-54\zeta_2$$

$$+\frac{2072}{3}\zeta_3-\frac{2732}{15}\zeta_2^2\Big)\Big]S_1+C_F\Big\{C_A\Big[\frac{32813}{108}+\frac{1321}{3}\zeta_2-\frac{11908}{81}\zeta_3-\frac{1172}{9}\zeta_2^2$$

$$+\frac{352}{9}\zeta_2\zeta_3+\frac{1496}{3}\zeta_5\Big]+T_FN_F\Big(-\frac{1189}{27}-\frac{188}{3}\zeta_2-\frac{16672}{81}\zeta_3+\frac{1028}{45}\zeta_2^2\Big)\Big\}$$

$$+C_F^2\Big[\frac{617}{12}-173\zeta_2+\frac{5564}{9}\zeta_3+\frac{193}{5}\zeta_2^2-\frac{1040}{9}\zeta_2\zeta_3-\frac{2384}{3}\zeta_5\Big]+C_A^2\Big[-\frac{31172}{81}$$

$$-\frac{478}{9}\zeta_2-\frac{10988}{81}\zeta_3+\frac{163}{5}\zeta_2^2-\frac{344}{9}\zeta_2\zeta_3-16\zeta_5\Big]+C_AT_FN_F\Big[\frac{13120}{81}-\frac{16}{9}\zeta_2$$

$$+\frac{17824}{81}\zeta_3-\frac{4}{5}\zeta_2^2\Big]\Big\}, \tag{1332}$$

$$\Delta C_{g_1,g}^{(3),d_{abc}} \simeq \frac{1}{N}\frac{d_{abc}d^{abc}N_F^2}{N_A}\Big\{\Big[128+128\zeta_2-256\zeta_3\Big]S_1^2+48+\Big[-64$$

$$-128\zeta_2+768\zeta_3-\frac{1536}{5}\zeta_2^2\Big]S_1+256\zeta_2-512\zeta_3+\frac{1728}{5}\zeta_2^2-640\zeta_5\Big\}. \tag{1333}$$

8.3 The large N_F limit

There are also some predictions on the large N_F behaviour of deep–inelastic Wilson coefficients. The $O(C_FT_F^2N_F^2)$ contribution to the flavor non–singlet Wilson coefficient of the structure function $F_2(x,Q^2)$ agrees with Eq. (15a) of [96], which can be brought into a more compact form.

For the structure function $F_L^{\rm NS}(N,a_s)$ the generating functional of the Wilson coefficients reads [97].

$$G_L(g)=C_F\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial^k}{\partial x^k}\Big[\frac{ge^{5/3x}\Gamma(x+N)}{(1-x)(2-x)(1+x+n)\Gamma(1+x)\Gamma(N)}\Big]_{x=(4/3)T_FN_Fg}. \tag{1334}$$

The $O(a_s^k)$ Wilson coefficient is given by its kth expansion coefficient in the variable g, $f_L^{(k)}(g=a_s)$ to be multiplied by $(-1)^{k+1}$. This prediction agrees with the corresponding contributions in our direct calculation to three loop order and implies a modification of [97].

9 Conclusions

We have calculated the unpolarized massless three–loop Wilson coefficients of the structure functions $F_2(x,Q^2)$ and $F_L(x,Q)$ for pure photon exchange and for the charged current structure function $xF_3(x,Q^2)$ to three–loop order. In the polarized case, we calculated the Wilson

coefficients to the structure function $g_1(x, Q^2)$ to three–loop order in the Larin scheme and for the non–singlet case also in the $\overline{\text{MS}}$ scheme. We applied the forward Compton amplitude for the respective scattering cross section and followed the other technical calculation steps having been described in our previous calculations of the three–loop anomalous dimensions in Refs. [27, 28]. By using the method of arbitrarily high Mellin moments [61] we have obtained a sufficiently large basis to compute the Wilson coefficients without any reference to further structural assumptions. In parallel, we also used the method of differential equations [80] since we face only first order factorizable problems, which, even further, relate to harmonic polylogarithms only. This method works for any basis of master integrals. The Wilson coefficients depend on 60 harmonic sums, weighted by rational functions in the Mellin variable N, after applying the algebraic relations, and on 31 harmonic sums by also applying the structural relations. We confirm former results on F_2, F_L and xF_3 at three–loop order in Refs. [15, 17] for the first time and all earlier results at one– and two–loop order, also for the structure function $g_1(x, Q^2)$. The three–loop results for the structure function $g_1(x, Q^2)$ are new. Concerning the polarized case we would like to remark that one may perfectly work in the Larin scheme, expressing the Wilson coefficients, the anomalous dimensions and the parton densities at the starting scale in this scheme, representing the polarized deep–inelastic structure functions or any other observables. For convenience, we also provide the expansions of the Wilson coefficients in the small x and large x region. The latter information may be of relevance studying the high energy Regge limit and the soft region, respectively. We also checked some predictions in the large N_F limit. The present results are of importance for the application in experimental and phenomenological analyzes of precision deep–inelastic data and in particular for precision measurements of the strong coupling constant.

A $Z_5^{\text{NS}}(N)$

In the following we calculate the function $Z_5^{\text{NS}}(N)$ needed to transform the non–singlet Wilson coefficient of the structure function g_1 from the Larin into the $\overline{\text{MS}}$ scheme to three–loop order. To two–loop order it has been calculated in [26] in unrenormalized form.[5] In [29] we provided the expansion coefficients needed to two loop order at the required depth in ε for the present calculation in the unrenormalized case.

The practical approach consists in deriving $Z_5^{\text{NS}}(N)$ in renormalized form, since it provides a finite renormalization. It is given by

$$Z_5^{\text{NS}}(N) \;\; = \;\; \frac{A_{qq}^{\text{NS,phys,ac,ren}}}{A_{qq}^{\text{NS,phys,Larin,ren}}} = 1 + a_s z_5^{(1),\text{NS}} + a_s^2 z_5^{(2),\text{NS}} + a_s^3 z_5^{(3),\text{NS}} + O(\hat{a}_s^4), \quad (1335)$$

where the OMEs are calculated for anticommuting γ_5 and in the Larin scheme. The gauge parameter ξ is defined in [27]. Note that in the Larin scheme one has to add the physical and EOM expansion coefficients projected in Ref. [29], although the two projections are orthogonal.

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[5]Note that Eq. (A11) in Ref. [26] contains typographical errors.

For odd integers N it is given by

$$z_5^{(1),\text{NS,odd}} = -C_F \frac{8}{N(1+N)} \tag{1336}$$

$$z_5^{(2),\text{NS,odd}} = C_F T_F N_F \frac{16(-3-N+5N^2)}{9N^2(1+N)^2} + C_F^2 \left[\frac{16(1+2N)}{N^2(1+N)^2} S_1 + \frac{16}{N(1+N)} S_2 \right.$$

$$\left. + \frac{8Q_2}{N^3(1+N)^3} + \frac{32}{N(1+N)} S_{-2} \right] - C_A C_F \left[\frac{4Q_5}{9N^3(1+N)^3} + \frac{16}{N(1+N)} S_{-2} \right],$$
$$\tag{1337}$$

$$z_5^{(3),\text{NS,odd}} = C_F T_F^2 N_F^2 \frac{128Q_1}{81N^3(1+N)^3} + C_A \left\{ C_F T_F N_F \left[-\frac{32Q_3}{9N^3(1+N)^3} S_1 + \frac{16Q_{13}}{81N^4(1+N)^4} \right.\right.$$

$$-\frac{64}{3N(1+N)} S_3 + \frac{128(-3+4N+10N^2)}{27N^2(1+N)^2} S_{-2} - \frac{128}{9N(1+N)} S_{-3}$$

$$\left. -\frac{256}{9N(1+N)} S_{-2,1} + \frac{128}{3N(1+N)} \zeta_3 \right] + C_F^2 \left[\frac{64Q_4}{27N^3(1+N)^3} S_2 \right.$$

$$+\frac{8Q_{15}}{27(N-1)N^5(1+N)^5(2+N)} + \left(-\frac{16Q_{11}}{27N^4(1+N)^4} - \frac{512}{3N(1+N)} S_3 \right.$$

$$\left. -\frac{3584}{3N(1+N)} S_{-2,1} \right) S_1 - \frac{32(-30-7N+5N^2)}{9N^2(1+N)^2} S_3 - \frac{320}{3N(1+N)} S_4$$

$$+\left(\frac{32Q_{12}}{27(N-1)N^3(1+N)^3(2+N)} + \frac{64(-10+19N+11N^2)S_1}{3N^2(1+N)^2} \right.$$

$$\left. +\frac{256}{3N(1+N)} S_2 \right) S_{-2} - \frac{512}{3N(1+N)} S_{-2}^2 + \left(-\frac{64(78-N+5N^2)}{9N^2(1+N)^2} \right.$$

$$\left. +\frac{768}{N(1+N)} S_1 \right) S_{-3} + \frac{1472}{3N(1+N)} S_{-4} - \frac{256(-33+16N+13N^2)}{9N^2(1+N)^2} S_{-2,1}$$

$$+\frac{1024}{3N(1+N)} S_{3,1} - \frac{3712}{3N(1+N)} S_{-2,2} - \frac{1280}{N(1+N)} S_{-3,1} + \frac{7168}{3N(1+N)} S_{-2,1,1}$$

$$\left. \left. -\frac{96(-2+5N+5N^2)}{N^2(1+N)^2} \zeta_3 \right] \right\} + C_F^2 T_F N_F \left[\frac{8Q_7}{27N^4(1+N)^4} + \frac{256(3+N-5N^2)}{27N^2(1+N)^2} \right.$$

$$\times S_2 + \frac{128(12+17N-14N^3+3N^4)}{27N^3(1+N)^3} S_1 + \frac{512}{9N(1+N)} S_3 + \frac{256}{9N(1+N)} S_{-3}$$

$$\left. -\frac{256(-3+4N+10N^2)}{27N^2(1+N)^2} S_{-2} + \frac{512}{9N(1+N)} S_{-2,1} - \frac{128}{3N(1+N)} \zeta_3 \right]$$

$$+C_A^2 C_F \left[-\frac{4Q_{16}}{81(N-1)N^5(1+N)^5(2+N)} + \left(\frac{16Q_9}{9N^4(1+N)^4} + \frac{256}{3N(1+N)} S_3 \right.\right.$$

$$\left. +\frac{1024}{3N(1+N)} S_{-2,1} \right) S_1 - \frac{16(8+N+N^2)}{3N^2(1+N)^2} S_3 + \frac{256}{3N(1+N)} S_4$$

$$+\left(-\frac{64Q_{10}}{27(N-1)N^3(1+N)^3(2+N)}-\frac{256(-1+N+N^2)}{3N^2(1+N)^2}S_1\right)S_{-2}$$

$$+\frac{64}{N(1+N)}S_{-2}^2+\left(\frac{64(21+N+N^2)}{9N^2(1+N)^2}-\frac{512}{3N(1+N)}S_1\right)S_{-3}-\frac{320}{3N(1+N)}S_{-4}$$

$$-\frac{512}{3N(1+N)}S_{3,1}+\frac{128(-21+10N+10N^2)}{9N^2(1+N)^2}S_{-2,1}+\frac{1024}{3N(1+N)}S_{-2,2}$$

$$+\frac{1024}{3N(1+N)}S_{-3,1}-\frac{2048}{3N(1+N)}S_{-2,1,1}+\frac{32(-2+5N+5N^2)}{N^2(1+N)^2}\zeta_3\Bigg]$$

$$+C_F^3\Bigg[-\frac{8Q_{14}}{3(N-1)N^5(1+N)^5(2+N)}+\left(\frac{16Q_8}{3N^4(1+N)^4}-\frac{256(1+2N)}{3N^2(1+N)^2}S_2\right.$$

$$+\frac{1024}{N(1+N)}S_{-2,1}\Bigg)S_1-\frac{128(1+3N+3N^2)}{3N^3(1+N)^3}S_1^2-\frac{32(3+11N-4N^2+4N^3)}{3N^2(1+N)^3}S_2$$

$$-\frac{128}{3N(1+N)}S_2^2-\frac{64(2+5N+N^2)S_3}{3N^2(1+N)^2}-\frac{128}{3N(1+N)}S_4$$

$$+\left(-\frac{64Q_6}{3(N-1)N^3(1+N)^3(2+N)}-\frac{128(-2+11N+3N^2)}{3N^2(1+N)^2}S_1\right.$$

$$\left.-\frac{512}{3N(1+N)}S_2\right)S_{-2}+\frac{256}{3N(1+N)}S_{-2}^2+\left(\frac{128(12-N+N^2)}{3N^2(1+N)^2}\right.$$

$$\left.-\frac{2560}{3N(1+N)}S_1\right)S_{-3}-\frac{1664}{3N(1+N)}S_{-4}+\frac{512(-4+2N+N^2)}{3N^2(1+N)^2}S_{-2,1}$$

$$-\frac{512}{3N(1+N)}S_{3,1}+\frac{3328}{3N(1+N)}S_{-2,2}+\frac{3584}{3N(1+N)}S_{-3,1}-\frac{2048}{N(1+N)}S_{-2,1,1}$$

$$+\frac{64(-2+5N+5N^2)}{N^2(1+N)^2}\zeta_3\Bigg] \tag{1338}$$

with

$$Q_1 = N^4+12N^3+7N^2-4N-3, \tag{1339}$$
$$Q_2 = 2N^4+N^3+8N^2+5N+2, \tag{1340}$$
$$Q_3 = 3N^4+6N^3+5N^2+2N+2, \tag{1341}$$
$$Q_4 = 85N^4+104N^3+13N^2-6N+18, \tag{1342}$$
$$Q_5 = 103N^4+140N^3+58N^2+21N+36, \tag{1343}$$
$$Q_6 = N^6-3N^5+9N^3-33N^2-6N+8, \tag{1344}$$
$$Q_7 = 17N^6+207N^5-685N^4-691N^3-312N^2+80N+48, \tag{1345}$$
$$Q_8 = 22N^6+50N^5+41N^4-132N^3-153N^2-104N-40, \tag{1346}$$
$$Q_9 = 24N^6+72N^5+44N^4-32N^3-35N^2-7N-12, \tag{1347}$$
$$Q_{10} = 85N^6+222N^5-38N^4-336N^3-92N^2+69N+36, \tag{1348}$$
$$Q_{11} = 165N^6-185N^5-1034N^4-1285N^3-895N^2-726N-396, \tag{1349}$$
$$Q_{12} = 349N^6+861N^5-152N^4-1263N^3-665N^2+222N+216, \tag{1350}$$
$$Q_{13} = 485N^6+643N^5+253N^4+85N^3+326N^2-96N-144, \tag{1351}$$

$$
\begin{aligned}
Q_{14} &= 24N^{10} + 99N^9 + 259N^8 + 308N^7 - 186N^6 - 853N^5 - 1153N^4 - 82N^3 \\
&\quad +344N^2 + 328N + 144, \quad (1352) \\
Q_{15} &= 845N^{10} + 3292N^9 + 7545N^8 + 11366N^7 - 121N^6 - 19168N^5 - 19017N^4 \\
&\quad -2522N^3 + 5420N^2 + 4008N + 1440, \quad (1353) \\
Q_{16} &= 6087N^{10} + 24679N^9 + 32532N^8 + 11838N^7 - 18471N^6 - 38727N^5 \\
&\quad -37968N^4 - 12190N^3 + 11772N^2 + 8352N + 1728. \quad (1354)
\end{aligned}
$$

To derive the above relations one has to apply the relations [60]

$$
S_{n_1,\ldots,n_p}\left(\frac{N}{2}\right) = 2^{n_1+\ldots+n_p-p} \sum_{\pm} S_{\pm n_1,\ldots,\pm n_p}(N). \quad (1355)
$$

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