

# Adler function, Bjorken Sum Rule and Crewther-Broadhurst-Kataev relation with generic fermion representations at order $O(\alpha_s^4)$

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## Abstract

We compute the nonsinglet Adler  $D$ -function and the coefficient function for Bjorken polarized sum rules  $S^{Bjp}$  at order  $O(\alpha_s^4)$  in an extended QCD model with arbitrary number of fermion representations. The Crewther-Broadhurst-Kataev (CBK) relation in this order is confirmed.

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## 1. Introduction

The Crewther-Broadhurst-Kataev (CBK) relation [1, 2] demonstrates a non-trivial connection between two (at first sight seemingly unrelated) important physical quantities, namely the (non-singlet) Adler  $D$ -function

$$D(L, a) = 1 + 3 C_F a + \sum_{i=2}^{\infty} d_i(L) a^i(\mu^2) \quad (1)$$

and the (non-singlet) coefficient function for the Bjorken polarized sum rules

$$S^{Bjp}(L, a) = 1 - 3 C_F a + \sum_{i=2}^{\infty} c_i(L) a^i(\mu^2). \quad (2)$$

Here  $L = \ln \frac{\mu^2}{Q^2}$ ,  $\mu$  is the normalization scale in  $\overline{\text{MS}}$ -scheme [3, 4] (which we will assume throughout the paper) and  $a = \frac{g_s^2}{16\pi^2} = \frac{\alpha_s}{4\pi}$  (precise definitions for both functions and color factors involved will be given in Sections 2.2, 2.3 and 2.1 correspondingly).

The functions (1) and (2) are very well studied in perturbative QCD. Due to works [5–14] they are known to impressively high order  $\alpha_s^4$ . The CBK relation

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connecting both functions reads<sup>1</sup> :

$$D(a) C^{Bjp}(a) = 1 + \beta(a) K(a), \quad K(a) = a K_1 + a^2 K_2 + a^3 K_3 + \dots \quad (3)$$

Here

$$\beta(a) = \mu^2 \frac{d}{d\mu^2} \ln a(\mu) = \sum_{i \geq 1} \beta_i a^i \quad (4)$$

is the QCD  $\beta$ -function describing the *running* of the coupling constant  $a$  with respect to a change of the normalization scale  $\mu$  and with its first term

$$\beta_1 = -\frac{11}{3} C_A + \frac{4}{3} T_F n_f$$

being responsible for asymptotic freedom of QCD. The term proportional the  $\beta$ -function responsible for deviation from the limit of exact conformal invariance, with the deviation starting in order  $\alpha_s^2$ , and was suggested [2] on the basis of  $\mathcal{O}(\alpha_s^3)$  calculations of  $D(a)$  [8, 15] and  $C^{Bjp}(a)$  [16]. The original relation without this term was first proposed in [1].

The fact that the CBK relation is valid up to maximally known order in  $\alpha_s$  is highly non-trivial. Indeed, a simple counting of available color factors shows that fulfillment of (3) sets as many as 6 constraints at the sum  $d_4 + c_4$  and all of them are met identically. At lower orders the number of constraints is 2 and 3 for the sums  $d_2 + c_2$  and  $d_3 + c_3$  correspondingly (see discussions in [2, 14] and Section 4).

Some formal arguments in favour of (3) were suggested in [17, 18]. Unfortunately, these considerations can not replace a real proof. Such a proof should demonstrate at least how it works in detail and in which renormalization schemes it holds<sup>2</sup>. Finally, it would be highly desirable if the future proof would clarify a way of computing the factor  $K(a)$  *directly* that is without previous calculations of  $D(a)$  and  $C^{Bjp}(a)$ .

In the present work we use an extended QCD (eQCD) model with arbitrary number of fermion representations in order to subject the CBK relation to one more non-trivial test. We compute both components  $D(a)$  and  $C^{Bjp}(a)$  within the extended QCD to order  $\alpha_s^4$  and demonstrate the validity of the resulting CBK relation. Let us stress that the knowledge of both  $D(a)$  and  $C^{Bjp}(a)$  in QCD with multiple fermion representations provides important ingredients to obtain the so-called  $\beta$ -expansion representation [20–23] for observables. This representation allows one to apply the extended BLM (eBLM) approach to optimize the PT series [20, 23, 24]. The approach suggests a way to resum the non-conformal parts of various QCD observables into the scale of the coupling

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<sup>1</sup>We omit direct indication on  $L$ -dependence in places where it can not lead to misunderstandings.

<sup>2</sup>It has been shown in [19] that the CBK relation ceases to take place in the 't Hooft  $\overline{\text{MS}}$ -based scheme.

in a unique way for any optimization task. Note that there exists an alternative method known as the Principle of Maximum Conformality (PMC) [25–27]. In this approach content of  $\beta$ -expansion as well as results of optimization in general differ from those in eBLM.

## 2. Preliminaries

### 2.1. QCD-like Lagrangian and notations for color factors

The Lagrangian of a (massless) QCD-like model extended to include several fermion representations of the gauge group (to be referred as QCDe) is given by (our notations essentially follow those of [28])

$$\begin{aligned} \mathcal{L}_{\text{QCDe}} &= -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{2\lambda} (\partial_\mu A^{a\mu})^2 + \partial_\mu \bar{c}^a \partial^\mu c^a + g_s f^{abc} \partial_\mu \bar{c}^a A^{b\mu} c^c \\ &+ \sum_{r=1}^{N_{\text{rep}}} \sum_{q=1}^{n_{f,r}} \left\{ \frac{i}{2} \bar{\psi}_{q,r} \overleftrightarrow{\not{D}} \psi_{q,r} + g_s \bar{\psi}_{q,r} \not{A}^a T^{a,r} \psi_{q,r} \right\}, \end{aligned} \quad (5)$$

with

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c, \quad [T^{a,r}, T^{b,r}] = i f^{abc} T^{c,r}, \quad (6)$$

and  $f^{abc}$  being the structure constants of the gauge group. The index  $r$  specifies the fermion representation and the index  $q$  the fermion flavour,  $\psi_{q,r}$  is the corresponding fermion field. The number of fermion flavours in representation  $r$  is  $n_{f,r}$  for any of the  $N_{\text{rep}}$  fermion representations.

For every fermion representation  $r$  we have two quadratic Casimir operators  $C_{F,r}$  and  $T_{F,r}$

$$\delta_{ij} C_{F,r} = T_{ik}^{a,r} T_{kj}^{a,r}, \quad T_{F,r} \delta^{ab} = \mathbf{Tr} (T^{a,r} T^{b,r}) = T_{ij}^{a,r} T_{ji}^{b,r}. \quad (7)$$

The dimension of  $r$  will be denoted as  $d_{F,r}$ . As for gluon (adjoint) representation we use the standard notation  $C_A$  and  $N_A$  for the corresponding quadratic Casimir operator and dimension of the gluon representation. The standard QCD corresponds to the case of  $N_{\text{rep}} = 1$ . If  $N_{\text{rep}} > 1$  we will consider the first fermion representation as a special one in what follows with

$$C_{F,1} \equiv C_F, \quad d_{F,1} \equiv d_F, \quad n_{f,1} \equiv n_f, \quad T_{F,1} \equiv T_F \quad \text{and} \quad T^{a,1} \equiv T^a.$$

Let us stress that all external operators (like the EM current) which appear later are assumed to involve only fermion fields  $\psi_{q,1}$  which we will refer also as  $\psi_q$ .

In addition to quadratic Casimir operators we need also quartic ones which are expressed in terms of symmetric tensors (see [29] for details)

$$d_R^{a_1 a_2 a_3 a_4} = \frac{1}{n!} \sum_{\text{perm } \pi} \mathbf{Tr} \{ T^{a_{\pi(1)},R} T^{a_{\pi(2)},R} T^{a_{\pi(3)},R} T^{a_{\pi(4)},R} \}, \quad (8)$$

where  $R$  can be any fermion representation,  $R = F, r$  ( $r = 1 \dots N_{\text{rep}}$ ) or the adjoint representation,  $R = A$  where  $T_{bc}^{a,A} = -i f^{abc}$ .

The following quartic Casimir operators appear in our results at order  $\alpha_s^4$ :

$$\tilde{d}_{FA} = \frac{d_F^{abcd} d_A^{abcd}}{d_F}, \quad \tilde{d}_{FF,r} = \frac{d_F^{abcd} d_{F,r}^{abcd}}{d_F}, \quad (9)$$

with  $d_F^{abcd} \equiv d_{F,1}^{abcd}$  and  $\tilde{d}_{FF} \equiv \frac{d_F^{abcd} d_{F,1}^{abcd}}{d_F}$ .

### 2.2. Adler function in QCDe

We start with the (non-singlet) polarization function  $\Pi(L, a)$  of the vector current  $j_\alpha = \bar{\psi}_q \gamma_\alpha \psi_q$  and defined as

$$(-g_{\alpha\beta} q^2 + q_\alpha q_\beta) \Pi^{\text{NS}}(L, a) = i \int d^4 x e^{iq \cdot x} \langle 0 | T j_\alpha(x) j_\beta(0) | 0 \rangle^{\text{NS}}, \quad (10)$$

where  $Q^2 = -q^2$ ,  $L = \ln \frac{\mu^2}{Q^2}$ . It is understood that the rhs of (10) includes only non-singlet diagrams, that is those with both external currents belonging to one and the same quark loop. The Adler function is defined as (the normalization factor below is conventionally fixed by the requirement that in Born approximation the Adler function starts from one) [30]

$$d_F n_f D(L, a_s) = -12 \pi^2 Q^2 \frac{d}{dQ^2} \Pi^{\text{NS}}(L, a). \quad (11)$$

It is worthwhile to note that the Adler function unlike the polarization operator (10) is scale invariant due to the derivative in  $Q^2$  which kills a quadratic UV divergence of (10) around integration region  $x \approx 0$ .

### 2.3. Bjorken function in QCDe

The most convenient for us definition of the coefficient function  $S^{Bjp}$  comes from the following Operator Product Expansion (OPE):

$$\int T[j_\alpha(x) j_\beta(0)] e^{iqx} dx |_{q^2 \rightarrow -\infty} \approx \frac{q^\sigma}{q^2} C_{\text{NS}}^{Bjp}(L, a) \epsilon_{\alpha\beta\rho\sigma} A_\rho(0) \quad (12)$$

+ ... (singlet and other terms),

where  $A_\rho = \bar{\psi}_q \gamma_\rho \gamma_5 \psi_q$  is the axial vector current. The function  $C_{\text{NS}}^{Bjp}$  is by definition contributed by non-singlet diagrams only (see Fig. 1). In what follows we will not write index NS explicitly. Singlet contributions to OPE (12) were discussed in [31].

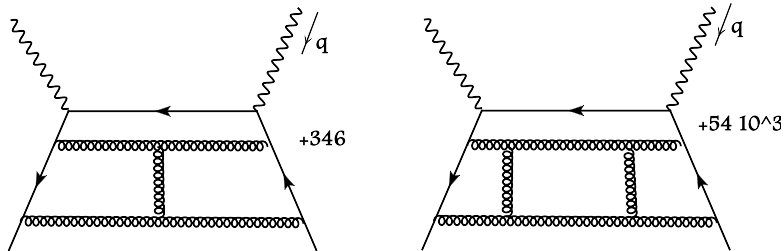


Figure 1: Examples of diagrams contributing to the coefficient function  $C_{NS}^{Bjp}$  at three and four loops.

There is a technical subtlety in definition of the  $\gamma_5$  matrix appearing in the definition of the axial current. Following works [16, 14] we will use so-called Larin's approach [32]. It means that renormalized axial current is defined as

$$A_\rho \equiv z_A^{\text{NS}} \frac{1}{6} \epsilon_{\alpha\beta\sigma\rho} [\bar{\psi}_q \gamma_{[\alpha\beta\sigma]} \psi_q]^{\text{MS}}, \quad \gamma_{[\alpha\beta\sigma]} \equiv \frac{1}{2} (\gamma_\alpha \gamma_\beta \gamma_\sigma - \gamma_\sigma \gamma_\beta \gamma_\alpha), \quad (13)$$

where  $[\bar{\psi}_q \gamma_{[\alpha\beta\sigma]} \psi_q]^{\text{MS}}$  stands for the  $\overline{\text{MS}}$ -renormalized current. The (finite) factor  $z_A^{\text{NS}}$  is chosen in such a way to effectively restore the anticommutativity of  $\gamma_5$  (see corresponding discussion in [32, 33]).

#### 2.4. Color factors

For future reference let us describe color factors which appear in all three components of the CBK relation. First, we note an obvious fact that one and the same collection of color factors may appear in  $d_n$ ,  $c_n$ . Second, due to the prefactor  $\beta(a)$  in (3), the same set of color factors describes coefficient  $K_{n+1}$ . This is true for both QCD and QCDe cases [14]. Another important fact is that transition from QCD to QCDe does touch only  $n_f$ -dependent color factors<sup>3</sup>. The corresponding modifications are shown in Table 1. Here we use the following notations:

$$\mathbf{nT} \equiv \sum_i n_{f,i} T_{F,i}, \quad \mathbf{nTC1} \equiv \sum_i n_{f,i} T_{F,i} C_{F,i}, \quad \mathbf{nTC2} \equiv \sum_i n_{f,i} T_{F,i} C_{F,i}^2. \quad (14)$$

Note that if a color structure in the left column of Table 1 does not proliferate then the corresponding contributions should be identical in QCD and QCDe results.

In order to transform a QCDe result to the corresponding one in the standard

<sup>3</sup>This means that contributions proportional to  $n_f$ -independent color factors are identical in both cases.

QCD	QCDe
$\alpha_s^2$	
$n_f T_f$	<b>nT</b>
$\alpha_s^3$	
$C_F^2 n_f T_f$	$C_F^2 \mathbf{nT}, C_f \mathbf{nTC1}$
$C_F n_f^2 T_f^2$	$C_F (\mathbf{nT})^2$
$C_F C_A n_f T_f$	$C_F C_A \mathbf{nT}$
$\alpha_s^4$	
$C_F^3 n_f T_f$	$C_F^3 \mathbf{nT}, C_f^2 \mathbf{nTC1}, C_f \mathbf{nTC2}$
$C_F^2 n_f^2 T_f^2$	$C_F (\mathbf{nT})^2, C_f \mathbf{nTC1}$
$C_F^2 C_A n_f T_f$	$C_F^2 C_A \mathbf{nT}, C_F C_A \mathbf{nTC1}$
$C_F n_f^3 T_f^3$	$C_F (\mathbf{nT})^3$
$C_F C_A n_f^2 T_f^2$	$C_F C_A (\mathbf{nT})^2$
$C_F C_A^2 n_f T_f$	$C_F C_A^2 (\mathbf{nT})$
$n_f \tilde{d}_{FF}$	$\sum_r n_{f,r} \tilde{d}_{FF,r}$

Table 1: Proliferation of the  $n_f$ -dependent color factors in the QCDe-model.

QCD one should make the following replacements:

$$\begin{aligned}
\mathbf{nT} &\rightarrow n_f T_F, & \mathbf{nTC1} &\rightarrow n_f T_F C_F, \\
\mathbf{nTC2} &\rightarrow n_f T_F C_F^2, & \sum_r n_{f,r} \tilde{d}_{FF,r} &\rightarrow n_f \tilde{d}_{FF}.
\end{aligned} \tag{15}$$

An inspection of Table 1 clearly shows that in the case of the QCDe-model the number of extra constraints imposed by the CBK relation on the combinations  $d_3 + c_3$  and  $d_4 + c_4$  is increased from 3 and 6 to 4 and 9 correspondingly.

### 3. Calculation and results

#### 3.1. Results for $D$ and $S^{Bjp}$

We have computed the functions  $D$  and  $S^{Bjp}$  to order  $\mathcal{O}(\alpha_s^4)$  using essentially the same methods as in [14] (for a short review see [34]). All momentum diagrams have been generated with QGRAF [35] and reduced to master integrals (well known from [36, 37]) with the help of the  $1/D$  expansion [38, 39].

For calculation of color factors we have employed a generalization of the FORM [40] package COLOR [29] developed by M. Zoller [28]. Below we present our results for the Adler function and the coefficient function  $S^{Bjp}$  as defined by (1,12).

Note that we set  $\mu^2 = Q^2$ ; the full dependence on  $\mu$  can be easily restored by expressing  $a(Q^2)$  by  $a(\mu^2)$  with the help of the standard RG evolution equation for  $a$  (the  $\beta$ -function for QCD is known at four loops from [28]).

$$d_1 = 3C_F, \quad (16)$$

$$d_2 = -\frac{3}{2}C_F^2 + C_F C_A \left( \frac{123}{2} - 44\zeta_3 \right) - 2C_F(\mathbf{nT})(11 - 8\zeta_3), \quad (17)$$

$$\begin{aligned} d_3 = & -\frac{69}{2}C_F^3 + \\ & C_F^2 \left[ C_A (-127 - 572\zeta_3 + 880\zeta_5) + (\mathbf{nT})(72 + 208\zeta_3 - 320\zeta_5) \right] + \\ & C_F C_A^2 \left( \frac{90445}{54} - \frac{10948}{9}\zeta_3 - \frac{440}{3}\zeta_5 \right) + \\ & C_F C_A (\mathbf{nT}) \left( -\frac{31040}{27} + \frac{7168}{9}\zeta_3 + \frac{160}{3}\zeta_5 \right) + \\ & C_F (\mathbf{nT})^2 \left( \frac{4832}{27} - \frac{1216}{9}\zeta_3 \right) + C_F (\mathbf{nTC1})(-101 + 96\zeta_3), \quad (18) \end{aligned}$$

$$\begin{aligned} d_4 = & C_F^4 \left( \frac{4157}{8} + 96\zeta_3 \right) + \\ & C_F^3 \left[ C_A (-2024 - 278\zeta_3 + 18040\zeta_5 - 18480\zeta_7) \right. \\ & \quad \left. - \mathbf{nT} (-298 + 56\zeta_3 + 6560\zeta_5 - 6720\zeta_7) \right] + \\ & C_F^2 \left[ C_A^2 \left( -\frac{592141}{72} - \frac{87850}{3}\zeta_3 + \frac{104080}{3}\zeta_5 + 9240\zeta_7 \right) + \right. \\ & \quad \left. C_A (\mathbf{nT}) \left( \frac{67925}{9} + \frac{61912}{3}\zeta_3 - \frac{83680}{3}\zeta_5 - 3360\zeta_7 \right) + \right. \end{aligned}$$

$$\begin{aligned}
& (\mathbf{nT})^2 \left( -\frac{13466}{9} - \frac{10240}{3}\zeta_3 + \frac{16000}{3}\zeta_5 \right) + \mathbf{nTC1}(251 + 576\zeta_3 - 960\zeta_5) \Big] + \\
C_F & \left[ C_A^3 \left( \frac{52207039}{972} - \frac{912446}{27}\zeta_3 - \frac{155990}{9}\zeta_5 + 4840\zeta_3^2 - 1540\zeta_7 \right) \right. \\
& C_A^2(\mathbf{nT}) \left( -\frac{4379861}{81} + \frac{275488}{9}\zeta_3 + \frac{150440}{9}\zeta_5 - 1408\zeta_3^2 + 560\zeta_7 \right) + \\
& C_A(\mathbf{nT})^2 \left( \frac{1363372}{81} - \frac{83624}{9}\zeta_3 - \frac{43520}{9}\zeta_5 - 128\zeta_3^2 \right) + \\
& C_A(\mathbf{nTC1}) \left( -\frac{375193}{54} + 7792\zeta_3 + 400\zeta_5 - 2112\zeta_3^2 \right) + \\
& (\mathbf{nT})^3 \left( -\frac{392384}{243} + \frac{25984}{27}\zeta_3 + \frac{1280}{3}\zeta_5 \right) + \\
& (\mathbf{nT})(\mathbf{nTC1}) \left( \frac{63250}{27} - 2784\zeta_3 + 768\zeta_3^2 \right) + \\
& \left. \mathbf{nTC2} \left( \frac{355}{3} + 272\zeta_3 - 480\zeta_5 \right) \right] - \\
16 & \left[ \sum_r n_{f,r} \tilde{d}_{\text{FF},r} \cdot (13 + 16\zeta_3 - 40\zeta_5) + \tilde{d}_{\text{FA}} \cdot (-3 + 4\zeta_3 + 20\zeta_5) \right]. \tag{19}
\end{aligned}$$

The results for  $c_k$  of the Bjorken SR in QCDe,

$$c_1 = -3C_F, \tag{20}$$

$$c_2 = \frac{21}{2}C_F^2 - 23C_A C_F + 8C_F(\mathbf{nT}), \tag{21}$$

$$\begin{aligned}
c_3 & = -\frac{3}{2}C_F^3 + C_F^2 \left[ C_A \left( \frac{1241}{9} - \frac{176}{3}\zeta_3 \right) - \mathbf{nT} \left( \frac{664}{9} - \frac{64}{3}\zeta_3 \right) \right] + \\
& C_F C_A^2 \left( -\frac{10874}{27} + \frac{440}{3}\zeta_5 \right) + \\
& C_F C_A(\mathbf{nT}) \left( \frac{7070}{27} + 48\zeta_3 - \frac{160}{3}\zeta_5 \right) - \\
& C_F(\mathbf{nT})^2 \frac{920}{27} + C_F(\mathbf{nTC1})(59 - 48\zeta_3), \tag{22}
\end{aligned}$$



$$\begin{aligned}
c_4 = & -C_F^4 \left( \frac{4823}{8} + 96\zeta_3 \right) + \\
& C_F^3 \left[ -C_A \left( \frac{3707}{18} + \frac{7768}{3}\zeta_3 - \frac{16720}{3}\zeta_5 \right) + \right. \\
& \quad \left. \mathbf{nT} \left( \frac{5912}{9} + \frac{3296}{3}\zeta_3 - \frac{6080}{3}\zeta_5 \right) \right] + \\
& C_F^2 \left[ C_A^2 \left( \frac{1071641}{216} + \frac{25456}{9}\zeta_3 - \frac{22000}{9}\zeta_5 - 6160\zeta_7 \right) - \right. \\
& \quad C_A(\mathbf{nT}) \left( \frac{106081}{27} + \frac{9104}{9}\zeta_3 - \frac{8000}{9}\zeta_5 - 2240\zeta_7 \right) + \\
& \quad \left. (\mathbf{nT})^2 \left( \frac{16114}{27} - \frac{512}{3}\zeta_3 \right) - \mathbf{nTC1} \left( \frac{1399}{3} - 400\zeta_3 \right) \right] + \\
& C_F \left[ C_A^3 \left( -\frac{8004277}{972} + \frac{4276}{9}\zeta_3 + \frac{25090}{9}\zeta_5 - \frac{968}{3}\zeta_3^2 + 1540\zeta_7 \right) + \right. \\
& \quad C_A^2(\mathbf{nT}) \left( \frac{1238827}{162} + 236\zeta_3 - \frac{14840}{9}\zeta_5 + \frac{704}{3}\zeta_3^2 - 560\zeta_7 \right) - \\
& \quad C_A(\mathbf{nT})^2 \left( \frac{165283}{81} + \frac{688}{9}\zeta_3 - \frac{320}{3}\zeta_5 + \frac{128}{3}\zeta_3^2 \right) + \\
& \quad C_A(\mathbf{nTC1}) \left( \frac{124759}{54} - 1280\zeta_3 - 400\zeta_5 \right) + \\
& \quad \frac{38720}{243}(\mathbf{nT})^3 - (\mathbf{nT})(\mathbf{nTC1}) \left( \frac{19294}{27} - 480\zeta_3 \right) - \\
& \quad \left. \mathbf{nTC2} \left( \frac{292}{3} + 296\zeta_3 - 480\zeta_5 \right) \right] + \\
& 16 \left[ \sum_r n_{f,r} \tilde{d}_{\text{FF},r} \cdot (13 + 16\zeta_3 - 40\zeta_5) + \tilde{d}_{\text{FA}} \cdot (-3 + 4\zeta_3 + 20\zeta_5) \right].
\end{aligned} \tag{23}$$

#### 4. CBK relation in QCDe

Using the color structures of  $d_2$  and  $d_3$  as templates we find that the CBK relation (3) is indeed fulfilled identically with the following values for the coefficients  $K_i$ :

$$K_1 = C_F \left( -\frac{21}{2} + 12 \zeta_3 \right), \quad (24)$$

$$K_2 = C_F^2 \left( \frac{397}{6} + 136 \zeta_3 - 240 \zeta_5 \right) + C_F C_A \left( -\frac{629}{2} + \frac{884}{3} \zeta_3 \right) \\ + (C_F \mathbf{nT}) \left( \frac{326}{3} - \frac{304}{3} \zeta_3 \right), \quad (25)$$

$$K_3 = C_F^3 \left( \frac{2471}{12} + 488 \zeta_3 - 5720 \zeta_5 + 5040 \zeta_7 \right) \\ + C_F^2 C_A \left( \frac{99757}{36} + \frac{16570}{3} \zeta_3 - \frac{24880}{3} \zeta_5 - 840 \zeta_7 \right) \\ + C_F C_A^2 \left( -\frac{406043}{36} + \frac{72028}{9} \zeta_3 - 1232 \zeta_3^2 + \frac{11900}{3} \zeta_5 \right) \\ + C_F C_A (\mathbf{nT}) \left( \frac{67520}{9} - \frac{40336}{9} \zeta_3 - \frac{8000}{3} \zeta_5 - 128 \zeta_3^2 \right) \\ + C_F (\mathbf{nT})^2 \left( -\frac{9824}{9} + \frac{6496}{9} \zeta_3 + 320 \zeta_5 \right) \\ + C_F^2 (\mathbf{nT}) \left( -\frac{11573}{9} - 2288 \zeta_3 + 4000 \zeta_5 \right) \\ + C_F (\mathbf{nTC1}) \left( \frac{1713}{2} - 1380 \zeta_3 + 576 \zeta_3^2 \right). \quad (26)$$

As expected from Table 1 and relations (15) and (15) coefficient  $K_2$  in QCD is essentially identical to the one in QCD (that is after identification  $\mathbf{nT}$  with  $n_f T_f$ ). Coefficient  $K_3$  in QCDe is different from the case of QCD only by 2 last terms. All constraints imposed by the CBK relation are fulfilled.

## 5. Conclusion

We have computed the nonsinglet Adler  $D$ -function and the coefficient function for Bjorken polarized sum rules  $S^{Bjp}$  at order  $O(\alpha^4)$  in the extended QCD model. The CBK relation is confirmed.

These results have been extensively used for construction and analyzing explicit expressions for the elements of the  $\{\beta\}$ -expansion for the nonsinglet Adler  $D$ -function and Bjorken polarized sum rules  $S^{Bjp}$  in the N<sup>4</sup>LO and higher orders in [41].

They may be also useful for renormalization group analysis of the  $D$  and  $S^{Bjp}$  functions in large- $N_c$  and large- $N_f$  limits [42, 43].

For readers's convenience all our results are collected in an ancillary file.

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