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The Two-Loop Massless Off-Shell QCD Operator Matrix Elements to Finite Terms

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Abstract

We calculate the unpolarized and polarized two-loop massless off-shell operator matrix elements in QCD to $O(\varepsilon)$ in the dimensional parameter in an automated way. Here we use the method of arbitrary high Mellin moments and difference ring theory, based on integration-by-parts relations. This method also constitutes one way to compute the QCD anomalous dimensions. The presented higher order contributions to these operator matrix elements occur as building blocks in the corresponding higher order calculations up to four-loop order. All contributing quantities can be expressed in terms of harmonic sums in Mellin- N space or by harmonic polylogarithms in z -space. We also perform comparisons to the literature.

1 Introduction

The unpolarized and polarized anomalous dimensions of the local twist-2 operators in Quantum Chromodynamics (QCD) play a fundamental role in the description of the scaling violations of the deep-inelastic structure functions. Their measurement provides one of the safest ways to measure the strong coupling constant $\alpha_s(M_Z^2) = 4\pi a_s$ [1]. While at first order, there is a wide variety of possibilities to calculate the anomalous dimensions, see e.g. [2], at higher orders only a few efficient methods are known. These are based either on the calculation of massless off-shell operator matrix elements (OMEs) [3–20], the calculation of the forward Compton amplitude of a space-like virtual gauge boson on a massless on-shell parton [21–25] and on massive on-shell OMEs [26–41]. All these methods have advantages and disadvantages and form complimentary ways to compute the anomalous dimensions.

To perform phenomenological analyses of the deep-inelastic world data also the process dependent massless and massive Wilson coefficients have to be calculated [21, 26, 33, 39, 41–70]. This is also necessary to account for the specific scaling violations implied by massive quarks, such as charm and bottom.

In this paper we present the results for the unpolarized and polarized massless off-shell OMEs up to two-loop order and to corrections of $O(\varepsilon)$ (at $O(a_s)$ or $O(\varepsilon^2)$), with the dimensional parameter $\varepsilon = D - 4$, including non-gauge invariant contributions. A part of these expansion coefficients of the various OMEs contribute to the physical OMEs up to four-loop order. Others emerge due to the kinematic breaking of gauge invariance both in the unpolarized and polarized off-shell case. One characteristic is the emergence of additional OMEs related to the breaking of the equation of motion (eom) and of new non-gauge invariant OMEs, also with new unphysical anomalous dimensions. These quantities play a role in the calculation of the unpolarized anomalous dimensions due to mixing. We extend earlier work of Refs. [19, 20] and perform the calculation in an automated way, applying methods which have been developed by us solving a series of massive three-loop problems and other applications during the last decade.

The theoretical basis for these calculations has been laid out in a series of papers describing the situation holding in the off-shell case, breaking gauge invariance, cf. [4, 15, 19, 20, 71–78]. Compared to the method of the forward Compton amplitude, the method of massless off-shell OMEs needs no precautions as reference to Higgs and gravitational subsidiary fields. The use of massive on-shell OMEs, as also the method of the forward Compton amplitude do not encounter gauge invariance problems, on the other hand. However, the massive OMEs allow to derive only the contributions $\propto T_F$ of the anomalous dimensions, except of going to one order higher in the coupling constant. A new challenge in the present approach is to master the breaking of gauge invariance to the respective perturbative order. We present the results in Mellin- N space, because the expressions are more compact than in momentum fraction z space. We also perform a detailed comparison to the literature [15, 19, 20, 79] and correct results given there, including Feynman rules, and also all non-gauge invariant terms. A by-product of the present calculation is the calculation of the (by now well-known) unpolarized physical anomalous dimensions to two-loop order. We present all contributing expansion coefficients up to $O(a_s^2)$ to the depth being needed in future four-loop calculations, extending the level previously attempted in Refs. [19, 20].

The paper is organized as follows. In Section 2 we give a brief outline of the formalism, including the renormalization, and the main steps of the calculation of the different off-shell OMEs. We then turn to the calculation of the OMEs of the so-called alien operators in Section 3, which also lead to additional anomalous dimensions. These operator matrix elements contribute via mixing to the renormalization of the unpolarized singlet operators, which we discuss in Section 4. In Sections 5–7 we present the expansion coefficients of the unpolarized and the

polarized standard OMEs and those for transversity for non-negative powers in the dimensional parameter ε . In Section 8 we compare to and correct partial results given previously in Refs. [19, 20]. Section 9 contains the conclusions. In Appendix A we provide Feynman rules for the alien operators, if not previously being presented in Refs. [38, 80, 81], and list the contributing polynomials appearing in the standard OMEs in Appendix B. Our results both in Mellin- N space and z -space, are given in ancillary files in computer-readable form. In z -space we use the decomposition given in [82], Eqs. (45,46).

2 The Formalism

The massless off-shell OMEs are defined by

$$\hat{A}_{ij}^l = \langle j(p) | O_i^l | j(p) \rangle, \quad \text{with } i, j = q, g, \quad (1)$$

where l further labels the type of the OME and p denotes the off-shell momentum with $p^2 < 0$. The twist-2 local physical operators¹ are given by

$$O_{q,r;\mu_1\dots\mu_N}^{\text{NS}} = i^{N-1} \mathbf{S} \left[\bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{trace terms}, \quad (2)$$

$$O_{q;\mu_1\dots\mu_N}^{\text{S}} = i^{N-1} \mathbf{S} \left[\bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi \right] - \text{trace terms}, \quad (3)$$

$$O_{g;\mu_1\dots\mu_N}^{\text{S}} = 2i^{N-2} \mathbf{SSp} \left[F_{\mu_1\alpha}^a D_{\mu_2} \dots D_{\mu_{N-1}} F_{\mu_N}^{\alpha,a} \right] - \text{trace terms}, \quad (4)$$

in the unpolarized case. In the polarized case the operators are

$$O_{q,r;\mu_1\dots\mu_N}^{\text{NS}} = i^{N-1} \mathbf{S} \left[\bar{\psi} \gamma_5 \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{trace terms}, \quad (5)$$

$$O_{q;\mu_1\dots\mu_N}^{\text{S}} = i^{N-1} \mathbf{S} \left[\bar{\psi} \gamma_5 \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi \right] - \text{trace terms}, \quad (6)$$

$$O_{g;\mu_1\dots\mu_N}^{\text{S}} = 2i^{N-2} \mathbf{SSp} \left[\frac{1}{2} \varepsilon_{\mu_1\alpha\beta\gamma} F^{\beta\gamma,a} D_{\mu_2} \dots D_{\mu_{N-1}} F_{\mu_N}^{\alpha,a} \right] - \text{trace terms}. \quad (7)$$

For transversity [83] the following local non-singlet operator contributes

$$O_{q,r;\mu\mu_1\dots\mu_N}^{\text{NS}} = i^{N-1} \mathbf{S} \left[\bar{\psi} \sigma_{\mu\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{trace terms}, \quad (8)$$

where $\sigma_{\mu\nu} = (i/2)[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu]$. Here Δ denotes a light-like vector, $\Delta \cdot \Delta = 0$. ψ denotes the quark field, λ_r , $r \in [1, N_F^2 - 1]$ the $SU(N_F)$ matrices, γ_μ a Dirac matrix, D_μ the covariant derivative and A_ν^a the gluon fields, with $F_{\mu\nu}^a$ the gluonic field strength tensor and $\varepsilon_{\alpha\beta\gamma\delta}$ the Levi-Civita pseudo-tensor. For further notations we refer to Ref. [82].

¹The unphysical local operators are defined in Section 3 below.

Furthermore, the relative normalization between the quark–singlet and gluon external states has to be fixed. We adopt the same convention as in the massive on–shell case [80] by demanding 4–momentum conservation

$$\int_0^1 dx x [\Sigma(x) + G(x)] = 1, \quad (9)$$

where Σ is the quark singlet distribution

$$\Sigma(x) = \sum_{k=1}^{N_F} [q_k(x) + \bar{q}_k(x)], \quad (10)$$

with q the quark, \bar{q} the antiquark distributions and G is the gluon distribution. In this way the operators are also fixed in the polarized case, replacing γ_μ by $\gamma_\mu\gamma_5$, etc., cf. [81].

We apply the following projectors to determine the different contributions to the OMEs. For external quark fields one uses in the unpolarized case

$$\hat{A}_{iq} = \left[\not{\Delta} \hat{A}_{iq}^{\text{phys}} + \not{p} \frac{\Delta \cdot p}{p^2} \hat{A}_{iq}^{\text{eom}} \right] (\Delta \cdot p)^{N-1}, \quad i = q, g. \quad (11)$$

The quarkonic projections are obtained by

$$\hat{A}_{iq}^{\text{phys}} = \frac{1}{4(\Delta \cdot p)^N} \text{tr} \left[\left(\not{p} - \frac{p^2}{\Delta \cdot p} \not{\Delta} \right) \hat{A}_{iq} \right] \quad (12)$$

$$\hat{A}_{iq}^{\text{eom}} = \frac{1}{(4\Delta \cdot p)^N} \text{tr} \left[\not{\Delta} \hat{A}_{iq} \right]. \quad (13)$$

For external gluons the decomposition is as follows [19]

$$\hat{A}_{ig,\mu\nu} = \hat{A}_{ig}^{\text{phys}} T_{\mu\nu}^{(1)} + \hat{A}_{ig}^{\text{eom}} T_{\mu\nu}^{(2)} + \hat{A}_{ig}^{\text{ngi}} T_{\mu\nu}^{(3)}, \quad (14)$$

where

$$T_{\mu\nu}^{(1)} = \frac{1}{2} [1 + (-1)^N] \left[g_{\mu\nu} - \frac{p_\mu \Delta_\nu + \Delta_\mu p_\nu}{\Delta \cdot p} + \frac{\Delta_\mu \Delta_\nu p^2}{(\Delta \cdot p)^2} \right] (\Delta \cdot p)^N, \quad (15)$$

$$T_{\mu\nu}^{(2)} = \frac{1}{2} [1 + (-1)^N] \left[\frac{p_\mu p_\nu}{p^2} - \frac{p_\mu \Delta_\nu + \Delta_\mu p_\nu}{\Delta \cdot p} + \frac{\Delta_\mu \Delta_\nu p^2}{(\Delta \cdot p)^2} \right] (\Delta \cdot p)^N, \quad (16)$$

$$T_{\mu\nu}^{(3)} = \frac{1}{2} [1 + (-1)^N] \left[-\frac{p_\mu \Delta_\nu + \Delta_\mu p_\nu}{2\Delta \cdot p} + \frac{\Delta_\mu \Delta_\nu p^2}{(\Delta \cdot p)^2} \right] (\Delta \cdot p)^N. \quad (17)$$

Later one more tensor structure is needed

$$T_{\mu\nu}^{(4)} = \frac{1}{2} [1 + (-1)^N] \left[\frac{p_\mu \Delta_\nu + \Delta_\mu p_\nu}{2\Delta \cdot p} \right] (\Delta \cdot p)^N, \quad (18)$$

with [19]

$$\begin{aligned} p^\mu T_{\mu\nu}^{(i)} &= 0, & (i = 1, 2), & & p^\mu T_{\mu\nu}^{(i)} &\neq 0, & (i = 3, 4) \\ p^\mu p^\nu T_{\mu\nu}^{(i)} &= 0, & (i = 1, 2, 3), & & p^\mu p^\nu T_{\mu\nu}^{(4)} &\neq 0. \end{aligned} \quad (19)$$

This tensor decomposition implies the choice of a physical gauge for the external gluon lines, with a propagator

$$D^{\mu\nu}(k^2) = i \frac{d^{\mu\nu}(k)}{k^2 + i0}, \quad d_{\mu\nu}(k) = -g^{\mu\nu} - n^2 \frac{k^\mu k^\nu}{(k \cdot n)^2} + \frac{n^\mu k^\nu + n^\nu k^\mu}{k \cdot n}, \quad (20)$$

and $n^2 \leq 0$.

The resulting Ward identities are

$$p^\mu \hat{A}_{qg,\mu\nu} = \frac{1}{2} [1 + (-1)^N] \left[-p_\nu + \frac{\Delta_\nu p^2}{\Delta \cdot p} \right] (\Delta \cdot p)^N \hat{A}_{qg}^{\text{ngi}}, \quad (21)$$

$$p^\mu p^\nu \hat{A}_{qg,\mu\nu} = 0. \quad (22)$$

The gluonic projections are given by

$$\hat{A}_{ig}^{\text{phys}} = \frac{1}{D-2} \left[g_{\mu\nu} + \frac{p^2}{(\Delta \cdot p)^2} \Delta_\mu \Delta_\nu - \frac{p_\mu \Delta_\nu + p_\nu \Delta_\mu}{\Delta \cdot p} \right] \hat{A}_{ig}^{\mu\nu}, \quad (23)$$

$$\hat{A}_{ig}^{\text{eom}} = \frac{p^2}{(\Delta \cdot p)^2} \Delta_\mu \Delta_\nu \hat{A}_{ig}^{\mu\nu}, \quad (24)$$

$$\hat{A}_{ig}^{\text{ngi}} = \left[\frac{p_\mu p_\nu}{4p^2} - \frac{p_\mu \Delta_\nu + p_\nu \Delta_\mu}{2\Delta \cdot p} \right] \hat{A}_{ig}^{\mu\nu}, \quad (25)$$

$$\hat{A}_{ig}^{\text{wi}} = \frac{p_\mu p_\nu}{4p^2} \hat{A}_{ig}^{\mu\nu}. \quad (26)$$

In the polarized case the OMEs $\Delta \hat{A}_{iq}^1$, $i = q, g$ are given by

$$\Delta \hat{A}_{iq} = \left[\gamma_5 \not{\Delta} \Delta \hat{A}_{iq}^{\text{phys}} + \gamma_5 \not{p} \frac{\Delta \cdot p}{p^2} \Delta \hat{A}_{iq}^{\text{eom}} \right] (\Delta \cdot p)^{N-1}, \quad (27)$$

and the OMEs with external gluon lines read

$$\Delta \hat{A}_{ig,\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} \frac{\Delta^\alpha p^\beta}{\Delta \cdot p} \Delta \hat{A}_{ig}^{\text{phys}}, \quad i = q, g. \quad (28)$$

To treat γ_5 in $D = 4 + \varepsilon$ dimensions we use the Larin scheme [20,86] and perform the replacement

$$\not{p} \gamma_5 = \frac{i}{6} \varepsilon_{\mu\nu\rho\sigma} p^\mu \gamma^\nu \gamma^\rho \gamma^\sigma, \quad (29)$$

while contracting the occurring ε -tensors in D dimensions using

$$\varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\alpha\lambda\tau\gamma} = -\text{Det} [g_\omega^\beta], \quad \beta = \alpha, \lambda, \tau, \gamma; \quad \omega = \mu, \nu, \rho, \sigma. \quad (30)$$

In this scheme the projections onto the different terms are given by

$$\begin{aligned} \Delta \hat{A}_{iq}^{\text{phys}} &= -\frac{1}{4(D-2)(D-3)} \varepsilon_{\mu\nu\rho\sigma} p^\rho \Delta^\sigma \text{tr} \left[\not{p} \gamma^\mu \gamma^\nu \Delta \hat{A}_{iq} \right] (\Delta \cdot p)^{-N-1} \\ &\quad - \frac{p^2}{4(D-2)(D-3)} \varepsilon_{\mu\nu\rho\sigma} p^\rho \Delta^\sigma \text{tr} \left[\not{\Delta} \gamma^\mu \gamma^\nu \Delta \hat{A}_{iq} \right] (\Delta \cdot p)^{-N-2}, \end{aligned} \quad (31)$$

$$\Delta \hat{A}_{iq}^{\text{eom}} = \frac{p^2}{4(D-2)(D-3)} \varepsilon_{\mu\nu\rho\sigma} p^\rho \Delta^\sigma \text{tr} \left[\not{\Delta} \gamma^\mu \gamma^\nu \Delta \hat{A}_{iq} \right] (\Delta \cdot p)^{-N-2}, \quad (32)$$

$$\Delta \hat{A}_{iq}^{\text{phys}} = \frac{1}{(D-2)(D-3)} \varepsilon_{\mu\nu\rho\sigma} \Delta^\rho p^\sigma (\Delta \cdot p)^{-N-1} \Delta \hat{A}_{iq}^{\mu\nu}. \quad (33)$$

There is no mixing between the physical and the ngi and the alien operators, because no symmetric rank two tensor contributes.

For transversity the tensor decomposition for the OME reads

$$\hat{A}_{qq,\mu}^{\text{NS,tr}} = \Delta^\rho \sigma_{\mu\rho} \tilde{A}_{qq}^{\text{NS,tr}} + c_1 \Delta_\mu + c_2 p_\mu + c_3 \gamma_\mu \not{p} + c_4 \Delta_\mu \not{\Delta} \not{p} + c_5 p_\mu \not{\Delta} \not{p}. \quad (34)$$

The physical OME can be extracted using the following projector

$$\tilde{A}_{qq}^{\text{NS,tr}} = \frac{i}{4(D-2)} \text{tr} \left[\left(-p^\mu \not{\Delta} \not{p} + \Delta \cdot p \gamma^\mu \not{p} + ip^2 \Delta_\rho \sigma^{\mu\rho} \right) \hat{A}_{qq,\mu}^{\text{NS,tr}} \right] (\Delta \cdot p)^{-N-2}. \quad (35)$$

In the following we will insert the wave function renormalization [84] and perform a partial renormalization of the coupling constant [85] and the gauge parameter [84], cf. [82] for details. We will then denote the partly renormalized OMEs by $(\Delta) \tilde{A}_{ij}$.

The following representations hold for \tilde{A}_{ij} and $(\Delta) \tilde{A}_{ij}$. The different partly renormalized OMEs are given by

$$\tilde{A}_{ij}^{\text{phys}} = \delta_{ij} + \sum_{k=1}^{\infty} S_\varepsilon^k \left(\frac{-p^2}{\mu^2} \right)^{k\varepsilon/2} a_s^k \tilde{A}_{ij}^{\text{phys},k} \quad (36)$$

$$\tilde{A}_{ij}^{\text{eom}} = \sum_{k=1}^{\infty} S_\varepsilon^k \left(\frac{-p^2}{\mu^2} \right)^{k\varepsilon/2} a_s^k \tilde{A}_{ij}^{\text{eom},k} \quad (37)$$

$$\tilde{A}_{ij}^{\text{ngi}} = \sum_{k=1}^{\infty} S_\varepsilon^k \left(\frac{-p^2}{\mu^2} \right)^{k\varepsilon/2} a_s^k \tilde{A}_{ij}^{\text{ngi},k}, \quad (38)$$

with μ the factorization scale and the spherical factor S_ε is given by

$$S_\varepsilon = \exp \left[\frac{\varepsilon}{2} (\gamma_E - \ln(4\pi)) \right], \quad (39)$$

where γ_E is the Euler-Mascheroni number. Analogous relations hold in the polarized case and the OMEs, cf. [82, 83], and those of transversity, which are flavor non-singlet quantities. The structure of the partial amplitudes up to two-loop order is

$$\tilde{A}_{ij}^{\text{phys},1} = \frac{1}{\varepsilon} \gamma_{ij}^{(0)} + a_{ij}^{(1,0)} + \varepsilon a_{ij}^{(1,1)} + \varepsilon^2 a_{ij}^{(1,2)} + O(\varepsilon^3), \quad ij = qq \text{ NS}, qq, gg, gg, \quad (40)$$

$$\tilde{A}_{ij}^{\text{eom},1} = b_{ij}^{(1,0)} b + \varepsilon b_{ij}^{(1,1)} + \varepsilon^2 b_{ij}^{(1,2)} + O(\varepsilon^3) \quad ij = qq \text{ NS}, qq, gg, gg, \quad (41)$$

$$\tilde{A}_{ij}^{\text{ngi},1} = \frac{1}{\varepsilon} \gamma_{gA}^{(0)} + c_{gA}^{(1,0)} + \varepsilon c_{gA}^{(1,1)} + \varepsilon^2 c_{gA}^{(1,2)} + O(\varepsilon^3), \quad ij = gg, \quad (42)$$

$$\tilde{A}_{ij}^{\text{phys},2} = \frac{1}{\varepsilon^2} a_{ij}^{(2,-2)} + \frac{1}{\varepsilon} a_{ij}^{(2,-1)} + a_{ij}^{(2,0)} + \varepsilon a_{ij}^{(2,1)} + O(\varepsilon^2), \quad ij = qq \text{ NS}, qq \text{ PS}, qq, gg, gg, \quad (43)$$

$$\tilde{A}_{ij}^{\text{eom},2} = \frac{1}{\varepsilon} b_{ij}^{(2,-1)} + b_{ij}^{(2,0)} + \varepsilon b_{ij}^{(2,1)} + O(\varepsilon^2), \quad ij = qq \text{ NS}, qq \text{ PS}, qq, gg, gg, \quad (44)$$

$$\tilde{A}_{ij}^{\text{ngi},2} = \frac{1}{\varepsilon^2} c_{ij}^{(2,-2)} + \frac{1}{\varepsilon} c_{ij}^{(2,-1)} + c_{ij}^{(2,0)} + \varepsilon c_{ij}^{(2,1)} + O(\varepsilon^2), \quad ij = qq, gg. \quad (45)$$

Here the contributions $e^{(k,-1(-2))}$ with $e = a, b, c$ contain the LO and NLO anomalous dimensions, cf. [19, 20, 81, 82]. Structures which lead to new anomalous dimensions are dealt with in Section 3. In the polarized case there are no ngi contributions, but the eom terms for $ij = qq$ NS, PS and qq to two-loop order. All results in the polarized case are presented in the Larin scheme [20, 86].

There are also eom contributions in the transversity cases, which we will not deal with in the present paper, since the tensor composition in this case is even richer, cf. [83], but the anomalous

dimensions come only from the physical part, cf. [82]. Note that our definitions differ in part from those in [19,20]. The corresponding mapping is obtained by performing the renormalization, which is carried out to 2-loop order in the present paper. The anomalous dimensions obey the following expansion in the strong coupling constant

$$\gamma_{ij}^a = \sum_{k=1}^{\infty} a_s^k \gamma_{ij}^{(k-1),a}. \quad (46)$$

Let us now turn to technical aspects of the calculation. The Feynman diagrams for the different operator insertions are generated by `QGRAF` [80,87]. The spinor and Lorentz-algebra is performed using `FORM` [88]. The different operator insertions are resummed using generating functions [89] either for even or odd integer moments, cf. [82], implied by the respective crossing relations [90,91]. In this way the local operators reappear in terms of propagators and one may derive the integration-by-parts relations [92] for the corresponding quantities, for which we use the package `Crusher` [93]. For part of the calculation we performed the reduction using `Litered` [94] and solved the differential equations by using the method of Refs. [95,96]. The general solution followed the route described in Refs. [81,82]. The relations between master integrals obtained by the reduction using `Crusher` allow to generate a sufficiently large number of Mellin moments by using the method of arbitrary high moments [97] for the different color and zeta factors, which formed the basis for the method of guessing [98,99], implemented in `Sage` [100,101], to determine the corresponding recurrences. Those were solved by applying difference-ring theory [102–111] as implemented in the package `Sigma` [112,113]. The generated recurrences in the present case factorize all at first order, unlike the case in a series of massive higher order calculations. First one obtains solutions in terms of also cyclotomic harmonic sums [114] because of separating even and odd moments. All the results can be written in terms of harmonic sums [115,116]

$$S_{b,\bar{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\bar{a}}(k), \quad S_{\emptyset} = 1, \quad a, b_i \in \mathbb{Z} \setminus \{0\}, \quad (47)$$

for which one maps first

$$S_{b,\bar{a}}(2N+1) = \frac{1}{2N+1} S_{\bar{a}}(2N+1) + S_{b,\bar{a}}(2N) \quad (48)$$

recursively and then applies the `HarmonicSums` [114–122] command `Synchronize` and finally the command `TransformToBasis[expr, Online → True]` is applied. All results are now obtained in terms of harmonic sums of argument N only.

In the following section we turn now to the calculation of the matrix elements of the so-called alien operators, which contribute via mixing to the (ultraviolet) operator renormalization in the unpolarized case and discuss in a subsequent section the mixing relations to two-loop order in explicit form.

3 The OMEs of the alien operators

These operator matrix elements contribute in the unpolarized case and have been discussed in detail in Refs. [15,19,79]. The OMEs containing gluonic operators mix with these OMEs. The operators are given by

$$O_A^{\mu_1, \dots, \mu_N} = i^{N-2} \mathbf{SSp} \left[F_{\alpha}^{a, \mu_1} D_{\alpha} \partial^{\mu_2} \dots \partial^{\mu_{N-1}} A_a^{\mu_N} \right]$$

$$\begin{aligned}
& +igf^{abc}F_{\alpha}^{a,\mu_1} \sum_{i=2}^{N-1} \kappa_i \partial^{\alpha} \left\{ (\partial^{\mu_2} \dots \partial^{\mu_{i-2}} A_b^{\mu_{i-1}}) (\partial^{\mu_2} \dots \partial^{\mu_{N-1}} A_c^{\mu_N}) \right\} \\
& +O(g^3) \Big] - \text{trace terms} , \tag{49}
\end{aligned}$$

$$\begin{aligned}
O_{\omega}^{\mu_1, \dots, \mu_N} & = i^{N-2} \mathbf{SSp} \left[\xi^a \partial^{\mu_1} \dots \partial^{\mu_{N-1}} \bar{\omega}^a \right. \\
& \left. -igf^{abc} \xi^a \sum_{i=2}^{N-1} \eta_i \partial^{\mu_1} \left\{ (\partial^{\mu_2} \dots \partial^{\mu_{i-2}} \bar{\omega}_b) (\partial^{\mu_2} \dots \partial^{\mu_{N-1}} A_c^{\mu_N}) \right\} \right. \\
& \left. +O(g^3) \right] - \text{trace terms} , \tag{50}
\end{aligned}$$

$$O_B^{\mu_1, \dots, \mu_N} = i^{N-1} \mathbf{S} \left[g \bar{\psi}_k \gamma_{\mu_1} (T_a)^{kl} A^{a, \mu_2} \partial^{\mu_3} \dots \partial_{\mu_N} \psi_l + O(g^3) \right] - \text{trace terms}, \tag{51}$$

with

$$\kappa_i = \frac{(-1)^i}{8} + \frac{3}{8} \left[\frac{(n-2)!}{(i-1)!(n-i-1)!} - \frac{(n-2)!}{i!(N-i-2)!} \right], \tag{52}$$

$$\eta_i = \frac{(-1)^i}{4} + \frac{1}{4} \left[3 \frac{(n-2)!}{(i-1)!(n-i-1)!} + \frac{(n-2)!}{i!(N-i-2)!} \right], \tag{53}$$

and ξ , $\bar{\omega}$ denote the ghost and antighost respectively. The Feynman rules for these operator insertions are summarized in Appendix A.

To one loop order the OMEs are given by, after performing partial renormalization,

$$\tilde{A}_{\text{Aq}}^{\text{phys}} = a_s S_{\varepsilon} \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[\frac{1}{\varepsilon} \gamma_{\text{Aq}}^{(0)} + a_{\text{Aq}}^{(1,0)} + \varepsilon a_{\text{Aq}}^{(1,1)} + \varepsilon^2 a_{\text{Aq}}^{(1,2)} \right], \tag{54}$$

$$\tilde{A}_{\text{Aq}}^{\text{eom}} = a_s S_{\varepsilon} \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[b_{\text{Aq}}^{(1,0)} + \varepsilon b_{\text{Aq}}^{(1,1)} + \varepsilon^2 b_{\text{Aq}}^{(1,2)} \right], \tag{55}$$

$$\tilde{A}_{\text{Bq}}^{\text{phys}} = a_s S_{\varepsilon} \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[-\frac{1}{\varepsilon} \gamma_{\text{Aq}}^{(0)} + a_{\text{Bq}}^{(1,0)} + \varepsilon a_{\text{Bq}}^{(1,1)} + \varepsilon^2 a_{\text{Bq}}^{(1,2)} \right], \tag{56}$$

$$\tilde{A}_{\text{Bq}}^{\text{eom}} = a_s S_{\varepsilon} \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[b_{\text{Bq}}^{(1,0)} + \varepsilon b_{\text{Bq}}^{(1,1)} + \varepsilon^2 b_{\text{Bq}}^{(1,2)} \right], \tag{57}$$

$$\tilde{A}_{\text{Ag}}^{\text{phys}} = a_s S_{\varepsilon} \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[\frac{1}{\varepsilon} \gamma_{\text{Ag}}^{(0)} + a_{\text{Ag}}^{(1,0)} + \varepsilon a_{\text{Ag}}^{(1,1)} + \varepsilon^2 a_{\text{Ag}}^{(1,2)} \right], \tag{58}$$

$$\tilde{A}_{\text{Ag}}^{\text{eom}} = a_s S_{\varepsilon} \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[b_{\text{Ag}}^{(1,0)} + \varepsilon b_{\text{Ag}}^{(1,1)} + \varepsilon^2 b_{\text{Ag}}^{(1,2)} \right], \tag{59}$$

$$\tilde{A}_{\text{Ag}}^{\text{ngi}} = 1 + a_s S_{\varepsilon} \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[\frac{1}{\varepsilon} \gamma_{\text{AA}}^{(0)} c_{\text{Ag}}^{(1,0)} + \varepsilon c_{\text{Ag}}^{(1,1)} + \varepsilon^2 c_{\text{Ag}}^{(1,2)} \right], \tag{60}$$

$$\tilde{A}_{\text{Ag}}^{\text{wi}} = a_s S_{\varepsilon} \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[d_{\text{Ag}}^{(1,0)} + \varepsilon d_{\text{Ag}}^{(1,1)} + \varepsilon^2 d_{\text{Ag}}^{(1,2)} \right], \tag{61}$$

$$\tilde{A}_{\omega g}^{\text{phys}} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[-\frac{1}{\varepsilon} \gamma_{Ag}^{(0)} + a_{\omega g}^{(1,0)} + \varepsilon a_{\omega g}^{(1,1)} + \varepsilon^2 a_{\omega g}^{(1,2)} \right], \quad (62)$$

$$\tilde{A}_{\omega g}^{\text{eom}} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[+b_{\omega g}^{(1,0)} + \varepsilon b_{\omega g}^{(1,1)} + \varepsilon^2 b_{\omega g}^{(1,2)} \right], \quad (63)$$

$$\tilde{A}_{\omega g}^{\text{ngi}} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[\frac{1}{\varepsilon} \gamma_{\omega A}^{(0)} + c_{\omega A}^{(1,0)} + \varepsilon c_{\omega A}^{(1,1)} + \varepsilon^2 c_{\omega A}^{(1,2)} \right], \quad (64)$$

$$\tilde{A}_{\omega g}^{\text{wi}} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[d_{Ag}^{(1,0)} + \varepsilon d_{Ag}^{(1,1)} + \varepsilon^2 d_{Ag}^{(1,2)} \right], \quad (65)$$

$$\tilde{A}_{qg}^{\text{ngi}} = a_s^2 S_\varepsilon^2 \left(\frac{-p^2}{\mu^2} \right)^\varepsilon \left[\frac{1}{\varepsilon^2} \gamma_{qg}^{(0)} \gamma_{gA}^{(0)} + \frac{1}{\varepsilon} \gamma_{qg}^{(0)} a_{gA}^{(1,0)} + c_{qg}^{(2,0)} + \varepsilon c_{qg}^{(2,1)} \right], \quad (66)$$

$$\tilde{A}_{gg}^{\text{ngi}} = a_s S_\varepsilon \left(\frac{-p^2}{\mu^2} \right)^{\varepsilon/2} \left[\frac{1}{\varepsilon} \gamma_{gA}^{(0)} + c_{gg}^{(1,0)} + \varepsilon c_{gg}^{(1,1)} + \varepsilon^2 c_{gg}^{(1,2)} \right]. \quad (67)$$

Here we listed also those contributions to the OMEs given in Section 2 for which new anomalous dimensions contribute to two-loop order. The latter read

$$\gamma_{Aq}^{(0)} = -\frac{8C_F}{(N-1)N}, \quad (68)$$

$$\gamma_{AA}^{(0)} = -C_A \left[\frac{16 + 46N + N^2 - 12N^3 - 3N^4}{2(N-1)N(1+N)(2+N)} + 2\xi + 6S_1 \right], \quad (69)$$

$$\gamma_{Ag}^{(0)} = -\frac{2C_A}{(1+N)(2+N)}, \quad (70)$$

$$\gamma_{\omega A}^{(0)} = -\frac{C_A(N-1)(4+N)(6+N)}{6N(1+N)(2+N)}, \quad (71)$$

$$\gamma_{gA}^{(0)} = \frac{4C_A}{(N-1)N}. \quad (72)$$

The higher order expansion terms are given by

$$a_{Ag}^{(1,0)} = C_A \left(\xi^2 \frac{1}{2N} + \xi \frac{(1-2N)}{(N-1)N} - \frac{Q_9}{(N-1)N(1+N)^2(2+N)^2} + \frac{S_1}{(1+N)(2+N)} \right), \quad (73)$$

$$a_{Ag}^{(1,1)} = C_A \left(-\xi^2 \left[\frac{N+1}{4N^2} + \frac{S_1}{4N} \right] + \xi \left[\frac{1-N+N^3}{2(N-1)^2N^2} + \frac{(-1+2N)S_1}{2(N-1)N} \right] \right. \\ \left. + \frac{S_1 Q_9}{2(N-1)N(1+N)^2(2+N)^2} - \frac{Q_{17}}{(N-1)^2N^2(1+N)^3(2+N)^3} - \frac{S_1^2}{4(1+N)(2+N)} \right. \\ \left. - \frac{3S_2}{4(1+N)(2+N)} + \frac{\zeta_2}{4(1+N)(2+N)} \right), \quad (74)$$

$$a_{Ag}^{(1,2)} = C_A \left(\xi^2 \left[\frac{1+N}{8N^3} + \frac{(1+N)S_1}{8N^2} + \frac{S_1^2}{16N} + \frac{3S_2}{16N} - \frac{\zeta_2}{16N} \right] + \xi \left[\frac{1-2N+N^2-N^3}{4(N-1)^3N^3} \right. \right. \\ \left. \left. + \frac{(-1+N-N^3)S_1}{4(N-1)^2N^2} + \frac{(1-2N)S_1^2}{8(N-1)N} - \frac{3(-1+2N)S_2}{8(N-1)N} + \frac{(-1+2N)\zeta_2}{8(N-1)N} \right] \right)$$

$$\begin{aligned}
& -\frac{S_1^2 Q_9}{8(N-1)N(1+N)^2(2+N)^2} - \frac{3S_2 Q_9}{8(N-1)N(1+N)^2(2+N)^2} \\
& + \frac{Q_{20}}{2(N-1)^3 N^3 (1+N)^4 (2+N)^4} + \left(\frac{Q_{17}}{2(N-1)^2 N^2 (1+N)^3 (2+N)^3} \right. \\
& \left. + \frac{3S_2}{8(1+N)(2+N)} \right) S_1 + \frac{S_1^3}{24(1+N)(2+N)} + \frac{7S_3}{12(1+N)(2+N)} \\
& + \left(\frac{Q_9}{8(N-1)N(1+N)^2(2+N)^2} - \frac{S_1}{8(1+N)(2+N)} \right) \zeta_2 \\
& - \frac{7\zeta_3}{12(1+N)(2+N)} \Big), \tag{75}
\end{aligned}$$

$$b_{Ag}^{(1,0)} = C_A \left(\xi \left[\frac{-1+9N-6N^2}{2(N-1)N} + \frac{3}{2} S_1 \right] - \frac{2(4+7N+2N^2)}{N(1+N)(2+N)} + \frac{(-2+N)\xi^2}{4N} \right), \tag{76}$$

$$\begin{aligned}
b_{Ag}^{(1,1)} = & C_A \left(\xi^2 \left[\frac{-2+2N+6N^2-5N^3}{8(N-1)N^2} + \frac{(1+N)S_1}{4N} \right] + \xi \left[\frac{Q_7}{4(N-1)^2 N^2} \right. \right. \\
& \left. \left. + \frac{(1-3N)S_1}{4(N-1)N} - \frac{3}{8} S_1^2 - \frac{9}{8} S_2 \right] - \frac{Q_3}{N^2(1+N)^2(2+N)^2} + \frac{(4+7N+2N^2)S_1}{N(1+N)(2+N)} \right), \tag{77}
\end{aligned}$$

$$\begin{aligned}
b_{Ag}^{(1,2)} = & C_A \left(\xi^2 \left[\frac{Q_{10}}{16(N-1)^2 N^3} + \frac{(2-2N-N^3)S_1}{16(N-1)N^2} + \frac{(-1-N)S_1^2}{16N} - \frac{3(1+N)S_2}{16N} \right. \right. \\
& \left. \left. + \frac{(2-N)\zeta_2}{32N} \right] + \xi \left[\frac{Q_{11}}{8(N-1)^3 N^3} + \left(\frac{1+8N-13N^2+6N^3}{8(N-1)^2 N^2} + \frac{9}{16} S_2 \right) S_1 \right. \right. \\
& \left. \left. + \frac{(-1+3N)S_1^2}{16(N-1)N} + \frac{1}{16} S_1^3 + \frac{3(-1+3N)S_2}{16(N-1)N} + \frac{7}{8} S_3 + \left(\frac{1-9N+6N^2}{16(N-1)N} - \frac{3}{16} S_1 \right) \zeta_2 \right] \right. \\
& \left. + \frac{S_1 Q_3}{2N^2(1+N)^2(2+N)^2} + \frac{Q_{13}}{N^3(1+N)^3(2+N)^3} + \frac{(-4-7N-2N^2)S_1^2}{4N(1+N)(2+N)} \right. \\
& \left. - \frac{3(4+7N+2N^2)S_2}{4N(1+N)(2+N)} + \frac{(4+7N+2N^2)\zeta_2}{4N(1+N)(2+N)} \right), \tag{78}
\end{aligned}$$

$$\begin{aligned}
c_{Ag}^{(1,0)} = & C_A \left(\frac{\xi^2}{2} + \xi \left[\frac{-2+2N-N^2}{2(N-1)N} - S_1 \right] + \frac{Q_{16}}{12(N-1)^2 N^2 (1+N)^2 (2+N)^2} \right. \\
& \left. + \frac{(8+20N-N^2-3N^3)S_1}{2(N-1)N(1+N)(2+N)} + \frac{3}{2} S_1^2 + \frac{9}{2} S_2 \right), \tag{79}
\end{aligned}$$

$$\begin{aligned}
c_{Ag}^{(1,1)} = & C_A \left(-\frac{\xi^2}{2} + \xi \left[\frac{-2+8N-7N^2+2N^3}{4(N-1)^2 N^2} + \frac{(2-2N+N^2)S_1}{4(N-1)N} + \frac{1}{4} S_1^2 + \frac{3}{4} S_2 + \frac{1}{4} \zeta_2 \right] \right. \\
& \left. + \frac{Q_{21}}{72(N-1)^3 N^3 (1+N)^3 (2+N)^3} + \left(\frac{Q_{15}}{8(N-1)^2 N^2 (1+N)^2 (2+N)^2} - \frac{9}{4} S_2 \right) S_1 \right. \\
& \left. + \frac{(-8-20N+N^2+3N^3)S_1^2}{8(N-1)N(1+N)(2+N)} - \frac{1}{4} S_1^3 + \frac{3(-8-20N+N^2+3N^3)S_2}{8(N-1)N(1+N)(2+N)} - \frac{7}{2} S_3 \right)
\end{aligned}$$

$$+ \left(\frac{Q_1}{16(N-1)N(1+N)(2+N)} + \frac{3}{4}S_1 \right) \zeta_2, \quad (80)$$

$$\begin{aligned} c_{Ag}^{(1,2)} = & C_A \left(\xi^2 \left[\frac{1}{2} - \frac{\zeta_2}{16} \right] + \xi \left[\frac{Q_2}{8(N-1)^3N^3} + \left(\frac{2-8N+7N^2-2N^3}{8(N-1)^2N^2} - \frac{3}{8}S_2 \right) S_1 - \frac{7}{12}S_3 \right. \right. \\ & + \frac{(-2+2N-N^2)S_1^2}{16(N-1)N} - \frac{3(2-2N+N^2)S_2}{16(N-1)N} + \left. \left(\frac{2-2N+N^2}{16(N-1)N} + \frac{S_1}{8} \right) \zeta_2 - \frac{1}{24}S_1^3 \right. \\ & \left. - \frac{7}{12}\zeta_3 \right] - \frac{3S_2Q_{15}}{32(N-1)^2N^2(1+N)^2(2+N)^2} + \frac{Q_{23}}{432(N-1)^4N^4(1+N)^4(2+N)^4} \\ & + \left(\frac{Q_{19}}{16(N-1)^3N^3(1+N)^3(2+N)^3} - \frac{3(-8-20N+N^2+3N^3)S_2}{16(N-1)N(1+N)(2+N)} + \frac{7}{4}S_3 \right) S_1 \\ & + \left(-\frac{Q_{15}}{32(N-1)^2N^2(1+N)^2(2+N)^2} + \frac{9}{16}S_2 \right) S_1^2 + \frac{(8+20N-N^2-3N^3)S_1^3}{48(N-1)N(1+N)(2+N)} \\ & + \frac{1}{32}S_1^4 + \frac{27}{32}S_2^2 - \frac{7(-8-20N+N^2+3N^3)S_3}{24(N-1)N(1+N)(2+N)} + \frac{45}{16}S_4 \\ & + \left(-\frac{Q_{16}}{96(N-1)^2N^2(1+N)^2(2+N)^2} + \frac{(-8-20N+N^2+3N^3)S_1}{16(N-1)N(1+N)(2+N)} - \frac{3}{16}S_1^2 \right. \\ & \left. - \frac{9}{16}S_2 \right) \zeta_2 + \left(-\frac{7Q_1}{48(N-1)N(1+N)(2+N)} - \frac{7}{4}S_1 \right) \zeta_3, \quad (81) \end{aligned}$$

$$d_{Ag}^{(1,0)} = -\frac{C_A}{4N}, \quad (82)$$

$$d_{Ag}^{(1,1)} = C_A \left(\frac{1}{8N^2} + \frac{S_1}{8N} \right), \quad (83)$$

$$d_{Ag}^{(1,2)} = C_A \left(-\frac{1}{16N^3} - \frac{S_1}{16N^2} - \frac{S_1^2}{32N} - \frac{3S_2}{32N} + \frac{\zeta_2}{32N} \right), \quad (84)$$

$$a_{Aq}^{(1,0)} = C_F \left(-\frac{2(-2+N)(-1+3N)}{(N-1)^2N^2} + \frac{(-4+N)\xi}{2(N-1)N} + \frac{4S_1}{(N-1)N} \right), \quad (85)$$

$$\begin{aligned} a_{Aq}^{(1,1)} = & C_F \left(\xi(N-4) \left[\frac{(1-3N+N^2)}{4(N-1)^2N^2} - \frac{S_1}{4(N-1)N} \right] - \frac{(N-2)(-1+4N-6N^2+N^3)}{(N-1)^3N^3} \right. \\ & \left. + \frac{(N-2)(-1+3N)S_1}{(N-1)^2N^2} - \frac{S_1^2}{(N-1)N} - \frac{3S_2}{(N-1)N} + \frac{\zeta_2}{(N-1)N} \right), \quad (86) \end{aligned}$$

$$\begin{aligned} a_{Aq}^{(1,2)} = & C_F \left(\xi(N-4) \left[-\frac{(-1+4N-6N^2+2N^3)}{8(N-1)^3N^3} - \frac{(1-3N+N^2)S_1}{8(N-1)^2N^2} + \frac{S_1^2}{16(N-1)N} \right. \right. \\ & \left. + \frac{3S_2}{16(N-1)N} - \frac{\zeta_2}{16(N-1)N} \right] + \frac{(N-2)Q_4}{2(N-1)^4N^4} + \left(\frac{3S_2}{2(N-1)N} \right. \\ & \left. + \frac{(N-2)(-1+4N-6N^2+N^3)}{2(N-1)^3N^3} \right) S_1 - \frac{(N-2)(-1+3N)S_1^2}{4(N-1)^2N^2} + \frac{S_1^3}{6(N-1)N} \end{aligned}$$

$$-\frac{3(N-2)(-1+3N)S_2}{4(N-1)^2N^2} + \frac{7S_3}{3(N-1)N} + \left(\frac{(N-2)(-1+3N)}{4(N-1)^2N^2} - \frac{S_1}{2(N-1)N} \right) \times \zeta_2 - \frac{7\zeta_3}{3(N-1)N}, \quad (87)$$

$$b_{Aq}^{(1,0)} = -\xi \frac{C_F}{N}, \quad (88)$$

$$b_{Aq}^{(1,1)} = \xi C_F \left(\frac{1-N}{2N^2} + \frac{S_1}{2N} \right), \quad (89)$$

$$b_{Aq}^{(1,2)} = \xi C_F \left(\frac{N-1}{4N^3} + \frac{(N-1)S_1}{4N^2} - \frac{S_1^2}{8N} - \frac{3S_2}{8N} + \frac{\zeta_2}{8N} \right), \quad (90)$$

$$a_{Bq}^{(1,0)} = C_F \left(\xi \frac{2}{(N-1)N} + \frac{4(1-3N+N^2)}{(N-1)^2N^2} - \frac{4S_1}{(N-1)N} \right), \quad (91)$$

$$a_{Bq}^{(1,1)} = C_F \left(\xi \left[\frac{1-3N+N^2}{(N-1)^2N^2} - \frac{S_1}{(N-1)N} \right] + \frac{2(1-4N+6N^2-2N^3)}{(N-1)^3N^3} - \frac{2(1-3N+N^2)S_1}{(N-1)^2N^2} + \frac{S_1^2}{(N-1)N} + \frac{3S_2}{(N-1)N} - \frac{\zeta_2}{(N-1)N} \right), \quad (92)$$

$$a_{Bq}^{(1,2)} = C_F \left(\xi \left[\frac{1-4N+6N^2-2N^3}{2(N-1)^3N^3} + \frac{(-1+3N-N^2)S_1}{2(N-1)^2N^2} + \frac{S_1^2}{4(N-1)N} + \frac{3S_2}{4(N-1)N} - \frac{\zeta_2}{4(N-1)N} \right] + \frac{Q_5}{(N-1)^4N^4} + \left(\frac{-1+4N-6N^2+2N^3}{(N-1)^3N^3} - \frac{3S_2}{2(N-1)N} \right) S_1 + \frac{(1-3N+N^2)S_1^2}{2(N-1)^2N^2} - \frac{S_1^3}{6(N-1)N} + \frac{3(1-3N+N^2)S_2}{2(N-1)^2N^2} - \frac{7S_3}{3(N-1)N} + \left(\frac{-1+3N-N^2}{2(N-1)^2N^2} + \frac{S_1}{2(N-1)N} \right) \zeta_2 + \frac{7\zeta_3}{3(N-1)N} \right), \quad (93)$$

$$a_{\omega g}^{(1,0)} = C_A \left(\frac{-4-N+N^2}{(1+N)^2(2+N)^2} - \frac{S_1}{(1+N)(2+N)} \right), \quad (94)$$

$$a_{\omega g}^{(1,1)} = C_A \left(\frac{8+5N-3N^2-2N^3}{(1+N)^3(2+N)^3} + \frac{(4+N-N^2)S_1}{2(1+N)^2(2+N)^2} + \frac{S_1^2}{4(1+N)(2+N)} + \frac{3S_2}{4(1+N)(2+N)} - \frac{\zeta_2}{4(1+N)(2+N)} \right), \quad (95)$$

$$a_{\omega g}^{(1,2)} = C_A \left(\frac{Q_6}{(1+N)^4(2+N)^4} + \left(\frac{-8-5N+3N^2+2N^3}{2(1+N)^3(2+N)^3} - \frac{3S_2}{8(1+N)(2+N)} \right) S_1 + \frac{(-4-N+N^2)S_1^2}{8(1+N)^2(2+N)^2} - \frac{S_1^3}{24(1+N)(2+N)} + \frac{3(-4-N+N^2)S_2}{8(1+N)^2(2+N)^2} - \frac{7S_3}{12(1+N)(2+N)} + \left(\frac{4+N-N^2}{8(1+N)^2(2+N)^2} + \frac{S_1}{8(1+N)(2+N)} \right) \zeta_2 \right)$$

$$+ \frac{7\zeta_3}{12(1+N)(2+N)} \Big), \quad (96)$$

$$b_{\omega g}^{(1,0)} = \frac{2C_A}{(1+N)(2+N)}, \quad (97)$$

$$b_{\omega g}^{(1,1)} = C_A \left(\frac{-4 - N + N^2}{(1+N)^2(2+N)^2} - \frac{S_1}{(1+N)(2+N)} \right), \quad (98)$$

$$b_{\omega g}^{(1,2)} = C_A \left(\frac{8 + 5N - 3N^2 - 2N^3}{(1+N)^3(2+N)^3} + \frac{(4 + N - N^2)S_1}{2(1+N)^2(2+N)^2} + \frac{S_1^2}{4(1+N)(2+N)} \right. \\ \left. + \frac{3S_2}{4(1+N)(2+N)} - \frac{\zeta_2}{4(1+N)(2+N)} \right), \quad (99)$$

$$c_{\omega g}^{(1,0)} = C_A \left(-\frac{Q_{12}}{36N^2(1+N)^2(2+N)^2} + \frac{(-4 + 2N + N^2)S_1}{2N(1+N)(2+N)} \right), \quad (100)$$

$$c_{\omega g}^{(1,1)} = C_A \left(-\frac{S_1 Q_8}{8N^2(1+N)^2(2+N)^2} + \frac{Q_{18}}{216N^3(1+N)^3(2+N)^3} + \frac{(4 - 2N - N^2)S_1^2}{8N(1+N)(2+N)} \right. \\ \left. - \frac{3(-4 + 2N + N^2)S_2}{8N(1+N)(2+N)} + \frac{(N-1)(4+N)(6+N)\zeta_2}{48N(1+N)(2+N)} \right), \quad (101)$$

$$c_{\omega g}^{(1,2)} = C_A \left(\frac{S_1^2 Q_8}{32N^2(1+N)^2(2+N)^2} + \frac{3S_2 Q_8}{32N^2(1+N)^2(2+N)^2} + \frac{Q_{22}}{1296N^4(1+N)^4(2+N)^4} \right. \\ \left. + \left(\frac{Q_{14}}{16N^3(1+N)^3(2+N)^3} + \frac{3(-4 + 2N + N^2)S_2}{16N(1+N)(2+N)} \right) S_1 + \frac{(-4 + 2N + N^2)S_1^3}{48N(1+N)(2+N)} \right. \\ \left. + \frac{7(-4 + 2N + N^2)S_3}{24N(1+N)(2+N)} + \left(\frac{Q_{12}}{288N^2(1+N)^2(2+N)^2} + \frac{(4 - 2N - N^2)S_1}{16N(1+N)(2+N)} \right) \zeta_2 \right. \\ \left. - \frac{7(N-1)(4+N)(6+N)\zeta_3}{144N(1+N)(2+N)} \right), \quad (102)$$

$$d_{\omega g}^{(1,0)} = \frac{C_A}{4N}, \quad (103)$$

$$d_{\omega g}^{(1,1)} = C_A \left(-\frac{1}{8N^2} - \frac{S_1}{8N} \right), \quad (104)$$

$$d_{\omega g}^{(1,2)} = C_A \left(\frac{1}{16N^3} + \frac{S_1}{16N^2} + \frac{S_1^2}{32N} + \frac{3S_2}{32N} - \frac{\zeta_2}{32N} \right), \quad (105)$$

with the polynomials

$$Q_1 = -3N^4 - 12N^3 + N^2 + 46N + 16, \quad (106)$$

$$Q_2 = -3N^4 + 13N^3 - 19N^2 + 10N - 2, \quad (107)$$

$$Q_3 = -N^4 - 13N^3 - 30N^2 - 24N - 8, \quad (108)$$

$$Q_4 = 2N^4 - 10N^3 + 10N^2 - 5N + 1, \quad (109)$$

$$Q_5 = 3N^4 - 10N^3 + 10N^2 - 5N + 1, \quad (110)$$

$$Q_6 = 3N^4 + 10N^3 + 4N^2 - 17N - 16, \quad (111)$$

$$Q_7 = 12N^4 - 30N^3 + 25N^2 - 8N - 1, \quad (112)$$

$$Q_8 = -3N^5 - 14N^4 - 11N^3 - 24N^2 - 60N - 16, \quad (113)$$

$$Q_9 = -2N^5 - 13N^4 - 40N^3 - 53N^2 - 28N - 8, \quad (114)$$

$$Q_{10} = 12N^5 - 26N^4 + 17N^3 - 6N^2 + 4N - 2, \quad (115)$$

$$Q_{11} = -24N^6 + 72N^5 - 64N^4 - 3N^3 + 29N^2 - 7N - 1, \quad (116)$$

$$Q_{12} = -8N^6 - 21N^5 + 22N^4 + 3N^3 + 184N^2 + 540N + 144, \quad (117)$$

$$Q_{13} = N^6 - 6N^5 - 38N^4 - 71N^3 - 66N^2 - 36N - 8, \quad (118)$$

$$Q_{14} = -3N^7 - 23N^6 - 15N^5 + 7N^4 - 194N^3 - 372N^2 - 168N - 32, \quad (119)$$

$$Q_{15} = -3N^7 + 12N^6 + 94N^5 + 16N^4 - 231N^3 - 164N^2 - 44N + 32, \quad (120)$$

$$Q_{16} = -16N^8 - 55N^7 - 68N^6 - 154N^5 + 64N^4 + 629N^3 + 428N^2 + 132N - 96, \quad (121)$$

$$Q_{17} = -N^8 - 7N^7 + 3N^6 + 69N^5 + 148N^4 + 154N^3 + 78N^2 - 4N - 8, \quad (122)$$

$$Q_{18} = -52N^9 - 468N^8 - 1635N^7 - 2655N^6 - 3027N^5 - 2061N^4 + 4822N^3 \\ + 10044N^2 + 4536N + 864, \quad (123)$$

$$Q_{19} = 11N^{10} + 34N^9 - 157N^8 - 422N^7 + 69N^6 + 798N^5 + 1025N^4 + 670N^3 \\ - 308N^2 - 56N + 64, \quad (124)$$

$$Q_{20} = -3N^{11} - 33N^{10} - 85N^9 - 7N^8 + 363N^7 + 897N^6 + 1085N^5 + 527N^4 - 96N^3 \\ - 88N^2 + 16N + 16, \quad (125)$$

$$Q_{21} = 92N^{12} + 552N^{11} + 729N^{10} - 1226N^9 - 1623N^8 + 3246N^7 + 2599N^6 \\ - 5526N^5 - 10329N^4 - 6766N^3 + 2772N^2 + 504N - 576, \quad (126)$$

$$Q_{22} = 320N^{12} + 3840N^{11} + 19840N^{10} + 57357N^9 + 99480N^8 + 114390N^7 \\ + 103660N^6 + 40845N^5 - 103420N^4 - 176904N^3 - 103680N^2 - 34992N \\ - 5184, \quad (127)$$

$$Q_{23} = -544N^{16} - 4352N^{15} - 10880N^{14} + 513N^{13} + 43712N^{12} + 41684N^{11} \\ - 67480N^{10} - 119386N^9 - 6592N^8 + 133644N^7 + 221992N^6 + 105361N^5 \\ - 75664N^4 - 1512N^3 + 21600N^2 + 1296N - 3456. \quad (128)$$

The eom part of A_{Bq} vanishes. In the above expressions the QCD color factor are $C_F = (N_c^2 - 1)/(2N_c)$, $C_A = N_c$, $T_F = 1/2$ for $SU(N_c)$ and $N_c = 3$ for QCD; N_F denotes the number of massless quark flavors. ζ_k , $k \in \mathbb{N}$, $k \geq 2$ are the values of Riemann's ζ function at integer arguments.

4 The operator mixing in the unpolarized singlet case

In the calculation of the off-shell OMEs in the unpolarized singlet case mixing between the physical and alien operators occurs, which we discuss in the following. In Mellin N -space the Z -factor for the (ultraviolet) renormalization of the local operators up to $O(a_s^2)$ reads

$$Z_{ij}^S = \delta_{ij} + a_s S_\epsilon \frac{\gamma_{ij}^{(0)}}{\epsilon} + a_s^2 S_\epsilon^2 \left[\frac{1}{\epsilon^2} \left(\frac{1}{2} \gamma_{il}^{(0)} \gamma_{lj}^{(0)} + \beta_0 \gamma_{ij}^{(0)} \right) + \frac{1}{2\epsilon} \gamma_{ij}^{(1)} \right] + O(a_s^3). \quad (129)$$

The renormalized physical OMEs are obtained by²

$$A_{ij}^{\text{phys}} = (Z_{ik}^S)^{-1} \tilde{A}_{kj}, \quad (130)$$

with³

$$\tilde{A}_{qq}^{\text{PS}} = \tilde{A}_{qq}^{\text{PS,phys}}, \quad (131)$$

$$\tilde{A}_{gg} = \tilde{A}_{gg}^{\text{phys}}, \quad (132)$$

$$\tilde{A}_{gq} = \tilde{A}_{gq}^{\text{phys}} + \eta(\tilde{A}_{Aq} + \tilde{A}_{Bq}), \quad (133)$$

$$\tilde{A}_{gg} = \tilde{A}_{gg}^{\text{phys}} - \frac{\eta}{2}(\tilde{A}_{Ag} + \tilde{A}_{\omega g}), \quad (134)$$

and

$$\eta = -a_s S_\varepsilon \frac{\gamma_{gA}^{(0)}}{\varepsilon} + O(a_s^2). \quad (135)$$

The individual contributions \tilde{A}_{ij} are given by

$$\tilde{A}_{qq}^{\text{PS}} = a_s^2 S_\varepsilon^2 \left[\frac{1}{2\varepsilon^2} \gamma_{gg}^{(0)} \gamma_{qq}^{(0)} + \frac{1}{\varepsilon} \left(\gamma_{gg}^{(0)} a_{qq}^{(1,0)} + \frac{\gamma_{qq}^{\text{PS,(1)}}}{2} \right) + a_{qq}^{\text{PS,(2,0)}} + \varepsilon a_{qq}^{\text{PS,(2,1)}} \right] + O(a_s^3), \quad (136)$$

$$\begin{aligned} \tilde{A}_{gg} &= a_s S_\varepsilon \left[\frac{\gamma_{gg}^{(0)}}{\varepsilon} + a_{gg}^{(1,0)} + \varepsilon a_{gg}^{(1,1)} + \varepsilon^2 a_{gg}^{(1,2)} \right] + a_s^2 S_\varepsilon^2 \left[\frac{1}{2\varepsilon^2} \gamma_{gg}^{(0)} \left(\gamma_{gg}^{(0)} + \gamma_{qq}^{(0)} + 2\beta_0 \right) \right. \\ &\quad \left. + \frac{1}{\varepsilon} \left(\gamma_{gg}^{(0)} a_{gg}^{(1,0)} + \gamma_{qq}^{(0)} a_{qq}^{(1,0)} + \frac{\gamma_{gg}^{(1)}}{2} \right) + a_{gg}^{(2,0)} + \varepsilon a_{gg}^{(2,1)} \right] + O(a_s^3), \end{aligned} \quad (137)$$

$$\begin{aligned} \tilde{A}_{gq} &= a_s S_\varepsilon \left[\frac{\gamma_{gq}^{(0)}}{\varepsilon} + a_{gq}^{(1,0)} + \varepsilon a_{gq}^{(1,1)} + \varepsilon^2 a_{gq}^{(1,2)} \right] + a_s^2 S_\varepsilon^2 \left[\frac{1}{2\varepsilon^2} \gamma_{gq}^{(0)} \left(\gamma_{gg}^{(0)} + \gamma_{qq}^{(0)} + 2\beta_0 \right) \right. \\ &\quad \left. + \frac{1}{\varepsilon} \left(\gamma_{gg}^{(0)} a_{gq}^{(1,0)} + \gamma_{gq}^{(0)} a_{qq}^{(1,0)} + \frac{\gamma_{gq}^{(1)}}{2} \right) + \frac{\delta_{gq}^{(-1)}}{\varepsilon} + a_{gq}^{(2,0)} + \delta_{gq}^{(0)} + \varepsilon a_{gq}^{(2,1)} + \varepsilon \delta_{gq}^{(1)} \right] + O(a_s^3), \end{aligned} \quad (138)$$

$$\begin{aligned} \tilde{A}_{gg} &= 1 + a_s S_\varepsilon \left[\frac{\gamma_{gg}^{(0)}}{\varepsilon} + a_{gg}^{(1,0)} + \varepsilon a_{gg}^{(1,1)} + \varepsilon^2 a_{gg}^{(1,2)} \right] + a_s^2 S_\varepsilon^2 \left[\frac{1}{2\varepsilon^2} \left(\gamma_{gg}^{(0)} \left(\gamma_{gg}^{(0)} + 2\beta_0 \right) \right. \right. \\ &\quad \left. \left. + \gamma_{gq}^{(0)} \gamma_{qq}^{(0)} \right) + \frac{1}{\varepsilon} \left(\gamma_{gg}^{(0)} a_{gg}^{(1,0)} + \gamma_{gq}^{(0)} a_{qq}^{(1,0)} + \frac{\gamma_{gg}^{(1)}}{2} \right) + \frac{\delta_{gg}^{(-1)}}{\varepsilon} + a_{gg}^{(2,0)} + \delta_{gg}^{(0)} \right. \\ &\quad \left. + \varepsilon a_{gg}^{(2,1)} + \varepsilon \delta_{gg}^{(1)} \right] + O(a_s^3). \end{aligned} \quad (139)$$

In the flavor non-singlet cases, including transversity [82], and in the polarized singlet case [81] no mixing with the alien operators occurs. To two-loop order, these terms contribute in the

²Note that we focus on the physical projection of the OMEs in this section, since the QCD anomalous dimensions are extracted from them. Similar relations can also be derived for the other projections.

³The occurrence of the factor of $-1/2$ in (134) is due to a convention used in Ref. [19], which is, however, not correctly implemented in Ref. [19].

renormalization of \tilde{A}_{gq} and \tilde{A}_{gg} , since one has to consider the combinations (133, 134) in this case. Because of this the additional contributions

$$\delta_{gq}^{(i-1)} = -\gamma_{gA}^{(0)} \left(a_{Aq}^{(1,i)} + a_{Bq}^{(1,i)} \right), \quad i \geq 0, \quad (140)$$

$$\delta_{gg}^{(i-1)} = \frac{\gamma_{gA}^{(0)}}{2} \left(a_{Ag}^{(1,i)} + a_{\omega g}^{(1,i)} \right), \quad i \geq 0 \quad (141)$$

are present. The anomalous dimensions are obtained from the pole terms of (136–139).

5 The unpolarized OMEs

We now turn to the calculation of contributions to the unpolarized off-shell OMEs for non-negative order in the dimensional parameter ε . They are gauge dependent in general and are given by⁴

$$a_{qq}^{(1,0),\text{NS}} = C_F \left[-\frac{P_{39}}{N^2(1+N)^2} + \frac{2S_1}{N(1+N)} - 2S_1^2 - 6S_2 + \xi \left(-\frac{1}{N} + S_1 \right) \right], \quad (142)$$

$$\begin{aligned} a_{qq}^{(1,1),\text{NS}} = & C_F \left[\xi \left(\frac{1-N+2N^2}{2N^2} + \frac{(1-N)S_1}{2N} - \frac{1}{4}S_1^2 - \frac{3}{4}S_2 \right) + \frac{P_{299}}{N^3(1+N)^3} \right. \\ & + \left(-\frac{(N-1)(-1-N+N^2)}{N^2(1+N)^2} + 3S_2 \right) S_1 - \frac{S_1^2}{2N(1+N)} + \frac{1}{3}S_1^3 - \frac{3S_2}{2N(1+N)} \\ & \left. + \frac{14}{3}S_3 + \left(\frac{2+3N+3N^2}{4N(1+N)} - S_1 \right) \zeta_2 \right], \quad (143) \end{aligned}$$

$$\begin{aligned} a_{qq}^{(1,2),\text{NS}} = & C_F \left[\xi \left(\frac{-1+N-4N^3}{4N^3} + \left(\frac{N-1}{4N^2} + \frac{3S_2}{8} \right) S_1 + \frac{(N-1)S_1^2}{8N} + \frac{1}{24}S_1^3 \right) \right. \\ & + \frac{3(N-1)S_2}{8N} + \frac{7}{12}S_3 + \left(\frac{1}{8N} - \frac{S_1}{8} \right) \zeta_2 \left. \right) + \frac{P_{469}}{2N^4(1+N)^4} \\ & + \left(-\frac{(N-1)(1+2N-2N^3+N^4)}{2N^3(1+N)^3} + \frac{3S_2}{4N(1+N)} - \frac{7}{3}S_3 \right) S_1 \\ & - \left(\frac{(N-1)(1+N-N^2)}{4N^2(1+N)^2} + \frac{3}{4}S_2 \right) S_1^2 + \frac{S_1^3}{12N(1+N)} - \frac{1}{24}S_1^4 \\ & + \frac{3(N-1)(-1-N+N^2)S_2}{4N^2(1+N)^2} - \frac{9}{8}S_2^2 + \frac{7S_3}{6N(1+N)} \\ & - \frac{15}{4}S_4 + \left(\frac{P_{39}}{8N^2(1+N)^2} - \frac{S_1}{4N(1+N)} + \frac{1}{4}S_1^2 + \frac{3}{4}S_2 \right) \zeta_2 \\ & \left. + \left(\frac{7}{3}S_1 - \frac{7(2+3N+3N^2)}{12N(1+N)} \right) \zeta_3 \right], \quad (144) \end{aligned}$$

⁴The polynomials are listed in Appendix B.

$$\begin{aligned}
a_{qq}^{(2,0),\text{NS}} = & \xi^2 \left[C_A C_F \left(\frac{1+2N}{4N} + \frac{(2-3N)S_1}{4N} - \frac{1}{4}S_1^2 - \frac{1}{4}S_2 \right) + C_F^2 \left(\frac{1-N}{N} - \frac{S_1}{N} + \frac{1}{2}S_1^2 \right. \right. \\
& \left. \left. + \frac{1}{2}S_2 \right) \right] + \xi \left[C_F^2 \left(\frac{P_{186}}{N^3(1+N)^2} + \left(\frac{P_{132}}{N^2(1+N)^2} - 16S_2 \right) S_1 - 4S_1^3 \right. \right. \\
& \left. \left. + \frac{(18+7N-5N^2)S_1^2}{2N(1+N)} + \frac{(22+25N+13N^2)S_2}{2N(1+N)} \right) + C_A C_F \left(\frac{-2-3N-7N^2}{2N^2} \right. \right. \\
& \left. \left. + \left(\frac{-2+11N+12N^2}{2N(2+N)} - \frac{(N-1)NS_2}{(1+N)(2+N)} \right) S_1 + \frac{5}{4}S_1^2 + \frac{(2+13N)S_2}{4(2+N)} \right. \right. \\
& \left. \left. - \frac{(N-1)NS_3}{(1+N)(2+N)} + \frac{2(N-1)NS_{2,1}}{(1+N)(2+N)} - \frac{6(N-1)N}{(1+N)(2+N)} \zeta_3 \right) \right] \\
& + C_F \left(N_F T_F \left(\left(\frac{4(-21+29N+73N^2+41N^3)}{27N(1+N)^2} + \frac{8}{3}S_2 \right) S_1 + \frac{P_{286}}{54N^3(1+N)^3} \right. \right. \\
& \left. \left. + \frac{2(-6+17N+17N^2)S_1^2}{9N(1+N)} + \frac{2(-6+37N+37N^2)S_2}{9N(1+N)} + \frac{8}{9}S_1^3 - \frac{8}{9}S_3 \right) \right) \\
& + C_A \left(\frac{S_2 P_{166}}{18N^2(1+N)^2(2+N)} + \frac{P_{376}}{216N^3(1+N)^3} + \left(\frac{P_{379}}{27N^3(1+N)^3(2+N)} \right. \right. \\
& \left. \left. - \frac{2(22+33N+14N^2)S_2}{3(1+N)(2+N)} - 6S_3 - 4S_{2,1} \right) S_1 + \left(-\frac{11(-6+23N+23N^2)}{18N(1+N)} \right. \right. \\
& \left. \left. + S_2 \right) S_1^2 + \frac{2(72+58N+33N^2+2N^3)S_3}{9N(1+N)(2+N)} - \frac{22}{9}S_1^3 \right. \\
& \left. - 6S_2^2 - 29S_4 + \left(-\frac{4(-3+N)}{N(1+N)^2} + \frac{16S_1}{N(1+N)} - 8S_1^2 - 16S_2 \right) S_{-2} - 4S_{-2}^2 \right. \\
& \left. + \left(\frac{20}{N(1+N)} - 32S_1 \right) S_{-3} - 28S_{-4} + \frac{4N^2 S_{2,1}}{(1+N)(2+N)} + 6S_{3,1} - \frac{8S_{-2,1}}{N(1+N)} \right. \\
& \left. + 8S_{-2,2} + 24S_{-3,1} + 6S_{2,1,1} + 16S_{-2,1,1} + \left(-\frac{12(2-N+N^2)}{N(2+N)} + 12S_1 \right) \zeta_3 \right) \\
& + C_F^2 \left[\frac{S_2 P_{173}}{N^2(1+N)^2(2+N)} + \frac{P_{486}}{8N^4(1+N)^4} + \left(-\frac{2P_{414}}{N^3(1+N)^3(2+N)} \right. \right. \\
& \left. \left. - \frac{2(26+11N)(2+N+N^2)S_2}{N(1+N)(2+N)} + \frac{256}{3}S_3 \right) S_1 + \left(\frac{P_{29}}{N^2(1+N)^2} + 48S_2 \right) S_1^2 \right. \\
& \left. - \frac{2(14+3N+3N^2)S_1^3}{3N(1+N)} + \frac{14}{3}S_1^4 + 30S_2^2 - \frac{2(164+232N+237N^2+75N^3)S_3}{3N(1+N)(2+N)} \right. \\
& \left. + 48S_4 + \left(\frac{8(-3+N)}{N(1+N)^2} - \frac{32S_1}{N(1+N)} + 16S_1^2 + 32S_2 \right) S_{-2} + 8S_{-2}^2 \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{40}{N(1+N)} + 64S_1 \right) S_{-3} + 56S_{-4} - \frac{4(-2+N)(2+N+N^2)S_{2,1}}{N(1+N)(2+N)} + \frac{16S_{-2,1}}{N(1+N)} \\
& - 16S_{-2,2} - 48S_{-3,1} - 32S_{-2,1,1} + \left(-\frac{(2+3N+3N^2)^2}{2N^2(1+N)^2} + \frac{4(2+3N+3N^2)S_1}{N(1+N)} \right. \\
& \left. - 8S_1^2 \right) \zeta_2 - \frac{48N}{(1+N)(2+N)} \zeta_3 \Big], \tag{145}
\end{aligned}$$

$$\begin{aligned}
a_{qq}^{(2,1),\text{NS}} = & \xi^2 \left[C_F^2 \left(\frac{-1-N+3N^2}{2N^2} + \left(\frac{1-2N+3N^2}{2N^2} - \frac{5S_2}{4} \right) S_1 + \frac{(3-2N)S_1^2}{4N} - \frac{1}{4}S_1^3 \right. \right. \\
& + \left. \frac{(5-2N)S_2}{4N} - S_3 \right) + C_A C_F \left(\frac{-1-2N-13N^2}{8N^2} + \left(\frac{-2+5N^2}{8N^2} + \frac{5S_2}{8} \right) S_1 \right. \\
& + \left. \frac{(-3+4N)S_1^2}{8N} + \frac{1}{8}S_1^3 + \frac{(-5+7N)S_2}{8N} + \frac{1}{2}S_3 + \left(\frac{1}{8N} - \frac{S_1}{8} \right) \zeta_2 \right) \Big] \\
& + \xi \left[C_A C_F \left(\frac{6+3N+10N^2+42N^3}{4N^3} + \frac{S_2 P_{17}}{8N^2(2+N)^2} + \frac{S_{2,1} P_{47}}{N(1+N)^2(2+N)^2} \right. \right. \\
& + \frac{S_3 P_{183}}{2N(1+N)^2(2+N)^2} + \left(\frac{S_2 P_{176}}{8N(1+N)^2(2+N)^2} + \frac{P_{180}}{4N^3(2+N)^2} \right. \\
& + \left. \frac{3(N-1)NS_3}{(1+N)(2+N)} - \frac{2(N-1)NS_{2,1}}{(1+N)(2+N)} \right) S_1 + \left(\frac{6-23N-31N^2}{8N(2+N)} \right. \\
& + \left. \frac{3(N-1)NS_2}{4(1+N)(2+N)} \right) S_1^2 - \frac{5}{8}S_1^3 + \frac{(-1-N-N^2)S_2^2}{(1+N)(2+N)} + \frac{(-4-13N+5N^2)S_4}{4(1+N)(2+N)} \\
& + \frac{(2+7N-3N^2)S_{3,1}}{(1+N)(2+N)} + \frac{(N-1)NS_{2,1,1}}{(1+N)(2+N)} + \left(-\frac{5}{8N} + \frac{5S_1}{8} \right) \zeta_2 \\
& + \frac{9(N-1)N\zeta_2^2}{5(1+N)(2+N)} + \left(\frac{3P_{227}}{2N(1+N)^2(2+N)^2} + \frac{6(1+N+N^2)S_1}{(1+N)(2+N)} \right) \zeta_3 \Big) \\
& + C_F^2 \left(\frac{S_2 P_{13}}{4N^2(1+N)^2} + \frac{P_{410}}{2N^4(1+N)^3} + \left(\frac{P_{292}}{2N^3(1+N)^3} + \frac{(-70-25N+19N^2)S_2}{4N(1+N)} \right. \right. \\
& + \left. \frac{52}{3}S_3 \right) S_1 + \left(\frac{P_{23}}{4N^2(1+N)^2} + 12S_2 \right) S_1^2 + \frac{7(-6-N+3N^2)S_1^3}{12N(1+N)} + \frac{7}{6}S_1^4 \\
& + \frac{15}{2}S_2^2 + \frac{(-66-73N-45N^2)S_3}{6N(1+N)} + \left(\frac{-2-3N-3N^2}{2N^2(1+N)} + \frac{(6+7N+3N^2)S_1}{2N(1+N)} \right. \\
& \left. - 2S_1^2 \right) \zeta_2 + \left(\frac{3(2+3N+3N^2)}{N(1+N)} - 12S_1 \right) \zeta_3 \Big] \\
& + C_F \left[C_A \left(-\frac{2S_{2,1}P_{325}}{N^2(1+N)^2(2+N)^2} + \frac{S_3 P_{371}}{27N^2(1+N)^2(2+N)^2} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{P_{432}}{2592N^4(1+N)^4} + \left(\frac{S_2 P_{373}}{36N^2(1+N)^2(2+N)^2} + \frac{P_{550}}{162N^4(1+N)^4(2+N)^2} + 5S_2^2 \right. \\
& + \left. \frac{2(-18+29N+57N^2+28N^3)S_3}{3N(1+N)(2+N)} + \frac{49}{2}S_4 + \frac{4(-2+3N)S_{2,1}}{N(2+N)} - 2S_{2,1,1} \right) S_1 \\
& + \left(\frac{P_{428}}{108N^3(1+N)^3(2+N)} + S_3 + \frac{(12+104N+147N^2+58N^3)S_2}{6N(1+N)(2+N)} + 2S_{2,1} \right) S_1^2 \\
& + \left(\frac{-198+491N+491N^2}{108N(1+N)} - \frac{1}{2}S_2 \right) S_1^3 + \frac{11}{12}S_1^4 + \left(\frac{P_{488}}{108N^3(1+N)^3(2+N)^2} \right. \\
& + \left. 18S_3 + 4S_{2,1} - 8S_{-2,1} \right) S_2 + \frac{(-96+106N+231N^2+65N^3)S_2^2}{12N(1+N)(2+N)} \\
& + \frac{(-324-64N+147N^2+70N^3)S_4}{6N(1+N)(2+N)} + 62S_5 + \left(-\frac{2P_{83}}{N^3(1+N)^3} \right. \\
& + \left. \left(\frac{8(-1-3N+N^2)}{N^2(1+N)^2} + 16S_2 \right) S_1 - \frac{8S_1^2}{N(1+N)} + \frac{8}{3}S_1^3 - \frac{16S_2}{N(1+N)} + \frac{64}{3}S_3 \right. \\
& + \left. 8S_{2,1} \right) S_{-2} + \left(-\frac{4}{N(1+N)} + 4S_1 \right) S_{-2}^2 + \left(\frac{2(-2-15N+5N^2)}{N^2(1+N)^2} \right. \\
& - \left. \frac{32S_1}{N(1+N)} + 16S_1^2 + 20S_2 + 16S_{-2} \right) S_{-3} + \left(-\frac{42}{N(1+N)} + 56S_1 \right) S_{-4} + 54S_{-5} \\
& + 16S_{2,3} + 20S_{2,-3} - \frac{2(-2-7N-9N^2+N^3)S_{3,1}}{N(1+N)(2+N)} - 13S_{4,1} \\
& - \frac{4(-2-3N+N^2)S_{-2,1}}{N^2(1+N)^2} + \frac{4S_{-2,2}}{N(1+N)} - 12S_{-2,3} + \frac{12S_{-3,1}}{N(1+N)} - 28S_{-4,1} \\
& - \frac{2(-6+3N+9N^2+2N^3)S_{2,1,1}}{N(1+N)(2+N)} - 8S_{2,1,-2} - 18S_{2,2,1} - 6S_{3,1,1} + \frac{8S_{-2,1,1}}{N(1+N)} \\
& - 8S_{-2,2,1} - 24S_{-3,1,1} - 3S_{2,1,1,1} - 16S_{-2,1,1,1} + \left(\frac{310+793N+1128N^2+513N^3}{72N(1+N)^2} \right. \\
& - \left. \frac{11(-6+23N+23N^2)S_1}{36N(1+N)} - \frac{11}{6}S_1^2 - \frac{11}{6}S_2 - 2S_3 + \left(\frac{2}{N(1+N)} - 4S_1 \right) S_{-2} \right. \\
& - \left. 2S_{-3} + 4S_{-2,1} \right) \zeta_2 + \left(\frac{18(2-N+N^2)}{5N(2+N)} - \frac{18}{5}S_1 \right) \zeta_2^2 + \left(\frac{P_{265}}{2N^2(1+N)(2+N)^2} \right. \\
& - \left. \frac{2(22+33N+8N^2)S_1}{(1+N)(2+N)} - 3S_1^2 + 3S_2 + 12S_{-2} \right) \zeta_3 + N_F T_F \left(\frac{S_2 P_6}{27N^2(1+N)^2} \right. \\
& + \left. \frac{P_{492}}{648N^4(1+N)^4} + \left(-\frac{2P_{278}}{81N^2(1+N)^3} - \frac{5(-6+13N+13N^2)S_2}{9N(1+N)} - \frac{16}{3}S_3 \right) \right. \\
& \times S_1 + \left. \left(\frac{P_{14}}{27N^2(1+N)^2} - \frac{10S_2}{3} \right) S_1^2 + \frac{(18-31N-31N^2)S_1^3}{27N(1+N)} - \frac{1}{3}S_1^4 - \frac{7}{3}S_2^2 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{8(-9+28N+28N^2)S_3}{27N(1+N)} - \frac{10}{3}S_4 + \left(\frac{-14-53N-96N^2-45N^3}{18N(1+N)^2} \right. \\
& + \left. \frac{(-6+17N+17N^2)S_1}{9N(1+N)} + \frac{2}{3}S_1^2 + \frac{2}{3}S_2 \right) \zeta_2 + \left(-\frac{2(2+3N+3N^2)}{N(1+N)} + 8S_1 \right) \\
& \times \zeta_3 \left. \right] + C_F^2 \left(\frac{4S_{2,1}P_{210}}{N(1+N)^2(2+N)^2} + \frac{S_3P_{364}}{3N^2(1+N)^2(2+N)^2} + \frac{P_{520}}{32N^5(1+N)^5} \right. \\
& + \left(\frac{S_2P_{352}}{2N^2(1+N)^2(2+N)^2} + \frac{P_{541}}{N^4(1+N)^4(2+N)^2} - 47S_2^2 \right. \\
& + \left. \frac{2(76+60N+45N^2+11N^3)S_3}{N(1+N)(2+N)} - 128S_4 - \frac{16NS_{2,1}}{(1+N)(2+N)} \right) S_1 \\
& + \left(\frac{P_{412}}{2N^3(1+N)^3(2+N)} + \frac{(116+80N+45N^2+11N^3)S_2}{2N(1+N)(2+N)} - 60S_3 \right) S_1^2 \\
& + \left(\frac{P_{123}}{6N^2(1+N)^2} - \frac{56S_2}{3} \right) S_1^3 + \frac{(10+N+N^2)S_1^4}{4N(1+N)} - S_1^5 + \left(-\frac{232}{3}S_3 + 16S_{-2,1} \right. \\
& + \left. \frac{P_{480}}{2N^3(1+N)^3(2+N)^2} \right) S_2 + \frac{(388+416N+361N^2+111N^3)S_4}{2N(1+N)(2+N)} \\
& + \frac{(204+196N+125N^2+47N^3)S_2^2}{4N(1+N)(2+N)} - 108S_5 + \left(\frac{4P_{83}}{N^3(1+N)^3} \right. \\
& + \left(-\frac{16(-1-3N+N^2)}{N^2(1+N)^2} - 32S_2 \right) S_1 + \frac{16S_1^2}{N(1+N)} - \frac{16}{3}S_1^3 + \frac{32S_2}{N(1+N)} - \frac{128}{3}S_3 \\
& - 16S_{2,1} \left. \right) S_{-2} + \left(\frac{8}{N(1+N)} - 8S_1 \right) S_{-2}^2 + \left(-\frac{4(-2-15N+5N^2)}{N^2(1+N)^2} \right. \\
& + \left. \frac{64S_1}{N(1+N)} - 32S_1^2 - 40S_2 - 32S_{-2} \right) S_{-3} + \left(\frac{84}{N(1+N)} - 112S_1 \right) S_{-4} - 108S_{-5} \\
& - 40S_{2,-3} + \frac{2(-12-7N^2+3N^3)S_{3,1}}{N(1+N)(2+N)} + \frac{8(-2-3N+N^2)S_{-2,1}}{N^2(1+N)^2} - \frac{8S_{-2,2}}{N(1+N)} \\
& + 24S_{-2,3} - \frac{24S_{-3,1}}{N(1+N)} + 56S_{-4,1} + \frac{4(-4+4N+N^2)S_{2,1,1}}{N(2+N)} + 16S_{2,1,-2} \\
& - \frac{16S_{-2,1,1}}{N(1+N)} + 16S_{-2,2,1} + 48S_{-3,1,1} + 32S_{-2,1,1,1} + \left(\frac{P_{360}}{8N^3(1+N)^3} + \left(\frac{P_{30}}{N^2(1+N)^2} \right. \right. \\
& + \left. \left. 16S_2 \right) S_1 - \frac{3(2+N+N^2)S_1^2}{N(1+N)} + 4S_1^3 - \frac{4(2+3N+3N^2)S_2}{N(1+N)} + 4S_3 \right. \\
& + \left. \left(-\frac{4}{N(1+N)} + 8S_1 \right) S_{-2} + 4S_{-3} - 8S_{-2,1} \right) \zeta_2 + \frac{72N\zeta_2^2}{5(1+N)(2+N)} \\
& + \left(\frac{P_{294}}{3N^2(1+N)^2(2+N)^2} - \frac{8(14+37N+36N^2+15N^3)S_1}{3N(1+N)(2+N)} + \frac{92}{3}S_1^2 - 12S_2 \right.
\end{aligned}$$

$$-24S_{-2})\zeta_3), \quad (146)$$

$$b_{qq}^{(1,0),\text{NS}} = C_F \left[-\frac{4}{1+N} + \xi \frac{2}{N} \right], \quad (147)$$

$$b_{qq}^{(1,1),\text{NS}} = C_F \left[\xi \left(\frac{N-1}{N^2} - \frac{S_1}{N} \right) + \left(-\frac{2(N-1)}{(1+N)^2} + \frac{2S_1}{1+N} \right) \right], \quad (148)$$

$$b_{qq}^{(1,2),\text{NS}} = C_F \left[\xi \left(\frac{1-N}{2N^3} + \frac{(1-N)S_1}{2N^2} + \frac{S_1^2}{4N} + \frac{3S_2}{4N} - \frac{\zeta_2}{4N} \right) + \frac{2(N-1)}{(1+N)^3} + \frac{(N-1)S_1}{(1+N)^2} - \frac{S_1^2}{2(1+N)} - \frac{3S_2}{2(1+N)} + \frac{\zeta_2}{2(1+N)} \right], \quad (149)$$

$$b_{qq}^{(2,0),\text{NS}} = C_F \left[N_F T_F \left(\frac{16(-1+6N+4N^2)}{9N(1+N)^2} + \frac{8(N-1)S_1}{3N(1+N)} \right) + C_A \left(\left(\frac{8S_2}{(1+N)(2+N)} - \frac{2(-12+8N+65N^2+23N^3)}{3N^2(1+N)(2+N)} \right) S_1 - \frac{4(-8+102N+77N^2)}{9N(1+N)^2} + \frac{4(-2+N)S_2}{N(2+N)} + \frac{8S_3}{(1+N)(2+N)} - \frac{16S_{2,1}}{(1+N)(2+N)} + \frac{48\zeta_3}{(1+N)(2+N)} \right) \right] \\ + C_F^2 \left(\frac{8P_{72}}{N^2(1+N)^3} + \left(-\frac{4P_{114}}{N^2(1+N)^2(2+N)} - \frac{16S_2}{(1+N)(2+N)} \right) S_1 + \frac{24S_1^2}{1+N} + \frac{8(2+7N+2N^2)S_2}{N(1+N)(2+N)} - \frac{16S_3}{(1+N)(2+N)} + \frac{32S_{2,1}}{(1+N)(2+N)} - \frac{96}{(1+N)(2+N)}\zeta_3 \right) + \xi \left[C_F^2 \left(\frac{2(3+N)(2+N+N^2+4N^3)}{N^3(1+N)^2} + \frac{2(-4+3N+9N^2)S_1}{N^2(1+N)} - \frac{12S_1^2}{N} - \frac{12S_2}{N} \right) + C_A C_F \left(\frac{2+3N}{N^2} + \left(\frac{2+7N}{N(2+N)} - \frac{2(N-1)S_2}{(1+N)(2+N)} \right) S_1 + \frac{6S_2}{2+N} - \frac{2(N-1)S_3}{(1+N)(2+N)} + \frac{4(N-1)S_{2,1}}{(1+N)(2+N)} - \frac{12(N-1)}{(1+N)(2+N)}\zeta_3 \right) \right] + \xi^2 \left[C_A C_F \left(-\frac{1}{2N} - \frac{S_1}{N} \right) + C_F^2 \left(-\frac{2}{N} + \frac{2S_1}{N} \right) \right], \quad (150)$$

$$b_{qq}^{(2,1),\text{NS}} = C_F^2 \left[-\frac{8S_3 P_{71}}{3N(1+N)^2(2+N)^2} + \frac{S_2 P_{266}}{N^2(1+N)^2(2+N)^2} + \frac{2P_{321}}{N^3(1+N)^4} + \left(-\frac{4S_2 P_{113}}{N(1+N)^2(2+N)^2} - \frac{4P_{403}}{N^3(1+N)^3(2+N)^2} + \frac{48S_3}{(1+N)(2+N)} - \frac{32S_{2,1}}{(1+N)(2+N)} \right) S_1 + \left(\frac{P_{137}}{N^2(1+N)^2(2+N)} + \frac{12S_2}{(1+N)(2+N)} \right) S_1^2 - \frac{28S_1^3}{3(1+N)} - \frac{8S_2^2}{(1+N)(2+N)} - \frac{16(-4+2N+5N^2+N^3)S_{2,1}}{N(1+N)^2(2+N)^2} \right]$$

$$\begin{aligned}
& + \frac{28S_4}{(1+N)(2+N)} - \frac{64S_{3,1}}{(1+N)(2+N)} + \frac{16S_{2,1,1}}{(1+N)(2+N)} + \left(\frac{8S_1}{1+N} \right. \\
& \left. - \frac{2(2+3N+3N^2)}{N(1+N)^2} \right) \zeta_2 + \frac{144\zeta_2^2}{5(1+N)(2+N)} + \left(\frac{48S_1}{(1+N)(2+N)} \right. \\
& \left. - \frac{24(4-12N-13N^2+N^4)}{N(1+N)^2(2+N)^2} \right) \zeta_3 \Big] + C_F \left[N_F T_F \left(-\frac{8P_{95}}{27N^2(1+N)^3} \right. \right. \\
& \left. \left. - \frac{4(-3-7N+33N^2+13N^3)S_1}{9N^2(1+N)^2} - \frac{2(N-1)S_1^2}{N(1+N)} - \frac{10(N-1)S_2}{3N(1+N)} \right. \right. \\
& \left. \left. + \frac{2(N-1)\zeta_2}{3N(1+N)} \right) + C_A \left(\frac{2P_{155}}{27N^2(1+N)^3} + \frac{S_2P_{262}}{6N^2(1+N)^2(2+N)^2} \right. \right. \\
& \left. \left. + \left(-\frac{2S_2P_{65}}{N(1+N)^2(2+N)^2} + \frac{P_{422}}{9N^3(1+N)^3(2+N)^2} - \frac{24S_3}{(1+N)(2+N)} \right. \right. \right. \\
& \left. \left. + \frac{16S_{2,1}}{(1+N)(2+N)} \right) S_1 + \left(\frac{-12+8N+65N^2+23N^3}{2N^2(1+N)(2+N)} - \frac{6S_2}{(1+N)(2+N)} \right) S_1^2 \right. \\
& \left. + \frac{4S_2^2}{(1+N)(2+N)} - \frac{4(3+2N)(-4-2N+N^2+N^3)S_3}{N(1+N)^2(2+N)^2} - \frac{14S_4}{(1+N)(2+N)} \right. \\
& \left. + \frac{8(-4+2N+5N^2+N^3)S_{2,1}}{N(1+N)^2(2+N)^2} + \frac{32S_{3,1}}{(1+N)(2+N)} - \frac{8S_{2,1,1}}{(1+N)(2+N)} \right. \\
& \left. + \frac{(5-17N)\zeta_2}{6N(1+N)} - \frac{72\zeta_2^2}{5(1+N)(2+N)} + \left(-\frac{12P_{70}}{N(1+N)^2(2+N)^2} \right. \right. \\
& \left. \left. - \frac{24S_1}{(1+N)(2+N)} \right) \zeta_3 \Big] + \xi \left[C_F^2 \left(\frac{P_{297}}{N^4(1+N)^3} + \left(\frac{22S_2}{N} \right. \right. \right. \\
& \left. \left. + \frac{(N-1)(-4-15N-11N^2+2N^3)}{N^3(1+N)^2} \right) S_1 + \frac{(16-3N-21N^2)S_1^2}{2N^2(1+N)} + \frac{14S_1^3}{3N} \right. \\
& \left. + \frac{(8-13N-27N^2)S_2}{2N^2(1+N)} + \frac{28S_3}{3N} + \left(\frac{2+3N+3N^2}{N^2(1+N)} - \frac{4S_1}{N} \right) \zeta_2 \right) \\
& \left. + C_A C_F \left(\frac{-6-3N-10N^2}{2N^3} + \frac{4S_{2,1}P_{66}}{N(1+N)^2(2+N)^2} - \frac{2S_3P_{103}}{N(1+N)^2(2+N)^2} \right. \right. \\
& \left. \left. + \left(\frac{S_2P_{40}}{N(1+N)^2(2+N)^2} + \frac{P_{59}}{2N^3(2+N)^2} + \frac{6(N-1)S_3}{(1+N)(2+N)} - \frac{4(N-1)S_{2,1}}{(1+N)(2+N)} \right) \right. \\
& \left. \times S_1 + \left(-\frac{3(2+7N)}{4N(2+N)} + \frac{3(N-1)S_2}{2(1+N)(2+N)} \right) S_1^2 + \frac{(1-N)S_2^2}{(1+N)(2+N)} \right. \\
& \left. + \frac{(-48-4N-84N^2-35N^3)S_2}{4N^2(2+N)^2} + \frac{7(N-1)S_4}{2(1+N)(2+N)} - \frac{8(N-1)S_{3,1}}{(1+N)(2+N)} \right. \\
& \left. + \frac{2(N-1)S_{2,1,1}}{(1+N)(2+N)} + \frac{5\zeta_2}{4N} + \frac{18(N-1)\zeta_2^2}{5(1+N)(2+N)} + \left(\frac{12P_{67}}{N(1+N)^2(2+N)^2} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{6(N-1)S_1}{(1+N)(2+N)} \zeta_3 \Big) \Big] + \xi^2 \left[C_F^2 \left(\frac{1+N}{N^2} + \frac{(-1+2N)S_1}{N^2} - \frac{3S_1^2}{2N} - \frac{5S_2}{2N} \right) \right. \\
& \left. + C_A C_F \left(\frac{1+2N}{4N^2} + \frac{S_1}{2N^2} + \frac{3S_1^2}{4N} + \frac{5S_2}{4N} - \frac{\zeta_2}{4N} \right) \right], \tag{151}
\end{aligned}$$

$$\begin{aligned}
a_{qq}^{(2,0),\text{PS}} &= C_F N_F T_F \left[-\frac{8S_1 P_{456}}{(N-1)^2 N^3 (1+N)^3 (2+N)^2} + \frac{4P_{574}}{(N-1)^3 N^4 (1+N)^4 (2+N)^3} \right. \\
& + \frac{8(2+N+N^2)^2 S_1^2}{(N-1)N^2(1+N)^2(2+N)} + \frac{16(2+N+N^2)^2 S_2}{(N-1)N^2(1+N)^2(2+N)} \\
& \left. - \frac{4(2+N+N^2)^2 \zeta_2}{(N-1)N^2(1+N)^2(2+N)} \right], \tag{152}
\end{aligned}$$

$$\begin{aligned}
a_{qq}^{(2,1),\text{PS}} &= C_F N_F T_F \left[\frac{4S_1^2 P_{456}}{(N-1)^2 N^3 (1+N)^3 (2+N)^2} + \frac{8S_2 P_{456}}{(N-1)^2 N^3 (1+N)^3 (2+N)^2} \right. \\
& + \frac{2P_{606}}{(N-1)^4 N^5 (1+N)^5 (2+N)^4} + \left(-\frac{4P_{574}}{(N-1)^3 N^4 (1+N)^4 (2+N)^3} \right. \\
& \left. - \frac{16(2+N+N^2)^2 S_2}{(N-1)N^2(1+N)^2(2+N)} \right) S_1 - \frac{8(2+N+N^2)^2 S_1^3}{3(N-1)N^2(1+N)^2(2+N)} \\
& - \frac{52(2+N+N^2)^2 S_3}{3(N-1)N^2(1+N)^2(2+N)} + \left(-\frac{2P_{456}}{(N-1)^2 N^3 (1+N)^3 (2+N)^2} \right. \\
& \left. + \frac{4(2+N+N^2)^2 S_1}{(N-1)N^2(1+N)^2(2+N)} \right) \zeta_2 + \frac{64(2+N+N^2)^2 \zeta_3}{3(N-1)N^2(1+N)^2(2+N)} \Big], \tag{153}
\end{aligned}$$

$$b_{qq}^{(2,0),\text{PS}} = C_F N_F T_F \left[\frac{16P_{226}}{N^3(1+N)^3(2+N)^2} - \frac{32(2+N+N^2)S_1}{N^2(1+N)^2(2+N)} \right], \tag{154}$$

$$\begin{aligned}
b_{qq}^{(2,1),\text{PS}} &= C_F N_F T_F \left[-\frac{16S_1 P_{226}}{N^3(1+N)^3(2+N)^2} + \frac{4P_{461}}{N^4(1+N)^4(2+N)^3} \right. \\
& \left. + \frac{16(2+N+N^2)S_1^2}{N^2(1+N)^2(2+N)} + \frac{32(2+N+N^2)S_2}{N^2(1+N)^2(2+N)} - \frac{8(2+N+N^2)\zeta_2}{N^2(1+N)^2(2+N)} \right], \tag{155}
\end{aligned}$$

$$a_{gg}^{(1,0)} = N_F T_F \left[\frac{4P_{194}}{N^2(1+N)^2(2+N)^2} + \frac{4(2+N+N^2)S_1}{N(1+N)(2+N)} \right], \tag{156}$$

$$\begin{aligned}
a_{gg}^{(1,1)} &= N_F T_F \left[-\frac{2S_1 P_{194}}{N^2(1+N)^2(2+N)^2} - \frac{2P_{391}}{N^3(1+N)^3(2+N)^3} - \frac{(2+N+N^2)S_1^2}{N(1+N)(2+N)} \right. \\
& \left. - \frac{3(2+N+N^2)S_2}{N(1+N)(2+N)} + \frac{(2+N+N^2)\zeta_2}{N(1+N)(2+N)} \right], \tag{157}
\end{aligned}$$

$$a_{gg}^{(1,2)} = N_F T_F \left[\frac{S_1^2 P_{194}}{2N^2(1+N)^2(2+N)^2} + \frac{3S_2 P_{194}}{2N^2(1+N)^2(2+N)^2} + \frac{P_{502}}{N^4(1+N)^4(2+N)^4} \right]$$

$$\begin{aligned}
& + \left(\frac{P_{391}}{N^3(1+N)^3(2+N)^3} + \frac{3(2+N+N^2)S_2}{2N(1+N)(2+N)} \right) S_1 + \frac{(2+N+N^2)S_1^3}{6N(1+N)(2+N)} \\
& + \frac{7(2+N+N^2)S_3}{3N(1+N)(2+N)} + \left(-\frac{P_{194}}{2N^2(1+N)^2(2+N)^2} + \frac{(-2-N-N^2)S_1}{2N(1+N)(2+N)} \right) \zeta_2 \\
& - \frac{7(2+N+N^2)\zeta_3}{3N(1+N)(2+N)} \Big], \tag{158}
\end{aligned}$$

$$\begin{aligned}
a_{qg}^{(2,0)} = & C_A N_F T_F \xi^2 \left[\frac{P_{193}}{N^2(1+N)^2(2+N)^2} - \frac{(2+N+N^2)S_1}{N(1+N)(2+N)} \right] \\
& + C_A N_F T_F \xi \left[\frac{4S_1 P_{207}}{N^2(1+N)^2(2+N)^2} - \frac{4P_{317}}{N^3(1+N)^2(2+N)^2} + \frac{6(2+N+N^2)S_1^2}{N(1+N)(2+N)} \right. \\
& + \left. \frac{10(2+N+N^2)S_2}{N(1+N)(2+N)} \right] + N_F^2 T_F^2 \left[-\frac{16P_{256}}{27N^2(1+N)^2(2+N)^2} - \frac{80(2+N+N^2)S_1}{9N(1+N)(2+N)} \right. \\
& + \left. \frac{8(2+N+N^2)\zeta_2}{3N(1+N)(2+N)} \right] + C_F N_F T_F \left[\frac{4S_2 P_{250}}{N^2(1+N)^2(2+N)^2} + \frac{4S_1^2 P_{252}}{N^2(1+N)^2(2+N)^2} \right. \\
& + \frac{2P_{536}}{N^4(1+N)^4(2+N)^3} + \left(-\frac{4P_{462}}{N^3(1+N)^3(2+N)^3} - \frac{28(2+N+N^2)S_2}{N(1+N)(2+N)} \right) S_1 \\
& - \frac{28(2+N+N^2)S_1^3}{3N(1+N)(2+N)} + \frac{64(2+N+N^2)S_3}{3N(1+N)(2+N)} - \frac{16(2+N+N^2)S_{2,1}}{N(1+N)(2+N)} \\
& + \left(-\frac{2(2+N+N^2)(2+3N+3N^2)}{N^2(1+N)^2(2+N)} + \frac{8(2+N+N^2)S_1}{N(1+N)(2+N)} \right) \zeta_2 \\
& - \frac{48(2+N+N^2)\zeta_3}{N(1+N)(2+N)} \Big] \\
& + C_A N_F T_F \left[\frac{8S_1^2 P_{107}}{(N-1)N^2(1+N)^2(2+N)^2} + \frac{8S_2 P_{398}}{(N-1)N^2(1+N)^2(2+N)^2(3+N)} \right. \\
& + \frac{4P_{604}}{27(N-1)^3 N^4(1+N)^4(2+N)^4(3+N)} + \left(-\frac{4(114+87N+70N^2+17N^3)S_2}{N(1+N)(2+N)(3+N)} \right. \\
& + \left. \frac{4P_{564}}{9(N-1)^2 N^3(1+N)^3(2+N)^3(3+N)} \right) S_1 - \frac{28(2+N+N^2)S_1^3}{3N(1+N)(2+N)} \\
& - \frac{8(258+203N+163N^2+40N^3)S_3}{3N(1+N)(2+N)(3+N)} + \left(-\frac{16P_{76}}{N(1+N)^2(2+N)^2(3+N)} \right. \\
& - \left. \frac{32(2+N+N^2)S_1}{N(1+N)(2+N)} \right) S_{-2} - \frac{40(2+N+N^2)S_{-3}}{N(1+N)(2+N)} + \frac{16(12+6N+5N^2+N^3)S_{2,1}}{N(1+N)(2+N)(3+N)} \\
& + \frac{16(2+N+N^2)S_{-2,1}}{N(1+N)(2+N)} + \left(-\frac{2(2+N+N^2)P_{115}}{3(N-1)N^2(1+N)^2(2+N)^2} + \frac{8(2+N+N^2)S_1}{N(1+N)(2+N)} \right)
\end{aligned}$$

$$\times \zeta_2 + \frac{24(6 + 7N + 3N^2)\zeta_3}{N(2 + N)(3 + N)} \Big], \quad (159)$$

$$\begin{aligned} a_{gg}^{(2,1)} = & C_A N_F T_F \xi^2 \left[-\frac{S_1 P_{193}}{2N^2(1 + N)^2(2 + N)^2} + \frac{P_{451}}{2N^3(1 + N)^3(2 + N)^3} + \frac{(2 + N + N^2)S_1^2}{4N(1 + N)(2 + N)} \right. \\ & + \left. \frac{3(2 + N + N^2)S_2}{4N(1 + N)(2 + N)} + \frac{(-2 - N - N^2)\zeta_2}{2N(1 + N)(2 + N)} \right] + C_A N_F T_F \xi \left[\frac{S_2 P_{190}}{N^2(1 + N)^2(2 + N)^2} \right. \\ & + \frac{S_1^2 P_{195}}{N^2(1 + N)^2(2 + N)^2} + \frac{P_{499}}{N^4(1 + N)^3(2 + N)^3} + \left(\frac{2P_{458}}{N^3(1 + N)^3(2 + N)^3} \right. \\ & - \left. \frac{13(2 + N + N^2)S_2}{N(1 + N)(2 + N)} \right) S_1 - \frac{7(2 + N + N^2)S_1^3}{3N(1 + N)(2 + N)} - \frac{38(2 + N + N^2)S_3}{3N(1 + N)(2 + N)} \\ & + \left(\frac{2(N - 1)(2 + N + N^2)}{N^2(1 + N)(2 + N)} + \frac{2(2 + N + N^2)S_1}{N(1 + N)(2 + N)} \right) \zeta_2 + \frac{12(2 + N + N^2)\zeta_3}{N(1 + N)(2 + N)} \Big] \\ & + N_F^2 T_F^2 \left[\frac{8S_1 P_{256}}{27N^2(1 + N)^2(2 + N)^2} + \frac{8P_{477}}{81N^3(1 + N)^3(2 + N)^3} + \frac{20(2 + N + N^2)S_1^2}{9N(1 + N)(2 + N)} \right. \\ & + \frac{20(2 + N + N^2)S_2}{3N(1 + N)(2 + N)} + \left(-\frac{4(3 + N)P_{100}}{9N^2(1 + N)^2(2 + N)^2} - \frac{4(2 + N + N^2)S_1}{3N(1 + N)(2 + N)} \right) \zeta_2 \\ & - \left. \frac{56(2 + N + N^2)\zeta_3}{9N(1 + N)(2 + N)} \right] + C_F N_F T_F \left(-\frac{16S_{2,1}P_{208}}{N^2(1 + N)^2(2 + N)^2(3 + N)} \right. \\ & - \frac{2P_{259}}{3N^2(1 + N)^2(2 + N)^2} S_1^3 - \frac{2P_{344}}{3N^2(1 + N)^2(2 + N)^2(3 + N)} S_3 \\ & + \frac{P_{597}}{N^5(1 + N)^5(2 + N)^4(3 + N)} + \frac{P_{509}}{N^3(1 + N)^3(2 + N)^3(3 + N)} S_2 \\ & + \left(-\frac{2S_2 P_{347}}{N^2(1 + N)^2(2 + N)^2(3 + N)} - \frac{2P_{575}}{N^4(1 + N)^4(2 + N)^4(3 + N)} \right. \\ & + \left. \frac{4(2 + N + N^2)S_3}{N(1 + N)(2 + N)} + \frac{16(2 + N + N^2)S_{2,1}}{N(1 + N)(2 + N)} \right) S_1 + \left(\frac{P_{473}}{N^3(1 + N)^3(2 + N)^3} \right. \\ & + \left. \frac{19(2 + N + N^2)S_2}{N(1 + N)(2 + N)} \right) S_1^2 + \frac{5(2 + N + N^2)S_1^4}{2N(1 + N)(2 + N)} + \frac{15(2 + N + N^2)S_2^2}{2N(1 + N)(2 + N)} \\ & - \frac{57(2 + N + N^2)S_4}{N(1 + N)(2 + N)} + \frac{48(2 + N + N^2)S_{3,1}}{N(1 + N)(2 + N)} - \frac{8(2 + N + N^2)S_{2,1,1}}{N(1 + N)(2 + N)} \\ & + \left(\frac{2P_{245}}{N^2(1 + N)^2(2 + N)^2} S_1 + \frac{P_{394}}{N^3(1 + N)^3(2 + N)^2} - \frac{6(2 + N + N^2)S_1^2}{N(1 + N)(2 + N)} \right. \\ & - \left. \frac{2(2 + N + N^2)S_2}{N(1 + N)(2 + N)} \right) \zeta_2 + \frac{72(2 + N + N^2)\zeta_2^2}{5N(1 + N)(2 + N)} + \left(-\frac{16P_{339}}{3N^2(1 + N)^2(2 + N)^2(3 + N)} \right. \\ & + \left. \frac{16(2 + N + N^2)S_1}{3N(1 + N)(2 + N)} \right) \zeta_3 \Big) + C_A N_F T_F \left(-\frac{4S_1^3 P_{131}}{3(N - 1)N^2(1 + N)^2(2 + N)^2} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{8P_{324}S_{2,1}}{N^2(1+N)^2(2+N)^2(3+N)^2} - \frac{4S_3P_{466}}{3(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} \\
& + \frac{P_{571}S_2}{3(N-1)^2N^3(1+N)^3(2+N)^3(3+N)^2} \\
& - \frac{2P_{619}}{81(N-1)^4N^5(1+N)^5(2+N)^5(3+N)^2} \\
& - \left(\frac{4P_{449}S_2}{(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} - \frac{8(102+73N+59N^2+14N^3)S_3}{N(1+N)(2+N)(3+N)} \right. \\
& + \frac{2P_{607}}{27(N-1)^3N^4(1+N)^4(2+N)^4(3+N)^2} + \left. \frac{8(18+7N+6N^2+N^3)S_{2,1}}{N(1+N)(2+N)(3+N)} \right) S_1 \\
& + \left(\frac{P_{555}}{9(N-1)^2N^3(1+N)^3(2+N)^3(3+N)} + \frac{(246+181N+146N^2+35N^3)S_2}{N(1+N)(2+N)(3+N)} \right) \\
& \times S_1^2 + \frac{5(2+N+N^2)S_1^4}{2N(1+N)(2+N)} + \frac{(426+387N+308N^2+79N^3)S_2^2}{2N(1+N)(2+N)(3+N)} \\
& + \frac{(1182+929N+746N^2+183N^3)S_4}{N(1+N)(2+N)(3+N)} + \left(\frac{32S_1P_{206}}{N^2(1+N)^2(2+N)^2(3+N)} \right. \\
& + \frac{8P_{503}}{N^3(1+N)^3(2+N)^3(3+N)^2} + \frac{16(2+N+N^2)S_1^2}{N(1+N)(2+N)} + \left. \frac{32(2+N+N^2)S_2}{N(1+N)(2+N)} \right) S_{-2} \\
& + \frac{8(2+N+N^2)S_{-2}^2}{N(1+N)(2+N)} + \left(\frac{8P_{249}}{N^2(1+N)^2(2+N)^2(3+N)} + \frac{64(2+N+N^2)S_1}{N(1+N)(2+N)} \right) \\
& \times S_{-3} + \frac{84(2+N+N^2)S_{-4}}{N(1+N)(2+N)} - \frac{4(102+53N+44N^2+9N^3)S_{3,1}}{N(1+N)(2+N)(3+N)} \\
& + \frac{16(-4-12N-5N^2+N^3)S_{-2,1}}{N^2(1+N)^2(2+N)^2} - \frac{8(2+N+N^2)S_{-2,2}}{N(1+N)(2+N)} \\
& - \frac{24(2+N+N^2)S_{-3,1}}{N(1+N)(2+N)} - \frac{4(N-1)(6+3N+N^2)S_{2,1,1}}{N(1+N)(2+N)(3+N)} - \frac{16(2+N+N^2)}{N(1+N)(2+N)} \\
& \times S_{-2,1,1} + \left(\frac{P_{336}S_1}{3(N-1)N^2(1+N)^2(2+N)^2} + \frac{P_{542}}{9(N-1)^2N^3(1+N)^3(2+N)^3} \right. \\
& - \frac{6(2+N+N^2)S_1^2}{N(1+N)(2+N)} - \frac{14(2+N+N^2)S_2}{N(1+N)(2+N)} - \left. \frac{4(2+N+N^2)S_{-2}}{N(1+N)(2+N)} \right) \zeta_2 \\
& - \frac{36(6+7N+3N^2)\zeta_2^2}{5N(2+N)(3+N)} + \left(\frac{2P_{485}}{9(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} \right. \\
& - \left. \frac{8(96+116N+91N^2+25N^3)S_1}{3N(1+N)(2+N)(3+N)} \right) \zeta_3, \tag{160}
\end{aligned}$$

$$b_{gg}^{(1,0)} = \frac{16N_F T_F}{(1+N)(2+N)}, \tag{161}$$

$$b_{gg}^{(1,1)} = N_F T_F \left[\frac{8(-4-N+N^2)}{(1+N)^2(2+N)^2} - \frac{8S_1}{(1+N)(2+N)} \right], \tag{162}$$

$$b_{qg}^{(1,2)} = N_F T_F \left[-\frac{8(-8-5N+3N^2+2N^3)}{(1+N)^3(2+N)^3} - \frac{4(-4-N+N^2)S_1}{(1+N)^2(2+N)^2} + \frac{2S_1^2}{(1+N)(2+N)} \right. \\ \left. + \frac{6S_2}{(1+N)(2+N)} - \frac{2\zeta_2}{(1+N)(2+N)} \right], \quad (163)$$

$$b_{qg}^{(2,0)} = -\frac{4C_A N_F T_F \xi^2}{(1+N)(2+N)} + C_A N_F T_F \xi \left[\frac{2P_{253}}{(N-1)^2 N(1+N)^2(2+N)^2} \right. \\ \left. + \frac{8(-2-3N+N^2)S_1}{(N-1)N(1+N)(2+N)} \right] - \frac{320N_F^2 T_F^2}{9(1+N)(2+N)} \\ + C_F N_F T_F \left(-\frac{16P_{223}}{N^2(1+N)^3(2+N)^2} + \frac{32(4-4N+11N^2+9N^3)S_1}{N(1+N)^2(2+N)^2} \right. \\ \left. - \frac{96S_1^2}{(1+N)(2+N)} - \frac{32S_2}{(1+N)(2+N)} \right) + C_A N_F T_F \\ \times \left[\frac{8P_{481}}{9N^3(1+N)^3(2+N)^3(3+N)} + \left(-\frac{16(-2+N)P_{68}}{N^2(1+N)^2(2+N)^2(3+N)} \right. \right. \\ \left. \left. - \frac{16(N-1)S_2}{(1+N)(2+N)(3+N)} \right) S_1 - \frac{16(9+N)S_2}{(1+N)(2+N)(3+N)} \right. \\ \left. + \frac{16(7+N)S_3}{(1+N)(2+N)(3+N)} + \left(-\frac{32(6+7N+6N^2+N^3)}{N(1+N)^2(2+N)(3+N)} + \frac{64S_1}{(1+N)(2+N)} \right) \right. \\ \left. \times S_{-2} + \frac{32S_{-3}}{(1+N)(2+N)} + \frac{32(N-1)S_{2,1}}{(1+N)(2+N)(3+N)} - \frac{64S_{-2,1}}{(1+N)(2+N)} \right. \\ \left. - \frac{192\zeta_3}{(2+N)(3+N)} \right], \quad (164)$$

$$b_{qg}^{(2,1)} = C_A N_F T_F \xi^2 \left[\frac{2(8+7N+N^2)}{(1+N)^2(2+N)^2} + \frac{2S_1}{(1+N)(2+N)} \right] \\ + C_A N_F T_F \xi \left[-\frac{2S_1 P_{251}}{(N-1)^2 N(1+N)^2(2+N)^2} + \frac{P_{539}}{2(N-1)^3 N^3(1+N)^3(2+N)^3} \right. \\ \left. - \frac{8(-1-2N+N^2)S_1^2}{(N-1)N(1+N)(2+N)} - \frac{4(-4-7N+3N^2)S_2}{(N-1)N(1+N)(2+N)} \right. \\ \left. - \frac{2(2+N+N^2)\zeta_2}{(N-1)N(1+N)(2+N)} \right] + N_F^2 T_F^2 \left(\frac{32(116+99N+13N^2)}{27(1+N)^2(2+N)^2} \right. \\ \left. + \frac{160S_1}{9(1+N)(2+N)} - \frac{16\zeta_2}{3(1+N)(2+N)} \right) + C_F N_F T_F \\ \times \left(-\frac{8P_{505}}{N^3(1+N)^4(2+N)^3(3+N)} - \frac{16S_2 P_{130}}{N(1+N)^2(2+N)^2(3+N)} \right. \\ \left. + \left(\frac{16P_{404}}{N^2(1+N)^3(2+N)^3(3+N)} + \frac{16(17+11N)S_2}{(1+N)(2+N)(3+N)} \right) S_1 \right)$$

$$\begin{aligned}
& -\frac{32(2-5N+6N^2+6N^3)S_1^2}{N(1+N)^2(2+N)^2} + \frac{112S_1^3}{3(1+N)(2+N)} - \frac{64(15+N)S_3}{3(1+N)(2+N)(3+N)} \\
& -\frac{64(-5+N)S_{2,1}}{(1+N)(2+N)(3+N)} + \left(\frac{8(2+3N+3N^2)}{N(1+N)^2(2+N)} - \frac{32S_1}{(1+N)(2+N)} \right) \zeta_2 \\
& + \frac{192(1+3N)\zeta_3}{(1+N)(2+N)(3+N)} \Big) + C_{AN_F T_F} \left(\frac{4\zeta_2 P_{122}}{3N^2(1+N)^2(2+N)^2} \right. \\
& + \frac{8S_3 P_{214}}{N(1+N)^2(2+N)^2(3+N)^2} + \frac{4S_2 P_{327}}{N^2(1+N)^2(2+N)^2(3+N)^2} \\
& - \frac{2P_{582}}{27N^4(1+N)^4(2+N)^4(3+N)^2} + \left(-\frac{8P_{511}}{9N^3(1+N)^3(2+N)^3(3+N)^2} \right. \\
& - \frac{16(-64-47N-3N^2+2N^3)S_2}{(1+N)(2+N)^2(3+N)^2} + \frac{32(-3+N)S_3}{(1+N)(2+N)(3+N)} \\
& \left. - \frac{64S_{2,1}}{(2+N)(3+N)} \right) S_1 + \left(\frac{4P_{235}}{N^2(1+N)^2(2+N)^2(3+N)} \right. \\
& + \frac{4(3+5N)S_2}{(1+N)(2+N)(3+N)} \Big) S_1^2 - \frac{8(5+3N)S_2^2}{(1+N)(2+N)(3+N)} \\
& - \frac{4(85+19N)S_4}{(1+N)(2+N)(3+N)} + \left(\frac{32(12+2N+N^2+4N^3+N^4)S_1}{N(1+N)^2(2+N)^2(3+N)} \right. \\
& + \frac{32P_{314}}{N^2(1+N)^3(2+N)^2(3+N)^2} - \frac{32S_1^2}{(1+N)(2+N)} - \frac{64S_2}{(1+N)(2+N)} \Big) S_{-2} \\
& - \frac{16S_{-2}^2}{(1+N)(2+N)} + \left(\frac{32P_{78}}{N(1+N)^2(2+N)^2(3+N)} - \frac{128S_1}{(1+N)(2+N)} \right) S_{-3} \\
& - \frac{112S_{-4}}{(1+N)(2+N)} + \frac{16(-74-43N+8N^2+5N^3)S_{2,1}}{(1+N)(2+N)^2(3+N)^2} + \frac{96S_{-3,1}}{(1+N)(2+N)} \\
& - \frac{16(-1+5N)S_{3,1}}{(1+N)(2+N)(3+N)} + \frac{64(3+2N)S_{-2,1}}{(1+N)^2(2+N)^2} + \frac{32S_{-2,2}}{(1+N)(2+N)} \\
& + \frac{64S_{2,1,1}}{(1+N)(3+N)} + \frac{64S_{-2,1,1}}{(1+N)(2+N)} + \frac{288\zeta_2^2}{5(2+N)(3+N)} \\
& + \left(-\frac{48P_{102}}{N(1+N)(2+N)^2(3+N)^2} + \frac{48(N-1)S_1}{(1+N)(2+N)(3+N)} \right) \zeta_3 \Big), \tag{165}
\end{aligned}$$

$$\begin{aligned}
c_{qg}^{(2,0)} &= C_{AN_F T_F} \left[\frac{8S_1 P_{323}}{(N-1)^2 N^3 (1+N)^2 (2+N)^2} - \frac{4P_{531}}{(N-1)^3 N^4 (1+N)^3 (2+N)^3} \right. \\
& - \frac{8(2+N+N^2)S_1^2}{(N-1)N^2(1+N)(2+N)} - \frac{16(2+N+N^2)S_2}{(N-1)N^2(1+N)(2+N)} \\
& \left. + \frac{4(2+N+N^2)\zeta_2}{(N-1)N^2(1+N)(2+N)} \right], \tag{166}
\end{aligned}$$

$$c_{qg}^{(2,1)} = C_{AN_F T_F} \left[-\frac{4S_1^2 P_{323}}{(N-1)^2 N^3 (1+N)^2 (2+N)^2} - \frac{8S_2 P_{323}}{(N-1)^2 N^3 (1+N)^2 (2+N)^2} \right]$$

$$\begin{aligned}
& + \frac{P_{598}}{(N-1)^4 N^5 (1+N)^4 (2+N)^4} + \left(\frac{4P_{531}}{(N-1)^3 N^4 (1+N)^3 (2+N)^3} \right. \\
& \left. + \frac{16(2+N+N^2)S_2}{(N-1)N^2(1+N)(2+N)} \right) S_1 + \frac{8(2+N+N^2)S_1^3}{3(N-1)N^2(1+N)(2+N)} \\
& + \frac{52(2+N+N^2)S_3}{3(N-1)N^2(1+N)(2+N)} + \left(\frac{2P_{323}}{(N-1)^2 N^3 (1+N)^2 (2+N)^2} \right. \\
& \left. - \frac{4(2+N+N^2)S_1}{(N-1)N^2(1+N)(2+N)} \right) \zeta_2 - \frac{64(2+N+N^2)\zeta_3}{3(N-1)N^2(1+N)(2+N)}, \tag{167}
\end{aligned}$$

$$a_{gq}^{(1,0)} = C_F \left[\frac{2P_{192}}{(N-1)^2 N^2 (1+N)^2} + \frac{2(2+N+N^2)S_1}{(N-1)N(1+N)} \right], \tag{168}$$

$$\begin{aligned}
a_{gq}^{(1,1)} = C_F & \left[-\frac{S_1 P_{192}}{(N-1)^2 N^2 (1+N)^2} + \frac{P_{447}}{(N-1)^3 N^3 (1+N)^3} - \frac{(2+N+N^2)S_1^2}{2(N-1)N(1+N)} \right. \\
& \left. - \frac{3(2+N+N^2)S_2}{2(N-1)N(1+N)} + \frac{(2+N+N^2)\zeta_2}{2(N-1)N(1+N)} \right], \tag{169}
\end{aligned}$$

$$\begin{aligned}
a_{gq}^{(1,2)} = C_F & \left[\frac{S_1^2 P_{192}}{4(N-1)^2 N^2 (1+N)^2} + \frac{3S_2 P_{192}}{4(N-1)^2 N^2 (1+N)^2} + \frac{P_{529}}{2(N-1)^4 N^4 (1+N)^4} \right. \\
& + \left(-\frac{P_{447}}{2(N-1)^3 N^3 (1+N)^3} + \frac{3(2+N+N^2)S_2}{4(N-1)N(1+N)} \right) S_1 + \frac{(2+N+N^2)S_1^3}{12(N-1)N(1+N)} \\
& + \frac{7(2+N+N^2)S_3}{6(N-1)N(1+N)} + \left(-\frac{P_{192}}{4(N-1)^2 N^2 (1+N)^2} - \frac{(2+N+N^2)S_1}{4(N-1)N(1+N)} \right) \zeta_2 \\
& \left. - \frac{7(2+N+N^2)\zeta_3}{6(N-1)N(1+N)} \right], \tag{170}
\end{aligned}$$

$$\begin{aligned}
a_{gq}^{(2,0)} = \xi & \left[C_F^2 \left(-\frac{2S_1 P_{200}}{(N-1)^2 N^2 (1+N)^2} - \frac{2P_{320}}{(N-1)^2 N^3 (1+N)^2} + \frac{3(2+N+N^2)S_1^2}{(N-1)N(1+N)} \right. \right. \\
& \left. + \frac{5(2+N+N^2)S_2}{(N-1)N(1+N)} \right) + C_A C_F \left(-\frac{P_{50}}{2(-2+N)(N-1)^3 N^2} - \frac{2S_1}{(N-1)^2 N} \right. \\
& \left. + \frac{2(-3+N)S_{-2}}{(-2+N)(N-1)(1+N)} \right) \left. \right] + C_F \left[N_F T_F \left(-\frac{8S_1 P_{220}}{9(N-1)^2 N^2 (1+N)^2} \right. \right. \\
& \left. - \frac{8(2+N+N^2)S_2}{3(N-1)N(1+N)} - \frac{8P_{464}}{27(N-1)^3 N^3 (1+N)^3} + \frac{4(2+N+N^2)\zeta_2}{3(N-1)N(1+N)} \right) \\
& + C_A \left(\frac{4S_2 P_{254}}{3(N-1)N^2 (1+N)^2 (2+N)} + \frac{2S_1^2 P_{334}}{(N-1)^2 N^2 (1+N)^2 (2+N)} \right. \\
& \left. + \frac{P_{603}}{27(-2+N)(N-1)^4 N^4 (1+N)^4 (2+N)^3} + \left(-\frac{2P_{540}}{9(N-1)^3 N^3 (1+N)^3 (2+N)^2} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{2(12+28N+23N^2+9N^3)S_2}{(N-1)N(1+N)(2+N)} \Big) S_1 - \frac{4(-32-8N-3N^2+N^3)S_3}{3(N-1)N(1+N)(2+N)} \\
& -\frac{14(2+N+N^2)S_1^3}{3(N-1)N(1+N)} + \left(\frac{4P_{62}}{(-2+N)(N-1)N(1+N)^2} - \frac{16(2+N+N^2)S_1}{(N-1)N(1+N)} \right) \\
& \times S_{-2} - \frac{20(2+N+N^2)S_{-3}}{(N-1)N(1+N)} - \frac{16(6+2N+N^2)S_{2,1}}{(N-1)N(1+N)(2+N)} + \frac{8(2+N+N^2)S_{-2,1}}{(N-1)N(1+N)} \\
& + \left(-\frac{(2+N+N^2)P_{115}}{3(N-1)^2N^2(1+N)^2(2+N)} + \frac{4(2+N+N^2)S_1}{(N-1)N(1+N)} \right) \zeta_2 \\
& - \frac{12(12+8N+3N^2)\zeta_3}{N(1+N)(2+N)} \Big] + C_F^2 \left[\frac{2S_1^2P_{97}}{(N-1)N^2(1+N)^2} + \frac{2S_2P_{108}}{(N-1)N^2(1+N)^2} \right. \\
& + \frac{P_{507}}{(N-1)^2N^4(1+N)^4} + \left(\frac{4P_{335}}{(N-1)^2N^3(1+N)^3} - \frac{2(38+23N+15N^2)S_2}{(N-1)N(1+N)} \right) S_1 \\
& - \frac{14(2+N+N^2)S_1^3}{3(N-1)N(1+N)} - \frac{4(74+43N+31N^2)S_3}{3(N-1)N(1+N)} + \frac{32S_{2,1}}{(N-1)N} \\
& \left. + \left(-\frac{(2+N+N^2)(2+3N+3N^2)}{(N-1)N^2(1+N)^2} + \frac{4(2+N+N^2)S_1}{(N-1)N(1+N)} \right) \zeta_2 + \frac{48\zeta_3}{1+N} \right], \quad (171)
\end{aligned}$$

$$\begin{aligned}
a_{gq}^{(2,1)} = & \xi \left[C_A C_F \left(\frac{S_1 P_{50}}{2(-2+N)(N-1)^3 N^2} + \frac{P_{397}}{8(-2+N)^2(N-1)^4 N^3} + \frac{S_1^2 + 2S_2}{(N-1)^2 N} \right. \right. \\
& + \frac{(3-N)S_3}{(-2+N)(N-1)(1+N)} + \left(\frac{P_{60}}{(-2+N)^2(N-1)N(1+N)^2} \right. \\
& - \left. \frac{2(-3+N)S_1}{(-2+N)(N-1)(1+N)} \right) S_{-2} - \frac{4(-3+N)S_{-3}}{(-2+N)(N-1)(1+N)} - \frac{\zeta_2}{2(N-1)^2 N} \\
& \left. - \frac{3(-3+N)\zeta_3}{(-2+N)(N-1)(1+N)} \right) + C_F^2 \left(\frac{S_1^2 P_{224}}{2(N-1)^2 N^2(1+N)^2} \right. \\
& + \frac{S_2 P_{238}}{2(N-1)^2 N^2(1+N)^2} + \frac{P_{500}}{(N-1)^3 N^4(1+N)^3} + \left(\frac{P_{459}}{(N-1)^3 N^3(1+N)^3} \right. \\
& - \left. \frac{13(2+N+N^2)S_2}{2(N-1)N(1+N)} \right) S_1 - \frac{7(2+N+N^2)S_1^3}{6(N-1)N(1+N)} - \frac{19(2+N+N^2)S_3}{3(N-1)N(1+N)} \\
& \left. + \left(\frac{-2-N-N^2}{(N-1)N^2(1+N)} + \frac{(2+N+N^2)S_1}{(N-1)N(1+N)} \right) \zeta_2 + \frac{6(2+N+N^2)\zeta_3}{(N-1)N(1+N)} \right] \\
& + C_F (N_F T_F \left(-\frac{2S_2 P_{215}}{9(N-1)^2 N^2(1+N)^2} + \frac{2S_1^2 P_{220}}{9(N-1)^2 N^2(1+N)^2} \right. \\
& - \frac{4P_{562}}{81(N-1)^4 N^4(1+N)^4} + \left(\frac{4P_{464}}{27(N-1)^3 N^3(1+N)^3} + \frac{4(2+N+N^2)S_2}{3(N-1)N(1+N)} \right) S_1 \\
& \left. + \frac{16(2+N+N^2)S_3}{3(N-1)N(1+N)} + \left(\frac{2P_{216}}{9(N-1)^2 N^2(1+N)^2} - \frac{2(2+N+N^2)S_1}{3(N-1)N(1+N)} \right) \zeta_2 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{100(2+N+N^2)\zeta_3}{9(N-1)N(1+N)} + C_A \left(\frac{S_1^3 P_{296}}{3(N-1)^2 N^2 (1+N)^2 (2+N)} \right. \\
& + \frac{4S_{-2,1} P_{205}}{(N-1)^2 N^2 (1+N)^2} - \frac{2S_3 P_{474}}{3(-2+N)(N-1)^2 N^2 (1+N)^2 (2+N)^2} \\
& - \frac{4S_{2,1} P_{332}}{(N-1)^2 N^2 (1+N)^2 (2+N)^2} + \frac{S_2 P_{543}}{18(N-1)^3 N^3 (1+N)^3 (2+N)^2} \\
& + \frac{P_{618}}{324(-2+N)^2 (N-1)^5 N^5 (1+N)^5 (2+N)^4} \\
& + \left(-\frac{2S_2 P_{419}}{3(N-1)^2 N^2 (1+N)^2 (2+N)^2} + \frac{P_{602}}{27(-2+N)(N-1)^4 N^4 (1+N)^4 (2+N)^3} \right. \\
& + \frac{4(-16+8N+9N^2+5N^3)S_3}{(N-1)N(1+N)(2+N)} + \frac{4(28+12N+7N^2+N^3)S_{2,1}}{(N-1)N(1+N)(2+N)} \Big) S_1 \\
& + \left(\frac{P_{545}}{18(N-1)^3 N^3 (1+N)^3 (2+N)^2} + \frac{(20+68N+57N^2+23N^3)S_2}{2(N-1)N(1+N)(2+N)} \right) S_1^2 \\
& + \frac{5(2+N+N^2)S_1^4}{4(N-1)N(1+N)} + \frac{(188+124N+85N^2+23N^3)S_2^2}{4(N-1)N(1+N)(2+N)} \\
& + \frac{(-124-12N+5N^2+11N^3)S_4}{2(N-1)N(1+N)(2+N)} + \left(-\frac{8S_1 P_{308}}{(-2+N)(N-1)^2 N^2 (1+N)^2} \right. \\
& - \frac{4P_{450}}{(-2+N)^2 (N-1)^2 N^3 (1+N)^3} + \frac{8(2+N+N^2)S_1^2}{(N-1)N(1+N)} + \frac{16(2+N+N^2)S_2}{(N-1)N(1+N)} \Big) S_{-2} \\
& + \frac{4(2+N+N^2)S_{-2}^2}{(N-1)N(1+N)} + \left(-\frac{2P_{329}}{(-2+N)(N-1)^2 N^2 (1+N)^2} + \frac{32(2+N+N^2)S_1}{(N-1)N(1+N)} \right) \\
& \times S_{-3} + \frac{42(2+N+N^2)S_{-4}}{(N-1)N(1+N)} + \frac{2(108+44N+25N^2+3N^3)S_{3,1}}{(N-1)N(1+N)(2+N)} \\
& - \frac{4(2+N+N^2)S_{-2,2}}{(N-1)N(1+N)} - \frac{12(2+N+N^2)S_{-3,1}}{(N-1)N(1+N)} \\
& - \frac{2(36+20N+13N^2+3N^3)S_{2,1,1}}{(N-1)N(1+N)(2+N)} - \frac{8(2+N+N^2)}{(N-1)N(1+N)} \\
& \times S_{-2,1,1} + \left(\frac{S_1 P_{350}}{6(N-1)^2 N^2 (1+N)^2 (2+N)} + \frac{P_{537}}{18(N-1)^3 N^3 (1+N)^3 (2+N)^2} \right. \\
& - \frac{3(2+N+N^2)S_1^2}{(N-1)N(1+N)} + \frac{(-2-N-N^2)S_2}{(N-1)N(1+N)} - \frac{2(2+N+N^2)S_{-2}}{(N-1)N(1+N)} \Big) \zeta_2 \\
& + \frac{18(12+8N+3N^2)\zeta_2^2}{5N(1+N)(2+N)} + \left(\frac{P_{440}}{9(-2+N)(N-1)^2 N^2 (1+N)^2 (2+N)^2} \right. \\
& + \frac{4(-64+8N+15N^2+11N^3)S_1}{3(N-1)N(1+N)(2+N)} \Big) \zeta_3 \Big) + C_F^2 \left(\frac{S_1^3 P_{32}}{3(N-1)N^2(1+N)^2} \right. \\
& + \frac{S_3 P_{170}}{3(N-1)N^2(1+N)^2(2+N)} + \frac{S_2 P_{465}}{2(N-1)^2 N^3(1+N)^3(2+N)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{P_{573}}{2(N-1)^3 N^5 (1+N)^5} + \left(\frac{S_2 P_{179}}{(N-1)N^2(1+N)^2(2+N)} \right. \\
& - \frac{2P_{559}}{(N-1)^3 N^4 (1+N)^4 (2+N)} + \frac{2(62+43N+19N^2)S_3}{(N-1)N(1+N)} - \frac{32S_{2,1}}{(N-1)N} \Big) S_1 \\
& + \left(\frac{P_{400}}{2(N-1)^2 N^3 (1+N)^3} + \frac{(86+55N+31N^2)S_2}{2(N-1)N(1+N)} \right) S_1^2 + \frac{5(2+N+N^2)S_1^4}{4(N-1)N(1+N)} \\
& + \frac{(110+39N+71N^2)S_2^2}{4(N-1)N(1+N)} + \frac{(286+171N+115N^2)S_4}{2(N-1)N(1+N)} \\
& + \frac{8(26+N+3N^2)S_{2,1}}{(N-1)N(1+N)(2+N)} - \frac{64S_{3,1}}{(N-1)N} + \frac{16S_{2,1,1}}{(N-1)N} + \left(\frac{S_1 P_{92}}{(N-1)N^2(1+N)^2} \right. \\
& + \frac{P_{319}}{2(N-1)N^3(1+N)^3} - \frac{3(2+N+N^2)S_1^2}{(N-1)N(1+N)} - \frac{7(2+N+N^2)S_2}{(N-1)N(1+N)} \Big) \zeta_2 \\
& + \left(\frac{2P_{263}}{3(N-1)N^2(1+N)^2(2+N)} - \frac{8(16-N+17N^2)S_1}{3(N-1)N(1+N)} \right) \zeta_3 - \frac{72\zeta_2^2}{5(1+N)} \Big), \quad (172)
\end{aligned}$$

$$b_{gq}^{(1,0)} = -\frac{4C_F}{N(1+N)}, \quad (173)$$

$$b_{gq}^{(1,1)} = C_F \left[-\frac{2(N-1)(1+2N)}{N^2(1+N)^2} + \frac{2S_1}{N(1+N)} \right], \quad (174)$$

$$\begin{aligned}
b_{gq}^{(1,2)} = C_F & \left[-\frac{(N-1)(-1-3N-3N^2+N^3)}{N^3(1+N)^3} + \frac{(N-1)(1+2N)S_1}{N^2(1+N)^2} - \frac{S_1^2}{2N(1+N)} \right. \\
& \left. - \frac{3S_2}{2N(1+N)} + \frac{\zeta_2}{2N(1+N)} \right], \quad (175)
\end{aligned}$$

$$\begin{aligned}
b_{gq}^{(2,0)} = \xi & \left[C_F^2 \left(-\frac{4P_{221}}{(N-1)^2 N^3 (1+N)^2} + \frac{4(4+3N+N^2)S_1}{(N-1)N^2(1+N)} \right) \right. \\
& + C_A C_F \left(\frac{P_{49}}{(-2+N)(N-1)^2 N^3} + \frac{4S_1}{(N-1)N^2} - \frac{4(N-1)S_{-2}}{(-2+N)N(1+N)} \right) \Big] \\
& + C_F \left[N_F T_F \left(-\frac{16(-9-19N+2N^2)}{9N^2(1+N)^2} + \frac{16S_1}{3N(1+N)} \right) \right. \\
& + C_A \left(-\frac{2P_{475}}{9(-2+N)(N-1)N^3(1+N)^3(2+N)^2} + \left(-\frac{4(-2+N)S_2}{N(1+N)(2+N)} \right. \right. \\
& \left. \left. - \frac{4P_{268}}{3(N-1)N^3(1+N)^2(2+N)} \right) S_1 + \frac{24S_1^2}{N(1+N)} + \frac{4(-2+5N+5N^2)S_2}{N^2(1+N)(2+N)} \right. \\
& \left. - \frac{4(-2+N)S_3}{N(1+N)(2+N)} + \frac{8(-3+N)S_{-2}}{(-2+N)N(1+N)} + \frac{8(-2+N)S_{2,1}}{N(1+N)(2+N)} \right. \\
& \left. \left. - \frac{24(-2+N)\zeta_3}{N(1+N)(2+N)} \right) \right] + C_F^2 \left(\frac{4P_{243}}{(N-1)^2 N^2 (1+N)^3} - \frac{4(-4+9N+3N^2)S_1}{(N-1)N^2(1+N)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{16S_2}{N(1+N)} \Big), \tag{176} \\
b_{gg}^{(2,1)} = & \xi \left[C_F^2 \left(\frac{2S_1 P_{240}}{(N-1)^2 N^3 (1+N)^2} - \frac{2P_{460}}{(N-1)^3 N^4 (1+N)^3} - \frac{(8+7N+N^2)S_1^2}{(N-1)N^2(1+N)} \right. \right. \\
& + \frac{(-16-13N-3N^2)S_2}{(N-1)N^2(1+N)} + \left. \frac{2(2+N+N^2)\zeta_2}{(N-1)N^2(1+N)} \right) \\
& + C_A C_F \left(-\frac{S_1 P_{49}}{(-2+N)(N-1)^2 N^3} + \frac{P_{392}}{4(-2+N)^2(N-1)^3 N^4} - \frac{2S_1^2}{(N-1)N^2} \right. \\
& - \frac{4S_2}{(N-1)N^2} + \frac{2(N-1)S_3}{(-2+N)N(1+N)} + \left(-\frac{2P_{204}}{(-2+N)^2(N-1)N^2(1+N)^2} \right. \\
& + \left. \frac{4(N-1)S_1}{(-2+N)N(1+N)} \right) S_{-2} + \frac{8(N-1)S_{-3}}{(-2+N)N(1+N)} + \frac{\zeta_2}{(N-1)N^2} \\
& + \left. \frac{6(N-1)\zeta_3}{(-2+N)N(1+N)} \right) \Big] + C_F (N_F T_F \left(-\frac{8P_{139}}{27N^3(1+N)^3} \right. \\
& + \left. \frac{8(-9-19N+2N^2)S_1}{9N^2(1+N)^2} - \frac{4S_1^2}{3N(1+N)} - \frac{28S_2}{3N(1+N)} + \frac{4\zeta_2}{N(1+N)} \right) \\
& + C_A \left(-\frac{4S_3 P_{258}}{3(-2+N)N^2(1+N)^2(2+N)^2} + \frac{S_2 P_{365}}{3(N-1)N^3(1+N)^2(2+N)^2} \right. \\
& + \frac{P_{580}}{54(-2+N)^2(N-1)N^4(1+N)^4(2+N)^3} + \left(-\frac{2S_2 P_{138}}{N^2(1+N)^2(2+N)^2} \right. \\
& - \frac{2P_{513}}{9(-2+N)(N-1)N^4(1+N)^3(2+N)^2} + \frac{12(-2+N)S_3}{N(1+N)(2+N)} \\
& - \left. \frac{8(N-2)S_{2,1}}{N(1+N)(2+N)} \right) S_1 + \left(\frac{P_{275}}{3(N-1)N^3(1+N)^2(2+N)} + \frac{3(N-2)S_2}{N(1+N)(2+N)} \right) S_1^2 \\
& - \frac{28S_1^3}{3N(1+N)} - \frac{2(-2+N)S_2^2}{N(1+N)(2+N)} + \frac{7(-2+N)S_4}{N(1+N)(2+N)} \\
& + \left(\frac{4(-3+N)P_{58}}{(-2+N)^2(N-1)N^2(1+N)^2} - \frac{8(-3+N)S_1}{(-2+N)N(1+N)} \right) S_{-2} \\
& - \frac{16(-3+N)S_{-3}}{(-2+N)N(1+N)} + \frac{8(-14-23N-3N^2+3N^3)S_{2,1}}{N(1+N)^2(2+N)^2} - \frac{16(-2+N)S_{3,1}}{N(1+N)(2+N)} \\
& + \frac{4(-2+N)S_{2,1,1}}{N(1+N)(2+N)} + \left(\frac{P_{34}}{(N-1)N^2(1+N)^2(2+N)} + \frac{8S_1}{N(1+N)} \right) \zeta_2 \\
& + \frac{36(-2+N)\zeta_2^2}{5N(1+N)(2+N)} + \left(-\frac{24P_{239}}{(-2+N)N^2(1+N)^2(2+N)^2} \right. \\
& + \left. \frac{12(-2+N)S_1}{N(1+N)(2+N)} \right) \zeta_3 \Big) + C_F^2 \left(\frac{S_2 P_{36}}{(N-1)N^2(1+N)^2(2+N)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{2P_{472}}{(N-1)^3 N^3 (1+N)^4} + \left(-\frac{2P_{341}}{(N-1)^2 N^3 (1+N)^2 (2+N)} - \frac{48S_2}{N(1+N)(2+N)} \right) \\
& \times S_1 + \frac{(-12 + 7N + 32N^2 + 5N^3)S_1^2}{(N-1)N^2(1+N)^2} - \frac{24(4+N)S_3}{N(1+N)(2+N)} - \frac{16(-4+N)S_{2,1}}{N(1+N)(2+N)} \\
& - \frac{4(2+N+N^2)\zeta_2}{(N-1)N(1+N)^2} + \frac{96(N-1)\zeta_3}{N(1+N)(2+N)} \Big), \tag{177}
\end{aligned}$$

$$\begin{aligned}
a_{gg}^{(1,0)} & = -\frac{C_A \xi^2}{4} + C_A \xi \left(\frac{-1+N}{N} + S_1 \right) + C_A \left(-\frac{P_{442}}{9(N-1)^2 N^2 (1+N)^2 (2+N)^2} \right. \\
& \left. + \frac{8(1+N+N^2)S_1}{(N-1)N(1+N)(2+N)} - 2S_1^2 - 6S_2 \right) - \frac{20N_F T_F}{9}, \tag{178}
\end{aligned}$$

$$\begin{aligned}
a_{gg}^{(1,1)} & = \frac{C_A \xi^2}{4} + C_A \xi \left(\frac{1+N}{2N^2} + \frac{(1-N)S_1}{2N} - \frac{1}{4}S_1^2 - \frac{3}{4}S_2 \right) + C_A \left(\right. \\
& \frac{P_{570}}{27(N-1)^3 N^3 (1+N)^3 (2+N)^3} + \left(\frac{P_{395}}{(N-1)^2 N^2 (1+N)^2 (2+N)^2} + 3S_2 \right) \\
& \times S_1 - \frac{2(1+N+N^2)S_1^2}{(N-1)N(1+N)(2+N)} + \frac{1}{3}S_1^3 - \frac{6(1+N+N^2)S_2}{(N-1)N(1+N)(2+N)} + \frac{14}{3}S_3 \\
& \left. + \left(\frac{P_{115}}{12(N-1)N(1+N)(2+N)} - S_1 \right) \zeta_2 \right) + N_F T_F \left(\frac{56}{27} - \frac{\zeta_2}{3} \right), \tag{179}
\end{aligned}$$

$$\begin{aligned}
a_{gg}^{(1,2)} & = C_A \xi^2 \left[-\frac{1}{4} + \frac{\zeta_2}{32} \right] + C_A \xi \left[\frac{-N-1}{4N^3} + \left(\frac{-1-N}{4N^2} + \frac{3S_2}{8} \right) S_1 + \frac{(N-1)S_1^2}{8N} + \frac{1}{24}S_1^3 \right. \\
& \left. + \frac{3(N-1)S_2}{8N} + \frac{7}{12}S_3 + \left(\frac{1-N}{8N} - \frac{S_1}{8} \right) \zeta_2 \right] + C_A \left(-\frac{1}{24}S_1^4 \right. \\
& \left. + \frac{P_{608}}{162(N-1)^4 N^4 (1+N)^4 (2+N)^4} - \frac{3S_2 P_{395}}{4(N-1)^2 N^2 (1+N)^2 (2+N)^2} \right. \\
& \left. + \left(\frac{P_{530}}{2(N-1)^3 N^3 (1+N)^3 (2+N)^3} + \frac{3(1+N+N^2)S_2}{(N-1)N(1+N)(2+N)} - \frac{7}{3}S_3 \right) S_1 \right. \\
& \left. + \left(-\frac{P_{395}}{4(N-1)^2 N^2 (1+N)^2 (2+N)^2} - \frac{3}{4}S_2 \right) S_1^2 + \frac{(1+N+N^2)S_1^3}{3(N-1)N(1+N)(2+N)} \right. \\
& \left. - \frac{9}{8}S_2^2 + \frac{14(1+N+N^2)S_3}{3(N-1)N(1+N)(2+N)} - \frac{15}{4}S_4 + \left(\frac{P_{442}}{72(N-1)^2 N^2 (1+N)^2 (2+N)^2} \right. \right. \\
& \left. \left. + \frac{(-1-N-N^2)S_1}{(N-1)N(1+N)(2+N)} + \frac{1}{4}S_1^2 + \frac{3}{4}S_2 \right) \zeta_2 + \left(-\frac{7P_{115}}{36(N-1)N(1+N)(2+N)} \right. \right. \\
& \left. \left. + \frac{7}{3}S_1 \right) \zeta_3 \right) + N_F T_F \left(-\frac{164}{81} + \frac{5\zeta_2}{18} + \frac{7\zeta_3}{9} \right), \tag{180}
\end{aligned}$$

$$a_{gg}^{(2,0)} = \frac{C_A^2 \xi^4}{16} + C_A^2 \xi^3 \left[\frac{4-3N}{16N} - \frac{S_1}{4} \right] + \xi^2 \left[\frac{2C_A N_F T_F}{3} + C_A^2 \left(\frac{S_1 P_{105}}{4N^2(1+N)(2+N)} \right) \right]$$

$$\begin{aligned}
& + \frac{P_{556}}{12(-2+N)(N-1)^2N^3(1+N)^2(2+N)^2(3+N)} + \frac{3}{4}S_1^2 + \frac{3}{2}S_2 \\
& + \left. \frac{(4-N+N^2)S_{-2}}{4(-2+N)N(1+N)(3+N)} - \frac{1}{8}\zeta_3 \right) + \xi \left[C_{AN_F T_F} \left(\frac{2(18+16N+7N^2)}{9N^2} \right. \right. \\
& \left. \left. - \frac{4(-3+8N)S_1}{9N} - \frac{2}{3}S_1^2 - \frac{14}{3}S_2 \right) + C_A^2 \left(\frac{S_3P_{48}}{2N(1+N)(2+N)(3+N)} \right. \right. \\
& \left. \left. - \frac{S_{2,1}P_{48}}{N(1+N)(2+N)(3+N)} + \frac{S_2P_{359}}{12(N-1)N^2(1+N)(2+N)(3+N)} \right. \right. \\
& \left. \left. + \frac{P_{577}}{18(-2+N)(N-1)^3N^3(1+N)^2(2+N)^2(3+N)} - \frac{3\zeta_3P_{88}}{2N(1+N)(2+N)(3+N)} \right. \right. \\
& \left. \left. + \left(\frac{S_2P_{21}}{2N(1+N)(2+N)(3+N)} + \frac{P_{547}}{18(N-1)^2N^3(1+N)^2(2+N)^2(3+N)} \right) S_1 \right. \right. \\
& \left. \left. + \frac{(142+323N+2N^2-35N^3)S_1^2}{12(N-1)(1+N)(2+N)} - 4S_1^3 + \frac{(-1-15N-N^2+N^3)S_{-2}}{(-2+N)(N-1)N(1+N)(3+N)} \right) \right] \\
& + C_{FN_F T_F} \left(-\frac{8S_1P_{401}}{(N-1)N^3(1+N)^3(2+N)^2} + \frac{P_{572}}{3(N-1)N^4(1+N)^4(2+N)^3} \right. \\
& \left. + \frac{8(2+N+N^2)^2S_1^2}{(N-1)N^2(1+N)^2(2+N)} + \frac{16(2+N+N^2)^2S_2}{(N-1)N^2(1+N)^2(2+N)} \right. \\
& \left. + \frac{64S_{-2}}{(N-1)N(1+N)(2+N)} - \frac{4(2+N+N^2)^2\zeta_2}{(N-1)N^2(1+N)^2(2+N)} + 16\zeta_3 \right) \\
& + C_{AN_F T_F} \left(\frac{2S_1^2P_{144}}{9(N-1)N(1+N)(2+N)} + \frac{2S_2P_{145}}{3(N-1)N(1+N)(2+N)} \right. \\
& \left. - \frac{4P_{583}}{81(N-1)^3N^3(1+N)^3(2+N)^3} + \left(\frac{4P_{476}}{9(N-1)^2N^2(1+N)^2(2+N)^2} \right. \right. \\
& \left. \left. + \frac{8(3+2N)S_2}{2+N} \right) S_1 + \frac{16}{9}S_1^3 + \frac{8(47+28N)S_3}{9(2+N)} - \frac{32S_{-2}}{(N-1)N(1+N)(2+N)} \right. \\
& \left. - \frac{16(N-1)S_{2,1}}{3(2+N)} + \left(\frac{4P_{115}}{9(N-1)N(1+N)(2+N)} - \frac{16}{3}S_1 \right) \zeta_2 - \frac{8(8+N)\zeta_3}{2+N} \right) \\
& + C_A^2 \left(-\frac{4S_1^3P_{116}}{9(N-1)N(1+N)(2+N)} + \frac{S_3P_{163}}{9(N-1)N(1+N)(2+N)(3+N)} \right. \\
& \left. + \frac{2S_{2,1}P_{255}}{3(N-1)N(1+N)(2+N)(3+N)} + \frac{S_2P_{433}}{6(N-1)N^2(1+N)^2(2+N)^2(3+N)} \right. \\
& \left. + \frac{P_{616}}{162(-2+N)(N-1)^4N^4(1+N)^4(2+N)^4(3+N)} + \left(\frac{238}{3}S_3 - 4S_{2,1} \right. \right. \\
& \left. \left. + \frac{S_2P_{174}}{(N-1)N(1+N)(2+N)(3+N)} + \frac{P_{585}}{9(N-1)^3N^3(1+N)^3(2+N)^3(3+N)} \right) S_1 \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{P_{434}}{18(N-1)^2 N^2 (1+N)^2 (2+N)^2} + 49S_2 \right) S_1^2 + \frac{14}{3} S_1^4 + 24S_2^2 + 19S_4 \\
& + \left(\frac{2P_{408}}{(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)} - \frac{8(10+9N+6N^2-N^3)S_1}{(N-1)N(1+N)(2+N)} \right. \\
& \left. + 8S_1^2 + 16S_2 \right) S_{-2} + 4S_{-2}^2 + \left(\frac{4(-22-21N-18N^2+N^3)}{(N-1)N(1+N)(2+N)} + 32S_1 \right) S_{-3} \\
& + 28S_{-4} + 6S_{3,1} + \frac{8(6+5N+2N^2-N^3)S_{-2,1}}{(N-1)N(1+N)(2+N)} - 8S_{-2,2} - 24S_{-3,1} + 6S_{2,1,1} \\
& - 16S_{-2,1,1} + \left(\frac{4S_1 P_{115}}{3(N-1)N(1+N)(2+N)} - \frac{P_{115}^2}{18(N-1)^2 N^2 (1+N)^2 (2+N)^2} \right. \\
& \left. - 8S_1^2 \right) \zeta_2 + \left(\frac{-72-24N-83N^2-25N^3}{N(2+N)(3+N)} + 12S_1 \right) \zeta_3 \\
& + N_F^2 T_F^2 \left(\frac{848}{81} - \frac{8\zeta_2}{9} \right), \tag{181}
\end{aligned}$$

$$\begin{aligned}
a_{gg}^{(2,1)} & = -\frac{1}{8} C_A^2 \xi^4 + C_A^2 \xi^3 \left[\frac{-4-12N-N^2}{32N^2} + \frac{(-1+3N)S_1}{8N} + \frac{1}{16} S_1^2 + \frac{3}{16} S_2 + \frac{1}{16} \zeta_2 \right] \\
& + \xi^2 \left[C_A^2 \left(\frac{S_3 P_{25}}{24(-2+N)N(1+N)(3+N)} + \frac{\zeta_3 P_{46}}{24(-2+N)N(1+N)(3+N)} \right. \right. \\
& + \frac{S_1^2 P_{187}}{8(N-1)N^2(1+N)(2+N)} + \frac{S_2 P_{191}}{2(N-1)N^2(1+N)(2+N)} \\
& + \frac{P_{609}}{144(-2+N)^2(N-1)^3 N^4(1+N)^3(2+N)^3(3+N)^2} \\
& + \left. \left(\frac{P_{557}}{8(-2+N)(N-1)^2 N^3(1+N)^2(2+N)^2(3+N)} - \frac{5}{4} S_2 \right) S_1 - \frac{5}{24} S_1^3 \right. \\
& + \left. \left(\frac{P_{202}}{4(-2+N)^2(N-1)N^2(1+N)^2(3+N)^2} + \frac{(-4+N-N^2)S_1}{4(-2+N)N(1+N)(3+N)} \right) S_{-2} \right. \\
& + \frac{(-4+N-N^2)S_{-3}}{2(-2+N)N(1+N)(3+N)} + \left. \left(\frac{P_{182}}{24(N-1)N^2(1+N)(2+N)} \right. \right. \\
& \left. + \frac{3}{8} S_1 \right) \zeta_2 + \frac{3}{80} \zeta_2^2 \left. \right) + C_A N_F T_F \left(-\frac{2}{3} + \frac{\zeta_2}{12} \right) \\
& + \xi \left[C_A N_F T_F \left(\frac{-126-144N-50N^2-53N^3}{27N^3} + \left(\frac{2(-27-24N+25N^2)}{27N^2} \right. \right. \right. \\
& + \left. \left. \frac{7S_2}{3} \right) S_1 + \frac{(-3+8N)S_1^2}{9N} + \frac{1}{9} S_1^3 + \frac{7(-3+8N)S_2}{9N} + \frac{62}{9} S_3 + \left(\frac{4-3N}{6N} - \frac{2S_1}{3} \right) \right. \\
& \left. \times \zeta_2 - 4\zeta_3 \right) + C_A^2 \left(\frac{S_{3,1} P_{43}}{2N(1+N)(2+N)(3+N)} + \frac{S_4 P_{54}}{8N(1+N)(2+N)(3+N)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{9\zeta_2^2 P_{88}}{20N(1+N)(2+N)(3+N)} + \frac{S_{2,1,1} P_{91}}{2N(1+N)(2+N)(3+N)} \\
& + \frac{S_2^2 P_{135}}{4N(1+N)(2+N)(3+N)} + \frac{S_{2,1} P_{452}}{2(N-1)N(1+N)^2(2+N)^2(3+N)^2} \\
& + \frac{S_3 P_{525}}{36(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} \\
& + \frac{S_2 P_{553}}{72(N-1)^2 N^3(1+N)^2(2+N)^2(3+N)^2} \\
& + \frac{P_{613}}{108(-2+N)^2(N-1)^4 N^4(1+N)^3(2+N)^3(3+N)^2} \\
& + \left(\frac{S_{2,1} P_{44}}{N(1+N)(2+N)(3+N)} + \frac{S_3 P_{153}}{6N(1+N)(2+N)(3+N)} \right) \\
& + \frac{P_{605}}{108(-2+N)(N-1)^3 N^4(1+N)^3(2+N)^3(3+N)^2} \\
& + \frac{S_2 P_{508}}{24(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} \Big) S_1 + \left(\frac{S_2 P_{150}}{8N(1+N)(2+N)(3+N)} \right. \\
& + \left. \frac{P_{524}}{72(N-1)^2 N^3(1+N)^2(2+N)^2(3+N)} \right) S_1^2 + \frac{7}{6} S_1^4 \\
& - \frac{(370 + 773N - 34N^2 - 101N^3) S_1^3}{72(N-1)(1+N)(2+N)} + \left(\frac{P_{311}}{(-2+N)^2(N-1)N^2(1+N)^2(3+N)^2} \right. \\
& + \left. \frac{(1 + 15N + N^2 - N^3) S_1}{(-2+N)(N-1)N(1+N)(3+N)} \right) S_{-2} + \frac{2(1 + 15N + N^2 - N^3) S_{-3}}{(N-2)(N-1)N(1+N)(3+N)} \\
& + \left(\frac{P_{351}}{24(N-1)^2 N^2(1+N)(2+N)} + \frac{(26 + 181N + 70N^2 + 11N^3) S_1}{24(N-1)(1+N)(2+N)} - 2S_1^2 \right) \zeta_2 \\
& + \left(\frac{P_{534}}{(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} \right. \\
& - \left. \frac{3S_1 P_{90}}{2N(1+N)(2+N)(3+N)} \right) \zeta_3 \Big] + C_F N_F T_F \left(-\frac{4S_3 P_{120}}{3(N-1)N^2(1+N)^2(2+N)} \right. \\
& - \frac{4\zeta_3 P_{343}}{3(N-1)N^2(1+N)^2(2+N)} + \frac{4S_1^2 P_{401}}{(N-1)N^3(1+N)^3(2+N)^2} \\
& + \frac{8S_2 P_{401}}{(N-1)N^3(1+N)^3(2+N)^2} + \frac{P_{610}}{36(N-2)(N-1)N^5(1+N)^5(2+N)^4(3+N)} \\
& + \left(-\frac{4P_{532}}{(N-1)N^4(1+N)^4(2+N)^3} - \frac{16(2+N+N^2)^2 S_2}{(N-1)N^2(1+N)^2(2+N)} \right) S_1 \\
& - \frac{8(2+N+N^2)^2 S_1^3}{3(N-1)N^2(1+N)^2(2+N)} + \left(\frac{8P_{399}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)} \right. \\
& - \left. \frac{64S_1}{(N-1)N(1+N)(2+N)} \right) S_{-2} - \frac{128S_{-3}}{(N-1)N(1+N)(2+N)}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{P_{501}}{(N-1)N^3(1+N)^3(2+N)^2} + \frac{4(2+N+N^2)^2 S_1}{(N-1)N^2(1+N)^2(2+N)} \right) \zeta_2 - \frac{24}{5} \zeta_2^2 \\
& + C_A N_F T_F \left(\frac{S_1^3 P_{26}}{9(N-1)N(1+N)(2+N)} + \frac{8S_{2,1} P_{330}}{9(N-1)N(1+N)(2+N)^2(3+N)} \right. \\
& - \frac{4S_3 P_{358}}{9(N-1)N(1+N)(2+N)^2(3+N)} + \frac{S_2 P_{495}}{9(N-1)^2 N^2 (1+N)^2 (2+N)^2 (3+N)} \\
& + \frac{P_{615}}{54(-2+N)(N-1)^4 N^4 (1+N)^4 (2+N)^4 (3+N)} \\
& + \left(\frac{S_2 P_{291}}{3(N-1)N(1+N)(2+N)^2(3+N)} - \frac{2P_{589}}{27(N-1)^3 N^3 (1+N)^3 (2+N)^3 (3+N)} \right. \\
& - \frac{8(23+25N)S_3}{9(2+N)} - \frac{16S_{2,1}}{2+N} \left. \right) S_1 + \left(\frac{P_{438}}{27(N-1)^2 N^2 (1+N)^2 (2+N)^2} \right. \\
& - \left. \frac{2(11+10N)S_2}{3(2+N)} \right) S_1^2 - \frac{4}{9} S_1^4 - \frac{4(17+7N)S_2^2}{3(2+N)} - \frac{2(29+18N)S_4}{2+N} \\
& + \left(\frac{32S_1}{(N-1)N(1+N)(2+N)} - \frac{4P_{399}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)} \right) \\
& \times S_{-2} + \frac{64S_{-3}}{(N-1)N(1+N)(2+N)} + \frac{8(-2+N)S_{3,1}}{2+N} + \frac{8(7+2N)S_{2,1,1}}{3(2+N)} \\
& + \left(\frac{S_1 P_{128}}{3(N-1)N(1+N)(2+N)} + \frac{P_{439}}{9(N-1)^2 N^2 (1+N)^2 (2+N)^2} + 2S_1^2 + 6S_2 \right) \zeta_2 \\
& + \frac{12(8+N)\zeta_2^2}{5(2+N)} + \left(-\frac{2P_{367}}{27(N-1)N(1+N)(2+N)^2(3+N)} + \frac{8(109+41N)S_1}{9(2+N)} \right) \\
& \times \zeta_3 \left. \right) + C_A^2 \left(\frac{S_1^4 P_{117}}{9(N-1)N(1+N)(2+N)} + \frac{S_{2,1,1} P_{178}}{3(N-1)N(1+N)(2+N)(3+N)} \right. \\
& + \frac{S_{3,1} P_{185}}{(N-1)N(1+N)(2+N)(3+N)} + \frac{S_2^2 P_{274}}{6(N-1)N(1+N)(2+N)(3+N)} \\
& + \frac{S_4 P_{279}}{4(N-1)N(1+N)(2+N)(3+N)} + \frac{8S_{-2,1} P_{307}}{(N-1)^2 N^2 (1+N)^2 (2+N)^2} \\
& - \frac{2S_{2,1} P_{538}}{9(N-1)^2 N^2 (1+N)^2 (2+N)^2 (3+N)^2} \\
& + \frac{S_3 P_{566}}{9(-2+N)(N-1)^2 N^2 (1+N)^2 (2+N)^2 (3+N)^2} \\
& + \frac{P_{620}}{648(-2+N)^2 (N-1)^5 N^5 (1+N)^5 (2+N)^5 (3+N)^2} \\
& + \left(\frac{S_3 P_{281}}{9(N-1)N(1+N)(2+N)(3+N)} + \frac{S_2 P_{549}}{12(N-1)^2 N^2 (1+N)^2 (2+N)^2 (3+N)^2} \right. \\
& + \left. \frac{P_{617}}{27(-2+N)(N-1)^4 N^4 (1+N)^4 (2+N)^4 (3+N)^2} - 42S_2^2 - \frac{207}{2} S_4 \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{2(-12 + 21N + 7N^2)S_{2,1}}{N(2+N)(3+N)} - 2S_{2,1,1} \Big) S_1 + \left(\frac{S_2 P_{276}}{12(N-1)N(1+N)(2+N)(3+N)} \right. \\
& + \left. \frac{P_{592}}{108(N-1)^3 N^3 (1+N)^3 (2+N)^3 (3+N)} - 59S_3 + 2S_{2,1} \right) S_1^2 \\
& + \left(\frac{P_{483}}{36(N-1)^2 N^2 (1+N)^2 (2+N)^2} - \frac{115}{6} S_2 \right) S_1^3 - S_1^5 \\
& + \left(\frac{P_{601}}{36(N-1)^3 N^3 (1+N)^3 (2+N)^3 (3+N)^2} - \frac{178}{3} S_3 + 4S_{2,1} + 8S_{-2,1} \right) S_2 - 46S_5 \\
& + \left(-\frac{2P_{576}}{(-2+N)^2 (N-1)^2 N^3 (1+N)^3 (2+N)^3 (3+N)^2} \right. \\
& + \left. \left(-\frac{2P_{471}}{(-2+N)(N-1)^2 N^2 (1+N)^2 (2+N)^2 (3+N)} - 16S_2 \right) S_1 \right. \\
& - \frac{4(-10 - 9N - 6N^2 + N^3)S_1^2}{(N-1)N(1+N)(2+N)} - \frac{8}{3} S_1^3 - \frac{8(-10 - 9N - 6N^2 + N^3)S_2}{(N-1)N(1+N)(2+N)} - \frac{64}{3} S_3 \\
& - 8S_{2,1} \Big) S_{-2} + \left(-\frac{2(-10 - 9N - 6N^2 + N^3)}{(N-1)N(1+N)(2+N)} - 4S_1 \right) S_{-2}^2 \\
& + \left(-\frac{4P_{467}}{(-2+N)(N-1)^2 N^2 (1+N)^2 (2+N)^2 (3+N)} - 16S_1^2 - 20S_2 - 16S_{-2} \right. \\
& - \left. \frac{16(-10 - 9N - 6N^2 + N^3)S_1}{(N-1)N(1+N)(2+N)} \right) S_{-3} - \left(\frac{14(-14 - 13N - 10N^2 + N^3)}{(N-1)N(1+N)(2+N)} \right. \\
& + 56S_1 \Big) S_{-4} - \frac{4(6 + 5N + 2N^2 - N^3)S_{-2,2}}{(N-1)N(1+N)(2+N)} - 54S_{-5} + 16S_{2,3} - 20S_{2,-3} - 13S_{4,1} \\
& + 12S_{-2,3} + \frac{12(-6 - 5N - 2N^2 + N^3)S_{-3,1}}{(N-1)N(1+N)(2+N)} + 28S_{-4,1} + 8S_{2,1,-2} - 18S_{2,2,1} \\
& - 6S_{3,1,1} + \frac{8(-6 - 5N - 2N^2 + N^3)S_{-2,1,1}}{(N-1)N(1+N)(2+N)} + 8S_{-2,2,1} + 24S_{-3,1,1} - 3S_{2,1,1,1} \\
& + 16S_{-2,1,1,1} + \left(\frac{S_2 P_{22}}{2(N-1)N(1+N)(2+N)} + \frac{S_1^2 P_{33}}{2(N-1)N(1+N)(2+N)} \right. \\
& + \left. \frac{P_{581}}{36(N-1)^3 N^3 (1+N)^3 (2+N)^3} + \left(\frac{P_{437}}{12(N-1)^2 N^2 (1+N)^2 (2+N)^2} + 16S_2 \right) \right. \\
& \times S_1 + 4S_1^3 + 2S_3 + \left(-\frac{8(1+N+N^2)}{(N-1)N(1+N)(2+N)} + 4S_1 \right) S_{-2} + 2S_{-3} - 4S_{-2,1} \Big) \\
& \times \zeta_2 + \left(\frac{3(72 + 24N + 83N^2 + 25N^3)}{10N(2+N)(3+N)} - \frac{18}{5} S_1 \right) \zeta_2^2 \\
& + \left(\frac{P_{568}}{54(-2+N)(N-1)^2 N^2 (1+N)^2 (2+N)^2 (3+N)^2} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{S_1 P_{162}}{9(N-1)N(1+N)(2+N)(3+N)} + \frac{83}{3} S_1^2 - 9S_2 - 12S_{-2} \Big) \zeta_3 \Big) \\
& + N_F^2 T_F^2 \left(-\frac{1184}{81} + \frac{20\zeta_2}{9} + \frac{56\zeta_3}{27} \right), \tag{182}
\end{aligned}$$

$$b_{gg}^{(1,0)} = \frac{2C_A(2-N+N^2)}{N(1+N)(2+N)} - \frac{C_A \xi}{N-1}, \tag{183}$$

$$b_{gg}^{(1,1)} = C_A \left[\xi \left(\frac{1+N-N^2}{2(N-1)^2 N} + \frac{S_1}{2(N-1)} \right) - \frac{P_{196}}{N^2(1+N)^2(2+N)^2} - \frac{(2-N+N^2)S_1}{N(1+N)(2+N)} \right], \tag{184}$$

$$\begin{aligned}
b_{gg}^{(1,2)} & = C_A \xi \left[\frac{1-4N+2N^2}{4(N-1)^3 N^2} + \frac{(-1-N+N^2)S_1}{4(N-1)^2 N} - \frac{S_1^2}{8(N-1)} - \frac{3S_2}{8(N-1)} + \frac{\zeta_2}{8(N-1)} \right] \\
& + C_A \left(\frac{S_1 P_{196}}{2N^2(1+N)^2(2+N)^2} + \frac{P_{388}}{2N^3(1+N)^3(2+N)^3} + \frac{(2-N+N^2)S_1^2}{4N(1+N)(2+N)} \right. \\
& \left. + \frac{3(2-N+N^2)S_2}{4N(1+N)(2+N)} - \frac{(2-N+N^2)\zeta_2}{4N(1+N)(2+N)} \right), \tag{185}
\end{aligned}$$

$$\begin{aligned}
b_{gg}^{(2,0)} & = \frac{C_A^2 \xi^3}{4(N-1)} + C_A^2 \xi^2 \left[\frac{P_{444}}{8(-2+N)(N-1)^2 N^3(1+N)(2+N)(3+N)} \right. \\
& \left. + \frac{(-4-N-2N^2)S_1}{4(N-1)N^2} + \frac{(-2-3N+N^2)S_{-2}}{2(-2+N)N(1+N)(3+N)} \right] + \xi \left[C_A N_F T_F \right. \\
& \times \left(-\frac{4(-3-10N+4N^2)}{9(N-1)^2 N} + \frac{4S_1}{3(N-1)} \right) + C_A^2 \left(\frac{S_2 P_{126}}{4(N-1)N^2(2+N)(3+N)} \right. \\
& \left. + \frac{P_{561}}{36(-2+N)(N-1)^3 N^3(1+N)^2(2+N)^2(3+N)} \right. \\
& \left. + \left(\frac{P_{385}}{3(N-1)^2 N^3(1+N)(2+N)(3+N)} + \frac{(-72-8N+3N^2+N^3)S_2}{2N(1+N)(2+N)(3+N)} \right) S_1 \right. \\
& \left. + \frac{3(3+8N)S_1^2}{4(N-1)N} + \frac{(-96-52N-21N^2-3N^3)S_3}{2N(1+N)(2+N)(3+N)} - \frac{2S_{-3}}{N} + \frac{4S_{-2,1}}{N} \right. \\
& \left. + \left(\frac{P_{219}}{(-2+N)(N-1)N^2(1+N)(3+N)} - \frac{4S_1}{N} \right) S_{-2} \right. \\
& \left. + \frac{(72+8N-3N^2-N^3)S_{2,1}}{N(1+N)(2+N)(3+N)} + \frac{3(-60+14N+15N^2+3N^3)\zeta_3}{N(1+N)(2+N)(3+N)} \right] \\
& + C_F N_F T_F \left(-\frac{32P_{234}}{(N-1)N^2(1+N)^3(2+N)^2} + \frac{64(2+N+N^2)S_1}{(N-1)N(1+N)^2(2+N)} \right) \\
& + C_A N_F T_F \left(-\frac{4P_{333}}{9(N-1)N^2(1+N)^2(2+N)^2} + \left(-\frac{4P_{225}}{3(N-1)^2 N^2(1+N)(2+N)} \right) \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{16S_2}{(1+N)(2+N)} \Big) S_1 + \frac{8(6-13N+N^2)S_2}{3(N-1)N(2+N)} + \frac{16S_3}{(1+N)(2+N)} \\
& - \frac{32S_{2,1}}{(1+N)(2+N)} + \frac{96\zeta_3}{(1+N)(2+N)} \Big) \\
& + C_A^2 \left(\frac{P_{578}}{9(-2+N)(N-1)^2N^3(1+N)^3(2+N)^3(3+N)} \right. \\
& + \frac{S_2P_{175}}{3(N-1)N^2(1+N)(2+N)(3+N)} + \left(\frac{P_{512}}{3(N-1)^2N^3(1+N)^2(2+N)^2(3+N)} \right. \\
& + \left. \frac{(18-22N-N^2+N^3)S_2}{N(1+N)(2+N)(3+N)} \right) S_1 + \frac{(42-26N+7N^2+5N^3)S_3}{N(1+N)(2+N)(3+N)} \\
& - \frac{12(2-N+N^2)S_1^2}{N(1+N)(2+N)} + \left(\frac{2P_{230}}{(-2+N)N^2(1+N)^2(2+N)(3+N)} \right. \\
& + \left. \frac{8(2-N+N^2)S_1}{N(1+N)(2+N)} \right) S_{-2} + \frac{4(2-N+N^2)S_{-3}}{N(1+N)(2+N)} - \frac{2(18-22N-N^2+N^3)S_{2,1}}{N(1+N)(2+N)(3+N)} \\
& - \left. \frac{8(2-N+N^2)S_{-2,1}}{N(1+N)(2+N)} - \frac{6(-6+20N+5N^2+N^3)\zeta_3}{N(1+N)(2+N)(3+N)} \right), \tag{186}
\end{aligned}$$

$$\begin{aligned}
b_{gg}^{(2,1)} = & C_A^2 \xi^3 \left[\frac{-1+N-N^2}{8(N-1)^2N} - \frac{S_1}{8(N-1)} \right] + C_A^2 \xi^2 \left[\frac{(8+5N+6N^2)S_1^2}{16(N-1)N^2} \right. \\
& + \frac{S_1P_{453}}{8(-2+N)(N-1)^2N^3(1+N)(2+N)(3+N)} \\
& + \frac{P_{596}}{32(-2+N)^2(N-1)^3N^4(1+N)^2(2+N)^2(3+N)^2} + \frac{(16+7N+8N^2)S_2}{16(N-1)N^2} \\
& + \frac{(2+3N-N^2)S_3}{4(-2+N)N(1+N)(3+N)} + \left(\frac{P_{302}}{2(-2+N)^2(N-1)N^2(1+N)^2(3+N)^2} \right. \\
& + \left. \frac{(2+3N-N^2)S_1}{2(-2+N)N(1+N)(3+N)} \right) S_{-2} + \frac{(2+3N-N^2)S_{-3}}{(-2+N)N(1+N)(3+N)} + \frac{(2+N)\zeta_2}{8N^2} \\
& - \left. \frac{3(-2-3N+N^2)\zeta_3}{4(-2+N)N(1+N)(3+N)} \right] + \xi \left[C_A N_F T_F \left(-\frac{2P_{127}}{27(N-1)^3N^2} \right. \right. \\
& + \left. \frac{2(-3-10N+4N^2)S_1}{9(N-1)^2N} - \frac{S_1^2}{3(N-1)} - \frac{7S_2}{3(N-1)} + \frac{2\zeta_2}{3(N-1)} \right) \\
& + C_A^2 \left(\frac{S_3P_{498}}{12(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} \right. \\
& + \frac{S_{2,1}P_{389}}{2(N-1)N(1+N)^2(2+N)^2(3+N)^2} + \frac{S_2P_{514}}{12(N-1)^2N^3(1+N)(2+N)^2(3+N)^2} \\
& + \left. \frac{P_{611}}{432(-2+N)^2(N-1)^4N^4(1+N)^3(2+N)^3(3+N)^2} \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{P_{588}}{36(-2+N)(N-1)^3N^4(1+N)^2(2+N)^2(3+N)^2} \right. \\
& + \frac{S_2P_{441}}{8(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} + \frac{(228+46N+3N^2-N^3)S_3}{2N(1+N)(2+N)(3+N)} \\
& \left. + \frac{(-60+14N+15N^2+3N^3)S_{2,1}}{N(1+N)(2+N)(3+N)} \right) S_1 \\
& + \left(\frac{P_{421}}{12(N-1)^2N^3(1+N)(2+N)(3+N)} + \frac{(192-20N-33N^2-7N^3)S_2}{8N(1+N)(2+N)(3+N)} \right) S_1^2 \\
& + \frac{(-48+36N+27N^2+5N^3)S_2^2}{4N(1+N)(2+N)(3+N)} + \frac{(816+628N+291N^2+45N^3)S_4}{8N(1+N)(2+N)(3+N)} \\
& - \frac{7(3+8N)S_1^3}{24(N-1)N} + \left(\frac{S_1P_{45}}{(-2+N)(N-1)N(1+N)(3+N)} \right. \\
& + \frac{P_{454}}{2(-2+N)^2(N-1)N^3(1+N)^2(3+N)^2} + \frac{2S_1^2}{N} + \frac{4S_2}{N} \left. \right) S_{-2} + \frac{S_{-2}^2}{N} \\
& + \left(\frac{P_{189}}{(-2+N)(N-1)N^2(1+N)(3+N)} + \frac{8S_1}{N} \right) S_{-3} + \frac{7S_{-4}}{N} \\
& + \frac{(-138-5N+12N^2+3N^3)S_{3,1}}{N(1+N)(2+N)(3+N)} - \frac{2(-2+N)S_{-2,1}}{(N-1)N^2} - \frac{2S_{-2,2}}{N} - \frac{6S_{-3,1}}{N} \\
& + \frac{(36-58N-39N^2-7N^3)S_{2,1,1}}{2N(1+N)(2+N)(3+N)} - \frac{4S_{-2,1,1}}{N} + \left(\frac{P_{171}}{24(N-1)^2N^2(1+N)(2+N)} \right. \\
& \left. + \frac{(3+8N)S_1}{4(N-1)N} \right) \zeta_2 - \frac{9(-60+14N+15N^2+3N^3)\zeta_2^2}{10N(1+N)(2+N)(3+N)} \\
& + \left(-\frac{3P_{504}}{2(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} \right. \\
& \left. - \frac{3(-72-8N+3N^2+N^3)S_1}{2N(1+N)(2+N)(3+N)} \right) \zeta_3 \left. \right] + C_F N_F T_F \left(\frac{32S_1P_{234}}{(N-1)N^2(1+N)^3(2+N)^2} \right. \\
& - \frac{16P_{533}}{(-2+N)(N-1)N^3(1+N)^4(2+N)^3(3+N)} - \frac{32(2+N+N^2)S_1^2}{(N-1)N(1+N)^2(2+N)} \\
& - \frac{64(2+N+N^2)S_2}{(N-1)N(1+N)^2(2+N)} + \frac{256S_{-2}}{(-2+N)(N-1)(1+N)(2+N)(3+N)} \\
& \left. + \frac{16(2+N+N^2)\zeta_2}{(N-1)N(1+N)^2(2+N)} \right) + C_A N_F T_F \left(\frac{32S_{2,1}P_{57}}{(N-1)N(1+N)^2(2+N)^2(3+N)} \right. \\
& - \frac{16S_3P_{305}}{3(N-1)N(1+N)^2(2+N)^2(3+N)} + \frac{S_2P_{411}}{9(N-1)^2N^2(1+N)(2+N)^2(3+N)} \\
& \left. + \frac{2P_{579}}{27(-2+N)(N-1)^2N^3(1+N)^3(2+N)^3(3+N)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{4S_2P_{306}}{3(N-1)N(1+N)^2(2+N)^2(3+N)} - \frac{48S_3}{(1+N)(2+N)} + \frac{32S_{2,1}}{(1+N)(2+N)} \right. \\
& + \frac{2P_{535}}{9(N-1)^3N^3(1+N)^2(2+N)^2(3+N)} \Big) S_1 + \left(\frac{P_{242}}{3(N-1)^2N^2(1+N)(2+N)} \right. \\
& - \frac{12S_2}{(1+N)(2+N)} \Big) S_1^2 - \frac{128S_{-2}}{(-2+N)(N-1)(1+N)(2+N)(3+N)} \\
& + \frac{8S_2^2}{(1+N)(2+N)} - \frac{28S_4}{(1+N)(2+N)} + \frac{64S_{3,1}}{(1+N)(2+N)} - \frac{16S_{2,1,1}}{(1+N)(2+N)} \\
& - \frac{(-2+N)(6-5N+5N^2)\zeta_2}{3(N-1)N(1+N)(2+N)} - \frac{144\zeta_2^2}{5(1+N)(2+N)} \\
& + \left(\frac{8P_{304}}{(N-1)N(1+N)^2(2+N)^2(3+N)} - \frac{48S_1}{(1+N)(2+N)} \right) \zeta_3 \\
& + C_A^2 \left(-\frac{4S_{-2,1}P_{201}}{N^2(1+N)^2(2+N)^2} + \frac{S_{2,1}P_{393}}{(N-1)N(1+N)^2(2+N)^2(3+N)^2} \right. \\
& + \frac{S_3P_{510}}{6(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} \\
& + \frac{P_{612}}{108(-2+N)^2(N-1)^3N^4(1+N)^4(2+N)^4(3+N)^2} \\
& + \frac{S_2P_{523}}{36(N-1)^2N^3(1+N)^2(2+N)^2(3+N)^2} \\
& + \left(\frac{P_{599}}{18(-2+N)(N-1)^3N^4(1+N)^2(2+N)^3(3+N)^2} \right. \\
& + \frac{S_2P_{482}}{6(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} + \frac{(-66+68N-N^2-5N^3)S_3}{N(1+N)(2+N)(3+N)} \\
& - \frac{2(-6+20N+5N^2+N^3)S_{2,1}}{N(1+N)(2+N)(3+N)} \Big) S_1 \\
& + \left(\frac{P_{494}}{12(N-1)^2N^3(1+N)^2(2+N)^2(3+N)} \right. \\
& + \frac{(-30+62N+11N^2+N^3)S_2}{4N(1+N)(2+N)(3+N)} \Big) S_1^2 - \frac{3(2+6N+3N^2+N^3)S_2^2}{2N(1+N)(2+N)(3+N)} \\
& + \frac{14(2-N+N^2)S_1^3}{3N(1+N)(2+N)} + \frac{(-438+206N-97N^2-59N^3)S_4}{4N(1+N)(2+N)(3+N)} \\
& + \left(\frac{P_{560}}{(-2+N)^2(N-1)N^3(1+N)^3(2+N)^2(3+N)^2} \right. \\
& + \frac{2S_1P_{315}}{(-2+N)N(1+N)^2(2+N)^2(3+N)} - \frac{4(2-N+N^2)S_1^2}{N(1+N)(2+N)} - \frac{8(2-N+N^2)S_2}{N(1+N)(2+N)} \Big) \\
& \times S_{-2} - \frac{2(2-N+N^2)S_{-2}^2}{N(1+N)(2+N)} + \left(\frac{2P_{396}}{(-2+N)N^2(1+N)^2(2+N)^2(3+N)} \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{16(2-N+N^2)S_1}{N(1+N)(2+N)} \Big) S_{-3} - \frac{14(2-N+N^2)S_{-4}}{N(1+N)(2+N)} + \frac{2(30-43N-4N^2+N^3)}{N(1+N)(2+N)(3+N)} \\
& \times S_{3,1} + \frac{4(2-N+N^2)S_{-2,2}}{N(1+N)(2+N)} + \frac{(18+16N+13N^2+5N^3)S_{2,1,1}}{N(1+N)(2+N)(3+N)} \\
& + \frac{12(2-N+N^2)S_{-3,1}}{N(1+N)(2+N)} + \frac{8(2-N+N^2)S_{-2,1,1}}{N(1+N)(2+N)} + \left(-\frac{4(2-N+N^2)S_1}{N(1+N)(2+N)} \right. \\
& \left. + \frac{P_{356}}{12(N-1)N^2(1+N)^2(2+N)^2} \right) \zeta_2 + \frac{9(-6+20N+5N^2+N^3)\zeta_2^2}{5N(1+N)(2+N)(3+N)} \\
& + \left(\frac{P_{506}}{(-2+N)(N-1)N^2(1+N)^2(2+N)^2(3+N)^2} \right. \\
& \left. - \frac{3(18-22N-N^2+N^3)S_1}{N(1+N)(2+N)(3+N)} \right) \zeta_3, \tag{187}
\end{aligned}$$

$$c_{gg}^{(1,0)} = C_A \left[\frac{2(1-4N+2N^2)}{(N-1)^2N^2} - \frac{2S_1}{(N-1)N} \right], \tag{188}$$

$$\begin{aligned}
c_{gg}^{(1,1)} = C_A \left[-\frac{P_{52}}{(N-1)^3N^3} + \frac{(-1+4N-2N^2)S_1}{(N-1)^2N^2} + \frac{S_1^2}{2(N-1)N} + \frac{3S_2}{2(N-1)N} \right. \\
\left. - \frac{\zeta_2}{2(N-1)N} \right], \tag{189}
\end{aligned}$$

$$\begin{aligned}
c_{gg}^{(1,2)} = C_A \left(-\frac{(1-3N+N^2)(-1+3N-5N^2+2N^3)}{2(N-1)^4N^4} + \left(\frac{P_{52}}{2(N-1)^3N^3} \right. \right. \\
\left. \left. - \frac{3S_2}{4(N-1)N} \right) S_1 + \frac{(1-4N+2N^2)S_1^2}{4(N-1)^2N^2} - \frac{S_1^3}{12(N-1)N} + \frac{3(1-4N+2N^2)S_2}{4(N-1)^2N^2} \right. \\
\left. - \frac{7S_3}{6(N-1)N} + \left(\frac{-1+4N-2N^2}{4(N-1)^2N^2} + \frac{S_1}{4(N-1)N} \right) \zeta_2 + \frac{7\zeta_3}{6(N-1)N} \right), \tag{190}
\end{aligned}$$

$$\begin{aligned}
c_{gg}^{(2,0)} = C_A^2 \xi \left[\frac{P_{402}}{4(-2+N)(N-1)^3N^3(1+N)^2} + \left(\frac{2-8N-21N^2+15N^4}{4(N-1)^2N^3(1+N)} + \frac{S_2}{2N(1+N)} \right) \right. \\
\left. \times S_1 - \frac{15S_1^2}{8(N-1)N} + \frac{(4-31N)S_2}{8(N-1)N^2} + \frac{S_3}{2N(1+N)} + \frac{S_{-2}}{(-2+N)(N-1)N(1+N)} \right. \\
\left. - \frac{S_{2,1}}{N(1+N)} + \frac{3\zeta_3}{N(1+N)} \right] + C_A N_F T_F \left(\frac{8P_{106}}{27(N-1)^3N^3} + \frac{8(-3+4N+2N^2)S_1}{9(N-1)^2N^2} \right. \\
\left. + \frac{8S_2}{3(N-1)N} - \frac{4\zeta_2}{3(N-1)N} \right) + C_A^2 \left(\frac{S_2P_4}{12(N-1)^2N^2(1+N)(2+N)} \right. \\
\left. + \frac{S_1^2P_{11}}{4(N-1)^2N^2(1+N)(2+N)} + \frac{P_{587}}{54(-2+N)(N-1)^4N^4(1+N)^3(2+N)^3} \right. \\
\left. + \left(\frac{P_{478}}{9(N-1)^3N^3(1+N)^2(2+N)^2} + \frac{(226+399N+125N^2)S_2}{4(N-1)N(1+N)(2+N)} \right) S_1 \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{77S_1^3}{12(N-1)N} + \frac{(214 + 411N + 125N^2)S_3}{6(N-1)N(1+N)(2+N)} + \left(-\frac{2(-5+2N)}{(-2+N)(N-1)N} \right. \\
& + \left. \frac{8S_1}{(N-1)N} \right) S_{-2} + \frac{10S_{-3}}{(N-1)N} - \frac{12S_{2,1}}{N(1+N)(2+N)} - \frac{4S_{-2,1}}{(N-1)N} \\
& + \left(\frac{P_{143}}{6(N-1)^2N^2(1+N)(2+N)} - \frac{11S_1}{2(N-1)N} \right) \zeta_2 + \frac{36\zeta_3}{N(1+N)(2+N)}, \quad (191)
\end{aligned}$$

$$\begin{aligned}
c_{gg}^{(2,1)} = & C_A^2 \xi \left(\frac{S_2 P_{20}}{16(N-1)^2 N^3 (1+N)} + \frac{S_3 P_{154}}{24(-2+N)(N-1)N^2(1+N)^2} \right. \\
& + \frac{P_{558}}{16(-2+N)^2(N-1)^4 N^4(1+N)^3} + \left(\frac{P_{446}}{4(-2+N)(N-1)^3 N^4(1+N)^2} \right. \\
& + \left. \frac{(-4+83N+146N^2+75N^3)S_2}{16(N-1)N^2(1+N)^2} - \frac{3S_3}{2N(1+N)} + \frac{S_{2,1}}{N(1+N)} \right) S_1 \\
& + \left(\frac{P_{24}}{16(N-1)^2 N^3(1+N)} - \frac{3S_2}{8N(1+N)} \right) S_1^2 + \frac{35S_1^3}{48(N-1)N} + \frac{S_2^2}{4N(1+N)} \\
& - \frac{7S_4}{8N(1+N)} + \left(\frac{-4-6N+N^2}{2(-2+N)^2 N^2(1+N)^2} - \frac{S_1}{(-2+N)(N-1)N(1+N)} \right) S_{-2} \\
& - \frac{2S_{-3}}{(-2+N)(N-1)N(1+N)} + \frac{(-5-2N-N^2)S_{2,1}}{2(N-1)N(1+N)^2} + \frac{2S_{3,1}}{N(1+N)} \\
& - \frac{S_{2,1,1}}{2N(1+N)} + \left(\frac{-5-8N+3N^2+8N^3}{8(N-1)^2 N^2(1+N)} - \frac{5S_1}{8(N-1)N} \right) \zeta_2 - \frac{9\zeta_2^2}{10N(1+N)} \\
& + \left(-\frac{3P_{82}}{2(-2+N)(N-1)N^2(1+N)^2} - \frac{3S_1}{2N(1+N)} \right) \zeta_3 \\
& + C_A N_F T_F \left(\frac{4P_{340}}{81(N-1)^4 N^4} + \left(-\frac{4P_{106}}{27(N-1)^3 N^3} - \frac{4S_2}{3(N-1)N} \right) S_1 \right. \\
& - \frac{2(-3+4N+2N^2)S_1^2}{9(N-1)^2 N^2} + \frac{2(21-44N+2N^2)S_2}{9(N-1)^2 N^2} - \frac{16S_3}{3(N-1)N} \\
& + \left. \left(-\frac{2(9-20N+2N^2)}{9(N-1)^2 N^2} + \frac{2S_1}{3(N-1)N} \right) \zeta_2 + \frac{100\zeta_3}{9(N-1)N} \right) \\
& + C_A^2 \left(\frac{S_1^3 P_{156}}{24(N-1)^2 N^2(1+N)(2+N)} + \frac{2S_{2,1} P_{309}}{(N-1)^2 N^2(1+N)^2(2+N)^2} \right. \\
& + \frac{S_3 P_{424}}{12(-2+N)(N-1)^2 N^2(1+N)^2(2+N)^2} + \frac{S_2 P_{436}}{36(N-1)^3 N^3(1+N)^2(2+N)^2} \\
& + \frac{P_{614}}{648(-2+N)^2(N-1)^5 N^5(1+N)^4(2+N)^4} + \left(\frac{S_2 P_{375}}{24(N-1)^2 N^2(1+N)^2(2+N)^2} \right. \\
& + \left. \frac{P_{590}}{54(-2+N)(N-1)^4 N^4(1+N)^3(2+N)^3} - \frac{3(70+165N+47N^2)S_3}{4(N-1)N(1+N)(2+N)} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{2(8-3N+N^2)S_{2,1}}{(N-1)N(1+N)(2+N)} \Big) S_1 + \left(\frac{P_{435}}{36(N-1)^3 N^3 (1+N)^2 (2+N)^2} \right. \\
& - \frac{(514+951N+293N^2)S_2}{16(N-1)N(1+N)(2+N)} \Big) S_1^2 - \frac{(1018+1287N+461N^2)S_2^2}{32(N-1)N(1+N)(2+N)} - \frac{55S_1^4}{32(N-1)N} \\
& + \frac{(-730-1515N-449N^2)S_4}{16(N-1)N(1+N)(2+N)} + \left(\frac{P_{236}}{(-2+N)^2(N-1)^2 N^3 (1+N)} \right. \\
& + \frac{2(-4+15N-11N^2+2N^3)S_1}{(-2+N)(N-1)^2 N^2} - \frac{4S_1^2}{(N-1)N} - \frac{8S_2}{(N-1)N} \Big) S_{-2} - \frac{2S_{-2}^2}{(N-1)N} \\
& + \left(\frac{2(-2+15N-16N^2+4N^3)}{(-2+N)(N-1)^2 N^2} - \frac{16S_1}{(N-1)N} \right) S_{-3} - \frac{3(10-5N+N^2)S_{3,1}}{(N-1)N(1+N)(2+N)} \\
& - \frac{21S_{-4}}{(N-1)N} + \frac{4(-1+2N)S_{-2,1}}{(N-1)^2 N^2} + \frac{2S_{-2,2}}{(N-1)N} + \frac{3(4+N+N^2)S_{2,1,1}}{(N-1)N(1+N)(2+N)} \\
& + \frac{6S_{-3,1}}{(N-1)N} + \frac{4S_{-2,1,1}}{(N-1)N} + \left(\frac{S_1 P_9}{12(N-1)^2 N^2 (1+N)(2+N)} \right. \\
& + \frac{P_{443}}{36(N-1)^3 N^3 (1+N)^2 (2+N)^2} + \frac{33S_1^2}{8(N-1)N} + \frac{37S_2}{8(N-1)N} + \frac{S_{-2}}{(N-1)N} \Big) \zeta_2 \\
& - \frac{54\zeta_2^2}{5N(1+N)(2+N)} + \left(\frac{P_{382}}{9(-2+N)(N-1)^2 N^2 (1+N)^2 (2+N)^2} \right. \\
& + \left. \frac{(176+129N+61N^2)S_1}{3(N-1)N(1+N)(2+N)} \right) \zeta_3 \Big). \tag{192}
\end{aligned}$$

6 The polarized OMEs

The expansion coefficients in the polarized flavor singlet case are calculated using the Larin scheme [20, 86], as the complete calculation of the massless off-shell OMEs in the polarized case. By this one obtains the anomalous dimensions in the Larin scheme, which are finally transformed into the $\overline{\text{M}}$ scheme [20, 25]. In the non-singlet case one could use a known Ward identity and obtain the anomalous dimension directly in the $\overline{\text{MS}}$ scheme. However, the consistent renormalization of the singlet case requires to calculate the non-singlet also in the Larin scheme.

$$\Delta a_{qq}^{(1,0),\text{NS}} = C_F \xi \left(-\frac{3}{N} + S_1 \right) + C_F \left(-\frac{P_{38}}{N^2(1+N)^2} + \frac{2S_1}{N(1+N)} - 2S_1^2 - 6S_2 \right), \tag{193}$$

$$\begin{aligned}
\Delta a_{qq}^{(1,1),\text{NS}} &= C_F \xi \left(\frac{3+3N+2N^2}{2N^2} + \frac{(3-N)S_1}{2N} - \frac{1}{4}S_1^2 - \frac{3}{4}S_2 \right) + C_F \left(\frac{P_{298}}{N^3(1+N)^3} \right. \\
&+ \left(-\frac{(1+3N)(1+N+N^2)}{N^2(1+N)^2} + 3S_2 \right) S_1 - \frac{S_1^2}{2N(1+N)} + \frac{1}{3}S_1^3 - \frac{3S_2}{2N(1+N)} \\
&+ \frac{14}{3}S_3 + \left(\frac{2+3N+3N^2}{4N(1+N)} - S_1 \right) \zeta_2 \Big), \tag{194}
\end{aligned}$$

$$\begin{aligned}
\Delta a_{qq}^{(1,2),\text{NS}} = & C_F \xi \left[\frac{-3 - 3N - 4N^3}{4N^3} + \left(-\frac{3(1+N)}{4N^2} + \frac{3S_2}{8} \right) S_1 + \frac{(-3+N)S_1^2}{8N} + \frac{1}{24} S_1^3 \right. \\
& + \frac{3(N-3)S_2}{8N} + \frac{7}{12} S_3 + \left. \left(\frac{3}{8N} - \frac{S_1}{8} \right) \zeta_2 \right] + C_F \left(\frac{P_{470}}{2N^4(1+N)^4} + \left(\frac{P_{228}}{2N^3(1+N)^3} \right. \right. \\
& + \frac{3S_2}{4N(1+N)} - \frac{7}{3} S_3 \left. \right) S_1 + \left(\frac{(1+3N)(1+N+N^2)}{4N^2(1+N)^2} - \frac{3}{4} S_2 \right) S_1^2 + \frac{S_1^3}{12N(1+N)} \\
& - \frac{1}{24} S_1^4 + \frac{3(1+3N)(1+N+N^2)S_2}{4N^2(1+N)^2} - \frac{9}{8} S_2^2 + \frac{7S_3}{6N(1+N)} - \frac{15}{4} S_4 \\
& + \left(\frac{P_{38}}{8N^2(1+N)^2} - \frac{S_1}{4N(1+N)} + \frac{1}{4} S_1^2 + \frac{3}{4} S_2 \right) \zeta_2 + \left(-\frac{7(2+3N+3N^2)}{12N(1+N)} \right. \\
& \left. + \frac{7}{3} S_1 \right) \zeta_3 \Big), \tag{195}
\end{aligned}$$

$$\begin{aligned}
\Delta a_{qq}^{(2,0),\text{NS}} = & \xi^2 \left[C_A C_F \left(\frac{3+2N}{4N} - \frac{3(-2+N)S_1}{4N} - \frac{1}{4} S_1^2 - \frac{1}{4} S_2 \right) + C_F^2 \left(\frac{3-N}{N} - \frac{3S_1}{N} + \frac{1}{2} S_1^2 \right. \right. \\
& \left. \left. + \frac{1}{2} S_2 \right) \right] + \xi \left[C_F^2 \left(-\frac{(3+N)P_{101}}{N^3(1+N)^2} + \frac{(42+31N-5N^2)S_1^2}{2N(1+N)} + \left(\frac{P_{133}}{N^2(1+N)^2} \right. \right. \right. \\
& \left. \left. - 16S_2 \right) S_1 - 4S_1^3 + \frac{(2+N)(23+13N)S_2}{2N(1+N)} \right) + C_A C_F \left(\frac{-6-9N-7N^2}{2N^2} \right. \\
& + \left(\frac{3(-2-N+4N^2)}{2N(2+N)} - \frac{(-2+N)(N-1)S_2}{(1+N)(2+N)} \right) S_1 + \frac{5}{4} S_1^2 + \frac{(-22+13N)S_2}{4(2+N)} \\
& \left. \left. - \frac{(-2+N)(N-1)S_3}{(1+N)(2+N)} + \frac{2(-2+N)(N-1)S_{2,1}}{(1+N)(2+N)} - \frac{6(-2+N)(N-1)\zeta_3}{(1+N)(2+N)} \right) \right] \\
& + C_F \left(N_F T_F \left(\frac{P_{285}}{54N^3(1+N)^3} + \left(\frac{4(-3+29N+55N^2+41N^3)}{27N(1+N)^2} + \frac{8}{3} S_2 \right) S_1 \right. \right. \\
& \left. \left. + \frac{2(-6+17N+17N^2)S_1^2}{9N(1+N)} + \frac{8}{9} S_1^3 + \frac{2(-6+37N+37N^2)S_2}{9N(1+N)} - \frac{8}{9} S_3 \right) \right) \\
& + C_A \left(\frac{S_2 P_{165}}{18N^2(1+N)^2(2+N)} + \frac{P_{551}}{216(N-1)N^4(1+N)^4(2+N)} \right. \\
& + \left(\frac{P_{380}}{27N^3(1+N)^3(2+N)} - \frac{2(34+33N+14N^2)S_2}{3(1+N)(2+N)} - 6S_3 - 4S_{2,1} \right) S_1 \\
& + \left(-\frac{11(-6+23N+23N^2)}{18N(1+N)} + S_2 \right) S_1^2 + \frac{2(72+22N+33N^2+2N^3)S_3}{9N(1+N)(2+N)} \\
& \left. - \frac{22}{9} S_1^3 - 6S_2^2 - 29S_4 + \left(\frac{4(-2-N-6N^2+N^3)}{(N-1)N(1+N)^2(2+N)} + \frac{16S_1}{N(1+N)} - 8S_1^2 \right) \right)
\end{aligned}$$

$$\begin{aligned}
& -16S_2)S_{-2} - 4S_{-2}^2 + \left(\frac{20}{N(1+N)} - 32S_1 \right) S_{-3} - 28S_{-4} + \frac{4(4+N^2)S_{2,1}}{(1+N)(2+N)} \\
& + 6S_{3,1} - \frac{8S_{-2,1}}{N(1+N)} + 8S_{-2,2} + 24S_{-3,1} + \left(-\frac{12(2+5N+N^3)}{N(1+N)(2+N)} + 12S_1 \right) \zeta_3 \\
& + 6S_{2,1,1} + 16S_{-2,1,1} \Big) + C_F^2 \left(\frac{P_{548}}{8(N-1)N^4(1+N)^4(2+N)} \right. \\
& + \frac{S_2 P_{172}}{N^2(1+N)^2(2+N)} + \left(-\frac{2(52+40N+37N^2+11N^3)S_2}{N(1+N)(2+N)} + \frac{256}{3} S_3 \right. \\
& \left. - \frac{2P_{415}}{N^3(1+N)^3(2+N)} \right) S_1 + \left(\frac{P_{27}}{N^2(1+N)^2} + 48S_2 \right) S_1^2 - \frac{2(14+3N+3N^2)S_1^3}{3N(1+N)} \\
& + \frac{14}{3} S_1^4 + 30S_2^2 - \frac{2(164+208N+237N^2+75N^3)S_3}{3N(1+N)(2+N)} + 48S_4 \\
& + \left(-\frac{8(-2-N-6N^2+N^3)}{(N-1)N(1+N)^2(2+N)} - \frac{32S_1}{N(1+N)} + 16S_1^2 + 32S_2 \right) S_{-2} + 8S_{-2}^2 \\
& + \left(-\frac{40}{N(1+N)} + 64S_1 \right) S_{-3} + 56S_{-4} - \frac{4(-4+8N-N^2+N^3)S_{2,1}}{N(1+N)(2+N)} \\
& + \frac{16S_{-2,1}}{N(1+N)} - 16S_{-2,2} - 48S_{-3,1} - 32S_{-2,1,1} + \left(-\frac{(2+3N+3N^2)^2}{2N^2(1+N)^2} \right. \\
& \left. + \frac{4(2+3N+3N^2)S_1}{N(1+N)} - 8S_1^2 \right) \zeta_2 - \frac{48(-2+N)\zeta_3}{(1+N)(2+N)}, \tag{196}
\end{aligned}$$

$$\begin{aligned}
\Delta a_{qq}^{(2,1),\text{NS}} = & \xi^2 \left[C_F^2 \left(\frac{3(-1-3N+N^2)}{2N^2} + \left(\frac{3(1+N^2)}{2N^2} - \frac{5S_2}{4} \right) S_1 + \frac{(9-2N)S_1^2}{4N} - \frac{1}{4} S_1^3 \right. \right. \\
& + \left. \frac{(15-2N)S_2}{4N} - S_3 \right) + C_A C_F \left(\frac{-3-12N-13N^2}{8N^2} + \left(\frac{-6-12N+5N^2}{8N^2} \right. \right. \\
& \left. \left. + \frac{5S_2}{8} \right) S_1 + \frac{(-9+4N)S_1^2}{8N} + \frac{1}{8} S_1^3 + \frac{(-15+7N)S_2}{8N} + \frac{1}{2} S_3 + \left(\frac{3}{8N} - \frac{S_1}{8} \right) \zeta_2 \right) \Big] \\
& + \xi \left[C_A C_F \left(\frac{3(6+7N+16N^2+14N^3)}{4N^3} + \frac{S_2 P_{18}}{8N^2(2+N)^2} + \frac{S_3 P_{184}}{2N(1+N)^2(2+N)^2} \right. \right. \\
& + \left(\frac{S_2 P_{177}}{8N(1+N)^2(2+N)^2} + \frac{P_{181}}{4N^3(2+N)^2} + \frac{3(-2+N)(N-1)S_3}{(1+N)(2+N)} \right. \\
& \left. \left. - \frac{2(-2+N)(N-1)S_{2,1}}{(1+N)(2+N)} \right) S_1 + \left(\frac{18+19N-31N^2}{8N(2+N)} + \frac{3(-2+N)(N-1)S_2}{4(1+N)(2+N)} \right) S_1^2 \right. \\
& + \frac{(-2-N^2)S_2^2}{(1+N)(2+N)} + \frac{(-5+N)(-2+5N)S_4}{4(1+N)(2+N)} + \frac{(12-44N-51N^2-7N^3)S_{2,1}}{N(1+N)^2(2+N)^2} \\
& \left. - \frac{3(2-5N+N^2)S_{3,1}}{(1+N)(2+N)} + \frac{(-2+N)(N-1)S_{2,1,1}}{(1+N)(2+N)} - \frac{5}{8} S_1^3 + \left(-\frac{15}{8N} + \frac{5S_1}{8} \right) \zeta_2 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{9(-2+N)(N-1)\zeta_2^2}{5(1+N)(2+N)} + \left(\frac{3P_{222}}{2N(1+N)^2(2+N)^2} + \frac{6(2+N^2)S_1}{(1+N)(2+N)} \right) \zeta_3 \\
& + C_F^2 \left(\frac{S_2 P_{12}}{4N^2(1+N)^2} + \frac{3P_{405}}{2N^4(1+N)^3} + \left(\frac{(-158-113N+19N^2)S_2}{4N(1+N)} \right. \right. \\
& \left. \left. + \frac{P_{293}}{2N^3(1+N)^3} + \frac{52}{3}S_3 \right) S_1 + \left(-\frac{3P_{110}}{4N^2(1+N)^2} + 12S_2 \right) S_1^2 \right. \\
& \left. + \frac{7(-14-9N+3N^2)S_1^3}{12N(1+N)} + \frac{7}{6}S_1^4 + \frac{15}{2}S_2^2 + \frac{(-122-129N-45N^2)S_3}{6N(1+N)} \right. \\
& \left. + \left(-\frac{3(2+3N+3N^2)}{2N^2(1+N)} + \frac{(14+15N+3N^2)S_1}{2N(1+N)} - 2S_1^2 \right) \zeta_2 \right. \\
& \left. + \left(\frac{3(2+3N+3N^2)}{N(1+N)} - 12S_1 \right) \zeta_3 \right] + C_F \left(C_A \left(-\frac{2S_{2,1}P_{326}}{N^2(1+N)^2(2+N)^2} \right. \right. \\
& \left. \left. + \frac{S_3 P_{425}}{27(N-1)N^2(1+N)^2(2+N)^2} + \frac{P_{593}}{2592(N-1)^2N^5(1+N)^5(2+N)^2} \right. \right. \\
& \left. \left. + \left(\frac{S_2 P_{374}}{36N^2(1+N)^2(2+N)^2} + \frac{P_{569}}{162(N-1)N^4(1+N)^4(2+N)^2} + 5S_2^2 \right. \right. \right. \\
& \left. \left. + \frac{2(-18+65N+57N^2+28N^3)S_3}{3N(1+N)(2+N)} + \frac{49}{2}S_4 + \frac{4(-2-3N+3N^2)S_{2,1}}{N(1+N)(2+N)} \right. \right. \\
& \left. \left. - 2S_{2,1,1} \right) S_1 + \left(\frac{P_{427}}{108N^3(1+N)^3(2+N)} + \frac{(12+140N+147N^2+58N^3)S_2}{6N(1+N)(2+N)} \right. \right. \\
& \left. \left. + S_3 + 2S_{2,1} \right) S_1^2 + \left(\frac{-198+491N+491N^2}{108N(1+N)} - \frac{1}{2}S_2 \right) S_1^3 + \frac{11}{12}S_1^4 \right. \\
& \left. + \left(\frac{P_{489}}{108N^3(1+N)^3(2+N)^2} + 18S_3 + 4S_{2,1} - 8S_{-2,1} \right) S_2 \right. \\
& \left. + \frac{(-96+58N+231N^2+65N^3)S_2^2}{12N(1+N)(2+N)} + \frac{(-324+20N+147N^2+70N^3)S_4}{6N(1+N)(2+N)} \right. \\
& \left. + 62S_5 + \left(-\frac{2P_{312}}{(N-1)^2N(1+N)^3(2+N)^2} + \left(-\frac{8P_{73}}{(N-1)N^2(1+N)^2(2+N)} \right. \right. \right. \\
& \left. \left. + 16S_2 \right) S_1 - \frac{8S_1^2}{N(1+N)} + \frac{8}{3}S_1^3 - \frac{16S_2}{N(1+N)} + \frac{64}{3}S_3 + 8S_{2,1} \right) S_{-2} \\
& \left. + \left(-\frac{4}{N(1+N)} + 4S_1 \right) S_{-2}^2 + \left(-\frac{2P_{99}}{(N-1)N^2(1+N)^2(2+N)} - \frac{32S_1}{N(1+N)} \right. \right. \\
& \left. \left. + 16S_1^2 + 20S_2 + 16S_{-2} \right) S_{-3} + \left(-\frac{42}{N(1+N)} + 56S_1 \right) S_{-4} + 54S_{-5} + 16S_{2,3} \right. \\
& \left. + 20S_{2,-3} - \frac{2(-2+9N-9N^2+N^3)S_{3,1}}{N(1+N)(2+N)} - 13S_{4,1} + \frac{4(2+N)(1+3N)S_{-2,1}}{N^2(1+N)^2} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{4S_{-2,2}}{N(1+N)} - 12S_{-2,3} + \frac{12S_{-3,1}}{N(1+N)} - \frac{2(-6-N+9N^2+2N^3)S_{2,1,1}}{N(1+N)(2+N)} \\
& - 28S_{-4,1} - 8S_{2,1,-2} - 18S_{2,2,1} - 6S_{3,1,1} + \frac{8S_{-2,1,1}}{N(1+N)} - 8S_{-2,2,1} - 24S_{-3,1,1} \\
& - 3S_{2,1,1,1} - 16S_{-2,1,1,1} + \left(\frac{P_{369}}{72N^3(1+N)^3} - \frac{11(-6+23N+23N^2)S_1}{36N(1+N)} \right. \\
& \left. - \frac{11}{6}S_1^2 - \frac{11}{6}S_2 - 2S_3 + \left(\frac{2}{N(1+N)} - 4S_1 \right) S_{-2} - 2S_{-3} + 4S_{-2,1} \right) \zeta_2 \\
& + \left(\frac{18(2+5N+N^3)}{5N(1+N)(2+N)} - \frac{18}{5}S_1 \right) \zeta_2^2 + \left(\frac{P_{418}}{2(N-1)N^2(1+N)^2(2+N)^2} \right. \\
& \left. - \frac{2(10+33N+8N^2)S_1}{(1+N)(2+N)} - 3S_1^2 + 3S_2 + 12S_{-2} \right) \zeta_3 + N_F T_F \left(\frac{S_2 P_7}{27N^2(1+N)^2} \right. \\
& \left. + \frac{P_{493}}{648N^4(1+N)^4} + \left(-\frac{2P_{277}}{81N^2(1+N)^3} - \frac{5(-6+13N+13N^2)S_2}{9N(1+N)} - \frac{16}{3}S_3 \right) \right. \\
& \times S_1 + \left(\frac{P_{15}}{27N^2(1+N)^2} - \frac{10S_2}{3} \right) S_1^2 + \frac{(18-31N-31N^2)S_1^3}{27N(1+N)} - \frac{1}{3}S_1^4 - \frac{7}{3}S_2^2 \\
& - \frac{8(-9+28N+28N^2)S_3}{27N(1+N)} - \frac{10}{3}S_4 + \left(\frac{-2-53N-108N^2-45N^3}{18N(1+N)^2} \right. \\
& \left. + \frac{(-6+17N+17N^2)S_1}{9N(1+N)} + \frac{2}{3}S_1^2 + \frac{2}{3}S_2 \right) \zeta_2 + \left(-\frac{2(2+3N+3N^2)}{N(1+N)} \right. \\
& \left. + 8S_1 \right) \zeta_3 \left. \right) + C_F^2 \left(\frac{4(3+N)S_{2,1}P_{64}}{N(1+N)^2(2+N)^2} + \frac{S_3 P_{423}}{3(N-1)N^2(1+N)^2(2+N)^2} \right. \\
& + \frac{P_{595}}{32(N-1)^2N^5(1+N)^5(2+N)^2} + \left(\frac{P_{563}}{(N-1)N^4(1+N)^4(2+N)^2} \right. \\
& + \frac{S_2 P_{353}}{2N^2(1+N)^2(2+N)^2} - 47S_2^2 + \frac{2(76+36N+45N^2+11N^3)S_3}{N(1+N)(2+N)} - 128S_4 \\
& \left. - \frac{16(-2+N)S_{2,1}}{(1+N)(2+N)} \right) S_1 + \left(\frac{(116+56N+45N^2+11N^3)S_2}{2N(1+N)(2+N)} \right. \\
& + \frac{P_{413}}{2N^3(1+N)^3(2+N)} - 60S_3 \left. \right) S_1^2 + \left(\frac{P_{124}}{6N^2(1+N)^2} - \frac{56S_2}{3} \right) S_1^3 \\
& + \frac{(10+N+N^2)S_1^4}{4N(1+N)} - S_1^5 + \left(\frac{P_{479}}{2N^3(1+N)^3(2+N)^2} - \frac{232}{3}S_3 + 16S_{-2,1} \right) S_2 \\
& + \frac{(204+228N+125N^2+47N^3)S_2^2}{4N(1+N)(2+N)} + \frac{(388+360N+361N^2+111N^3)S_4}{2N(1+N)(2+N)} \\
& - 108S_5 + \left(\frac{4P_{312}}{(N-1)^2N(1+N)^3(2+N)^2} + \left(\frac{16P_{73}}{(N-1)N^2(1+N)^2(2+N)} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -32S_2)S_1 + \frac{16S_1^2}{N(1+N)} - \frac{16}{3}S_1^3 + \frac{32S_2}{N(1+N)} - \frac{128}{3}S_3 - 16S_{2,1})S_{-2} \\
& + \left(\frac{8}{N(1+N)} - 8S_1 \right) S_{-2}^2 + \left(\frac{4P_{99}}{(N-1)N^2(1+N)^2(2+N)} + \frac{64S_1}{N(1+N)} \right. \\
& \left. - 32S_1^2 - 40S_2 - 32S_{-2} \right) S_{-3} + \left(\frac{84}{N(1+N)} - 112S_1 \right) S_{-4} - 108S_{-5} - 40S_{2,-3} \\
& + \frac{2(-12 + 32N - 7N^2 + 3N^3)S_{3,1}}{N(1+N)(2+N)} - \frac{8(2+N)(1+3N)S_{-2,1}}{N^2(1+N)^2} - \frac{8S_{-2,2}}{N(1+N)} \\
& + 24S_{-2,3} - \frac{24S_{-3,1}}{N(1+N)} + 56S_{-4,1} + \frac{4(-4 - 4N + 5N^2 + N^3)S_{2,1,1}}{N(1+N)(2+N)} + 16S_{2,1,-2} \\
& - \frac{16S_{-2,1,1}}{N(1+N)} + 16S_{-2,2,1} + 48S_{-3,1,1} + 32S_{-2,1,1,1} + \left(\frac{P_{361}}{8N^3(1+N)^3} \right. \\
& \left. + \left(\frac{P_{28}}{N^2(1+N)^2} + 16S_2 \right) S_1 - \frac{3(2+N+N^2)S_1^2}{N(1+N)} + 4S_1^3 - \frac{4(2+3N+3N^2)S_2}{N(1+N)} \right. \\
& \left. + 4S_3 + \left(-\frac{4}{N(1+N)} + 8S_1 \right) S_{-2} + 4S_{-3} - 8S_{-2,1} \right) \zeta_2 + \frac{72(-2+N)\zeta_2^2}{5(1+N)(2+N)} \\
& + \left(\frac{P_{386}}{3(N-1)N^2(1+N)^2(2+N)^2} - \frac{8(14+55N+36N^2+15N^3)S_1}{3N(1+N)(2+N)} + \frac{92}{3}S_1^2 \right. \\
& \left. - 12S_2 - 24S_{-2} \right) \zeta_3 \Big), \tag{197}
\end{aligned}$$

$$\Delta b_{qq}^{(1,0),\text{NS}} = \frac{2C_F\xi}{N} - \frac{4C_F}{1+N}, \tag{198}$$

$$\Delta b_{qq}^{(1,1),\text{NS}} = -C_F\xi \left(\frac{1+N}{N^2} + \frac{S_1}{N} \right) + C_F \left(\frac{2(3+N)}{(1+N)^2} + \frac{2S_1}{1+N} \right), \tag{199}$$

$$\begin{aligned}
\Delta b_{qq}^{(1,2),\text{NS}} &= C_F\xi \left(\frac{1+N}{2N^3} + \frac{(1+N)S_1}{2N^2} + \frac{S_1^2}{4N} + \frac{3S_2}{4N} - \frac{\zeta_2}{4N} \right) + C_F \left(-\frac{2(3+N)}{(1+N)^3} - \frac{(3+N)S_1}{(1+N)^2} \right. \\
&\left. - \frac{S_1^2}{2(1+N)} - \frac{3S_2}{2(1+N)} + \frac{\zeta_2}{2(1+N)} \right), \tag{200}
\end{aligned}$$

$$\begin{aligned}
\Delta b_{qq}^{(2,0),\text{NS}} &= \xi^2 \left[C_A C_F \left(-\frac{1}{2N} - \frac{S_1}{N} \right) + C_F^2 \left(-\frac{2}{N} + \frac{2S_1}{N} \right) \right] + \xi \left[C_F^2 \left(\frac{2P_{111}}{N^3(1+N)^2} \right. \right. \\
&+ \frac{2(-4-5N+N^2)S_1}{N^2(1+N)} - \frac{12S_1^2}{N} - \frac{12S_2}{N} \Big) + C_A C_F \left(\frac{2+3N}{N^2} + \left(\frac{2+7N}{N(2+N)} \right. \right. \\
&\left. \left. - \frac{2(N-1)S_2}{(1+N)(2+N)} \right) S_1 + \frac{6S_2}{2+N} - \frac{2(N-1)S_3}{(1+N)(2+N)} + \frac{4(N-1)S_{2,1}}{(1+N)(2+N)} \right. \\
&\left. \left. - \frac{12(N-1)\zeta_3}{(1+N)(2+N)} \right) \right] + C_F \left(N_F T_F \left(\frac{16(-1+6N+4N^2)}{9N(1+N)^2} + \frac{8(N-1)S_1}{3N(1+N)} \right) \right)
\end{aligned}$$

$$\begin{aligned}
& + C_A \left(-\frac{4P_{271}}{9(N-1)N(1+N)^3(2+N)} + \left(-\frac{2(-12+8N+65N^2+23N^3)}{3N^2(1+N)(2+N)} \right. \right. \\
& \left. \left. + \frac{8S_2}{(1+N)(2+N)} \right) S_1 + \frac{4(-2+N)S_2}{N(2+N)} + \frac{32S_{-2}}{(N-1)(1+N)(2+N)} \right. \\
& \left. + \frac{8S_3}{(1+N)(2+N)} - \frac{16S_{2,1}}{(1+N)(2+N)} + \frac{48\zeta_3}{(1+N)(2+N)} \right) \\
& + C_F^2 \left(-\frac{8P_{310}}{(N-1)N^2(1+N)^3(2+N)} + \left(-\frac{4P_{81}}{N^2(1+N)^2(2+N)} \right. \right. \\
& \left. \left. - \frac{16S_2}{(1+N)(2+N)} \right) S_1 + \frac{24S_1^2}{1+N} + \frac{8(2+7N+2N^2)S_2}{N(1+N)(2+N)} - \frac{16S_3}{(1+N)(2+N)} \right. \\
& \left. - \frac{64S_{-2}}{(N-1)(1+N)(2+N)} + \frac{32S_{2,1}}{(1+N)(2+N)} - \frac{96\zeta_3}{(1+N)(2+N)} \right), \tag{201}
\end{aligned}$$

$$\begin{aligned}
\Delta b_{qq}^{(2,1),\text{NS}} = & \xi^2 \left[C_F^2 \left(\frac{1+3N}{N^2} - \frac{S_1}{N^2} - \frac{3S_1^2}{2N} - \frac{5S_2}{2N} \right) + C_A C_F \left(\frac{1+4N}{4N^2} + \frac{(1+2N)S_1}{2N^2} + \frac{3S_1^2}{4N} \right. \right. \\
& \left. \left. + \frac{5S_2}{4N} - \frac{\zeta_2}{4N} \right) \right] + \xi \left[C_F^2 \left(\frac{P_{295}}{N^4(1+N)^3} + \left(\frac{P_{37}}{N^3(1+N)^2} + \frac{22S_2}{N} \right) S_1 \right. \right. \\
& \left. \left. + \frac{(16+21N+3N^2)S_1^2}{2N^2(1+N)} + \frac{14S_1^3}{3N} + \frac{(8+11N-3N^2)S_2}{2N^2(1+N)} + \frac{28S_3}{3N} \right. \right. \\
& \left. \left. + \left(\frac{2+3N+3N^2}{N^2(1+N)} - \frac{4S_1}{N} \right) \zeta_2 \right) + C_A C_F \left(\frac{-6-7N-16N^2}{2N^3} - \frac{4S_3 P_{89}}{N(1+N)^2(2+N)^2} \right. \right. \\
& \left. \left. + \left(\frac{P_{31}}{2N^3(2+N)^2} + \frac{S_2 P_{42}}{N(1+N)^2(2+N)^2} + \frac{6(N-1)S_3}{(1+N)(2+N)} - \frac{4(N-1)S_{2,1}}{(1+N)(2+N)} \right) \right. \\
& \left. \times S_1 - \left(\frac{3(2+7N)}{4N(2+N)} - \frac{3(N-1)S_2}{2(1+N)(2+N)} \right) S_1^2 - \frac{(48+4N+132N^2+59N^3)S_2}{4N^2(2+N)^2} \right. \\
& \left. + \frac{(1-N)S_2^2}{(1+N)(2+N)} + \frac{7(N-1)S_4}{2(1+N)(2+N)} + \frac{8(-1+4N+5N^2+N^3)S_{2,1}}{N(1+N)^2(2+N)^2} \right. \\
& \left. - \frac{8(N-1)S_{3,1}}{(1+N)(2+N)} + \frac{2(N-1)S_{2,1,1}}{(1+N)(2+N)} + \frac{5\zeta_2}{4N} + \frac{18(N-1)\zeta_2^2}{5(1+N)(2+N)} \right. \\
& \left. + \left(\frac{12P_{77}}{N(1+N)^2(2+N)^2} + \frac{6(N-1)S_1}{(1+N)(2+N)} \right) \zeta_3 \right] \\
& + C_F \left(N_F T_F \left[-\frac{8P_{140}}{27N^2(1+N)^3} - \frac{4(-3-13N+33N^2+19N^3)S_1}{9N^2(1+N)^2} \right. \right. \\
& \left. \left. - \frac{2(N-1)S_1^2}{N(1+N)} - \frac{10(N-1)S_2}{3N(1+N)} + \frac{2(N-1)\zeta_2}{3N(1+N)} \right] \right. \\
& \left. + C_A \left[-\frac{4S_3 P_{218}}{(N-1)N(1+N)^2(2+N)^2} + \frac{S_2 P_{247}}{6N^2(1+N)^2(2+N)^2} \right] \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{2P_{518}}{27(N-1)^2N^2(1+N)^4(2+N)^2} + \left(\frac{P_{484}}{9(N-1)N^3(1+N)^3(2+N)^2} \right. \\
& - \frac{2S_2P_{69}}{N(1+N)^2(2+N)^2} - \frac{24S_3}{(1+N)(2+N)} + \left. \frac{16S_{2,1}}{(1+N)(2+N)} \right) S_1 \\
& + \left(\frac{-12+8N+65N^2+23N^3}{2N^2(1+N)(2+N)} - \frac{6S_2}{(1+N)(2+N)} \right) S_1^2 + \frac{4S_2^2}{(1+N)(2+N)} \\
& - \frac{14S_4}{(1+N)(2+N)} + \left(-\frac{32(-3+5N+4N^2)}{(N-1)^2(1+N)^2(2+N)^2} - \frac{32S_1}{(N-1)(1+N)(2+N)} \right) \\
& \times S_{-2} - \frac{64S_{-3}}{(N-1)(1+N)(2+N)} + \frac{8(-4+6N+11N^2+3N^3)S_{2,1}}{N(1+N)^2(2+N)^2} \\
& + \frac{32S_{3,1}}{(1+N)(2+N)} - \frac{8S_{2,1,1}}{(1+N)(2+N)} + \frac{(5-17N)\zeta_2}{6N(1+N)} - \frac{72\zeta_2^2}{5(1+N)(2+N)} \\
& + \left(-\frac{12P_{213}}{(N-1)N(1+N)^2(2+N)^2} - \frac{24S_1}{(1+N)(2+N)} \right) \zeta_3 \Bigg] \\
& + C_F^2 \left(-\frac{8S_3P_{212}}{3(N-1)N(1+N)^2(2+N)^2} + \frac{S_2P_{264}}{N^2(1+N)^2(2+N)^2} \right. \\
& + \frac{2P_{528}}{(N-1)^2N^3(1+N)^4(2+N)^2} + \left(-\frac{4S_2P_{112}}{N(1+N)^2(2+N)^2} \right. \\
& + \frac{4P_{448}}{(N-1)N^3(1+N)^3(2+N)^2} + \frac{48S_3}{(1+N)(2+N)} - \left. \frac{32S_{2,1}}{(1+N)(2+N)} \right) S_1 \\
& + \left(\frac{P_{80}}{N^2(1+N)^2(2+N)} + \frac{12S_2}{(1+N)(2+N)} \right) S_1^2 - \frac{28S_1^3}{3(1+N)} - \frac{8S_2^2}{(1+N)(2+N)} \\
& + \frac{28S_4}{(1+N)(2+N)} + \left(\frac{64(-3+5N+4N^2)}{(N-1)^2(1+N)^2(2+N)^2} + \frac{64S_1}{(N-1)(1+N)(2+N)} \right) \\
& \times S_{-2} + \frac{128S_{-3}}{(N-1)(1+N)(2+N)} - \frac{16(-4+6N+11N^2+3N^3)S_{2,1}}{N(1+N)^2(2+N)^2} \\
& - \frac{64S_{3,1}}{(1+N)(2+N)} + \frac{16S_{2,1,1}}{(1+N)(2+N)} + \left(-\frac{2(2+3N+3N^2)}{N(1+N)^2} + \frac{8S_1}{1+N} \right) \zeta_2 \\
& + \frac{144\zeta_2^2}{5(1+N)(2+N)} + \left(-\frac{24P_{203}}{(N-1)N(1+N)^2(2+N)^2} \right. \\
& \left. + \frac{48S_1}{(1+N)(2+N)} \right) \zeta_3, \tag{202}
\end{aligned}$$

$$\begin{aligned}
\Delta a_{qq}^{(2,0),\text{PS}} & = C_F N_F T_F \left(-\frac{8S_1P_{87}}{N^3(1+N)^3} + \frac{4P_{328}}{N^4(1+N)^4} + \frac{8(N-1)(2+N)S_1^2}{N^2(1+N)^2} \right. \\
& \left. + \frac{16(N-1)(2+N)S_2}{N^2(1+N)^2} - \frac{4(N-1)(2+N)\zeta_2}{N^2(1+N)^2} \right), \tag{203}
\end{aligned}$$

$$\begin{aligned}
\Delta a_{qq}^{(2,1),\text{PS}} &= C_F N_F T_F \left(\frac{4S_1^2 P_{87}}{N^3(1+N)^3} + \frac{8S_2 P_{87}}{N^3(1+N)^3} + \frac{2P_{457}}{N^5(1+N)^5} + \left(-\frac{4P_{328}}{N^4(1+N)^4} \right. \right. \\
&\quad \left. \left. - \frac{16(N-1)(2+N)S_2}{N^2(1+N)^2} \right) S_1 - \frac{8(N-1)(2+N)S_1^3}{3N^2(1+N)^2} - \frac{52(N-1)(2+N)S_3}{3N^2(1+N)^2} \right. \\
&\quad \left. + \left(-\frac{2P_{87}}{N^3(1+N)^3} + \frac{4(N-1)(2+N)S_1}{N^2(1+N)^2} \right) \zeta_2 + \frac{64(N-1)(2+N)\zeta_3}{3N^2(1+N)^2} \right), \quad (204)
\end{aligned}$$

$$\Delta b_{qq}^{(2,0),\text{PS}} = C_F N_F T_F \left(-\frac{16(3+4N-6N^2+3N^3)}{N^3(1+N)^3} + \frac{32(N-1)S_1}{N^2(1+N)^2} \right), \quad (205)$$

$$\begin{aligned}
\Delta b_{qq}^{(2,1),\text{PS}} &= C_F N_F T_F \left(-\frac{8P_{232}}{N^4(1+N)^4} + \frac{16(3+4N-6N^2+3N^3)S_1}{N^3(1+N)^3} - \frac{16(N-1)S_1^2}{N^2(1+N)^2} \right. \\
&\quad \left. - \frac{32(N-1)S_2}{N^2(1+N)^2} + \frac{8(N-1)\zeta_2}{N^2(1+N)^2} \right), \quad (206)
\end{aligned}$$

$$\Delta a_{qg}^{(1,0)} = N_F T_F \left(-\frac{4(1+3N-N^2+N^3)}{N^2(1+N)^2} + \frac{4(N-1)S_1}{N(1+N)} \right), \quad (207)$$

$$\begin{aligned}
\Delta a_{qg}^{(1,1)} &= N_F T_F \left(-\frac{2P_{51}}{N^3(1+N)^3} + \frac{2(1+3N-N^2+N^3)S_1}{N^2(1+N)^2} + \frac{(1-N)S_1^2}{N(1+N)} - \frac{3(N-1)S_2}{N(1+N)} \right. \\
&\quad \left. + \frac{(N-1)\zeta_2}{N(1+N)} \right), \quad (208)
\end{aligned}$$

$$\begin{aligned}
\Delta a_{qg}^{(1,2)} &= N_F T_F \left(\frac{P_{198}}{N^4(1+N)^4} + \left(\frac{P_{51}}{N^3(1+N)^3} + \frac{3(N-1)S_2}{2N(1+N)} \right) S_1 + \frac{(N-1)S_1^3}{6N(1+N)} \right. \\
&\quad + \frac{(-1-3N+N^2-N^3)S_1^2}{2N^2(1+N)^2} - \frac{3(1+3N-N^2+N^3)S_2}{2N^2(1+N)^2} + \frac{7(N-1)S_3}{3N(1+N)} \\
&\quad \left. + \left(\frac{1+3N-N^2+N^3}{2N^2(1+N)^2} + \frac{(1-N)S_1}{2N(1+N)} \right) \zeta_2 - \frac{7(N-1)\zeta_3}{3N(1+N)} \right), \quad (209)
\end{aligned}$$

$$\begin{aligned}
\Delta a_{qg}^{(2,0)} &= \xi^2 C_A N_F T_F \left[\frac{1+5N-N^2-N^3}{N^2(1+N)^2} + \frac{(1-N)S_1}{N(1+N)} \right] + \xi C_A N_F T_F \left[\frac{6(N-1)S_1^2}{N(1+N)} \right. \\
&\quad \left. - \frac{4(N-1)(1+2N)(3+N^2)}{N^3(1+N)^2} + \frac{4(1-5N-N^2+N^3)S_1}{N^2(1+N)^2} + \frac{10(N-1)S_2}{N(1+N)} \right] \\
&\quad + N_F^2 T_F^2 \left(-\frac{16(-15-73N+15N^2+13N^3)}{27N^2(1+N)^2} - \frac{80(N-1)S_1}{9N(1+N)} + \frac{8(N-1)\zeta_2}{3N(1+N)} \right) \\
&\quad + C_F N_F T_F \left(\frac{2P_{407}}{N^4(1+N)^4} + \left(-\frac{4P_{248}}{N^3(1+N)^3} - \frac{28(N-1)S_2}{N(1+N)} \right) S_1 - \frac{28(N-1)S_1^3}{3N(1+N)} \right. \\
&\quad + \frac{4(1+3N)(7-4N+3N^2)S_1^2}{N^2(1+N)^2} + \frac{4(1+8N-N^2+8N^3)S_2}{N^2(1+N)^2} + \frac{64(N-1)S_3}{3N(1+N)} \\
&\quad \left. - \frac{16(N-1)S_{2,1}}{N(1+N)} + \left(-\frac{2(N-1)(2+3N+3N^2)}{N^2(1+N)^2} + \frac{8(N-1)S_1}{N(1+N)} \right) \zeta_2 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{48(N-1)\zeta_3}{N(1+N)} \Big) + C_{AN_F T_F} \left(\frac{4P_{516}}{27(N-1)N^4(1+N)^4(2+N)} \right. \\
& + \frac{8S_2 P_{63}}{N^2(1+N)^2(2+N)} + \left(\frac{4P_{362}}{9N^3(1+N)^3(2+N)} - \frac{4(-38+19N+17N^2)S_2}{N(1+N)(2+N)} \right) \\
& \times S_1 + \frac{8(-4+7N)S_1^2}{N^2(1+N)^2} - \frac{28(N-1)S_1^3}{3N(1+N)} - \frac{8(-86+43N+40N^2)S_3}{3N(1+N)(2+N)} \\
& + \left(-\frac{16(3-8N-N^2+2N^3)}{(N-1)N(1+N)^2(2+N)} - \frac{32(N-1)S_1}{N(1+N)} \right) S_{-2} - \frac{40(N-1)S_{-3}}{N(1+N)} \\
& + \frac{16(-4+2N+N^2)S_{2,1}}{N(1+N)(2+N)} + \frac{16(N-1)S_{-2,1}}{N(1+N)} - \left(\frac{2(N-1)(24+11N+11N^2)}{3N^2(1+N)^2} \right. \\
& \left. - \frac{8(N-1)S_1}{N(1+N)} \right) \zeta_2 + \frac{24(-2+3N)\zeta_3}{N(2+N)} \Big), \tag{210}
\end{aligned}$$

$$\begin{aligned}
\Delta a_{qg}^{(2,1)} = & \xi^2 C_{AN_F T_F} \left[\frac{P_{217}}{2N^3(1+N)^3} - \frac{(1+5N-N^2-N^3)S_1}{2N^2(1+N)^2} + \frac{(N-1)S_1^2}{4N(1+N)} \right. \\
& + \left. \frac{3(N-1)S_2}{4N(1+N)} + \frac{(1-N)\zeta_2}{2N(1+N)} \right] + \xi C_{AN_F T_F} \left[-\frac{2P_{316}}{N^4(1+N)^3} + \left(\frac{2P_{233}}{N^3(1+N)^3} \right. \right. \\
& \left. \left. - \frac{13(N-1)S_2}{N(1+N)} \right) S_1 + \frac{(-1+13N+N^2-N^3)S_1^2}{N^2(1+N)^2} - \frac{7(N-1)S_1^3}{3N(1+N)} \right. \\
& + \frac{(-3+23N+3N^2-3N^3)S_2}{N^2(1+N)^2} - \frac{38(N-1)S_3}{3N(1+N)} + \left(\frac{2(N-1)^2}{N^2(1+N)} \right. \\
& \left. + \frac{2(N-1)S_1}{N(1+N)} \right) \zeta_2 + \frac{12(N-1)\zeta_3}{N(1+N)} \Big] + N_F^2 T_F^2 \left(\frac{8(-15-73N+15N^2+13N^3)}{27N^2(1+N)^2} \right. \\
& \times S_1 + \frac{8P_{272}}{81N^3(1+N)^3} + \frac{20(N-1)S_1^2}{9N(1+N)} + \frac{20(N-1)S_2}{3N(1+N)} + \left(-\frac{4(N-1)S_1}{3N(1+N)} \right. \\
& \left. - \frac{4(-3-19N+3N^2+7N^3)}{9N^2(1+N)^2} \right) \zeta_2 - \frac{56(N-1)\zeta_3}{9N(1+N)} \Big) \\
& + C_{FN_F T_F} \left(-\frac{2S_3 P_{129}}{3N^2(1+N)^2(2+N)} + \frac{S_2 P_{348}}{N^3(1+N)^3(2+N)} \right. \\
& + \frac{P_{497}}{N^5(1+N)^5} + \left(-\frac{2S_2 P_{141}}{N^2(1+N)^2(2+N)} - \frac{2P_{468}}{N^4(1+N)^4(2+N)} + \frac{4(N-1)S_3}{N(1+N)} \right. \\
& \left. + \frac{16(N-1)S_{2,1}}{N(1+N)} \right) S_1 + \left(\frac{P_{260}}{N^3(1+N)^3} + \frac{19(N-1)S_2}{N(1+N)} \right) S_1^2 + \frac{5(N-1)S_1^4}{2N(1+N)} \\
& - \frac{2(17+40N-21N^2+20N^3)S_1^3}{3N^2(1+N)^2} - \frac{16(4+5N-4N^2+N^3)S_{2,1}}{N^2(1+N)^2(2+N)} \\
& + \frac{15(N-1)S_2^2}{2N(1+N)} - \frac{57(N-1)S_4}{N(1+N)} + \frac{48(N-1)S_{3,1}}{N(1+N)} - \frac{8(N-1)S_{2,1,1}}{N(1+N)}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{P_{197}}{N^3(1+N)^3} + \frac{2(4+11N-6N^2+7N^3)S_1}{N^2(1+N)^2} - \frac{6(N-1)S_1^2}{N(1+N)} - \frac{2(N-1)S_2}{N(1+N)} \right) \\
& \times \zeta_2 + \frac{72(N-1)\zeta_2^2}{5N(1+N)} + \left(-\frac{16P_{118}}{3N^2(1+N)^2(2+N)} + \frac{16(N-1)S_1}{3N(1+N)} \right) \zeta_3 \\
& + C_{ANFTF} \left(\frac{8S_{2,1}P_{84}}{N^2(1+N)^2(2+N)^2} - \frac{4S_3P_{338}}{3(N-1)N^2(1+N)^2(2+N)^2} \right. \\
& + \frac{S_2P_{383}}{3N^3(1+N)^3(2+N)^2} - \frac{2P_{591}}{81(N-1)^2N^5(1+N)^5(2+N)^2} \\
& + \left(-\frac{4S_2P_{209}}{N^2(1+N)^2(2+N)^2} - \frac{2P_{544}}{27(N-1)N^4(1+N)^4(2+N)^2} \right. \\
& + \left. \frac{8(-34+17N+14N^2)S_3}{N(1+N)(2+N)} - \frac{8(-6+3N+N^2)S_{2,1}}{N(1+N)(2+N)} \right) S_1 \\
& + \left(\frac{P_{290}}{9N^3(1+N)^3(2+N)} + \frac{(-82+41N+35N^2)S_2}{N(1+N)(2+N)} \right) S_1^2 - \frac{4(-8+15N)S_1^3}{3N^2(1+N)^2} \\
& + \frac{5(N-1)S_1^4}{2N(1+N)} + \frac{(-142+71N+79N^2)S_2^2}{2N(1+N)(2+N)} + \frac{(-394+197N+183N^2)S_4}{N(1+N)(2+N)} \\
& + \left(\frac{32S_1P_{61}}{(N-1)N^2(1+N)^2(2+N)} + \frac{8P_{313}}{(N-1)^2N(1+N)^3(2+N)^2} + \frac{16(N-1)S_1^2}{N(1+N)} \right. \\
& + \left. \frac{32(N-1)S_2}{N(1+N)} \right) S_{-2} + \frac{8(N-1)S_{-2}^2}{N(1+N)} + \left(\frac{8P_{104}}{(N-1)N^2(1+N)^2(2+N)} \right. \\
& + \left. \frac{64(N-1)S_1}{N(1+N)} \right) S_{-3} + \frac{84(N-1)S_{-4}}{N(1+N)} - \frac{4(-34+17N+9N^2)S_{3,1}}{N(1+N)(2+N)} \\
& + \frac{16(1+3N)S_{-2,1}}{N^2(1+N)^2} - \frac{8(N-1)S_{-2,2}}{N(1+N)} - \frac{24(N-1)S_{-3,1}}{N(1+N)} - \frac{4(2-N+N^2)S_{2,1,1}}{N(1+N)(2+N)} \\
& - \frac{16(N-1)S_{-2,1,1}}{N(1+N)} + \left(\frac{P_{273}}{9N^3(1+N)^3} - \frac{(48-61N-11N^3)S_1}{3N^2(1+N)^2} - \frac{6(N-1)S_1^2}{N(1+N)} \right. \\
& - \left. \frac{14(N-1)S_2}{N(1+N)} - \frac{4(N-1)S_{-2}}{N(1+N)} \right) \zeta_2 + \left(\frac{2P_{366}}{9(N-1)N^2(1+N)^2(2+N)^2} \right. \\
& - \left. \frac{8(-32+16N+25N^2)S_1}{3N(1+N)(2+N)} \right) \zeta_3 - \frac{36(-2+3N)\zeta_2^2}{5N(2+N)}, \tag{211}
\end{aligned}$$

$$\Delta a_{gq}^{(1,0)} = C_F \left(-\frac{2(-2-N+4N^2+N^3)}{N^2(1+N)^2} + \frac{2(2+N)S_1}{N(1+N)} \right), \tag{212}$$

$$\begin{aligned}
\Delta a_{gq}^{(1,1)} & = C_F \left(\frac{P_{53}}{N^3(1+N)^3} + \frac{(-2-N+4N^2+N^3)S_1}{N^2(1+N)^2} + \frac{(-2-N)S_1^2}{2N(1+N)} - \frac{3(2+N)S_2}{2N(1+N)} \right. \\
& \left. + \frac{(2+N)\zeta_2}{2N(1+N)} \right), \tag{213}
\end{aligned}$$

$$\begin{aligned}
\Delta a_{gq}^{(1,2)} = & C_F \left(\frac{P_{237}}{2N^4(1+N)^4} + \left(-\frac{P_{53}}{2N^3(1+N)^3} + \frac{3(2+N)S_2}{4N(1+N)} \right) S_1 + \frac{(2+N)S_1^3}{12N(1+N)} \right. \\
& + \frac{(2+N-4N^2-N^3)S_1^2}{4N^2(1+N)^2} - \frac{3(-2-N+4N^2+N^3)S_2}{4N^2(1+N)^2} + \frac{7(2+N)S_3}{6N(1+N)} \\
& \left. + \left(\frac{-2-N+4N^2+N^3}{4N^2(1+N)^2} + \frac{(-2-N)S_1}{4N(1+N)} \right) \zeta_2 - \frac{7(2+N)\zeta_3}{6N(1+N)} \right), \tag{214}
\end{aligned}$$

$$\begin{aligned}
\Delta a_{gq}^{(2,0)} = & \xi \left[C_F^2 \left(-\frac{6(2+N)(3+3N-N^2+N^3)}{N^3(1+N)^2} - \frac{2(10+15N+7N^2)S_1}{N^2(1+N)^2} + \frac{3(2+N)S_1^2}{N(1+N)} \right. \right. \\
& \left. \left. + \frac{5(2+N)S_2}{N(1+N)} \right) + C_A C_F \left(\frac{(2+3N)(2-N+N^2)}{(N-1)N^3(2+N)} + \frac{2(2+3N)S_{-2}}{(N-1)N(2+N)} \right) \right] \\
& + C_F \left(N_F T_F \left(-\frac{8P_{257}}{27N^3(1+N)^3} - \frac{8(6+7N+2N^3)S_1}{9N^2(1+N)^2} - \frac{8(2+N)S_2}{3N(1+N)} \right. \right. \\
& \left. \left. + \frac{4(2+N)\zeta_2}{3N(1+N)} \right) + C_A \left(\frac{2S_2 P_{134}}{3N^2(1+N)^2(2+N)} + \frac{2P_{515}}{27(N-1)N^4(1+N)^4(2+N)} \right. \right. \\
& \left. \left. + \left(-\frac{2P_{337}}{9N^3(1+N)^3(2+N)} - \frac{2(2+3N)(10+3N)S_2}{N(1+N)(2+N)} \right) S_1 - \frac{14(2+N)S_1^3}{3N(1+N)} \right. \right. \\
& \left. \left. + \frac{4(-1-2N+12N^2+3N^3)S_1^2}{N^2(1+N)^2} - \frac{4(-20+4N+N^2)S_3}{3N(1+N)(2+N)} \right. \right. \\
& \left. \left. + \left(-\frac{8(5+3N+2N^2)}{(N-1)(1+N)^2(2+N)} - \frac{16(2+N)S_1}{N(1+N)} \right) S_{-2} - \frac{20(2+N)S_{-3}}{N(1+N)} \right. \right. \\
& \left. \left. - \frac{64S_{2,1}}{N(1+N)(2+N)} + \frac{8(2+N)S_{-2,1}}{N(1+N)} + \left(-\frac{(2+N)(24+11N+11N^2)}{3N^2(1+N)^2} \right. \right. \right. \\
& \left. \left. \left. + \frac{4(2+N)S_1}{N(1+N)} \right) \zeta_2 - \frac{12(-4+12N+3N^2)\zeta_3}{N(1+N)(2+N)} \right) + C_F^2 \left(\frac{P_{409}}{N^4(1+N)^4} \right. \\
& \left. + \left(\frac{2P_{231}}{N^3(1+N)^3} - \frac{2(38+15N)S_2}{N(1+N)} \right) S_1 + \frac{2(2+N)(2+3N^2)S_1^2}{N^2(1+N)^2} - \frac{14(2+N)S_1^3}{3N(1+N)} \right. \\
& \left. + \frac{2(8+12N+22N^2+3N^3)S_2}{N^2(1+N)^2} - \frac{4(74+31N)S_3}{3N(1+N)} + \frac{32S_{2,1}}{N(1+N)} \right. \\
& \left. + \left(-\frac{(2+N)(2+3N+3N^2)}{N^2(1+N)^2} + \frac{4(2+N)S_1}{N(1+N)} \right) \zeta_2 + \frac{48\zeta_3}{1+N} \right), \tag{215}
\end{aligned}$$

$$\begin{aligned}
\Delta a_{gq}^{(2,1)} = & \xi \left[C_F^2 \left(\frac{3P_{199}}{N^4(1+N)^3} + \left(\frac{P_{241}}{N^3(1+N)^3} - \frac{13(2+N)S_2}{2N(1+N)} \right) S_1 + \frac{3(6+9N+5N^2)S_1^2}{2N^2(1+N)^2} \right. \right. \\
& \left. \left. - \frac{7(2+N)S_1^3}{6N(1+N)} + \frac{(38+57N+29N^2)S_2}{2N^2(1+N)^2} - \frac{19(2+N)S_3}{3N(1+N)} + \left(-\frac{3(2+N)}{N^2(1+N)} \right. \right. \right. \\
& \left. \left. \left. + \frac{(2+N)S_1}{N(1+N)} \right) \zeta_2 + \frac{6(2+N)\zeta_3}{N(1+N)} \right) + C_A C_F \left(\frac{P_{322}}{2(N-1)^2 N^4(2+N)^2} \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{(2+3N)(2-N+N^2)S_1}{(N-1)N^3(2+N)} + \frac{(-2-3N)S_3}{(N-1)N(2+N)} + \left(-\frac{2(10+3N+2N^2)}{(N-1)^2N(2+N)^2} \right. \\
& \left. -\frac{2(2+3N)S_1}{(N-1)N(2+N)} \right) S_{-2} - \frac{4(2+3N)S_{-3}}{(N-1)N(2+N)} - \frac{3(2+3N)\zeta_3}{(N-1)N(2+N)} \Big] \\
& + C_F \left(N_F T_F \left(\frac{4P_{416}}{81N^4(1+N)^4} + \left(\frac{4P_{257}}{27N^3(1+N)^3} + \frac{4(2+N)S_2}{3N(1+N)} \right) S_1 \right. \right. \\
& + \frac{2(6+7N+2N^3)S_1^2}{9N^2(1+N)^2} + \frac{2(42+41N-24N^2+4N^3)S_2}{9N^2(1+N)^2} + \frac{16(2+N)S_3}{3N(1+N)} \\
& \left. + \left(-\frac{2(18+17N-12N^2+N^3)}{9N^2(1+N)^2} - \frac{2(2+N)S_1}{3N(1+N)} \right) \zeta_2 - \frac{100(2+N)\zeta_3}{9N(1+N)} \right) \\
& + C_A \left(\frac{4S_{2,1}P_{96}}{N^2(1+N)^2(2+N)^2} - \frac{2S_3P_{345}}{3(N-1)N^2(1+N)^2(2+N)^2} \right. \\
& + \frac{S_2P_{390}}{18N^3(1+N)^3(2+N)^2} + \frac{P_{586}}{81(N-1)^2N^5(1+N)^5(2+N)^2} \\
& + \left(\frac{S_2P_{169}}{3N^2(1+N)^2(2+N)^2} + \frac{P_{526}}{27(N-1)N^4(1+N)^4(2+N)^2} \right. \\
& \left. + \frac{4(-4+20N+5N^2)S_3}{N(1+N)(2+N)} + \frac{4(20+4N+N^2)S_{2,1}}{N(1+N)(2+N)} \right) S_1 \\
& + \left(\frac{P_{363}}{18N^3(1+N)^3(2+N)} + \frac{(44+92N+23N^2)S_2}{2N(1+N)(2+N)} \right) S_1^2 + \frac{5(2+N)S_1^4}{4N(1+N)} \\
& - \frac{2(-5-6N+28N^2+7N^3)S_1^3}{3N^2(1+N)^2} + \frac{(156+92N+23N^2)S_2^2}{4N(1+N)(2+N)} \\
& + \frac{(-68+44N+11N^2)S_4}{2N(1+N)(2+N)} + \left(\frac{16S_1P_{75}}{(N-1)N^2(1+N)^2(2+N)} \right. \\
& \left. - \frac{4P_{303}}{(N-1)^2N(1+N)^3(2+N)^2} + \frac{8(2+N)S_1^2}{N(1+N)} + \frac{16(2+N)S_2}{N(1+N)} \right) S_{-2} \\
& + \frac{4(2+N)S_{-2}^2}{N(1+N)} + \left(\frac{4P_{109}}{(N-1)N^2(1+N)^2(2+N)} + \frac{32(2+N)S_1}{N(1+N)} \right) S_{-3} \\
& + \frac{42(2+N)S_{-4}}{N(1+N)} + \frac{2(76+12N+3N^2)S_{3,1}}{N(1+N)(2+N)} - \frac{8(2+N)(1+2N)S_{-2,1}}{N^2(1+N)^2} \\
& - \frac{4(2+N)S_{-2,2}}{N(1+N)} - \frac{12(2+N)S_{-3,1}}{N(1+N)} - \frac{2(28+12N+3N^2)S_{2,1,1}}{N(1+N)(2+N)} - \frac{8(2+N)S_{-2,1,1}}{N(1+N)} \\
& + \left(\frac{P_{269}}{18N^3(1+N)^3} + \frac{(24+22N+129N^2+35N^3)S_1}{6N^2(1+N)^2} - \frac{3(2+N)S_1^2}{N(1+N)} \right. \\
& \left. - \frac{(2+N)S_2}{N(1+N)} - \frac{2(2+N)S_{-2}}{N(1+N)} \right) \zeta_2 + \frac{18(-4+12N+3N^2)\zeta_2^2}{5N(1+N)(2+N)}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{P_{355}}{9(N-1)N^2(1+N)^2(2+N)^2} + \frac{4(-28+44N+11N^2)S_1}{3N(1+N)(2+N)} \right) \zeta_3 \Big) \\
& + C_F^2 \left(\frac{S_3 P_{41}}{3N^2(1+N)^2(2+N)} + \frac{S_2 P_{331}}{2N^3(1+N)^3(2+N)} + \frac{P_{496}}{2N^5(1+N)^5} \right. \\
& + \left. \left(\frac{S_2 P_{35}}{N^2(1+N)^2(2+N)} + \frac{P_{445}}{N^4(1+N)^4(2+N)} + \frac{2(62+19N)S_3}{N(1+N)} - \frac{32S_{2,1}}{N(1+N)} \right) \right. \\
& \times S_1 + \left. \left(\frac{P_{188}}{2N^3(1+N)^3} + \frac{(86+31N)S_2}{2N(1+N)} \right) S_1^2 - \frac{(2+N)(4-N+6N^2)S_1^3}{3N^2(1+N)^2} \right. \\
& + \frac{5(2+N)S_1^4}{4N(1+N)} + \frac{(110+71N)S_2^2}{4N(1+N)} + \frac{(286+115N)S_4}{2N(1+N)} - \frac{8(-3+N)(2+3N)S_{2,1}}{N(1+N)^2(2+N)} \\
& - \frac{64S_{3,1}}{N(1+N)} + \frac{16S_{2,1,1}}{N(1+N)} + \left(\frac{(2+N)P_{94}}{2N^3(1+N)^3} + \frac{(2+N)(2+N+3N^2)S_1}{N^2(1+N)^2} \right. \\
& - \left. \frac{3(2+N)S_1^2}{N(1+N)} - \frac{7(2+N)S_2}{N(1+N)} \right) \zeta_2 - \frac{72\zeta_2^2}{5(1+N)} + \left(-\frac{2P_{79}}{3N^2(1+N)^2(2+N)} \right. \\
& \left. - \frac{8(16+17N)S_1}{3N(1+N)} \right) \zeta_3 \Big), \tag{216}
\end{aligned}$$

$$\Delta b_{gq}^{(1,0)} = \frac{4C_F}{N(1+N)}, \tag{217}$$

$$\Delta b_{gq}^{(1,1)} = C_F \left(\frac{2(-1-2N+N^2)}{N^2(1+N)^2} - \frac{2S_1}{N(1+N)} \right), \tag{218}$$

$$\begin{aligned}
\Delta b_{gq}^{(1,2)} = & C_F \left(\frac{1+3N+3N^2-3N^3}{N^3(1+N)^3} + \frac{(1+2N-N^2)S_1}{N^2(1+N)^2} + \frac{S_1^2}{2N(1+N)} + \frac{3S_2}{2N(1+N)} \right. \\
& \left. - \frac{\zeta_2}{2N(1+N)} \right), \tag{219}
\end{aligned}$$

$$\begin{aligned}
\Delta b_{gq}^{(2,0)} = & \xi \left[C_F^2 \left(-\frac{4(2+N)(-3-3N+2N^2)}{N^3(1+N)^2} + \frac{4(4+3N)S_1}{N^2(1+N)} \right) \right. \\
& + C_A C_F \left(-\frac{2(1+N)(2-N+N^2)}{(N-1)N^3(2+N)} - \frac{4(1+N)S_{-2}}{(N-1)N(2+N)} \right) \Big] \\
& + C_F \left(N_F T_F \left(-\frac{16(9+22N+N^2)}{9N^2(1+N)^2} - \frac{16S_1}{3N(1+N)} \right) \right) \\
& + C_A \left(\frac{4P_{357}}{9(N-1)N^3(1+N)^3(2+N)} + \left(\frac{4P_{146}}{3N^3(1+N)^2(2+N)} \right. \right. \\
& + \left. \left. \frac{4(-2+N)S_2}{N(1+N)(2+N)} \right) S_1 - \frac{24S_1^2}{N(1+N)} - \frac{4(-2+5N+5N^2)S_2}{N^2(1+N)(2+N)} \right. \\
& + \left. \frac{4(-2+N)S_3}{N(1+N)(2+N)} + \frac{8(3+N^2)S_{-2}}{(N-1)N(1+N)(2+N)} - \frac{8(-2+N)S_{2,1}}{N(1+N)(2+N)} \right)
\end{aligned}$$

$$+ \frac{24(-2+N)\zeta_3}{N(1+N)(2+N)} \Big) \Big) + C_F^2 \left(\frac{4(-4-22N-5N^2+3N^3)}{N^2(1+N)^3} - \frac{4(-4+5N)S_1}{N^2(1+N)} - \frac{16S_2}{N(1+N)} \right), \quad (220)$$

$$\begin{aligned} \Delta b_{gq}^{(2,1)} = & \xi \left[C_F^2 \left(\frac{2P_{211}}{N^4(1+N)^3} - \frac{2(3+4N)(4+N-N^2)S_1}{N^3(1+N)^2} - \frac{(8+7N)S_1^2}{N^2(1+N)} - \frac{(16+13N)S_2}{N^2(1+N)} \right. \right. \\ & + \left. \frac{2(2+N)\zeta_2}{N^2(1+N)} \right) + C_A C_F \left(\frac{P_{301}}{(N-1)^2 N^4 (2+N)^2} + \frac{2(1+N)(2-N+N^2)S_1}{(N-1)N^3(2+N)} \right. \\ & + \left. \frac{2(1+N)S_3}{(N-1)N(2+N)} + \left(\frac{4(3+2N+N^2)}{(N-1)^2 N(2+N)^2} + \frac{4(1+N)S_1}{(N-1)N(2+N)} \right) \right. \\ & \left. \left. \right) S_{-2} + \frac{8(1+N)S_{-3}}{(N-1)N(2+N)} + \frac{6(1+N)\zeta_3}{(N-1)N(2+N)} \right] + C_F (N_F T_F \left(\frac{8P_{142}}{27N^3(1+N)^3} \right. \\ & + \left. \frac{8(9+22N+N^2)S_1}{9N^2(1+N)^2} + \frac{4S_1^2}{3N(1+N)} + \frac{28S_2}{3N(1+N)} - \frac{4\zeta_2}{N(1+N)} \right) \\ & + C_A \left(\frac{S_2 P_{167}}{3N^3(1+N)^2(2+N)^2} + \frac{2S_3 P_{261}}{3(N-1)N^2(1+N)^2(2+N)^2} \right. \\ & - \left. \frac{2P_{546}}{27(N-1)^2 N^4 (1+N)^4 (2+N)^2} + \left(-\frac{2P_{463}}{9(N-1)N^4(1+N)^3(2+N)^2} \right. \right. \\ & + \left. \frac{2S_2 P_{136}}{N^2(1+N)^2(2+N)^2} - \frac{12(-2+N)S_3}{N(1+N)(2+N)} + \frac{8(-2+N)S_{2,1}}{N(1+N)(2+N)} \right) S_1 \\ & + \left(\frac{P_{10}}{3N^3(1+N)^2(2+N)} - \frac{3(-2+N)S_2}{N(1+N)(2+N)} \right) S_1^2 + \frac{2(-2+N)S_2^2}{N(1+N)(2+N)} \\ & + \frac{28S_1^3}{3N(1+N)} - \frac{7(-2+N)S_4}{N(1+N)(2+N)} + \left(-\frac{8(3+N^2)(-3+5N+4N^2)}{(N-1)^2 N(1+N)^2(2+N)^2} \right. \\ & - \left. \frac{8(3+N^2)S_1}{(N-1)N(1+N)(2+N)} \right) S_{-2} - \frac{4(-24-42N-7N^2+5N^3)S_{2,1}}{N(1+N)^2(2+N)^2} \\ & - \frac{16(3+N^2)S_{-3}}{(N-1)N(1+N)(2+N)} + \frac{16(-2+N)S_{3,1}}{N(1+N)(2+N)} - \frac{4(-2+N)S_{2,1,1}}{N(1+N)(2+N)} \\ & + \left(\frac{16+11N+11N^2}{N^2(1+N)^2} - \frac{8S_1}{N(1+N)} \right) \zeta_2 - \frac{36(-2+N)\zeta_2^2}{5N(1+N)(2+N)} \\ & + \left(\frac{12P_{244}}{(N-1)N^2(1+N)^2(2+N)^2} - \frac{12(-2+N)S_1}{N(1+N)(2+N)} \right) \zeta_3 \Big) \\ & + C_F^2 \left(-\frac{2P_{246}}{N^3(1+N)^4} + \left(-\frac{2P_{74}}{N^3(1+N)^2(2+N)} + \frac{48S_2}{N(1+N)(2+N)} \right) S_1 \right. \\ & + \left. \frac{(-12-5N+11N^2)S_1^2}{N^2(1+N)^2} + \frac{(-40+30N+99N^2+49N^3)S_2}{N^2(1+N)^2(2+N)} \right) \end{aligned}$$

$$+ \frac{24(4+N)S_3}{N(1+N)(2+N)} + \frac{16(-4+N)S_{2,1}}{N(1+N)(2+N)} - \frac{4(2+N)\zeta_2}{N(1+N)^2} - \frac{96(N-1)\zeta_3}{N(1+N)(2+N)} \Big), \quad (221)$$

$$\Delta a_{gg}^{(1,0)} = -\frac{C_A \xi^2}{4} + \xi C_A \left(\frac{-1+N}{N} + S_1 \right) - \frac{20T_F N_F}{9} + C_A \left(-\frac{P_{16}}{9N^2(1+N)^2} + \frac{8S_1}{N(1+N)} - 2S_1^2 - 6S_2 \right), \quad (222)$$

$$\Delta a_{gg}^{(1,1)} = \frac{C_A \xi^2}{4} + C_A \xi \left(\frac{1+N}{2N^2} + \frac{(1-N)S_1}{2N} - \frac{1}{4}S_1^2 - \frac{3}{4}S_2 \right) + C_A \left(\frac{P_{288}}{27N^3(1+N)^3} + \left(\frac{-4-9N+2N^2-N^3}{N^2(1+N)^2} + 3S_2 \right) S_1 - \frac{2S_1^2}{N(1+N)} + \frac{1}{3}S_1^3 - \frac{6S_2}{N(1+N)} + \frac{14}{3}S_3 + \left(\frac{24+11N+11N^2}{12N(1+N)} - S_1 \right) \zeta_2 \right) + N_F T_F \left(\frac{56}{27} - \frac{\zeta_2}{3} \right), \quad (223)$$

$$\Delta a_{gg}^{(1,2)} = \xi^2 C_A \left[-\frac{1}{4} + \frac{\zeta_2}{32} \right] + \xi C_A \left[\frac{-1-N}{4N^3} + \left(\frac{-1-N}{4N^2} + \frac{3S_2}{8} \right) S_1 + \frac{(N-1)S_1^2}{8N} + \frac{1}{24}S_1^3 + \frac{3(N-1)S_2}{8N} + \frac{7}{12}S_3 + \left(\frac{1-N}{8N} - \frac{S_1}{8} \right) \zeta_2 \right] + C_A \left(\frac{P_{487}}{162N^4(1+N)^4} + \left(\frac{P_{56}}{2N^3(1+N)^3} + \frac{3S_2}{N(1+N)} - \frac{7}{3}S_3 \right) S_1 + \left(\frac{4+9N-2N^2+N^3}{4N^2(1+N)^2} - \frac{3}{4}S_2 \right) S_1^2 + \frac{S_1^3}{3N(1+N)} - \frac{1}{24}S_1^4 + \frac{3(4+9N-2N^2+N^3)S_2}{4N^2(1+N)^2} - \frac{9}{8}S_2^2 + \frac{14S_3}{3N(1+N)} - \frac{15}{4}S_4 + \left(\frac{P_{16}}{72N^2(1+N)^2} - \frac{S_1}{N(1+N)} + \frac{1}{4}S_1^2 + \frac{3}{4}S_2 \right) \zeta_2 + \left(\frac{7}{3}S_1 - \frac{7(24+11N+11N^2)}{36N(1+N)} \right) \zeta_3 \right) + N_F T_F \left(-\frac{164}{81} + \frac{5\zeta_2}{18} + \frac{7\zeta_3}{9} \right), \quad (224)$$

$$\Delta a_{gg}^{(2,0)} = \xi^4 \frac{C_A^2}{16} + \xi^3 C_A^2 \left[\frac{4-3N}{16N} - \frac{S_1}{4} \right] + \xi^2 \left[\frac{2C_A N_F T_F}{3} + C_A^2 \left(\frac{(-10+7N+9N^2)S_1}{4N(1+N)} + \frac{P_{384}}{12(N-1)N^3(1+N)^2(2+N)} + \frac{3}{4}S_1^2 + \frac{3}{2}S_2 + \frac{(2-N)S_{-2}}{4(N-1)N(2+N)} - \frac{1}{8}\zeta_3 \right) \right] + \xi \left[C_A N_F T_F \left(\frac{2(18+16N+7N^2)}{9N^2} - \frac{4(-3+8N)S_1}{9N} - \frac{2}{3}S_1^2 - \frac{14}{3}S_2 \right) + C_A^2 \left(\frac{S_2 P_{147}}{12N^2(1+N)(2+N)} + \frac{P_{417}}{18(N-1)N^3(1+N)^2(2+N)} + \left(\frac{P_{368}}{18N^3(1+N)^2(2+N)} + \frac{(-2-60N-96N^2-33N^3)S_2}{2N(1+N)(2+N)} \right) S_1 \right) \right]$$

$$\begin{aligned}
& + \frac{(216 + 37N - 35N^2)S_1^2}{12N(1+N)} - 4S_1^3 + \frac{(-2 + 4N - N^3)S_3}{2N(1+N)(2+N)} + \frac{(-7 + 3N)S_{-2}}{(N-1)N(2+N)} \\
& + \left. \frac{(2 - 4N + N^3)S_{2,1}}{N(1+N)(2+N)} - \frac{3(4 - 6N + 3N^2 + 3N^3)\zeta_3}{2N(1+N)(2+N)} \right] \\
& + C_F N_F T_F \left(\frac{P_{527}}{3(N-1)N^4(1+N)^4(2+N)} - \frac{8(2+N)(3+6N-4N^2+3N^3)}{N^3(1+N)^3} \right) \\
& \times S_1 + \frac{8(N-1)(2+N)S_1^2}{N^2(1+N)^2} + \frac{16(N-1)(2+N)S_2}{N^2(1+N)^2} - \frac{64S_{-2}}{(N-1)N(1+N)(2+N)} \\
& - \frac{4(N-1)(2+N)\zeta_2}{N^2(1+N)^2} + 16\zeta_3 \Big) + C_A N_F T_F \left(-\frac{2P_{491}}{81(N-1)N^3(1+N)^3(2+N)} \right. \\
& + \left. \left(\frac{4P_{267}}{9N^2(1+N)^2(2+N)} + \frac{8(3+2N)S_2}{2+N} \right) S_1 + \frac{2(-48+37N+37N^2)S_1^2}{9N(1+N)} \right. \\
& + \frac{16}{9}S_1^3 + \frac{2(-68+52N+127N^2+39N^3)S_2}{3N(1+N)(2+N)} + \frac{8(47+28N)S_3}{9(2+N)} \\
& + \frac{32S_{-2}}{(N-1)N(1+N)(2+N)} - \frac{16(N-1)S_{2,1}}{3(2+N)} + \left. \left(\frac{4(24+11N+11N^2)}{9N(1+N)} \right. \right. \\
& \left. \left. - \frac{16}{3}S_1 \right) \zeta_2 - \frac{8(8+N)\zeta_3}{2+N} \right) + C_A^2 \left(\frac{P_{552}}{162(N-1)N^4(1+N)^4(2+N)} \right. \\
& + \frac{S_2 P_{164}}{6N^2(1+N)^2(2+N)} + \left. \left(\frac{(-400-269N-114N^2-41N^3)S_2}{N(1+N)(2+N)} \right. \right. \\
& + \frac{P_{378}}{9N^3(1+N)^3(2+N)} + \frac{238}{3}S_3 - 4S_{2,1} \Big) S_1 + \left(\frac{P_3}{18N^2(1+N)^2} + 49S_2 \right) S_1^2 \\
& - \frac{4(84+11N+11N^2)S_1^3}{9N(1+N)} + \frac{(-3144-2561N-1650N^2-589N^3)S_3}{9N(1+N)(2+N)} \\
& + \frac{14}{3}S_1^4 + 24S_2^2 + 19S_4 + \left(\frac{8(8-13N-3N^2+2N^3)}{(N-1)N(1+N)^2(2+N)} + \frac{8(-7+N)S_1}{N(1+N)} + 8S_1^2 \right. \\
& + 16S_2 \Big) S_{-2} + \left(\frac{4(-19+N)}{N(1+N)} + 32S_1 \right) S_{-3} + \frac{2(48+11N+12N^2+13N^3)S_{2,1}}{3N(1+N)(2+N)} \\
& + 4S_{-2}^2 + 28S_{-4} + 6S_{3,1} - \frac{8(-3+N)S_{-2,1}}{N(1+N)} - 8S_{-2,2} - 24S_{-3,1} + 6S_{2,1,1} \\
& - 16S_{-2,1,1} + \left(-\frac{(24+11N+11N^2)^2}{18N^2(1+N)^2} + \frac{4(24+11N+11N^2)S_1}{3N(1+N)} - 8S_1^2 \right) \zeta_2 \\
& + \left(\frac{-24-8N-33N^2-25N^3}{N(1+N)(2+N)} + 12S_1 \right) \zeta_3 \Big) + N_F^2 T_F^2 \left(\frac{848}{81} - \frac{8\zeta_2}{9} \right), \tag{225}
\end{aligned}$$

$$\Delta a_{gg}^{(2,1)} = -\xi^4 \frac{1}{8} C_A^2 + \xi^3 C_A^2 \left[\frac{-4-12N-N^2}{32N^2} + \frac{(-1+3N)S_1}{8N} + \frac{1}{16}S_1^2 + \frac{3}{16}S_2 + \frac{1}{16}\zeta_2 \right]$$

$$\begin{aligned}
& +\xi^2 \left[C_A^2 \left(\frac{P_{565}}{72(N-1)^2 N^4 (1+N)^3 (2+N)^2} + \left(\frac{P_{387}}{8(N-1)N^3(1+N)^2(2+N)} \right. \right. \right. \\
& \left. \left. \left. - \frac{5}{4} S_2 \right) S_1 + \frac{(7-3N-6N^2)S_1^2}{8N(1+N)} - \frac{5}{24} S_1^3 + \frac{(-6+59N-28N^2-28N^3)S_3}{24(N-1)N(2+N)} \right. \\
& \left. + \frac{(4-2N-3N^2)S_2}{2N(1+N)} + \left(\frac{2-5N}{4(N-1)^2 N(2+N)^2} + \frac{(-2+N)S_1}{4(N-1)N(2+N)} \right) S_{-2} \right. \\
& \left. + \frac{(-2+N)S_{-3}}{2(N-1)N(2+N)} + \left(\frac{-21-13N-16N^2}{24N(1+N)} + \frac{3}{8} S_1 \right) \zeta_2 + \frac{3}{80} \zeta_2^2 \right. \\
& \left. + \frac{(-18+13N-2N^2-2N^3)\zeta_3}{24(N-1)N(2+N)} \right) + C_{AN_F T_F} \left(-\frac{2}{3} + \frac{\zeta_2}{12} \right) \Big] \\
& +\xi \left[C_{AN_F T_F} \left(\frac{-126-144N-50N^2-53N^3}{27N^3} + \left(\frac{2(-27-24N+25N^2)}{27N^2} \right. \right. \right. \\
& \left. \left. \left. + \frac{7S_2}{3} \right) S_1 + \frac{(-3+8N)S_1^2}{9N} + \frac{1}{9} S_1^3 + \frac{7(-3+8N)S_2}{9N} + \frac{62}{9} S_3 + \left(\frac{4-3N}{6N} \right. \right. \right. \\
& \left. \left. \left. - \frac{2S_1}{3} \right) \zeta_2 - 4\zeta_3 \right) + C_A^2 \left(\frac{S_{2,1}P_{229}}{2N(1+N)^2(2+N)^2} + \frac{S_2P_{377}}{72N^3(1+N)^2(2+N)^2} \right. \right. \\
& \left. \left. + \frac{S_3P_{381}}{36(N-1)N^2(1+N)^2(2+N)^2} + \frac{P_{554}}{108(N-1)^2 N^4 (1+N)^3 (2+N)^2} \right. \right. \\
& \left. \left. + \left(\frac{S_2P_{346}}{24N^2(1+N)^2(2+N)^2} + \frac{P_{522}}{108(N-1)N^4(1+N)^3(2+N)^2} \right. \right. \right. \\
& \left. \left. \left. + \frac{(18+178N+321N^2+116N^3)S_3}{6N(1+N)(2+N)} + \frac{(-2+2N-3N^2-2N^3)S_{2,1}}{N(1+N)(2+N)} \right) S_1 \right. \right. \\
& \left. \left. + \left(\frac{P_{287}}{72N^3(1+N)^2(2+N)} + \frac{(6+184N+294N^2+101N^3)S_2}{8N(1+N)(2+N)} \right) S_1^2 \right. \right. \\
& \left. \left. + \frac{(-504-67N+101N^2)S_1^3}{72N(1+N)} + \frac{7}{6} S_1^4 + \frac{(-2+56N+78N^2+25N^3)S_2^2}{4N(1+N)(2+N)} \right. \right. \\
& \left. \left. + \frac{(14-40N-18N^2+N^3)S_4}{8N(1+N)(2+N)} + \left(\frac{-5+17N}{(N-1)^2 N(2+N)^2} + \frac{(7-3N)S_1}{(N-1)N(2+N)} \right) \right. \right. \\
& \times S_{-2} - \frac{2(-7+3N)S_{-3}}{(N-1)N(2+N)} + \frac{(-8+18N+3N^2-3N^3)S_{3,1}}{2N(1+N)(2+N)} \\
& \left. + \frac{(2+2N+9N^2+4N^3)S_{2,1,1}}{2N(1+N)(2+N)} + \left(\frac{-96+37N-5N^2+54N^3}{24N^2(1+N)} \right. \right. \\
& \left. \left. + \frac{(144+59N+11N^2)S_1}{24N(1+N)} - 2S_1^2 \right) \zeta_2 + \frac{9(4-6N+3N^2+3N^3)\zeta_2^2}{20N(1+N)(2+N)} \right. \\
& \left. + \left(\frac{P_{406}}{(N-1)N^2(1+N)^2(2+N)^2} - \frac{3(-2+12N+12N^2+3N^3)S_1}{2N(1+N)(2+N)} \right) \zeta_3 \right] \Big]
\end{aligned}$$

$$\begin{aligned}
& + C_{FNFTF} \left(-\frac{4S_3P_{119}}{3(N-1)N^2(1+N)^2(2+N)} - \frac{4\zeta_3P_{342}}{3(N-1)N^2(1+N)^2(2+N)} \right. \\
& + \frac{P_{600}}{36(N-1)^2N^5(1+N)^5(2+N)^2} + \left(-\frac{4P_{455}}{(N-1)N^4(1+N)^4(2+N)} \right. \\
& \left. - \frac{16(N-1)(2+N)S_2}{N^2(1+N)^2} \right) S_1 + \frac{4(2+N)(3+6N-4N^2+3N^3)S_1^2}{N^3(1+N)^3} \\
& - \frac{8(N-1)(2+N)S_1^3}{3N^2(1+N)^2} + \frac{8(2+N)(3+6N-4N^2+3N^3)S_2}{N^3(1+N)^3} \\
& + \left(-\frac{32(4-11N-6N^2+N^3)}{(N-1)^2N(1+N)^2(2+N)^2} + \frac{64S_1}{(N-1)N(1+N)(2+N)} \right) S_{-2} \\
& + \frac{128S_{-3}}{(N-1)N(1+N)(2+N)} + \left(\frac{P_{300}}{N^3(1+N)^3} + \frac{4(N-1)(2+N)S_1}{N^2(1+N)^2} \right) \zeta_2 \\
& - \frac{24}{5} \zeta_2^2 \left. \right) + C_{ANFTF} \left(\frac{8S_{2,1}P_{93}}{9N(1+N)(2+N)^2} - \frac{4S_3P_{270}}{9(N-1)N(1+N)(2+N)^2} \right. \\
& + \frac{S_2P_{289}}{9N^2(1+N)^2(2+N)^2} + \frac{P_{584}}{54(N-1)^2N^4(1+N)^4(2+N)^2} + \left(-\frac{16S_{2,1}}{2+N} \right. \\
& + \frac{S_2P_{19}}{3N(1+N)(2+N)^2} - \frac{2P_{517}}{27(N-1)N^3(1+N)^3(2+N)^2} - \frac{8(23+25N)S_3}{9(2+N)} \left. \right) S_1 \\
& + \left(\frac{P_{168}}{27N^2(1+N)^2(2+N)} - \frac{2(11+10N)S_2}{3(2+N)} \right) S_1^2 + \frac{(32-17N-17N^2)S_1^3}{9N(1+N)} \\
& - \frac{4}{9} S_1^4 - \frac{4(17+7N)S_2^2}{3(2+N)} - \frac{2(29+18N)S_4}{2+N} + \left(\frac{16(4-11N-6N^2+N^3)}{(N-1)^2N(1+N)^2(2+N)^2} \right. \\
& \left. - \frac{32S_1}{(N-1)N(1+N)(2+N)} \right) S_{-2} - \frac{64S_{-3}}{(N-1)N(1+N)(2+N)} + \frac{8(N-2)S_{3,1}}{2+N} \\
& + \frac{8(7+2N)S_{2,1,1}}{3(2+N)} + \left(\frac{P_8}{9N^2(1+N)^2} + \frac{(-24+19N+19N^2)S_1}{3N(1+N)} + 2S_1^2 + 6S_2 \right) \\
& \times \zeta_2 + \frac{12(8+N)\zeta_2^2}{5(2+N)} + \left(-\frac{2P_{280}}{27(N-1)N(1+N)(2+N)^2} + \frac{8(109+41N)S_1}{9(2+N)} \right) \zeta_3 \left. \right) \\
& + C_A^2 \left(\frac{S_3P_{426}}{9(N-1)N^2(1+N)^2(2+N)^2} + \frac{P_{594}}{648(N-1)^2N^5(1+N)^5(2+N)^2} \right. \\
& - \frac{2S_{2,1}P_{349}}{9N^2(1+N)^2(2+N)^2} + \left(\frac{S_2P_{370}}{12N^2(1+N)^2(2+N)^2} - 42S_2^2 - \frac{207}{2}S_4 - 2S_{2,1,1} \right. \\
& + \frac{P_{567}}{27(N-1)N^4(1+N)^4(2+N)^2} + \frac{(4572+2891N+1227N^2+496N^3)S_3}{9N(1+N)(2+N)} \\
& \left. + \frac{2(-4+7N+7N^2)S_{2,1}}{N(1+N)(2+N)} \right) S_1 + \left(\frac{(2760+1709N+564N^2+211N^3)S_2}{12N(1+N)(2+N)} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{P_{429}}{108N^3(1+N)^3(2+N)} - 59S_3 + 2S_{2,1} \Big) S_1^2 + \left(\frac{P_{157}}{36N^2(1+N)^2} - \frac{115S_2}{6} \right) S_1^3 \\
& + \frac{(2024 + 1539N + 968N^2 + 361N^3)S_4}{4N(1+N)(2+N)} + \frac{(1056 + 809N + 414N^2 + 133N^3)S_2^2}{6N(1+N)(2+N)} \\
& + \frac{(90 + 11N + 11N^2)S_1^4}{9N(1+N)} - S_1^5 + \left(\frac{P_{490}}{36N^3(1+N)^3(2+N)^2} - \frac{178}{3}S_3 + 4S_{2,1} \right. \\
& \left. + 8S_{-2,1} \right) S_2 - 46S_5 + \left(-\frac{4P_{318}}{(N-1)^2N(1+N)^3(2+N)^2} + \left(-16S_2 \right. \right. \\
& \left. \left. - \frac{8P_{55}}{(N-1)N^2(1+N)^2(2+N)} \right) S_1 - \frac{4(-7+N)S_1^2}{N(1+N)} - \frac{8}{3}S_1^3 - \frac{8(-7+N)S_2}{N(1+N)} \right. \\
& \left. - \frac{64}{3}S_3 - 8S_{2,1} \right) S_{-2} + \left(-\frac{2(-7+N)}{N(1+N)} - 4S_1 \right) S_{-2}^2 + \left(-\frac{16(-7+N)S_1}{N(1+N)} - 16S_1^2 \right. \\
& \left. - 20S_2 - 16S_{-2} - \frac{4P_{98}}{(N-1)N^2(1+N)^2(2+N)} \right) S_{-3} + \left(-\frac{14(-11+N)}{N(1+N)} \right. \\
& \left. - 56S_1 \right) S_{-4} - 54S_{-5} + 16S_{2,3} - 20S_{2,-3} + \frac{(-20 + 20N + 7N^2 - 9N^3)S_{3,1}}{N(1+N)(2+N)} \\
& - 13S_{4,1} - \frac{8(3 + 8N + N^2)S_{-2,1}}{N^2(1+N)^2} + \frac{4(-3+N)S_{-2,2}}{N(1+N)} + 12S_{-2,3} + 28S_{-4,1} \\
& + \frac{12(-3+N)S_{-3,1}}{N(1+N)} + \frac{(-60 - 85N - 87N^2 - 26N^3)S_{2,1,1}}{3N(1+N)(2+N)} + 8S_{2,1,-2} - 18S_{2,2,1} \\
& - 6S_{3,1,1} + \frac{8(-3+N)S_{-2,1,1}}{N(1+N)} + 8S_{-2,2,1} + 24S_{-3,1,1} - 3S_{2,1,1,1} + 16S_{-2,1,1,1} \\
& + \left(\frac{P_{372}}{36N^3(1+N)^3} + \left(\frac{P_5}{12N^2(1+N)^2} + 16S_2 \right) S_1 + \frac{(-48 - 11N - 11N^2)S_1^2}{2N(1+N)} \right. \\
& \left. + 4S_1^3 + \frac{(-64 - 33N - 33N^2)S_2}{2N(1+N)} + 2S_3 + \left(-\frac{8}{N(1+N)} + 4S_1 \right) S_{-2} + 2S_{-3} \right. \\
& \left. - 4S_{-2,1} \right) \zeta_2 + \left(\frac{3(24 + 8N + 33N^2 + 25N^3)}{10N(1+N)(2+N)} - \frac{18}{5}S_1 \right) \zeta_2^2 + \left(\frac{83}{3}S_1^2 - 9S_2 \right. \\
& \left. + \frac{P_{430}}{54(N-1)N^2(1+N)^2(2+N)^2} - \frac{(1776 + 2665N + 2382N^2 + 713N^3)S_1}{9N(1+N)(2+N)} \right. \\
& \left. - 12S_{-2} \right) \zeta_3 \Big) + N_F^2 T_F^2 \left(-\frac{1184}{81} + \frac{20\zeta_2}{9} + \frac{56\zeta_3}{27} \right). \tag{226}
\end{aligned}$$

7 The Transversity OMEs

In the case of transversity various OMEs contribute [82,83]. Here we only consider the expansion coefficients contributing to the physical projection $A_{qq}^{\text{tr},\pm}$. The coefficients corresponding to the

non-negative powers in ε are given by

$$a_{qq}^{(1,0),\text{tr},+} = C_F \xi S_1 + C_F (7 - 2S_1^2 - 6S_2), \quad (227)$$

$$a_{qq}^{(1,1),\text{tr},+} = C_F \xi \left[\frac{-1 + 2N}{2N} - \frac{1}{2} S_1 - \frac{1}{4} S_1^2 - \frac{3}{4} S_2 \right] + C_F \left(\frac{-1 - 6N - 7N^2}{N(1+N)} + \frac{1}{3} S_1^3 + 3S_1 S_2 \right. \\ \left. + \frac{14}{3} S_3 + \left(\frac{3}{4} - S_1 \right) \zeta_2 \right), \quad (228)$$

$$a_{qq}^{(1,2),\text{tr},+} = C_F \xi \left[-\frac{(-1 + 2N)(1 + 2N)}{4N^2} + \left(\frac{1}{4N} + \frac{3S_2}{8} \right) S_1 + \frac{1}{8} S_1^2 + \frac{1}{24} S_1^3 + \frac{3}{8} S_2 + \frac{7}{12} S_3 \right. \\ \left. - \frac{1}{8} S_1 \zeta_2 \right] + C_F \left(\frac{P_{121}}{2N^2(1+N)^2} + \left(\frac{1-N}{2N(1+N)} - \frac{7S_3}{3} \right) S_1 - \frac{1}{24} S_1^4 - \frac{3}{4} S_1^2 S_2 \right. \\ \left. - \frac{9}{8} S_2^2 - \frac{15}{4} S_4 + \left(-\frac{7}{8} + \frac{S_1^2}{4} + \frac{3S_2}{4} \right) \zeta_2 + \left(-\frac{7}{4} + \frac{7S_1}{3} \right) \zeta_3 \right), \quad (229)$$

$$a_{qq}^{(2,0),\text{tr},+} = \xi^2 \left[C_A C_F \left(\frac{1}{2} - \frac{3}{4} S_1 - \frac{1}{4} S_1^2 - \frac{1}{4} S_2 \right) + C_F^2 \left(-1 + \frac{S_1^2}{2} + \frac{S_2}{2} \right) \right] + \xi \left[C_F^2 \left(\frac{3-7N}{N} \right. \right. \\ \left. \left. + \left(\frac{2(-2+11N)}{N} - 16S_2 \right) S_1 - \frac{5}{2} S_1^2 - 4S_1^3 + \frac{13}{2} S_2 \right) + C_A C_F \left(-\frac{7}{2} \right. \right. \\ \left. \left. + \left(\frac{3(3+2N)}{2+N} + \frac{(1-N)S_2}{2+N} \right) S_1 + \frac{5}{4} S_1^2 + \frac{(14+13N)S_2}{4(2+N)} + \frac{(1-N)S_3}{2+N} \right. \right. \\ \left. \left. + \frac{2(N-1)S_{2,1}}{2+N} - \frac{6(N-1)\zeta_3}{2+N} \right) \right] + C_F \left(N_F T_F \left(\frac{24 - 281N - 281N^2}{18N(1+N)} + \left(\frac{164}{27} \right. \right. \right. \\ \left. \left. + \frac{8S_2}{3} \right) S_1 + \frac{34}{9} S_1^2 + \frac{8}{9} S_1^3 + \frac{74}{9} S_2 - \frac{8}{9} S_3 \right) + C_A \left(\frac{-264 + 3625N + 3625N^2}{72N(1+N)} \right. \\ \left. + \left(\frac{108 - 1538N - 715N^2}{27N(2+N)} - \frac{2(19+14N)S_2}{3(2+N)} - 6S_3 - 4S_{2,1} \right) S_1 - 32S_{-3} S_1 \right. \\ \left. + \left(-\frac{253}{18} + S_2 \right) S_1^2 - \frac{22}{9} S_1^3 + \frac{(-1078 - 485N)S_2}{18(2+N)} - 6S_2^2 + \frac{2(31+2N)S_3}{9(2+N)} \right. \\ \left. - 29S_4 + \left(\frac{12}{N(1+N)} - 8S_1^2 - 16S_2 \right) S_{-2} - 4S_{-2}^2 - 28S_{-4} + \frac{4(N-1)S_{2,1}}{2+N} \right. \\ \left. + 6S_{3,1} + 8S_{-2,2} + 24S_{-3,1} + 6S_{2,1,1} + 16S_{-2,1,1} + \left(-\frac{12(N-1)}{2+N} + 12S_1 \right) \zeta_3 \right) \\ \left. + C_F^2 \left(\frac{48 + 493N + 541N^2}{8N(1+N)} + \left(-\frac{2(10 + 51N + 77N^2 + 28N^3)}{N(1+N)(2+N)} + \frac{256}{3} S_3 \right. \right. \right. \\ \left. \left. - \frac{2(26 + 11N)S_2}{2+N} \right) S_1 + 64S_{-3} S_1 + \left(-14 + 48S_2 \right) S_1^2 - 2S_1^3 + \frac{14}{3} S_1^4 \right)$$

$$\begin{aligned}
& -\frac{2(54+29N)S_2}{2+N} + 30S_2^2 - \frac{2(54+25N)S_3}{2+N} + 48S_4 + \left(-\frac{24}{N(1+N)} + 16S_1^2 \right. \\
& \left. + 32S_2 \right) S_{-2} + 8S_{-2}^2 + 56S_{-4} - \frac{4(-2+N)S_{2,1}}{2+N} - 16S_{-2,2} - 48S_{-3,1} - 32S_{-2,1,1} \\
& + \left(-\frac{9}{2} + 12S_1 - 8S_1^2 \right) \zeta_2 - \frac{48\zeta_3}{2+N}, \tag{230}
\end{aligned}$$

$$\begin{aligned}
a_{qq}^{(2,1),\text{tr}+} = & \xi^2 \left[C_F^2 \left(\frac{1+3N}{2N} + \left(\frac{-1+3N}{2N} - \frac{5S_2}{4} \right) S_1 - \frac{1}{2}S_1^2 - \frac{1}{4}S_1^3 - \frac{1}{2}S_2 - S_3 \right) \right. \\
& + C_A C_F \left(\frac{1-13N}{8N} + \left(\frac{2+5N}{8N} + \frac{5S_2}{8} \right) S_1 + \frac{1}{2}S_1^2 + \frac{1}{8}S_1^3 + \frac{7}{8}S_2 + \frac{1}{2}S_3 \right. \\
& \left. \left. - \frac{1}{8}S_1\zeta_2 \right) \right] + \xi \left(C_A C_F \left(\frac{-2-3N+42N^2}{4N^2} + \left(\frac{-4-72N-83N^2-21N^3}{4N(2+N)^2} \right. \right. \right. \\
& \left. \left. + \frac{(-132-244N-149N^2-33N^3)S_2}{8(1+N)(2+N)^2} + \frac{3(N-1)S_3}{2+N} - \frac{2(N-1)S_{2,1}}{2+N} \right) S_1 \right. \\
& \left. + \left(\frac{-44-31N}{8(2+N)} + \frac{3(N-1)S_2}{4(2+N)} \right) S_1^2 - \frac{5}{8}S_1^3 + \frac{(-160-218N-63N^2)S_2}{8(2+N)^2} \right. \\
& \left. + \frac{(-1-2N)S_2^2}{2(2+N)} + \frac{(-34-72N-52N^2-13N^3)S_3}{2(1+N)(2+N)^2} + \frac{(-11+5N)S_4}{4(2+N)} \right. \\
& \left. + \frac{(6+4N-N^2)S_{2,1}}{(1+N)(2+N)^2} - \frac{3(-2+N)S_{3,1}}{2+N} + \frac{(N-1)S_{2,1,1}}{2+N} + \frac{5}{8}S_1\zeta_2 + \frac{9(N-1)\zeta_2^2}{5(2+N)} \right. \\
& \left. + \left(\frac{3(N-1)(12+14N+3N^2)}{2(1+N)(2+N)^2} + \frac{3(1+2N)S_1}{2+N} \right) \zeta_3 \right) \\
& + C_F^2 \left(\frac{(1+3N)(-5+8N)}{2N^2} + \left(\frac{8-N-36N^2-31N^3}{2N^2(1+N)} + \frac{19}{4}S_2 + \frac{52}{3}S_3 \right) S_1 \right. \\
& \left. + \left(-\frac{3(-2+5N)}{2N} + 12S_2 \right) S_1^2 + \frac{7}{4}S_1^3 + \frac{7}{6}S_1^4 + \frac{(6-41N)S_2}{2N} + \frac{15}{2}S_2^2 - \frac{15}{2}S_3 \right. \\
& \left. + \left(\frac{3S_1}{2} - 2S_1^2 \right) \zeta_2 + \left(9 - 12S_1 \right) \zeta_3 \right) + C_F \left(C_A \left(\frac{S_3 P_{159}}{27N(1+N)(2+N)^2} \right. \right. \\
& \left. \left. + \frac{P_1}{864N^2(1+N)^2} + \left(\frac{(4564+8948N+5525N^2+1069N^3)S_2}{36(1+N)(2+N)^2} \right. \right. \right. \\
& \left. \left. + \frac{P_{284}}{162N^2(1+N)(2+N)^2} + 5S_2^2 + \frac{2(29+28N)S_3}{3(2+N)} + \frac{49}{2}S_4 + \frac{12S_{2,1}}{2+N} - 2S_{2,1,1} \right) S_1 \right. \\
& \left. + 4S_{-2}^2 S_1 + 56S_{-4} S_1 + \left(\frac{-324+3094N+1385N^2}{108N(2+N)} + S_3 + \frac{(89+58N)S_2}{6(2+N)} \right. \right. \\
& \left. \left. + 2S_{2,1} \right) S_1^2 + \left(\frac{491}{108} - \frac{S_2}{2} \right) S_1^3 + \frac{11}{12}S_1^4 + \left(18S_3 + 4S_{2,1} - 8S_{-2,1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{-1080 + 12232N + 11260N^2 + 2599N^3}{108N(2+N)^2} \Big) S_2 + \frac{(166 + 65N)S_2^2}{12(2+N)} \\
& + \frac{7(11 + 10N)S_4}{6(2+N)} + 62S_5 + \left(-\frac{12(1+2N)}{N^2(1+N)^2} + \left(-\frac{12}{N(1+N)} + 16S_2 \right) S_1 + \frac{8}{3}S_1^3 \right. \\
& + \left. \frac{64}{3}S_3 + 8S_{2,1} \right) S_{-2} + \left(-\frac{24}{N(1+N)} + 16S_1^2 + 20S_2 + 16S_{-2} \right) S_{-3} + 54S_{-5} \\
& - \frac{2(14 + 26N + 17N^2 + 3N^3)S_{2,1}}{(1+N)(2+N)^2} + 16S_{2,3} + 20S_{2,-3} - \frac{2(-10 + N)S_{3,1}}{2+N} \\
& - 13S_{4,1} - 12S_{-2,3} - 28S_{-4,1} - \frac{2(7 + 2N)S_{2,1,1}}{2+N} - 8S_{2,1,-2} - 18S_{2,2,1} - 6S_{3,1,1} \\
& - 8S_{-2,2,1} - 24S_{-3,1,1} - 3S_{2,1,1,1} - 16S_{-2,1,1,1} + \left(\frac{57}{8} - \frac{253}{36}S_1 - 4S_{-2}S_1 - \frac{11}{6}S_1^2 \right. \\
& - \left. \frac{11}{6}S_2 - 2S_3 - 2S_{-3} + 4S_{-2,1} \right) \zeta_2 + \left(\frac{18(N-1)}{5(2+N)} - \frac{18S_1}{5} \right) \zeta_2^2 \\
& + \left(\frac{3P_{125}}{2N(1+N)(2+N)^2} - \frac{2(25 + 8N)S_1}{2+N} - 3S_1^2 + 3S_2 + 12S_{-2} \right) \zeta_3 \Big) \\
& + N_F T_F \left(\frac{P_{161}}{216N^2(1+N)^2} + \left(-\frac{2(-27 + 253N + 226N^2)}{81N(1+N)} - \frac{65}{9}S_2 - \frac{16}{3}S_3 \right) S_1 \right. \\
& + \left(-\frac{73}{27} - \frac{10S_2}{3} \right) S_1^2 - \frac{31}{27}S_1^3 - \frac{1}{3}S_1^4 - \frac{155}{27}S_2 - \frac{7}{3}S_2^2 - \frac{224}{27}S_3 - \frac{10}{3}S_4 + \left(-\frac{5}{2} \right. \\
& + \left. \frac{17}{9}S_1 + \frac{2}{3}S_1^2 + \frac{2}{3}S_2 \right) \zeta_2 + \left(-6 + 8S_1 \right) \zeta_3 \Big) + C_F^2 \left(\frac{P_2}{32N^2(1+N)^2} \right. \\
& + \left. \frac{2S_3P_{149}}{3N(1+N)(2+N)^2} + \left(\frac{(112 + 214N + 131N^2 + 27N^3)S_2}{(1+N)(2+N)^2} \right. \right. \\
& + \left. \frac{P_{354}}{N^2(1+N)^2(2+N)^2} - 47S_2^2 + \frac{2(34 + 11N)S_3}{2+N} - 128S_4 - \frac{16S_{2,1}}{2+N} \right) S_1 \\
& - 8S_{-2}^2S_1 - 112S_{-4}S_1 + \left(\frac{30 + 41N + 63N^2 + 28N^3}{2N(1+N)(2+N)} + \frac{(34 + 11N)S_2}{2(2+N)} - 60S_3 \right) \\
& \times S_1^2 + \left(\frac{7}{3} - \frac{56S_2}{3} \right) S_1^3 + \frac{1}{4}S_1^4 - S_1^5 + \left(\frac{P_{152}}{2N(1+N)(2+N)^2} - \frac{232}{3}S_3 \right. \\
& + \left. 16S_{-2,1} \right) S_2 + \frac{(78 + 47N)S_2^2}{4(2+N)} + \frac{(250 + 111N)S_4}{2(2+N)} - 108S_5 + \left(\frac{24(1+2N)}{N^2(1+N)^2} \right. \\
& + \left(\frac{24}{N(1+N)} - 32S_2 \right) S_1 - \frac{16}{3}S_1^3 - \frac{128}{3}S_3 - 16S_{2,1} \Big) S_{-2} + \left(\frac{48}{N(1+N)} \right. \\
& - \left. 32S_1^2 - 40S_2 - 32S_{-2} \right) S_{-3} - 108S_{-5} + \frac{4(-2 + 3N + 5N^2 + N^3)S_{2,1}}{(1+N)(2+N)^2}
\end{aligned}$$

$$\begin{aligned}
& -40S_{2,-3} + \frac{2(-10+3N)S_{3,1}}{2+N} + 24S_{-2,3} + 56S_{-4,1} + \frac{4(4+N)S_{2,1,1}}{2+N} + 16S_{2,1,-2} \\
& + 16S_{-2,2,1} + 48S_{-3,1,1} + 32S_{-2,1,1,1} + \left(\frac{87}{8} + (-14+16S_2)S_1 + 8S_{-2}S_1 - 3S_1^2 \right. \\
& \left. + 4S_1^3 - 12S_2 + 4S_3 + 4S_{-3} - 8S_{-2,1} \right) \zeta_2 + \frac{72\zeta_2^2}{5(2+N)} + \left(-\frac{3P_{86}}{N(1+N)(2+N)^2} \right. \\
& \left. - \frac{8(7+5N)S_1}{2+N} + \frac{92}{3}S_1^2 - 12S_2 - 24S_{-2} \right) \zeta_3, \tag{231}
\end{aligned}$$

$$a_{qq}^{(1,0),\text{tr},-} = C_F \xi S_1 + C_F \left[7 - 2S_1^2 - 6S_2 \right], \tag{232}$$

$$\begin{aligned}
a_{qq}^{(1,1),\text{tr},-} &= C_F \xi \left[\frac{-1+2N}{2N} - \frac{1}{2}S_1 - \frac{1}{4}S_1^2 - \frac{3}{4}S_2 \right] + C_F \left(\frac{-1-6N-7N^2}{N(1+N)} + \frac{1}{3}S_1^3 + 3S_1S_2 \right. \\
& \left. + \frac{14}{3}S_3 + \left(\frac{3}{4} - S_1 \right) \zeta_2 \right), \tag{233}
\end{aligned}$$

$$\begin{aligned}
a_{qq}^{(1,2),\text{tr},-} &= C_F \xi \left[-\frac{(-1+2N)(1+2N)}{4N^2} + \left(\frac{1}{4N} + \frac{3S_2}{8} \right) S_1 + \frac{1}{8}S_1^2 + \frac{1}{24}S_1^3 + \frac{3}{8}S_2 + \frac{7}{12}S_3 \right. \\
& \left. - \frac{1}{8}S_1\zeta_2 \right] + C_F \left(\frac{P_{121}}{2N^2(1+N)^2} + \left(\frac{1-N}{2N(1+N)} - \frac{7S_3}{3} \right) S_1 - \frac{1}{24}S_1^4 - \frac{3}{4}S_1^2S_2 \right. \\
& \left. - \frac{9}{8}S_2^2 - \frac{15}{4}S_4 + \left(-\frac{7}{8} + \frac{S_1^2}{4} + \frac{3S_2}{4} \right) \zeta_2 + \left(-\frac{7}{4} + \frac{7S_1}{3} \right) \zeta_3 \right), \tag{234}
\end{aligned}$$

$$\begin{aligned}
a_{qq}^{(2,0),\text{tr},-} &= \xi^2 \left[C_A C_F \left(\frac{1}{2} - \frac{3}{4}S_1 - \frac{1}{4}S_1^2 - \frac{1}{4}S_2 \right) + C_F^2 \left(-1 + \frac{S_1^2}{2} + \frac{S_2}{2} \right) \right] + \xi \left[C_F^2 \left(\frac{3-7N}{N} \right. \right. \\
& \left. \left. + \left(\frac{2(-2+11N)}{N} - 16S_2 \right) S_1 - \frac{5}{2}S_1^2 - 4S_1^3 + \frac{13}{2}S_2 \right) + C_A C_F \left(-\frac{7}{2} \right. \right. \\
& \left. \left. + \left(\frac{3(3+2N)}{2+N} + \frac{(1-N)S_2}{2+N} \right) S_1 + \frac{5}{4}S_1^2 + \frac{(14+13N)S_2}{4(2+N)} + \frac{(1-N)S_3}{2+N} \right. \right. \\
& \left. \left. + \frac{2(N-1)S_{2,1}}{2+N} - \frac{6(N-1)\zeta_3}{2+N} \right) \right] + C_F \left(N_F T_F \left(\frac{24-281N-281N^2}{18N(1+N)} + \left(\frac{164}{27} \right. \right. \right. \\
& \left. \left. + \frac{8S_2}{3} \right) S_1 + \frac{34}{9}S_1^2 + \frac{8}{9}S_1^3 + \frac{74}{9}S_2 - \frac{8}{9}S_3 \right) + C_A \left(\frac{P_{283}}{72(N-1)N(1+N)^2(2+N)} \right. \\
& \left. + \left(\frac{-108-1538N-2253N^2-715N^3}{27N(1+N)(2+N)} - \frac{2(19+14N)S_2}{3(2+N)} - 6S_3 - 4S_{2,1} \right) S_1 \right. \\
& \left. - 32S_{-3}S_1 + \left(-\frac{253}{18} + S_2 \right) S_1^2 - \frac{22}{9}S_1^3 + \frac{(-1078-485N)S_2}{18(2+N)} - 6S_2^2 \right. \\
& \left. + \frac{2(31+2N)S_3}{9(2+N)} - 29S_4 + \left(-\frac{4}{(N-1)(2+N)} - 8S_1^2 - 16S_2 \right) S_{-2} - 4S_{-2}^2 \right)
\end{aligned}$$

$$\begin{aligned}
& -28S_{-4} + \frac{4(N-1)S_{2,1}}{2+N} + 6S_{3,1} + 8S_{-2,2} + 24S_{-3,1} + 6S_{2,1,1} + 16S_{-2,1,1} \\
& + \left(-\frac{12(N-1)}{2+N} + 12S_1 \right) \zeta_3 \Big) + C_F^2 \left(\frac{P_{282}}{8(N-1)N(1+N)^2(2+N)} \right. \\
& + \left(-\frac{2(2+47N+77N^2+28N^3)}{N(1+N)(2+N)} - \frac{2(26+11N)S_2}{2+N} + \frac{256}{3}S_3 \right) S_1 + 64S_{-3}S_1 \\
& + \left(-14 + 48S_2 \right) S_1^2 - 2S_1^3 + \frac{14}{3}S_1^4 - \frac{2(54+29N)S_2}{2+N} + 30S_2^2 - \frac{2(54+25N)S_3}{2+N} \\
& + 48S_4 + \left(\frac{8}{(N-1)(2+N)} + 16S_1^2 + 32S_2 \right) S_{-2} + 8S_{-2}^2 + 56S_{-4} - \frac{4(N-2)S_{2,1}}{2+N} \\
& - 16S_{-2,2} - 48S_{-3,1} - 32S_{-2,1,1} + \left(-\frac{9}{2} + 12S_1 - 8S_1^2 \right) \zeta_2 - \frac{48\zeta_3}{2+N} \Big), \quad (235)
\end{aligned}$$

$$\begin{aligned}
a_{qq}^{(2,1),\text{tr},-} = & \xi^2 \left[C_F^2 \left(\frac{1+3N}{2N} + \left(\frac{-1+3N}{2N} - \frac{5S_2}{4} \right) S_1 - \frac{1}{2}S_1^2 - \frac{1}{4}S_1^3 - \frac{1}{2}S_2 - S_3 \right) \right. \\
& + C_A C_F \left(\frac{1-13N}{8N} + \left(\frac{2+5N}{8N} + \frac{5S_2}{8} \right) S_1 + \frac{1}{2}S_1^2 + \frac{1}{8}S_1^3 + \frac{7}{8}S_2 + \frac{1}{2}S_3 \right. \\
& \left. \left. - \frac{1}{8}S_1\zeta_2 \right) \right] + \xi \left[C_A C_F \left(\frac{-2-3N+42N^2}{4N^2} + \left(\frac{-4-72N-83N^2-21N^3}{4N(2+N)^2} \right. \right. \right. \\
& + \left. \left. \frac{(-132-244N-149N^2-33N^3)S_2}{8(1+N)(2+N)^2} + \frac{3(N-1)S_3}{2+N} - \frac{2(N-1)S_{2,1}}{2+N} \right) S_1 \right. \\
& + \left(\frac{-44-31N}{8(2+N)} + \frac{3(N-1)S_2}{4(2+N)} \right) S_1^2 - \frac{5}{8}S_1^3 + \frac{(-160-218N-63N^2)S_2}{8(2+N)^2} \\
& + \frac{(-1-2N)S_2^2}{2(2+N)} + \frac{(-34-72N-52N^2-13N^3)S_3}{2(1+N)(2+N)^2} + \frac{(-11+5N)S_4}{4(2+N)} \\
& + \frac{(6+4N-N^2)S_{2,1}}{(1+N)(2+N)^2} - \frac{3(-2+N)S_{3,1}}{2+N} + \frac{(N-1)S_{2,1,1}}{2+N} + \frac{5}{8}S_1\zeta_2 + \frac{9(N-1)\zeta_2^2}{5(2+N)} \\
& + \left(\frac{3(N-1)(12+14N+3N^2)}{2(1+N)(2+N)^2} + \frac{3(1+2N)S_1}{2+N} \right) \zeta_3 \Big) \\
& + C_F^2 \left(\frac{(1+3N)(-5+8N)}{2N^2} + \left(\frac{8-N-36N^2-31N^3}{2N^2(1+N)} + \frac{19}{4}S_2 + \frac{52}{3}S_3 \right) S_1 \right. \\
& + \left(-\frac{3(-2+5N)}{2N} + 12S_2 \right) S_1^2 + \frac{7}{4}S_1^3 + \frac{7}{6}S_1^4 + \frac{(6-41N)S_2}{2N} + \frac{15}{2}S_2^2 - \frac{15}{2}S_3 \\
& + \left(\frac{3S_1}{2} - 2S_1^2 \right) \zeta_2 + \left(9 - 12S_1 \right) \zeta_3 \Big) \Big] + C_F \left(C_A \left(\frac{S_3 P_{158}}{27(N-1)(1+N)(2+N)^2} \right. \right. \\
& \left. \left. + \frac{P_{519}}{864(N-1)^2 N^3 (1+N)^3 (2+N)^2} + \left(\frac{P_{431}}{162(N-1)N^2(1+N)^2(2+N)^2} \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{(4564 + 8948N + 5525N^2 + 1069N^3)S_2}{36(1+N)(2+N)^2} + 5S_2^2 + \frac{2(29 + 28N)S_3}{3(2+N)} + \frac{49}{2}S_4 \\
& + \frac{12S_{2,1}}{2+N} - 2S_{2,1,1} \Big) S_1 + 4S_{-2}^2 S_1 + 56S_{-4}S_1 + \left(S_3 + \frac{(89 + 58N)S_2}{6(2+N)} + 2S_{2,1} \right. \\
& + \frac{108 + 2986N + 4479N^2 + 1385N^3}{108N(1+N)(2+N)} \Big) S_1^2 + \left(\frac{491}{108} - \frac{S_2}{2} \right) S_1^3 + \frac{11}{12}S_1^4 \\
& + \left(\frac{P_{160}}{108N(1+N)(2+N)^2} + 18S_3 + 4S_{2,1} - 8S_{-2,1} \right) S_2 + \frac{(166 + 65N)S_2^2}{12(2+N)} \\
& + \frac{7(11 + 10N)S_4}{6(2+N)} + 62S_5 + \left(-\frac{4(-7 - 2N + 2N^2 + N^3)}{(N-1)^2(1+N)(2+N)^2} + \left(\frac{4}{(N-1)(2+N)} \right. \right. \\
& + 16S_2 \Big) S_1 + \frac{8}{3}S_1^3 + \frac{64}{3}S_3 + 8S_{2,1} \Big) S_{-2} + \left(\frac{8}{(N-1)(2+N)} + 16S_1^2 + 20S_2 \right. \\
& + 16S_{-2} \Big) S_{-3} + 54S_{-5} - \frac{2(14 + 26N + 17N^2 + 3N^3)S_{2,1}}{(1+N)(2+N)^2} + 16S_{2,3} + 20S_{2,-3} \\
& - \frac{2(-10 + N)S_{3,1}}{2+N} - 13S_{4,1} - 12S_{-2,3} - 28S_{-4,1} - \frac{2(7 + 2N)S_{2,1,1}}{2+N} - 8S_{2,1,-2} \\
& - 18S_{2,2,1} - 6S_{3,1,1} - 8S_{-2,2,1} - 24S_{-3,1,1} - 3S_{2,1,1,1} - 16S_{-2,1,1,1} \\
& + \left(\frac{-8 + 57N + 57N^2}{8N(1+N)} - \frac{253}{36}S_1 - 4S_{-2}S_1 - \frac{11}{6}S_1^2 - \frac{11}{6}S_2 - 2S_3 - 2S_{-3} \right. \\
& + 4S_{-2,1} \Big) \zeta_2 + \left(\frac{18(N-1)}{5(2+N)} - \frac{18S_1}{5} \right) \zeta_2^2 + \left(-3S_1^2 + 3S_2 + 12S_{-2} \right. \\
& + \frac{3(3+N)(-12 - 8N + 11N^2 + 15N^3)}{2(N-1)(1+N)(2+N)^2} - \frac{2(25 + 8N)S_1}{2+N} \Big) \zeta_3 \Big) \\
& + N_F T_F \left(\frac{P_{161}}{216N^2(1+N)^2} + \left(-\frac{2(-27 + 253N + 226N^2)}{81N(1+N)} - \frac{65}{9}S_2 - \frac{16}{3}S_3 \right) S_1 \right. \\
& + \left(-\frac{73}{27} - \frac{10S_2}{3} \right) S_1^2 - \frac{31}{27}S_1^3 - \frac{1}{3}S_1^4 - \frac{155}{27}S_2 - \frac{7}{3}S_2^2 - \frac{224}{27}S_3 - \frac{10}{3}S_4 + \left(-\frac{5}{2} \right. \\
& + \frac{17}{9}S_1 + \frac{2}{3}S_1^2 + \frac{2}{3}S_2 \Big) \zeta_2 + \left(-6 + 8S_1 \right) \zeta_3 \Big) + C_F^2 \left(\frac{2S_3 P_{148}}{3(N-1)(1+N)(2+N)^2} \right. \\
& + \frac{P_{521}}{32(N-1)^2 N^3 (1+N)^3 (2+N)^2} + \left(\frac{(112 + 214N + 131N^2 + 27N^3)S_2}{(1+N)(2+N)^2} \right. \\
& + \frac{P_{420}}{(N-1)N^2(1+N)^2(2+N)^2} - 47S_2^2 + \frac{2(34 + 11N)S_3}{2+N} - 128S_4 - \frac{16S_{2,1}}{2+N} \Big) S_1 \\
& - 8S_{-2}^2 S_1 - 112S_{-4}S_1 + \left(\frac{14 + 33N + 63N^2 + 28N^3}{2N(1+N)(2+N)} + \frac{(34 + 11N)S_2}{2(2+N)} \right.
\end{aligned}$$

$$\begin{aligned}
& -60S_3 \Big) S_1^2 + \left(\frac{7}{3} - \frac{56S_2}{3} \right) S_1^3 + \frac{1}{4} S_1^4 - S_1^5 + \left(\frac{P_{151}}{2N(1+N)(2+N)^2} - \frac{232}{3} S_3 \right. \\
& + 16S_{-2,1} \Big) S_2 + \frac{(78+47N)S_2^2}{4(2+N)} + \frac{(250+111N)S_4}{2(2+N)} - 108S_5 \\
& + \left(\frac{8(-7-2N+2N^2+N^3)}{(N-1)^2(1+N)(2+N)^2} + \left(-\frac{8}{(N-1)(2+N)} - 32S_2 \right) S_1 - \frac{16}{3} S_1^3 \right. \\
& - \frac{128}{3} S_3 - 16S_{2,1} \Big) S_{-2} + \left(-\frac{16}{(N-1)(2+N)} - 32S_1^2 - 40S_2 - 32S_{-2} \right) S_{-3} \\
& - 108S_{-5} + \frac{4(-2+3N+5N^2+N^3)S_{2,1}}{(1+N)(2+N)^2} - 40S_{2,-3} + \frac{2(-10+3N)S_{3,1}}{2+N} \\
& + 24S_{-2,3} + 56S_{-4,1} + \frac{4(4+N)S_{2,1,1}}{2+N} + 16S_{2,1,-2} + 16S_{-2,2,1} + 48S_{-3,1,1} \\
& + 32S_{-2,1,1,1} + \left(\frac{16+87N+87N^2}{8N(1+N)} + \left(-14+16S_2 \right) S_1 + 8S_{-2}S_1 - 3S_1^2 + 4S_1^3 \right. \\
& - 12S_2 + 4S_3 + 4S_{-3} - 8S_{-2,1} \Big) \zeta_2 + \frac{72\zeta_2^2}{5(2+N)} + \left(\frac{92}{3} S_1^2 - 12S_2 - 24S_{-2} \right. \\
& \left. - \frac{3P_{85}}{(N-1)(1+N)(2+N)^2} - \frac{8(7+5N)S_1}{2+N} \right) \zeta_3. \tag{236}
\end{aligned}$$

In summary, the expressions shown in Sections 3–6 depend on the following 28 harmonic sums after algebraic reduction [118],

$$\left\{ S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_1, S_2, S_3, S_4, S_5, S_{-4,1}, S_{-3,1}, S_{-2,1}, S_{-2,2}, S_{-2,3}, S_{2,-3}, S_{2,1}, S_{2,3}, S_{3,1}, S_{4,1}, S_{-3,1,1}, S_{-2,1,1}, S_{-2,2,1}, S_{2,1,-2}, S_{2,1,1}, S_{2,2,1}, S_{3,1,1}, S_{-2,1,1,1}, S_{2,1,1,1} \right\}. \tag{237}$$

After applying also the structural relations [121] the following 13 sum contribute,

$$\{ S_1, S_{-2,1}, S_{2,1}, S_{-3,1}, S_{4,1}, S_{-4,1}, S_{-2,1,1}, S_{2,1,1}, S_{-3,1,1}, S_{-2,2,1}, S_{2,2,1}, S_{-2,1,1,1}, S_{2,1,1,1} \}. \tag{238}$$

8 Comparison to the literature

In Refs. [19, 20], extending earlier results in Refs. [15, 79], unpolarized and polarized OMEs have been calculated to $O(a_s^2 \varepsilon^0)$, i.e. one order in ε less than in the present calculation and the OMEs of transversity were not considered. The calculation has been carried out in z -space. This has been slightly before harmonic sums [115, 116] and harmonic polylogarithms [122] became the standard entities to represent different calculation steps and the final results in single scale calculations. Therefore classical polylogarithms and Nielsen integrals [123–128], partly with involved argument, were used. In [19, 20] the expansion coefficients of the completely unrenormalized OMEs are discussed, while we perform the renormalization of the strong coupling and the gauge parameter first. Furthermore, the gauge parameter there is $\tilde{\xi} = 1 - \xi$, compared to the present case. In the following we summarize a series of differences to the present results which we have observed, both due to typographical and combinatoric errors. We will mention main observations only and do not intend to give a complete list.

Comparing to [19, 20] one first observes that the concrete results for the non-singlet OMEs differ by a global minus sign. For all the other OMEs we find a relative minus sign at $O(\alpha_s)$, but agree at $O(\alpha_s^2)$. The formal representations in terms of anomalous dimensions, i.e. Eqs. (2.8, 2.16, 2.31, 2.34) of Ref. [19] and Eqs. (2.18, 2.24, 2.26, 2.29) of Ref. [20], are, however, correct. Furthermore, the expressions for $\Delta\hat{A}_{iq}^{\text{phys}}$ there are obtained as the sum $\Delta\hat{A}_{iq}^{\text{phys}} + \Delta\hat{A}_{iq}^{\text{com}}$ of the present results in the Larin scheme.

After adjusting the signs we find also some specific differences at the one-loop level. Let us define

$$\delta A_{ij}^{(k)}(x) = A_{ij}^{(k),\text{this calc.}}(x) - A_{ij}^{(k),[19]}(x). \quad (239)$$

$$\delta\Delta A_{ij}^{(k)}(x) = \Delta A_{ij}^{(k),\text{this calc.}}(x) - \Delta A_{ij}^{(k),[20]}(x). \quad (240)$$

We obtain the following differences

$$\delta A_{gg}^{(1)}(z) = -20C_A \frac{H_0 - H_1}{1-z} + 20\delta(1-z) - \frac{1}{12}C_A \varepsilon (3\xi\zeta_2 + 8\zeta_3)\delta(1-z) \quad (241)$$

$$\delta\Delta A_{gg}^{(1)}(z) = -\frac{1}{12}C_A \varepsilon (3\xi\zeta_2 + 8\zeta_3)\delta(1-z), \quad (242)$$

where the harmonic polylogarithms [122] are defined by $H_{\vec{a}} \equiv H_{\vec{a}}(z)$. Note that the $O(\varepsilon^0)$ contribution in (241) is not in agreement with the earlier calculation [15, 79], where the corresponding expansion has been given to $O(\varepsilon^0)$. It is therefore rather difficult to use these older results in current calculations.

9 Conclusions

We have calculated the unpolarized and polarized massless off-shell twist-2 operator matrix elements in QCD to two-loop orders in an automated way. These quantities are of interest because they allow the direct calculation of the QCD anomalous dimensions. Contrary to this, on shell massive operator matrix elements only allow to compute the anomalous dimensions of contributions $\propto T_F$ in the same order in perturbation theory and require the corrections of one more order in the coupling constant to obtain the complete anomalous dimensions. The off-shellness implies gauge variant expressions in general. In particular, the equation of motion is not valid anymore for these terms and non-gauge invariant contributions are present, which may mix with the gauge invariant contributions. Despite of these technical complications, the method allows for a direct calculations of the anomalous dimensions. The on-shell calculation of the forward Compton amplitude also allows to compute the anomalous dimensions contributing to the deep-inelastic structure functions. However, special arrangements are necessary for the gluonic contributions, which require auxiliary Higgs- and graviton fields, also complicating this method.

We compared our results to those in Refs. [19, 20] for the contributions to $O(a_s^2\varepsilon^0)$ and calculated newly the contributions of $O(a_s\varepsilon^2)$ and $O(a_s^2\varepsilon)$, which contribute to the calculation of the four-loop anomalous dimensions. In the comparison we have found a series of typographical and combinatoric errors in the previous calculations [19, 20]. In particular, as a by-product of the present calculation all unpolarized and the polarized anomalous dimensions and those for transversity are correctly obtained to two-loop order [8–18, 32, 34–36, 38, 81, 82]. Our calculation has been performed in an algorithmic manner using well established methods having been applied in various massless and massive three-loop calculations before, as e.g. [36, 38, 129]. The present formalism allows extensions to higher loop calculations. Here, however, also new structures

are expected to emerge and the complexity of the calculation will naturally be larger. The quantities to be computed are expressible in terms of the usual harmonic sums [115, 116] and no other function spaces as e.g. [114, 119, 120, 130] are required to derive and express the results such as the different expansion coefficients of the different massless OMEs and the anomalous dimensions.

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A Feynman rules

The alien operators introduce new Feynman rules for operator insertions. They have been given in Refs. [15, 19, 79]. We give them here fore completeness and correct typographical errors; particularly all sums have to start with the lower index 0.

$$O_{A,\mu\nu}^{ab}(p, -p) = \delta^{ab} \frac{1 + (-1)^N}{2} [2\Delta_\mu \Delta_\nu p^2 - (p_\mu \Delta_\nu + p_\nu \Delta_\mu) \Delta.p] (\Delta.p)^{N-2}, \quad (243)$$

$$O_{A,\mu\nu\lambda}^{abc}(p, q, k) = -ig \frac{1 + (-1)^N}{2} f^{abc} V_{\mu\nu\lambda}^{(3)}(p, q, k), \quad (244)$$

$$O_\omega^{ab}(p, -p) = -\delta^{ab} (\Delta.p)^N, \quad (245)$$

$$O_{\omega,\mu}^{abc}(p, q, k) = ig f^{abc} \tilde{V}_\mu^{(3)}(p, q, k), \quad (246)$$

$$\begin{aligned} V_{\mu\nu\lambda}^{(3)}(p, q, k) &= [\Delta_\mu \Delta_\nu (q_\lambda - p_\lambda) + \Delta_\mu \Delta_\lambda (p_\nu - k_\nu) + g_{\nu\lambda} \Delta_\mu (\Delta.k - \Delta.q)] (\Delta.p)^{N-2} \\ &\quad - \frac{1}{4} \Delta_\nu \Delta_\lambda (\Delta_\mu p^2 - p_\mu \Delta.p) \sum_{i=0}^{N-3} \left[(-\Delta.q)^i (\Delta.k)^{N-3-i} - 3(-\Delta.k)^i (\Delta.p)^{N-3-i} \right. \\ &\quad \left. - 3(-\Delta.p)^i (\Delta.q)^{N-3-i} \right] + \left\{ \begin{matrix} p \rightarrow q \rightarrow k \rightarrow p \\ \mu \rightarrow \nu \rightarrow \lambda \rightarrow \mu \end{matrix} \right\} + \left\{ \begin{matrix} p \rightarrow k \rightarrow q \rightarrow p \\ \mu \rightarrow \lambda \rightarrow \nu \rightarrow \mu \end{matrix} \right\} \end{aligned} \quad (247)$$

$$\begin{aligned} \tilde{V}_\mu^{(3)}(p, q, k) &= \Delta_\mu \left[\frac{1}{2} (\Delta.k)^{N-1} + \frac{1}{4} (\Delta.q - 3\Delta.k) (\Delta.q)^{N-2} + \frac{1}{4} (\Delta.k - \Delta.q) (\Delta.p)^{N-2} \right. \\ &\quad \left. - \frac{3}{4} (\Delta.k)^2 \sum_{i=0}^{N-3} (\Delta.k)^i (-\Delta.q)^{N-3-i} \right]. \end{aligned} \quad (248)$$

B The polynomials

In the following we list the polynomials occurring in Eqs. (142–192, 193–226) and (227–236).

$$P_1 = -102241N^4 - 197090N^3 - 96529N^2 - 1680N + 1584 \quad (249)$$

$$P_2 = -2895N^4 - 5822N^3 - 2863N^2 - 512N - 192 \quad (250)$$

$$\begin{aligned}
P_3 &= -521N^4 - 1258N^3 + 439N^2 - 2280N - 144 & (251) \\
P_4 &= -367N^4 - 602N^3 + 895N^2 + 866N - 648 & (252) \\
P_5 &= -263N^4 - 574N^3 + 97N^2 - 360N + 96 & (253) \\
P_6 &= -155N^4 - 265N^3 - 161N^2 + 75N + 18 & (254) \\
P_7 &= -155N^4 - 175N^3 - 89N^2 + 57N + 18 & (255) \\
P_8 &= -137N^4 - 289N^3 - 227N^2 - 291N - 60 & (256) \\
P_9 &= -127N^4 - 140N^3 + 331N^2 + 260N - 180 & (257) \\
P_{10} &= -101N^4 - 96N^3 + 260N^2 + 33N + 18 & (258) \\
P_{11} &= -97N^4 - 78N^3 + 333N^2 + 218N - 160 & (259) \\
P_{12} &= -82N^4 - 155N^3 - 126N^2 - 107N - 34 & (260) \\
P_{13} &= -82N^4 - 137N^3 - 22N^2 - 5N - 18 & (261) \\
P_{14} &= -73N^4 - 119N^3 - 103N^2 + 33N + 18 & (262) \\
P_{15} &= -73N^4 - 65N^3 - 31N^2 + 51N + 18 & (263) \\
P_{16} &= -67N^4 - 152N^3 - 31N^2 - 162N - 72 & (264) \\
P_{17} &= -63N^4 - 171N^3 - 52N^2 + 4N + 48 & (265) \\
P_{18} &= -63N^4 - 29N^3 + 260N^2 + 12N + 144 & (266) \\
P_{19} &= -55N^4 - 273N^3 - 366N^2 - 28N + 152 & (267) \\
P_{20} &= -49N^4 - 6N^3 + 67N^2 + 38N - 2 & (268) \\
P_{21} &= -33N^4 - 195N^3 - 348N^2 - 182N - 30 & (269) \\
P_{22} &= -33N^4 - 66N^3 - 31N^2 + 2N - 64 & (270) \\
P_{23} &= -30N^4 - 39N^3 + 6N^2 - 19N - 22 & (271) \\
P_{24} &= -29N^4 + 6N^3 + 47N^2 + 14N - 6 & (272) \\
P_{25} &= -28N^4 - 56N^3 + 137N^2 + 171N - 12 & (273) \\
P_{26} &= -17N^4 - 34N^3 + 49N^2 + 66N + 32 & (274) \\
P_{27} &= -14N^4 - 64N^3 - 59N^2 - 57N - 14 & (275) \\
P_{28} &= -14N^4 - 40N^3 - 27N^2 - 17N - 4 & (276) \\
P_{29} &= -14N^4 - 40N^3 + 13N^2 - 9N - 14 & (277) \\
P_{30} &= -14N^4 - 32N^3 - 3N^2 - N - 4 & (278) \\
P_{31} &= -13N^4 - 37N^3 - 18N^2 - 28N + 24 & (279) \\
P_{32} &= -13N^4 - 19N^3 - 50N^2 - 30N - 8 & (280) \\
P_{33} &= -11N^4 - 22N^3 - 37N^2 - 26N - 48 & (281) \\
P_{34} &= -11N^4 - 22N^3 - 5N^2 + 6N - 16 & (282) \\
P_{35} &= -9N^4 - 38N^3 - 104N^2 - 64N - 32 & (283) \\
P_{36} &= -9N^4 + 62N^3 + 125N^2 + 54N - 40 & (284) \\
P_{37} &= -8N^4 - 21N^3 - 10N^2 + 3N + 4 & (285) \\
P_{38} &= -7N^4 - 20N^3 - 15N^2 - 8N - 2 & (286) \\
P_{39} &= -7N^4 - 16N^3 - 3N^2 - 2 & (287) \\
P_{40} &= -5N^4 - 20N^3 - 33N^2 - 18N + 4 & (288) \\
P_{41} &= -3N^4 - 157N^3 - 558N^2 - 324N - 104 & (289) \\
P_{42} &= -3N^4 - 16N^3 - 35N^2 - 22N + 4 & (290)
\end{aligned}$$

$$\begin{aligned}
P_{43} &= -3N^4 - 6N^3 + 27N^2 + 46N - 120 & (291) \\
P_{44} &= -2N^4 - 9N^3 - 7N^2 + 4N - 30 & (292) \\
P_{45} &= -2N^4 - 6N^3 + 3N^2 + 20N + 17 & (293) \\
P_{46} &= -2N^4 - 4N^3 + N^2 + 21N - 36 & (294) \\
P_{47} &= -2N^4 - 3N^3 - 9N^2 - 8N + 4 & (295) \\
P_{48} &= -N^4 - 3N^3 + 4N^2 + 10N - 30 & (296) \\
P_{49} &= -N^4 - 2N^3 + 15N^2 - 22N + 12 & (297) \\
P_{50} &= -N^4 + 2N^3 + 3N^2 - 2N - 4 & (298) \\
P_{51} &= -N^4 + 4N^3 - 6N^2 - 4N - 1 & (299) \\
P_{52} &= -N^4 + 6N^3 - 10N^2 + 5N - 1 & (300) \\
P_{53} &= -N^4 + 7N^3 + 3N^2 - 3N - 2 & (301) \\
P_{54} &= N^4 - 15N^3 - 94N^2 - 106N + 210 & (302) \\
P_{55} &= N^4 - 12N^3 - 22N^2 + 21N + 6 & (303) \\
P_{56} &= N^4 - 9N^3 + 15N^2 + 13N + 4 & (304) \\
P_{57} &= N^4 - 6N^3 - 29N^2 - 20N + 6 & (305) \\
P_{58} &= N^4 - 6N^3 + 7N^2 - 2N - 4 & (306) \\
P_{59} &= N^4 - 5N^3 - 10N^2 - 28N + 24 & (307) \\
P_{60} &= N^4 - 4N^3 + 7N^2 - 4N + 8 & (308) \\
P_{61} &= N^4 - 2N^3 - 6N^2 + 4N + 1 & (309) \\
P_{62} &= N^4 - 2N^3 + 11N^2 - 24N - 14 & (310) \\
P_{63} &= N^4 - N^3 + 8N^2 + 25N - 14 & (311) \\
P_{64} &= N^4 + 3N^3 + 16N^2 + 4N - 16 & (312) \\
P_{65} &= N^4 + 4N^3 + 7N^2 - 4N - 12 & (313) \\
P_{66} &= N^4 + 4N^3 + 9N^2 + 6N - 2 & (314) \\
P_{67} &= N^4 + 5N^3 + 5N^2 + 3N + 4 & (315) \\
P_{68} &= N^4 + 8N^3 + 14N^2 + 5N + 6 & (316) \\
P_{69} &= N^4 + 8N^3 + 19N^2 + 4N - 12 & (317) \\
P_{70} &= N^4 + 10N^3 + 29N^2 + 20N - 4 & (318) \\
P_{71} &= N^4 + 20N^3 + 59N^2 + 70N + 36 & (319) \\
P_{72} &= 2N^4 + N^3 + 4N^2 - N - 1 & (320) \\
P_{73} &= 2N^4 + 2N^3 + N^2 - 7N - 2 & (321) \\
P_{74} &= 2N^4 + 5N^3 - 9N^2 + 30N + 8 & (322) \\
P_{75} &= 2N^4 + 5N^3 + 4N^2 - 4N - 2 & (323) \\
P_{76} &= 2N^4 + 7N^3 - 2N^2 - 9N + 6 & (324) \\
P_{77} &= 2N^4 + 7N^3 + 4N^2 + N + 4 & (325) \\
P_{78} &= 2N^4 + 14N^3 + 29N^2 + 31N + 24 & (326) \\
P_{79} &= 3N^4 - 237N^3 - 100N^2 + 20N - 64 & (327) \\
P_{80} &= 3N^4 - 45N^3 - 82N^2 + 22N + 12 & (328) \\
P_{81} &= 3N^4 - 9N^3 - 22N^2 + 10N + 4 & (329) \\
P_{82} &= 3N^4 - N^3 - 12N^2 + 6N + 2 & (330)
\end{aligned}$$

$$\begin{aligned}
P_{83} &= 3N^4 + 2N^3 + 11N^2 - 4N - 4 & (331) \\
P_{84} &= 3N^4 + 3N^3 + 12N^2 + 32N + 8 & (332) \\
P_{85} &= 3N^4 + 4N^3 - 23N^2 + 4N + 36 & (333) \\
P_{86} &= 3N^4 + 7N^3 - 32N^2 - 76N - 48 & (334) \\
P_{87} &= 3N^4 + 8N^3 - 8N^2 + 3N + 6 & (335) \\
P_{88} &= 3N^4 + 12N^3 + 3N^2 - 14N + 60 & (336) \\
P_{89} &= 3N^4 + 13N^3 + 20N^2 + 10N - 1 & (337) \\
P_{90} &= 3N^4 + 21N^3 + 48N^2 + 34N - 30 & (338) \\
P_{91} &= 4N^4 + 21N^3 + 29N^2 + 8N + 30 & (339) \\
P_{92} &= 5N^4 + 8N^3 + 19N^2 + 12N + 4 & (340) \\
P_{93} &= 5N^4 + 16N^3 + 13N^2 + 62N + 84 & (341) \\
P_{94} &= 5N^4 + 28N^3 + 22N^2 + 15N + 6 & (342) \\
P_{95} &= 5N^4 + 35N^3 + 61N^2 - 17N - 3 & (343) \\
P_{96} &= 6N^4 - 19N^3 - 56N^2 + 4N + 16 & (344) \\
P_{97} &= 6N^4 + 9N^3 + 23N^2 + 14N + 4 & (345) \\
P_{98} &= 7N^4 - 21N^3 - 61N^2 + 45N + 6 & (346) \\
P_{99} &= 7N^4 - 14N^3 - N^2 - 20N - 4 & (347) \\
P_{100} &= 7N^4 + 10N^3 + 13N^2 + 20N + 4 & (348) \\
P_{101} &= 7N^4 + 23N^3 + 19N^2 + 15N + 6 & (349) \\
P_{102} &= 7N^4 + 24N^3 - 9N^2 - 74N - 36 & (350) \\
P_{103} &= 7N^4 + 28N^3 + 39N^2 + 18N - 2 & (351) \\
P_{104} &= 8N^4 - 7N^3 - 36N^2 + 17N + 2 & (352) \\
P_{105} &= 9N^4 + 25N^3 + 4N^2 - 16N - 8 & (353) \\
P_{106} &= 10N^4 - 32N^3 + 109N^2 - 87N + 27 & (354) \\
P_{107} &= 10N^4 + 11N^3 + 7N^2 + 12N + 8 & (355) \\
P_{108} &= 10N^4 + 17N^3 + 77N^2 + 48N + 8 & (356) \\
P_{109} &= 10N^4 + 19N^3 + 23N^2 - 8N - 4 & (357) \\
P_{110} &= 10N^4 + 23N^3 + 38N^2 + 47N + 18 & (358) \\
P_{111} &= 10N^4 + 25N^3 + 22N^2 + 17N + 6 & (359) \\
P_{112} &= 10N^4 + 47N^3 + 69N^2 + 40N + 12 & (360) \\
P_{113} &= 10N^4 + 51N^3 + 81N^2 + 48N + 12 & (361) \\
P_{114} &= 11N^4 + 15N^3 - 6N^2 + 10N + 4 & (362) \\
P_{115} &= 11N^4 + 22N^3 + 13N^2 + 2N + 24 & (363) \\
P_{116} &= 11N^4 + 22N^3 + 73N^2 + 62N + 84 & (364) \\
P_{117} &= 11N^4 + 22N^3 + 79N^2 + 68N + 90 & (365) \\
P_{118} &= 12N^4 - 21N^3 + 2N^2 + 53N + 8 & (366) \\
P_{119} &= 13N^4 + 26N^3 - 63N^2 - 76N + 52 & (367) \\
P_{120} &= 13N^4 + 26N^3 + 89N^2 + 76N + 52 & (368) \\
P_{121} &= 14N^4 + 28N^3 + 11N^2 + 2N + 1 & (369) \\
P_{122} &= 14N^4 + 33N^3 + 31N^2 + 12 & (370)
\end{aligned}$$

$$\begin{aligned}
P_{123} &= 14N^4 + 56N^3 - 45N^2 + 25N + 34 & (371) \\
P_{124} &= 14N^4 + 112N^3 + 123N^2 + 137N + 34 & (372) \\
P_{125} &= 15N^4 + 71N^3 + 80N^2 - 4N - 48 & (373) \\
P_{126} &= 16N^4 + 101N^3 + 201N^2 + 366N - 144 & (374) \\
P_{127} &= 19N^4 - 44N^3 + 22N^2 + 93N - 27 & (375) \\
P_{128} &= 19N^4 + 38N^3 - 43N^2 - 62N - 24 & (376) \\
P_{129} &= 19N^4 + 80N^3 + 131N^2 - 142N - 112 & (377) \\
P_{130} &= 19N^4 + 82N^3 + 105N^2 + 50N + 24 & (378) \\
P_{131} &= 22N^4 + 23N^3 + 11N^2 + 24N + 16 & (379) \\
P_{132} &= 22N^4 + 35N^3 + 6N^2 + 3N + 6 & (380) \\
P_{133} &= 22N^4 + 41N^3 + 38N^2 + 37N + 14 & (381) \\
P_{134} &= 23N^4 + 133N^3 + 166N^2 + 74N + 84 & (382) \\
P_{135} &= 25N^4 + 153N^3 + 290N^2 + 166N - 30 & (383) \\
P_{136} &= 26N^4 + 93N^3 + 109N^2 + 44N - 4 & (384) \\
P_{137} &= 27N^4 + 27N^3 - 34N^2 + 22N + 12 & (385) \\
P_{138} &= 27N^4 + 94N^3 + 105N^2 + 40N - 4 & (386) \\
P_{139} &= 29N^4 - 8N^3 + 212N^2 + 186N + 63 & (387) \\
P_{140} &= 29N^4 + 95N^3 + 91N^2 - 23N - 3 & (388) \\
P_{141} &= 30N^4 + 31N^3 + 20N^2 + 109N + 26 & (389) \\
P_{142} &= 32N^4 + 61N^3 + 305N^2 + 213N + 63 & (390) \\
P_{143} &= 34N^4 + 71N^3 + 14N^2 - 41N + 30 & (391) \\
P_{144} &= 37N^4 + 74N^3 - 85N^2 - 122N - 48 & (392) \\
P_{145} &= 39N^4 + 88N^3 - 75N^2 - 120N - 28 & (393) \\
P_{146} &= 41N^4 + 54N^3 - 56N^2 + 21N - 6 & (394) \\
P_{147} &= 85N^4 + 315N^3 + 584N^2 + 606N + 12 & (395) \\
P_{148} &= 94N^4 + 352N^3 + 243N^2 - 367N - 358 & (396) \\
P_{149} &= 94N^4 + 446N^3 + 713N^2 + 418N + 72 & (397) \\
P_{150} &= 101N^4 + 597N^3 + 1066N^2 + 558N + 90 & (398) \\
P_{151} &= 116N^4 + 547N^3 + 807N^2 + 376N + 12 & (399) \\
P_{152} &= 116N^4 + 547N^3 + 823N^2 + 440N + 76 & (400) \\
P_{153} &= 116N^4 + 669N^3 + 1141N^2 + 552N + 270 & (401) \\
P_{154} &= 137N^4 - 24N^3 - 399N^2 - 262N + 48 & (402) \\
P_{155} &= 211N^4 + 1018N^3 + 1208N^2 - 118N - 24 & (403) \\
P_{156} &= 221N^4 + 170N^3 - 793N^2 - 510N + 360 & (404) \\
P_{157} &= 253N^4 + 674N^3 - 435N^2 + 1832N + 240 & (405) \\
P_{158} &= 799N^4 + 3574N^3 + 3099N^2 - 3358N - 3790 & (406) \\
P_{159} &= 799N^4 + 4373N^3 + 7256N^2 + 3250N - 648 & (407) \\
P_{160} &= 2599N^4 + 13859N^3 + 23924N^2 + 12880N + 648 & (408) \\
P_{161} &= 7865N^4 + 15346N^3 + 7529N^2 + 48N - 144 & (409) \\
P_{162} &= -713N^5 - 3808N^4 - 5290N^3 + 256N^2 + 1563N - 2664 & (410)
\end{aligned}$$

$$\begin{aligned}
P_{163} &= -589N^5 - 2828N^4 - 4094N^3 - 3388N^2 - 3357N - 5040 & (411) \\
P_{164} &= -531N^5 - 2210N^4 - 2251N^3 - 508N^2 - 752N + 768 & (412) \\
P_{165} &= -485N^5 - 2156N^4 - 2647N^3 - 556N^2 + 636N + 144 & (413) \\
P_{166} &= -485N^5 - 2084N^4 - 2647N^3 - 772N^2 + 492N + 144 & (414) \\
P_{167} &= -203N^5 - 706N^4 - 676N^3 - 293N^2 - 372N + 12 & (415) \\
P_{168} &= -185N^5 - 803N^4 - 763N^3 + 1070N^2 + 2241N + 378 & (416) \\
P_{169} &= -71N^5 - 599N^4 - 1230N^3 - 598N^2 + 128N - 72 & (417) \\
P_{170} &= -65N^5 - 282N^4 - 995N^3 - 2198N^2 - 1252N - 104 & (418) \\
P_{171} &= -59N^5 - 154N^4 - 91N^3 + 106N^2 - 6N - 12 & (419) \\
P_{172} &= -58N^5 - 248N^4 - 389N^3 - 293N^2 - 160N - 20 & (420) \\
P_{173} &= -58N^5 - 232N^4 - 253N^3 - 29N^2 - 16N - 20 & (421) \\
P_{174} &= -41N^5 - 196N^4 - 374N^3 - 604N^2 - 585N - 600 & (422) \\
P_{175} &= -37N^5 + 340N^3 - 246N^2 + 81N + 54 & (423) \\
P_{176} &= -33N^5 - 166N^4 - 321N^3 - 240N^2 - 52N - 16 & (424) \\
P_{177} &= -33N^5 - 150N^4 - 209N^3 + 48N^2 + 140N - 48 & (425) \\
P_{178} &= -26N^5 - 139N^4 - 181N^3 + 55N^2 - 33N - 252 & (426) \\
P_{179} &= -24N^5 - 97N^4 - 221N^3 - 450N^2 - 280N - 32 & (427) \\
P_{180} &= -21N^5 - 77N^4 - 51N^3 + 10N^2 + 28N - 24 & (428) \\
P_{181} &= -21N^5 - 37N^4 + 55N^3 + 54N^2 + 84N - 72 & (429) \\
P_{182} &= -16N^5 - 29N^4 - 2N^3 + 11N^2 - 12N + 12 & (430) \\
P_{183} &= -13N^5 - 52N^4 - 70N^3 - 27N^2 + 4N - 4 & (431) \\
P_{184} &= -13N^5 - 30N^4 + 30N^3 + 135N^2 + 88N - 12 & (432) \\
P_{185} &= -9N^5 - 11N^4 + 61N^3 - 33N^2 - 92N - 12 & (433) \\
P_{186} &= -7N^5 - 18N^4 - 26N^3 - 10N^2 - 5N - 6 & (434) \\
P_{187} &= -6N^5 - 9N^4 + 16N^3 + 9N^2 - 14N - 8 & (435) \\
P_{188} &= -4N^5 - 82N^4 - 99N^3 - 25N^2 - 42N - 24 & (436) \\
P_{189} &= -4N^5 - 15N^4 + 6N^3 + 67N^2 + 22N - 36 & (437) \\
P_{190} &= -3N^5 - 9N^4 + 21N^3 + 27N^2 - 8N + 12 & (438) \\
P_{191} &= -3N^5 - 5N^4 + 8N^3 + 6N^2 - 5N - 4 & (439) \\
P_{192} &= -2N^5 + N^4 - 3N^3 + 9N^2 + 5N - 2 & (440) \\
P_{193} &= -N^5 - 5N^4 - 5N^3 - 9N^2 - 16N - 4 & (441) \\
P_{194} &= -N^5 - 3N^4 - 9N^3 - 7N^2 + 8N + 4 & (442) \\
P_{195} &= -N^5 - 3N^4 + 15N^3 + 17N^2 - 8N + 4 & (443) \\
P_{196} &= -N^5 + 3N^4 + N^3 - 7N^2 + 8N + 4 & (444) \\
P_{197} &= -N^5 + 5N^4 - 18N^3 - 21N^2 - 23N - 6 & (445) \\
P_{198} &= -N^5 + 11N^4 - 10N^3 - 10N^2 - 5N - 1 & (446) \\
P_{199} &= N^5 - 4N^3 + 22N^2 + 35N + 14 & (447) \\
P_{200} &= N^5 + 4N^3 - 6N^2 - 5N - 2 & (448) \\
P_{201} &= N^5 + 5N^3 + 2N^2 - 20N - 8 & (449) \\
P_{202} &= N^5 - 23N^4 + 19N^3 + 51N^2 - 88N - 24 & (450)
\end{aligned}$$

$$\begin{aligned}
P_{203} &= N^5 - 5N^4 - 25N^3 - 7N^2 + 16N - 4 & (451) \\
P_{204} &= N^5 - 5N^4 + 5N^3 - 7N^2 + 10N + 4 & (452) \\
P_{205} &= N^5 - 3N^4 - 5N^3 - 13N^2 + 4 & (453) \\
P_{206} &= N^5 + 3N^4 + 9N^2 + 23N + 6 & (454) \\
P_{207} &= N^5 + 3N^4 - 3N^3 - 5N^2 - 4 & (455) \\
P_{208} &= N^5 + 3N^4 + 19N^3 + 15N^2 - 38N - 24 & (456) \\
P_{209} &= N^5 + 4N^4 + 25N^3 + 109N^2 + 100N - 52 & (457) \\
P_{210} &= N^5 + 6N^4 + 13N^3 + 16N^2 - 4N - 16 & (458) \\
P_{211} &= N^5 + 8N^4 + 12N^3 - 20N^2 - 35N - 14 & (459) \\
P_{212} &= N^5 + 13N^4 + 15N^3 - 19N^2 - 46N - 36 & (460) \\
P_{213} &= N^5 + 13N^4 + 31N^3 - N^2 - 24N + 4 & (461) \\
P_{214} &= N^5 + 61N^4 + 321N^3 + 547N^2 + 326N + 72 & (462) \\
P_{215} &= 2N^5 - 46N^4 - 7N^3 - 134N^2 - 25N + 42 & (463) \\
P_{216} &= 2N^5 - 19N^4 - N^3 - 59N^2 - 13N + 18 & (464) \\
P_{217} &= 2N^5 + 3N^4 - 4N^3 - 18N^2 - 6N - 1 & (465) \\
P_{218} &= 2N^5 + 5N^4 + 2N^3 - 3N^2 + 6N + 12 & (466) \\
P_{219} &= 2N^5 + 8N^4 - 3N^3 - 38N^2 - 9N + 24 & (467) \\
P_{220} &= 2N^5 + 8N^4 + 5N^3 + 16N^2 - N - 6 & (468) \\
P_{221} &= 3N^5 + 4N^3 - 14N^2 - 7N + 6 & (469) \\
P_{222} &= 3N^5 - 12N^4 - 79N^3 - 60N^2 - 20N - 48 & (470) \\
P_{223} &= 3N^5 - 8N^4 - 22N^3 - 9N^2 - 12N - 4 & (471) \\
P_{224} &= 3N^5 - 2N^4 + 10N^3 - 20N^2 - 13N - 2 & (472) \\
P_{225} &= 3N^5 + 4N^4 - 13N^3 - 36N^2 + 30N - 12 & (473) \\
P_{226} &= 3N^5 + 6N^4 + 8N^3 + 3N^2 - 24N - 12 & (474) \\
P_{227} &= 3N^5 + 8N^4 - 15N^3 - 32N^2 - 20N - 16 & (475) \\
P_{228} &= 3N^5 + 11N^4 + 10N^3 + 10N^2 + 5N + 1 & (476) \\
P_{229} &= 3N^5 + 14N^4 + 23N^3 + 16N^2 - 10N - 24 & (477) \\
P_{230} &= 3N^5 + 15N^4 - 17N^3 + 73N^2 + 54N + 48 & (478) \\
P_{231} &= 3N^5 + 40N^4 + 59N^3 + 28N^2 + 26N + 12 & (479) \\
P_{232} &= 4N^5 - 14N^4 + 24N^3 - 8N^2 - 19N - 7 & (480) \\
P_{233} &= 4N^5 + 3N^4 + 4N^3 + 10N^2 - 8N - 5 & (481) \\
P_{234} &= 4N^5 + 8N^4 + 11N^3 + 9N^2 - 12N - 4 & (482) \\
P_{235} &= 4N^5 + 21N^4 - 3N^3 - 60N^2 - 8N - 24 & (483) \\
P_{236} &= 5N^5 - 12N^4 - 19N^3 + 38N^2 + 12N - 16 & (484) \\
P_{237} &= 5N^5 - 10N^4 - 10N^3 + 5N + 2 & (485) \\
P_{238} &= 5N^5 - 2N^4 + 18N^3 - 32N^2 - 23N - 6 & (486) \\
P_{239} &= 5N^5 - N^4 - 20N^3 - 5N^2 - 4N - 4 & (487) \\
P_{240} &= 5N^5 + 4N^4 + 4N^3 - 28N^2 - 13N + 12 & (488) \\
P_{241} &= 5N^5 + 8N^4 + 12N^3 + 56N^2 + 85N + 34 & (489) \\
P_{242} &= 5N^5 + 24N^4 - 59N^3 - 88N^2 + 82N - 36 & (490)
\end{aligned}$$

$$\begin{aligned}
P_{243} &= 7N^5 + N^4 + 7N^3 - 33N^2 - 2N + 4 & (491) \\
P_{244} &= 7N^5 + 2N^4 - 19N^3 - 4N^2 - 6N - 4 & (492) \\
P_{245} &= 7N^5 + 22N^4 + 47N^3 + 32N^2 - 36N - 16 & (493) \\
P_{246} &= 7N^5 + 31N^4 - 84N^2 - 26N - 4 & (494) \\
P_{247} &= 7N^5 + 310N^4 + 1137N^3 + 1142N^2 + 140N - 120 & (495) \\
P_{248} &= 8N^5 - 11N^4 + 49N^3 - 31N^2 - 25N - 10 & (496) \\
P_{249} &= 8N^5 + 27N^4 - 6N^3 - 9N^2 + 64N + 12 & (497) \\
P_{250} &= 8N^5 + 31N^4 + 58N^3 + 43N^2 - 12N - 4 & (498) \\
P_{251} &= 9N^5 + 19N^4 - 25N^3 - 31N^2 - 24N - 44 & (499) \\
P_{252} &= 9N^5 + 27N^4 + 59N^3 + 33N^2 - 64N - 28 & (500) \\
P_{253} &= 13N^5 + 7N^4 - 29N^3 - 3N^2 - 40N - 44 & (501) \\
P_{254} &= 13N^5 + 31N^4 + 37N^3 - 46N^2 - 173N - 114 & (502) \\
P_{255} &= 13N^5 + 38N^4 - 4N^3 + 58N^2 + 111N + 72 & (503) \\
P_{256} &= 13N^5 + 67N^4 + 61N^3 + 119N^2 + 232N + 60 & (504) \\
P_{257} &= 16N^5 + 64N^4 - 49N^3 - N^2 + 105N + 54 & (505) \\
P_{258} &= 20N^5 - 27N^4 - 130N^3 + 48N^2 + 152N + 48 & (506) \\
P_{259} &= 20N^5 + 59N^4 + 130N^3 + 67N^2 - 156N - 68 & (507) \\
P_{260} &= 23N^5 - 28N^4 + 129N^3 - 114N^2 - 98N - 36 & (508) \\
P_{261} &= 25N^5 - 50N^4 - 231N^3 - 52N^2 + 116N + 48 & (509) \\
P_{262} &= 31N^5 + 358N^4 + 1065N^3 + 950N^2 + 44N - 120 & (510) \\
P_{263} &= 33N^5 + 24N^4 + 463N^3 + 276N^2 - 308N + 64 & (511) \\
P_{264} &= 33N^5 + 59N^4 - 132N^3 - 210N^2 + 40N + 40 & (512) \\
P_{265} &= 45N^5 + 225N^4 + 370N^3 + 244N^2 - 56N + 96 & (513) \\
P_{266} &= 49N^5 + 163N^4 + 84N^3 - 50N^2 + 72N + 40 & (514) \\
P_{267} &= 51N^5 + 207N^4 + 193N^3 - 210N^2 - 421N - 66 & (515) \\
P_{268} &= 53N^5 + 37N^4 - 122N^3 + 53N^2 + 117N + 6 & (516) \\
P_{269} &= 53N^5 + 47N^4 + 67N^3 - 197N^2 + 486N + 360 & (517) \\
P_{270} &= 67N^5 + 289N^4 + 179N^3 - 409N^2 - 238N + 220 & (518) \\
P_{271} &= 77N^5 + 256N^4 + 83N^3 - 380N^2 - 340N + 16 & (519) \\
P_{272} &= 80N^5 + 119N^4 - 152N^3 - 770N^2 - 264N - 45 & (520) \\
P_{273} &= 119N^5 + 188N^4 - 380N^3 + 184N^2 - 339N - 180 & (521) \\
P_{274} &= 133N^5 + 680N^4 + 1238N^3 + 1408N^2 + 1329N + 1692 & (522) \\
P_{275} &= 137N^5 + 67N^4 - 392N^3 + 155N^2 + 303N + 18 & (523) \\
P_{276} &= 211N^5 + 986N^4 + 2204N^3 + 4558N^2 + 4425N + 4032 & (524) \\
P_{277} &= 226N^5 + 165N^4 - 786N^3 - 632N^2 - 186N - 9 & (525) \\
P_{278} &= 226N^5 + 561N^4 + 348N^3 + 178N^2 - 168N - 63 & (526) \\
P_{279} &= 361N^5 + 1690N^4 + 2392N^3 + 2254N^2 + 2567N + 3504 & (527) \\
P_{280} &= 379N^5 + 922N^4 + 609N^3 + 3110N^2 + 1076N - 4152 & (528) \\
P_{281} &= 496N^5 + 2219N^4 + 3857N^3 + 6889N^2 + 7167N + 7020 & (529) \\
P_{282} &= 541N^5 + 1575N^4 + 461N^3 - 1415N^2 - 746N - 160 & (530)
\end{aligned}$$

$$\begin{aligned}
P_{283} &= 3625N^5 + 10875N^4 + 3505N^3 - 11691N^2 - 8282N + 816 & (531) \\
P_{284} &= 4148N^5 + 21847N^4 + 37963N^3 + 20318N^2 - 1188N - 648 & (532) \\
P_{285} &= -843N^6 - 3105N^5 - 4121N^4 - 1915N^3 + 40N^2 + 240N + 72 & (533) \\
P_{286} &= -843N^6 - 2721N^5 - 2681N^4 - 1051N^3 - 440N^2 - 48N + 72 & (534) \\
P_{287} &= -695N^6 - 2849N^5 - 4354N^4 - 4804N^3 - 4125N^2 - 603N + 54 & (535) \\
P_{288} &= -202N^6 - 606N^5 - 633N^4 + 41N^3 - 405N^2 - 351N - 108 & (536) \\
P_{289} &= -195N^6 - 1277N^5 - 2741N^4 - 978N^3 + 3881N^2 + 4532N + 900 & (537) \\
P_{290} &= -130N^6 - 381N^5 + 50N^4 - 411N^3 - 1054N^2 + 1854N + 828 & (538) \\
P_{291} &= -55N^6 - 383N^5 - 747N^4 + 59N^3 + 1434N^2 + 1012N - 168 & (539) \\
P_{292} &= -31N^6 - 95N^5 - 74N^4 + 14N^3 + 3N^2 - 19N - 6 & (540) \\
P_{293} &= -31N^6 - 69N^5 + 2N^4 + 66N^3 - N^2 - 33N - 14 & (541) \\
P_{294} &= -27N^6 - 54N^5 + 159N^4 + 462N^3 + 908N^2 + 584N - 160 & (542) \\
P_{295} &= -21N^6 - 77N^5 - 116N^4 - 104N^3 - 93N^2 - 55N - 14 & (543) \\
P_{296} &= -21N^6 - 21N^5 + 19N^4 + 115N^3 + 314N^2 + 22N - 116 & (544) \\
P_{297} &= -7N^6 - 25N^5 - 68N^4 - 48N^3 - 11N^2 - 19N - 14 & (545) \\
P_{298} &= -7N^6 - 24N^5 - 32N^4 - 17N^3 - 10N^2 - 5N - 1 & (546) \\
P_{299} &= -7N^6 - 20N^5 - 24N^4 - 5N^3 + 2N^2 - N - 1 & (547) \\
P_{300} &= -N^6 - 3N^5 - 9N^4 - 5N^3 + 4N^2 - 30N - 12 & (548) \\
P_{301} &= -N^6 + 2N^5 + 4N^4 + 6N^3 + 13N^2 + 12N - 12 & (549) \\
P_{302} &= -N^6 + 7N^5 - 5N^4 - 25N^3 + 38N^2 + 38N + 12 & (550) \\
P_{303} &= N^6 - 7N^5 - 15N^4 - 29N^3 - 50N^2 - 8N - 12 & (551) \\
P_{304} &= N^6 - 6N^5 - 80N^4 - 90N^3 + 235N^2 + 264N - 36 & (552) \\
P_{305} &= N^6 - 6N^5 - 65N^4 - 180N^3 - 200N^2 - 36N + 54 & (553) \\
P_{306} &= N^6 - 6N^5 - 56N^4 - 234N^3 - 461N^2 - 216N + 108 & (554) \\
P_{307} &= N^6 - 4N^5 - 18N^4 - 40N^3 - 35N^2 + 12N + 12 & (555) \\
P_{308} &= N^6 - 4N^5 + 7N^4 - 19N^3 + 18N^2 + 9N - 4 & (556) \\
P_{309} &= N^6 + 3N^5 + 17N^4 - 17N^3 - 74N^2 - 10N + 8 & (557) \\
P_{310} &= N^6 + 6N^5 + 12N^4 + 14N^3 + 14N^2 - 13N - 2 & (558) \\
P_{311} &= N^6 + 7N^5 + 13N^4 - 7N^3 + 36N^2 + 116N - 6 & (559) \\
P_{312} &= N^6 + 8N^5 - 58N^4 - 108N^3 + 21N^2 + 20N + 20 & (560) \\
P_{313} &= N^6 + 9N^5 - 56N^3 - 21N^2 + 55N - 36 & (561) \\
P_{314} &= N^6 + 21N^5 + 103N^4 + 221N^3 + 272N^2 + 186N + 36 & (562) \\
P_{315} &= 2N^6 - N^5 - 23N^4 + N^3 - 135N^2 - 280N + 68 & (563) \\
P_{316} &= 2N^6 + 2N^5 + 7N^4 - 12N^3 + 8N^2 + 18N + 7 & (564) \\
P_{317} &= 2N^6 + 7N^5 + 20N^4 + 24N^3 + 13N^2 + 24N + 12 & (565) \\
P_{318} &= 2N^6 + 16N^5 + 9N^4 - 87N^3 - 37N^2 + 99N - 74 & (566) \\
P_{319} &= 2N^6 + 16N^5 + 23N^4 + 42N^3 + 17N^2 + 16N + 12 & (567) \\
P_{320} &= 3N^6 + 3N^4 - 7N^3 + 8N^2 + 7N - 6 & (568) \\
P_{321} &= 3N^6 - 7N^5 + 11N^4 - 17N^3 + 2N^2 + 4N + 2 & (569) \\
P_{322} &= 3N^6 - 5N^5 - 7N^4 - 21N^3 - 26N^2 - 28N + 24 & (570)
\end{aligned}$$

$$\begin{aligned}
P_{323} &= 3N^6 + 3N^5 - 4N^4 - 17N^3 - 37N^2 - 8N + 12 & (571) \\
P_{324} &= 3N^6 + 19N^5 + 75N^4 + 157N^3 + 26N^2 - 192N - 72 & (572) \\
P_{325} &= 3N^6 + 20N^5 + 49N^4 + 52N^3 - 8N^2 - 48N - 16 & (573) \\
P_{326} &= 3N^6 + 20N^5 + 61N^4 + 96N^3 + 16N^2 - 64N - 16 & (574) \\
P_{327} &= 4N^6 + 11N^5 + 30N^4 + 105N^3 + 50N^2 - 96N - 144 & (575) \\
P_{328} &= 4N^6 + 18N^5 - 28N^4 + 24N^3 + 21N^2 - 21N - 14 & (576) \\
P_{329} &= 5N^6 - 17N^5 + 53N^4 - 143N^3 + 66N^2 + 60N - 8 & (577) \\
P_{330} &= 5N^6 + 26N^5 + 30N^4 + 40N^3 + 133N^2 + 234N + 396 & (578) \\
P_{331} &= 6N^6 - 204N^5 - 735N^4 - 827N^3 - 376N^2 - 236N - 96 & (579) \\
P_{332} &= 6N^6 + 9N^5 + 54N^4 + 41N^3 - 174N^2 - 96N + 16 & (580) \\
P_{333} &= 6N^6 + 41N^5 - 118N^4 - 171N^3 - 14N^2 - 212N - 72 & (581) \\
P_{334} &= 9N^6 + 9N^5 - 7N^4 - 47N^3 - 130N^2 - 6N + 52 & (582) \\
P_{335} &= 9N^6 + 10N^5 - 7N^4 - 29N^3 + 8N^2 + 7N - 6 & (583) \\
P_{336} &= 11N^6 + 33N^5 + 129N^4 + 131N^3 + 76N^2 + 100N + 96 & (584) \\
P_{337} &= 11N^6 + 276N^5 + 398N^4 + 45N^3 + 626N^2 - 240N - 288 & (585) \\
P_{338} &= 12N^6 - 3N^5 - 6N^4 + 413N^3 + 228N^2 - 684N + 112 & (586) \\
P_{339} &= 12N^6 + 39N^5 + 71N^4 + 45N^3 - 229N^2 - 298N - 48 & (587) \\
P_{340} &= 16N^6 - 222N^5 + 753N^4 - 1660N^3 + 1581N^2 - 846N + 189 & (588) \\
P_{341} &= 17N^6 + 6N^5 - 20N^4 - 36N^3 - N^2 - 6N - 8 & (589) \\
P_{342} &= 19N^6 + 57N^5 + 3N^4 - 89N^3 - 62N^2 - 8N - 64 & (590) \\
P_{343} &= 19N^6 + 57N^5 + 3N^4 - 89N^3 - 46N^2 + 8N - 64 & (591) \\
P_{344} &= 19N^6 + 175N^5 + 533N^4 + 473N^3 + 564N^2 + 1340N + 672 & (592) \\
P_{345} &= 24N^6 + 152N^5 + 59N^4 - 265N^3 + 70N^2 + 28N - 248 & (593) \\
P_{346} &= 29N^6 - 54N^5 - 1657N^4 - 5916N^3 - 7786N^2 - 3408N - 24 & (594) \\
P_{347} &= 30N^6 + 181N^5 + 473N^4 + 635N^3 + 77N^2 - 520N - 156 & (595) \\
P_{348} &= 37N^6 + 28N^5 + 9N^4 + 130N^3 - 32N^2 - 32N - 32 & (596) \\
P_{349} &= 40N^6 + 231N^5 + 484N^4 + 780N^3 + 691N^2 - 480N - 576 & (597) \\
P_{350} &= 47N^6 + 69N^5 + 21N^4 - 145N^3 - 500N^2 - 20N + 240 & (598) \\
P_{351} &= 54N^6 - 5N^5 - 125N^4 + 3N^3 + 107N^2 - 226N + 120 & (599) \\
P_{352} &= 54N^6 + 356N^5 + 729N^4 + 375N^3 + 2N^2 + 404N + 200 & (600) \\
P_{353} &= 54N^6 + 436N^5 + 1297N^4 + 1919N^3 + 1954N^2 + 1332N + 200 & (601) \\
P_{354} &= 56N^6 + 339N^5 + 718N^4 + 621N^3 + 208N^2 + 76N + 32 & (602) \\
P_{355} &= 59N^6 - 510N^5 + 2329N^4 + 5494N^3 - 1212N^2 - 1576N - 1344 & (603) \\
P_{356} &= 61N^6 + 36N^5 - 59N^4 - 6N^3 - 140N^2 - 456N + 96 & (604) \\
P_{357} &= 62N^6 + 399N^5 + 416N^4 + 6N^3 + 440N^2 - 657N - 378 & (605) \\
P_{358} &= 67N^6 + 490N^5 + 1046N^4 + 128N^3 - 1729N^2 - 1382N - 60 & (606) \\
P_{359} &= 85N^6 + 485N^5 + 959N^4 + 805N^3 + 480N^2 + 534N - 180 & (607) \\
P_{360} &= 87N^6 + 285N^5 + 293N^4 + 191N^3 + 88N^2 + 56N + 24 & (608) \\
P_{361} &= 87N^6 + 333N^5 + 485N^4 + 463N^3 + 216N^2 + 56N - 8 & (609) \\
P_{362} &= 94N^6 + 291N^5 + 4N^4 + 33N^3 + 460N^2 - 882N - 396 & (610)
\end{aligned}$$

$$\begin{aligned}
P_{363} &= 119N^6 + 1212N^5 + 1784N^4 + 171N^3 + 1706N^2 + 480N - 144 & (611) \\
P_{364} &= 188N^6 + 1084N^5 + 2143N^4 + 1539N^3 + 464N^2 + 572N + 160 & (612) \\
P_{365} &= 233N^6 + 593N^5 - 12N^4 - 497N^3 + 607N^2 + 792N + 12 & (613) \\
P_{366} &= 293N^6 + 231N^5 - 125N^4 + 661N^3 - 1152N^2 - 2308N + 1104 & (614) \\
P_{367} &= 379N^6 + 2059N^5 + 3375N^4 + 4937N^3 + 9470N^2 + 9948N + 13032 & (615) \\
P_{368} &= 380N^6 + 1499N^5 + 2113N^4 + 1870N^3 + 1191N^2 + 9N - 18 & (616) \\
P_{369} &= 513N^6 + 1845N^5 + 2269N^4 + 1187N^3 + 538N^2 + 288N + 144 & (617) \\
P_{370} &= 743N^6 + 4724N^5 + 9411N^4 + 7922N^3 + 9340N^2 + 10208N - 768 & (618) \\
P_{371} &= 799N^6 + 5280N^5 + 12295N^4 + 10866N^3 - 980N^2 - 6408N - 1728 & (619) \\
P_{372} &= 848N^6 + 2727N^5 + 4023N^4 + 5921N^3 + 3069N^2 + 3900N + 1440 & (620) \\
P_{373} &= 1069N^6 + 6630N^5 + 14503N^4 + 12330N^3 + 556N^2 - 3912N - 864 & (621) \\
P_{374} &= 1069N^6 + 6702N^5 + 15079N^4 + 13698N^3 + 844N^2 - 4776N - 864 & (622) \\
P_{375} &= 1087N^6 + 4415N^5 + 1621N^4 - 10867N^3 - 11540N^2 + 548N + 3504 & (623) \\
P_{376} &= 10875N^6 + 36321N^5 + 36361N^4 + 15419N^3 + 7336N^2 + 1248N - 792 & (624) \\
P_{377} &= -1859N^7 - 10827N^6 - 22970N^5 - 23466N^4 - 15995N^3 - 8151N^2 \\
&\quad - 144N + 36 & (625) \\
P_{378} &= -777N^7 - 3738N^6 - 5972N^5 + 10N^4 + 5057N^3 - 2044N^2 + 2112N \\
&\quad + 1008 & (626) \\
P_{379} &= -715N^7 - 3476N^6 - 5448N^5 - 2380N^4 + 1129N^3 + 174N^2 - 756N - 216 & (627) \\
P_{380} &= -715N^7 - 3062N^6 - 3450N^5 + 518N^4 + 2803N^3 + 1182N^2 + 108N + 216 & (628) \\
P_{381} &= -559N^7 - 2855N^6 - 5215N^5 - 3779N^4 + 1904N^3 + 7012N^2 + 4212N + 144 & (629) \\
P_{382} &= -128N^7 - 786N^6 - 1347N^5 + 5412N^4 + 5805N^3 - 7692N^2 - 1252N + 2256 & (630) \\
P_{383} &= -100N^7 - 485N^6 - 286N^5 + 985N^4 - 88N^3 - 710N^2 + 2568N + 984 & (631) \\
P_{384} &= -44N^7 - 138N^6 - 86N^5 + 12N^4 + 37N^3 + 174N^2 + 51N + 6 & (632) \\
P_{385} &= -41N^7 - 226N^6 - 220N^5 + 478N^4 + 123N^3 - 24N^2 + 162N - 108 & (633) \\
P_{386} &= -27N^7 + 45N^6 - 363N^5 - 2217N^4 - 274N^3 + 1980N^2 - 168N + 160 & (634) \\
P_{387} &= -12N^7 - 31N^6 + 14N^5 + 86N^4 + 34N^3 - 71N^2 - 24N - 4 & (635) \\
P_{388} &= -5N^7 + 20N^5 - 14N^4 - 35N^3 + 30N^2 + 28N + 8 & (636) \\
P_{389} &= -5N^7 - 46N^6 - 122N^5 + 48N^4 + 483N^3 + 678N^2 + 1700N + 1872 & (637) \\
P_{390} &= -5N^7 + 1382N^6 + 5380N^5 + 7451N^4 + 6436N^3 + 2228N^2 - 5952N - 3168 & (638) \\
P_{391} &= -N^7 - 12N^5 - 50N^4 - 35N^3 + 30N^2 + 28N + 8 & (639) \\
P_{392} &= -N^7 - 7N^6 + 87N^5 - 281N^4 + 482N^3 - 548N^2 + 384N - 112 & (640) \\
P_{393} &= -N^7 - 2N^6 - 2N^5 + 104N^4 + 987N^3 + 2102N^2 + 640N - 756 & (641) \\
P_{394} &= -N^7 + N^6 - 10N^5 + 15N^4 + 97N^3 + 102N^2 + 76N + 24 & (642) \\
P_{395} &= -N^7 + 4N^6 + 10N^5 + 6N^4 - 25N^3 - 62N^2 - 12N + 8 & (643) \\
P_{396} &= N^7 - 5N^6 - 43N^5 - 19N^4 - 126N^3 - 440N^2 - 200N - 144 & (644) \\
P_{397} &= N^7 + 3N^6 - 55N^5 + 165N^4 - 114N^3 - 172N^2 + 256N - 80 & (645) \\
P_{398} &= N^7 + 3N^6 + 22N^5 + 97N^4 + 75N^3 + 22N^2 + 104N + 60 & (646) \\
P_{399} &= N^7 + 5N^6 + 7N^5 - 33N^4 - 168N^3 - 100N^2 + 176N + 96 & (647) \\
P_{400} &= 2N^7 - 40N^6 - 53N^5 + 61N^4 + 149N^3 - 49N^2 - 46N + 24 & (648)
\end{aligned}$$

$$\begin{aligned}
P_{401} &= 4N^7 + 16N^6 + 37N^5 + 36N^4 + 9N^3 - 10N^2 - 52N - 24 & (649) \\
P_{402} &= 5N^7 - 4N^6 - 3N^5 - 5N^4 - 65N^3 + 67N^2 + 55N - 30 & (650) \\
P_{403} &= 6N^7 + 26N^6 + 60N^5 + 112N^4 + 91N^3 - 29N^2 - 24N - 4 & (651) \\
P_{404} &= 7N^7 + 13N^6 - 80N^5 - 133N^4 + 163N^3 + 226N^2 - 92N - 24 & (652) \\
P_{405} &= 8N^7 + 45N^6 + 105N^5 + 138N^4 + 116N^3 + 95N^2 + 55N + 14 & (653) \\
P_{406} &= 11N^7 + 55N^6 + 83N^5 + 40N^4 + 20N^3 - 38N^2 - 93N - 6 & (654) \\
P_{407} &= 12N^7 + 22N^6 + 6N^5 - 140N^4 - 167N^3 - 170N^2 - 77N - 14 & (655) \\
P_{408} &= 13N^7 + 33N^6 - 43N^5 - 77N^4 - 362N^3 - 868N^2 - 424N - 96 & (656) \\
P_{409} &= 14N^7 + 62N^6 + 192N^5 + 332N^4 + 253N^3 + 187N^2 + 112N + 28 & (657) \\
P_{410} &= 24N^7 + 79N^6 + 85N^5 + 86N^4 + 60N^3 + 17N^2 + 19N + 14 & (658) \\
P_{411} &= 25N^7 + 445N^6 + 2127N^5 - 355N^4 - 9232N^3 - 1938N^2 + 5688N - 1080 & (659) \\
P_{412} &= 28N^7 + 104N^6 + 224N^5 + 211N^4 - 80N^3 + 13N^2 + 106N + 48 & (660) \\
P_{413} &= 28N^7 + 122N^6 + 344N^5 + 537N^4 + 436N^3 + 425N^2 + 242N + 80 & (661) \\
P_{414} &= 28N^7 + 126N^6 + 236N^5 + 187N^4 - 14N^3 - 15N^2 + 24N + 12 & (662) \\
P_{415} &= 28N^7 + 128N^6 + 252N^5 + 241N^4 + 102N^3 + 101N^2 + 72N + 28 & (663) \\
P_{416} &= 32N^7 + 16N^6 + 246N^5 - 1198N^4 - 1034N^3 + 624N^2 + 1089N + 378 & (664) \\
P_{417} &= 43N^7 - 143N^6 - 1196N^5 - 1067N^4 + 676N^3 - 248N^2 + 909N + 738 & (665) \\
P_{418} &= 45N^7 + 249N^6 + 709N^5 + 1099N^4 - 186N^3 - 1204N^2 - 328N - 96 & (666) \\
P_{419} &= 49N^7 + 134N^6 + 51N^5 - 509N^4 - 1376N^3 - 825N^2 + 616N + 564 & (667) \\
P_{420} &= 56N^7 + 283N^6 + 383N^5 - 97N^4 - 465N^3 - 196N^2 - 28N - 32 & (668) \\
P_{421} &= 101N^7 + 550N^6 + 421N^5 - 1555N^4 - 258N^3 + 57N^2 - 504N + 324 & (669) \\
P_{422} &= 131N^7 + 1216N^6 + 3842N^5 + 4834N^4 + 1883N^3 - 398N^2 - 348N - 72 & (670) \\
P_{423} &= 188N^7 + 904N^6 + 1443N^5 + 636N^4 - 339N^3 - 1236N^2 - 1724N - 160 & (671) \\
P_{424} &= 361N^7 + 1201N^6 - 1633N^5 - 6185N^4 - 40N^3 + 6376N^2 + 1024N - 2544 & (672) \\
P_{425} &= 799N^7 + 4697N^6 + 7231N^5 - 2293N^4 - 12926N^3 - 4348N^2 + 6408N + 1728 & (673) \\
P_{426} &= 896N^7 + 4597N^6 + 5607N^5 - 2578N^4 - 5167N^3 - 2691N^2 - 3712N + 3696 & (674) \\
P_{427} &= 1385N^7 + 5386N^6 + 3624N^5 - 6814N^4 - 10313N^3 - 3408N^2 + 360N - 216 & (675) \\
P_{428} &= 1385N^7 + 6628N^6 + 10410N^5 + 5048N^4 - 1763N^3 - 96N^2 + 1872N + 648 & (676) \\
P_{429} &= 3001N^7 + 14078N^6 + 23382N^5 - 4526N^4 - 18883N^3 + 42660N^2 + 7128N \\
&\quad - 432 & (677) \\
P_{430} &= 3331N^7 + 16277N^6 + 25231N^5 + 25871N^4 + 21142N^3 - 20332N^2 - 35952N \\
&\quad - 23904 & (678) \\
P_{431} &= 4148N^7 + 21847N^6 + 33491N^5 - 1529N^4 - 34939N^3 - 15782N^2 - 108N + 648 & (679) \\
P_{432} &= -306723N^8 - 1247148N^7 - 1970434N^6 - 1388892N^5 - 354099N^4 - 91504N^3 \\
&\quad - 51504N^2 - 2736N + 4752 & (680) \\
P_{433} &= -531N^8 - 4334N^7 - 11238N^6 - 5704N^5 + 18857N^4 + 31974N^3 + 14368N^2 \\
&\quad - 5952N - 4608 & (681) \\
P_{434} &= -521N^8 - 2300N^7 + 350N^6 + 7048N^5 + 4991N^4 - 9164N^3 - 16628N^2 \\
&\quad - 480N + 3744 & (682) \\
P_{435} &= -436N^8 + 260N^7 + 4375N^6 + 680N^5 - 14813N^4 - 11410N^3 + 8516N^2 + 4512N
\end{aligned}$$

$$\begin{aligned}
& -3024 & (683) \\
P_{436} & = -392N^8 + 1192N^7 + 7397N^6 + 2398N^5 - 17569N^4 - 12656N^3 + 14578N^2 \\
& + 7680N - 4248 & (684) \\
P_{437} & = -263N^8 - 1100N^7 - 70N^6 + 3184N^5 + 2561N^4 - 2516N^3 - 4340N^2 - 144N \\
& + 960 & (685) \\
P_{438} & = -185N^8 - 803N^7 - 352N^6 + 2677N^5 + 2636N^4 + 970N^3 + 1897N^2 - 684N \\
& - 972 & (686) \\
P_{439} & = -137N^8 - 563N^7 - 346N^6 + 814N^5 + 695N^4 - 935N^3 - 1172N^2 + 156N \\
& + 192 & (687) \\
P_{440} & = -103N^8 + 663N^7 - 485N^6 + 1901N^5 + 460N^4 - 14164N^3 + 10688N^2 \\
& + 7136N - 8256 & (688) \\
P_{441} & = -70N^8 - 913N^7 - 4936N^6 - 14466N^5 - 24422N^4 - 23673N^3 - 13852N^2 \\
& - 4356N + 864 & (689) \\
P_{442} & = -67N^8 - 286N^7 - 62N^6 + 716N^5 + 577N^4 - 718N^3 - 1384N^2 - 216N \\
& + 144 & (690) \\
P_{443} & = -49N^8 - 583N^7 - 1331N^6 - 904N^5 + 757N^4 + 875N^3 - 2329N^2 - 1512N \\
& + 540 & (691) \\
P_{444} & = -9N^8 + 9N^7 + 81N^6 - 59N^5 - 140N^4 - 238N^3 + 428N^2 + 120N - 144 & (692) \\
P_{445} & = -7N^8 - 67N^7 - 257N^6 - 497N^5 - 426N^4 - 268N^3 - 334N^2 - 232N - 56 & (693) \\
P_{446} & = -N^8 - 7N^7 + 9N^6 + 22N^5 + 30N^4 - 76N^3 - 30N^2 + 27N - 6 & (694) \\
P_{447} & = -N^8 + 5N^7 - 9N^6 + 21N^5 - 15N^4 - 23N^3 + 3N^2 + 5N - 2 & (695) \\
P_{448} & = N^8 - 3N^7 - 33N^6 - 15N^5 + 157N^4 + 178N^3 - 61N^2 - 28N - 4 & (696) \\
P_{449} & = N^8 + 9N^7 + 71N^6 + 387N^5 + 816N^4 + 404N^3 + 184N^2 + 804N + 396 & (697) \\
P_{450} & = 2N^8 - 13N^7 + 58N^6 - 126N^5 + 34N^4 + 75N^3 + 46N^2 - 28N - 16 & (698) \\
P_{451} & = 2N^8 + 15N^7 + 44N^6 + 56N^5 + 56N^4 + 105N^3 + 118N^2 + 44N + 8 & (699) \\
P_{452} & = 3N^8 + 29N^7 + 102N^6 + 154N^5 + 69N^4 - 161N^3 - 110N^2 + 994N + 1224 & (700) \\
P_{453} & = 4N^8 - 13N^7 - 46N^6 + 96N^5 + 158N^4 + N^3 - 452N^2 - 84N + 144 & (701) \\
P_{454} & = 4N^8 - 3N^7 - 96N^6 + 4N^5 + 454N^4 - 221N^3 - 738N^2 + 468N + 576 & (702) \\
P_{455} & = 4N^8 + 4N^7 - 8N^6 + 8N^5 - 93N^4 - 198N^3 + 43N^2 + 84N + 28 & (703) \\
P_{456} & = 4N^8 + 10N^7 + 11N^6 - 19N^5 - 65N^4 - 63N^3 - 90N^2 - 4N + 24 & (704) \\
P_{457} & = 4N^8 + 24N^7 - 57N^6 + 109N^5 - 32N^4 - 141N^3 - 8N^2 + 75N + 30 & (705) \\
P_{458} & = 4N^8 + 27N^7 + 90N^6 + 208N^5 + 300N^4 + 257N^3 + 190N^2 + 140N + 40 & (706) \\
P_{459} & = 5N^8 + 2N^7 - 8N^6 + 3N^5 - 7N^4 + 14N^3 - 24N^2 - 11N + 10 & (707) \\
P_{460} & = 7N^8 - 8N^7 + 6N^6 - 45N^5 + 27N^4 + 68N^3 - 30N^2 - 23N + 14 & (708) \\
P_{461} & = 7N^8 + 13N^7 + 19N^6 + 27N^5 - 136N^4 - 118N^3 + 388N^2 + 392N + 112 & (709) \\
P_{462} & = 8N^8 + 37N^7 + 127N^6 + 127N^5 - 175N^4 - 104N^3 + 408N^2 + 304N + 80 & (710) \\
P_{463} & = 8N^8 + 604N^7 + 1583N^6 - 656N^5 - 1330N^4 + 2716N^3 - 837N^2 - 468N + 108 & (711) \\
P_{464} & = 10N^8 + 4N^7 + 90N^6 - 116N^5 + 387N^4 + 187N^3 - 145N^2 - 39N + 54 & (712) \\
P_{465} & = 12N^8 - 34N^7 - 287N^6 - 235N^5 + 347N^4 + 653N^3 - 180N^2 - 132N + 96 & (713) \\
P_{466} & = 12N^8 + 69N^7 + 272N^6 + 1450N^5 + 3076N^4 + 297N^3 - 784N^2 + 3792N
\end{aligned}$$

$$\begin{aligned}
& +1800 & (714) \\
P_{467} & = 14N^8 + 17N^7 - 104N^6 - 68N^5 - 252N^4 - 289N^3 + 678N^2 + 268N + 24 & (715) \\
P_{468} & = 14N^8 + 48N^7 + 83N^6 - 174N^5 - 518N^4 + 146N^3 + 241N^2 + 176N + 44 & (716) \\
P_{469} & = 14N^8 + 56N^7 + 81N^6 + 62N^5 + 13N^4 - 4N^3 - N^2 + 2N + 1 & (717) \\
P_{470} & = 14N^8 + 56N^7 + 89N^6 + 74N^5 + 29N^4 + 20N^3 + 15N^2 + 6N + 1 & (718) \\
P_{471} & = 17N^8 + 8N^7 - 188N^6 - 170N^5 - 153N^4 + 362N^3 + 1380N^2 + 88N - 192 & (719) \\
P_{472} & = 20N^8 - 23N^7 - 92N^5 + 148N^4 - 19N^3 - 4N^2 - 2N + 4 & (720) \\
P_{473} & = 23N^8 + 110N^7 + 373N^6 + 376N^5 - 398N^4 - 16N^3 + 1504N^2 + 1088N + 288 & (721) \\
P_{474} & = 25N^8 + 21N^7 - 62N^6 - 495N^5 - 209N^4 + 1768N^3 + 464N^2 - 1360N - 944 & (722) \\
P_{475} & = 31N^8 + 528N^7 + 318N^6 - 2460N^5 - 3105N^4 + 3012N^3 + 6860N^2 - 864N & \\
& \quad -1728 & (723) \\
P_{476} & = 51N^8 + 207N^7 + 64N^6 - 657N^5 - 598N^4 - 58N^3 - 177N^2 + 148N + 156 & (724) \\
P_{477} & = 80N^8 + 599N^7 + 1764N^6 + 2168N^5 + 2034N^4 + 4193N^3 + 5014N^2 + 1932N & \\
& \quad +360 & (725) \\
P_{478} & = 85N^8 - 413N^7 - 1792N^6 + 22N^5 + 5300N^4 + 3589N^3 - 3791N^2 - 1920N & \\
& \quad +1188 & (726) \\
P_{479} & = 116N^8 + 688N^7 + 1772N^6 + 2705N^5 + 2454N^4 + 1347N^3 + 784N^2 + 396N & \\
& \quad +176 & (727) \\
P_{480} & = 116N^8 + 738N^7 + 1936N^6 + 2615N^5 + 1502N^4 - 137N^3 - 192N^2 & \\
& \quad +60N + 48 & (728) \\
P_{481} & = 125N^8 + 1314N^7 + 5027N^6 + 9183N^5 + 9122N^4 + 4371N^3 + 846N^2 & \\
& \quad +1620N + 648 & (729) \\
P_{482} & = 136N^8 + 900N^7 + 1417N^6 - 1308N^5 - 4442N^4 - 3972N^3 - 3963N^2 & \\
& \quad -2268N - 324 & (730) \\
P_{483} & = 253N^8 + 1180N^7 - 518N^6 - 4088N^5 - 2571N^4 + 6716N^3 + 12532N^2 + 512N & \\
& \quad -2784 & (731) \\
P_{484} & = 269N^8 + 1889N^7 + 3838N^6 + 320N^5 - 6005N^4 - 4429N^3 + 170N^2 + 420N & \\
& \quad +72 & (732) \\
P_{485} & = 293N^8 + 1989N^7 + 6150N^6 + 11150N^5 + 8593N^4 + 1561N^3 + 12564N^2 & \\
& \quad +20628N + 10800 & (733) \\
P_{486} & = 541N^8 + 2100N^7 + 3278N^6 + 2212N^5 + 1357N^4 + 880N^3 + 528N^2 + 320N & \\
& \quad +112 & (734) \\
P_{487} & = 1214N^8 + 4856N^7 + 7284N^6 + 4937N^5 - 730N^4 + 1782N^3 + 2268N^2 + 1377N & \\
& \quad +324 & (735) \\
P_{488} & = 2599N^8 + 18886N^7 + 49546N^6 + 53608N^5 + 15163N^4 - 9566N^3 - 2256N^2 & \\
& \quad +3528N + 432 & (736) \\
P_{489} & = 2599N^8 + 18976N^7 + 44560N^6 + 25186N^5 - 41987N^4 - 65474N^3 - 33432N^2 & \\
& \quad -8136N - 3024 & (737) \\
P_{490} & = 3081N^8 + 21046N^7 + 53564N^6 + 46888N^5 - 26329N^4 - 55730N^3 - 29056N^2 & \\
& \quad -41472N - 12384 & (738)
\end{aligned}$$

$$P_{491} = 3412N^8 + 13756N^7 + 15943N^6 + 1681N^5 - 3785N^4 - 5141N^3 - 19998N^2 - 8676N - 2376 \quad (739)$$

$$P_{492} = 23595N^8 + 94860N^7 + 146258N^6 + 100860N^5 + 22875N^4 + 4400N^3 + 2928N^2 - 144N - 432 \quad (740)$$

$$P_{493} = 23595N^8 + 102732N^7 + 177746N^6 + 152124N^5 + 54363N^4 + 5360N^3 - 1680N^2 - 1872N - 432 \quad (741)$$

$$P_{494} = -235N^9 - 324N^8 + 503N^7 - 1189N^6 + 4599N^5 + 6718N^4 - 8011N^3 - 1233N^2 + 2952N - 324 \quad (742)$$

$$P_{495} = -195N^9 - 1472N^8 - 3139N^7 + 1553N^6 + 11481N^5 + 9208N^4 + 4501N^3 + 5003N^2 - 2856N - 3348 \quad (743)$$

$$P_{496} = -28N^9 - 154N^8 - 353N^7 - 719N^6 - 1138N^5 - 865N^4 - 720N^3 - 611N^2 - 300N - 60 \quad (744)$$

$$P_{497} = -16N^9 - 72N^8 - 29N^7 + 206N^6 + 835N^5 + 959N^4 + 985N^3 + 595N^2 + 203N + 30 \quad (745)$$

$$P_{498} = -5N^9 - 197N^8 - 1658N^7 - 6602N^6 - 9017N^5 + 15379N^4 + 57408N^3 + 56292N^2 + 12240N - 10368 \quad (746)$$

$$P_{499} = -4N^9 - 28N^8 - 111N^7 - 194N^6 - 117N^5 + 44N^4 + 290N^3 + 580N^2 + 424N + 112 \quad (747)$$

$$P_{500} = -3N^9 + 10N^8 - 8N^7 + 30N^6 - 36N^5 - 6N^4 + 62N^3 - 24N^2 - 23N + 14 \quad (748)$$

$$P_{501} = -N^9 - 6N^8 - 20N^7 - 38N^6 - 65N^5 - 60N^4 - 14N^3 + 20N^2 + 104N + 48 \quad (749)$$

$$P_{502} = -N^9 + 13N^8 + 30N^7 - 86N^6 - 277N^5 - 167N^4 + 112N^3 + 152N^2 + 80N + 16 \quad (750)$$

$$P_{503} = N^9 + 19N^8 + 108N^7 + 188N^6 - 213N^5 - 1111N^4 - 1512N^3 - 1192N^2 - 672N - 144 \quad (751)$$

$$P_{504} = 3N^9 + 44N^8 + 237N^7 + 342N^6 - 1093N^5 - 2454N^4 + 1513N^3 + 1948N^2 - 2556N - 288 \quad (752)$$

$$P_{505} = 4N^9 + 28N^8 + 176N^7 + 756N^6 + 1463N^5 + 1172N^4 + 403N^3 + 258N^2 + 116N + 24 \quad (753)$$

$$P_{506} = 8N^9 + 65N^8 + 302N^7 + 176N^6 - 3100N^5 - 4735N^4 + 4950N^3 + 6510N^2 - 648N + 1080 \quad (754)$$

$$P_{507} = 9N^9 + 17N^8 + 35N^7 - 42N^6 + 108N^5 - 22N^4 - 20N^3 + 31N^2 - 24N - 28 \quad (755)$$

$$P_{508} = 29N^9 + 91N^8 - 1840N^7 - 14648N^6 - 44203N^5 - 64085N^4 - 46458N^3 - 21846N^2 - 8568N + 1080 \quad (756)$$

$$P_{509} = 37N^9 + 287N^8 + 1021N^7 + 1977N^6 + 1334N^5 - 1200N^4 - 992N^3 + 1616N^2 + 1376N + 384 \quad (757)$$

$$P_{510} = 75N^9 + 170N^8 - 1700N^7 - 5464N^6 + 739N^5 + 15122N^4 + 16366N^3 + 4500N^2 - 4752N + 2592 \quad (758)$$

$$P_{511} = 76N^9 + 1002N^8 + 5101N^7 + 14103N^6 + 25360N^5 + 29769N^4 + 17235N^3 + 4158N^2 + 5508N + 1944 \quad (759)$$

$$P_{512} = 93N^9 + 152N^8 - 173N^7 + 367N^6 - 1609N^5 - 2430N^4 + 2577N^3 + 843N^2 - 1080N + 108 \quad (760)$$

$$P_{513} = 142N^9 - 354N^8 - 1086N^7 + 3057N^6 + 3105N^5 - 8769N^4 - 6553N^3 + 4014N^2 + 3636N + 216 \quad (761)$$

$$P_{514} = 158N^9 + 1721N^8 + 6837N^7 + 10006N^6 - 3098N^5 - 14157N^4 + 4347N^3 + 3798N^2 - 5076N + 648 \quad (762)$$

$$P_{515} = 173N^9 + 741N^8 + 423N^7 + 303N^6 - 1644N^5 - 6816N^4 + 5098N^3 + 3450N^2 - 3672N - 2376 \quad (763)$$

$$P_{516} = 203N^9 + 756N^8 + 15N^7 - 5034N^6 - 1836N^5 + 8223N^4 - 5618N^3 + 807N^2 + 3024N + 1188 \quad (764)$$

$$P_{517} = 294N^9 + 1836N^8 + 3347N^7 - 890N^6 - 9772N^5 - 10517N^4 + 2975N^3 + 12595N^2 + 4344N + 972 \quad (765)$$

$$P_{518} = 673N^9 + 4111N^8 + 7483N^7 - 1707N^6 - 18990N^5 - 16638N^4 + 3722N^3 + 10442N^2 - 16N + 552 \quad (766)$$

$$P_{519} = -102241N^{10} - 503813N^9 - 586422N^8 + 611142N^7 + 1447647N^6 + 282675N^5 - 722888N^4 - 354868N^3 + 29568N^2 - 10944N - 6912 \quad (767)$$

$$P_{520} = -2895N^{10} - 14571N^9 - 28822N^8 - 30262N^7 - 15467N^6 - 8303N^5 - 7456N^4 - 5600N^3 - 3296N^2 - 1760N - 480 \quad (768)$$

$$P_{521} = -2895N^{10} - 14507N^9 - 17370N^8 + 17754N^7 + 43745N^6 + 7693N^5 - 23608N^4 - 13244N^3 - 4736N^2 + 512N + 512 \quad (769)$$

$$P_{522} = -1300N^{10} - 6471N^9 - 5898N^8 + 15033N^7 + 23049N^6 - 5703N^5 - 8642N^4 + 7941N^3 - 3429N^2 - 3888N - 324 \quad (770)$$

$$P_{523} = -1037N^{10} - 5395N^9 - 9095N^8 - 14470N^7 + 5890N^6 + 124193N^5 + 92793N^4 - 103428N^3 - 55323N^2 + 35964N - 972 \quad (771)$$

$$P_{524} = -695N^{10} - 4934N^9 - 10384N^8 - 1718N^7 + 15950N^6 + 17014N^5 + 5092N^4 - 10254N^3 - 16335N^2 - 8316N + 1620 \quad (772)$$

$$P_{525} = -559N^{10} - 5091N^9 - 14922N^8 - 5760N^7 + 49539N^6 + 92367N^5 + 63754N^4 + 51540N^3 + 84276N^2 + 43416N - 12960 \quad (773)$$

$$P_{526} = -335N^{10} - 2167N^9 - 2175N^8 + 2226N^7 - 4038N^6 + 8349N^5 + 35642N^4 - 8030N^3 - 12408N^2 + 4536N + 4320 \quad (774)$$

$$P_{527} = -55N^{10} - 275N^9 - 392N^8 - 62N^7 + 289N^6 + 481N^5 - 1006N^4 - 2376N^3 + 516N^2 + 1008N + 336 \quad (775)$$

$$P_{528} = N^{10} + N^9 - 8N^8 + 30N^7 + 241N^6 + 585N^5 + 232N^4 - 460N^3 + 130N^2 + 8N + 8 \quad (776)$$

$$P_{529} = 3N^{10} - 15N^9 + 38N^8 - 68N^7 + 28N^6 + 50N^5 + 8N^4 - 20N^3 + 5N^2 + 5N - 2 \quad (777)$$

$$P_{530} = 5N^{10} + 2N^9 - 17N^8 - 66N^7 - 107N^6 + 172N^5 + 431N^4 + 92N^3 - 80N^2 - 16N + 16 \quad (778)$$

$$P_{531} = 5N^{10} + 6N^9 - 20N^8 - 44N^7 - 10N^6 + 205N^5 + 447N^4 + 161N^3 - 194N^2 - 36N + 56 \quad (779)$$

$$P_{532} = 6N^{10} + 32N^9 + 101N^8 + 175N^7 + 236N^6 + 379N^5 + 327N^4 + 208N^3 + 488N^2 + 400N + 112 \quad (780)$$

$$P_{533} = 7N^{10} + 23N^9 - 13N^8 - 105N^7 - 315N^6 - 592N^5 - 193N^4 + 100N^3 - 448N^2 - 208N - 48 \quad (781)$$

$$P_{534} = 11N^{10} + 99N^9 + 273N^8 + 6N^7 - 1002N^6 - 723N^5 + 145N^4 - 3090N^3 - 2307N^2 + 2592N + 540 \quad (782)$$

$$P_{535} = 11N^{10} + 191N^9 + 323N^8 - 1574N^7 - 1837N^6 + 2695N^5 + 1533N^4 - 2392N^3 - 2262N^2 + 1368N + 216 \quad (783)$$

$$P_{536} = 12N^{10} + 94N^9 + 350N^8 + 676N^7 + 1073N^6 + 1944N^5 + 2467N^4 + 1840N^3 + 1128N^2 + 512N + 112 \quad (784)$$

$$P_{537} = 20N^{10} + 331N^9 + 775N^8 + 664N^7 + 950N^6 + 2083N^5 + 1587N^4 + 738N^3 + 3436N^2 + 648N - 864 \quad (785)$$

$$P_{538} = 40N^{10} + 391N^9 + 1292N^8 + 1360N^7 - 775N^6 - 1709N^5 - 1826N^4 - 12102N^3 - 22635N^2 - 8100N + 2592 \quad (786)$$

$$P_{539} = 41N^{10} + 37N^9 - 76N^8 + 74N^7 - 243N^6 - 427N^5 + 1358N^4 + 1596N^3 + 200N^2 - 192N - 64 \quad (787)$$

$$P_{540} = 47N^{10} - 155N^9 - 479N^8 - 290N^7 - 1249N^6 + 2227N^5 + 9897N^4 + 2226N^3 - 6200N^2 - 1128N + 1584 \quad (788)$$

$$P_{541} = 56N^{10} + 456N^9 + 1470N^8 + 2491N^7 + 2381N^6 + 890N^5 - 529N^4 - 329N^3 + 162N^2 + 180N + 40 \quad (789)$$

$$P_{542} = 119N^{10} + 664N^9 + 829N^8 - 182N^7 - 1183N^6 - 512N^5 + 1647N^4 + 4518N^3 + 5356N^2 - 24N - 864 \quad (790)$$

$$P_{543} = 133N^{10} - 601N^9 - 949N^8 - 970N^7 - 7451N^6 + 2705N^5 + 28755N^4 + 8634N^3 - 24232N^2 - 5448N + 5904 \quad (791)$$

$$P_{544} = 149N^{10} + 838N^9 + 1068N^8 - 5247N^7 - 8718N^6 + 12867N^5 + 4483N^4 - 28114N^3 + 10794N^2 + 16848N + 5400 \quad (792)$$

$$P_{545} = 227N^{10} + 97N^9 - 497N^8 - 470N^7 - 3985N^6 + 7015N^5 + 31839N^4 + 9462N^3 - 14768N^2 - 3000N + 3888 \quad (793)$$

$$P_{546} = 265N^{10} + 2057N^9 + 8292N^8 + 10767N^7 - 7962N^6 - 6501N^5 + 9680N^4 - 23027N^3 - 15N^2 + 11736N + 5076 \quad (794)$$

$$P_{547} = 380N^{10} + 2639N^9 + 5326N^8 + 116N^7 - 9551N^6 - 8041N^5 - 484N^4 + 3666N^3 + 5589N^2 + 3492N - 540 \quad (795)$$

$$P_{548} = 541N^{10} + 2801N^9 + 5288N^8 + 4554N^7 + 2293N^6 - 11N^5 - 5338N^4 - 5232N^3 - 2800N^2 - 208N + 160 \quad (796)$$

$$P_{549} = 743N^{10} + 7696N^9 + 25621N^8 + 16138N^7 - 72099N^6 - 123940N^5 + 17403N^4 + 207482N^3 + 156396N^2 - 31536N - 38016 \quad (797)$$

$$P_{550} = 4148N^{10} + 32491N^9 + 99529N^8 + 149344N^7 + 106516N^6 + 8455N^5 - 38095N^4 - 6924N^3 + 20016N^2 + 15552N + 3888 \quad (798)$$

$$P_{551} = 10875N^{10} + 65463N^9 + 139552N^8 + 92374N^7 - 87581N^6 - 165293N^5 - 63158N^4 - 1040N^3 - 552N^2 - 13104N - 5184 \quad (799)$$

$$P_{552} = 24535N^{10} + 124997N^9 + 228788N^8 + 184556N^7 + 118649N^6 - 32641N^5 - 263792N^4 + 13608N^3 - 188532N^2 - 184248N - 57024 \quad (800)$$

$$P_{553} = -1859N^{11} - 18263N^{10} - 61876N^9 - 64472N^8 + 73286N^7 + 177838N^6 + 43720N^5 - 72000N^4 - 46503N^3 - 55251N^2 - 48816N + 1620 \quad (801)$$

$$P_{554} = -377N^{11} - 207N^{10} + 7199N^9 + 20907N^8 + 12912N^7 - 22608N^6 - 18769N^5 + 5463N^4 - 23573N^3 - 1287N^2 + 20880N + 9828 \quad (802)$$

$$P_{555} = -130N^{11} - 1031N^{10} - 2729N^9 - 2426N^8 + 2666N^7 + 7057N^6 - 8397N^5 - 29316N^4 - 30218N^3 - 23388N^2 + 1080N + 3888 \quad (803)$$

$$P_{556} = -44N^{11} - 226N^{10} - 21N^9 + 1281N^8 + 1182N^7 - 1707N^6 - 2068N^5 + 1228N^4 + 1635N^3 - 648N^2 + 252N + 432 \quad (804)$$

$$P_{557} = -12N^{11} - 55N^{10} + 22N^9 + 377N^8 + 238N^7 - 779N^6 - 624N^5 + 429N^4 + 472N^3 + 268N^2 - 336N - 288 \quad (805)$$

$$P_{558} = -N^{11} - 84N^{10} + 367N^9 - 180N^8 - 1321N^7 + 2558N^6 - 1007N^5 - 2224N^4 + 2306N^3 + 402N^2 - 888N + 280 \quad (806)$$

$$P_{559} = 2N^{11} + 5N^{10} + 31N^9 + 22N^8 - 39N^7 - 70N^6 - 65N^5 + 120N^4 + 15N^3 - 97N^2 + 28 \quad (807)$$

$$P_{560} = 3N^{11} + 30N^{10} - 117N^9 - 326N^8 + 145N^7 - 1822N^6 - 1399N^5 + 3918N^4 + 3624N^3 + 3192N^2 - 4176N - 2304 \quad (808)$$

$$P_{561} = 11N^{11} - 470N^{10} - 393N^9 + 2532N^8 - 5715N^7 - 1926N^6 + 29881N^5 - 2980N^4 - 9096N^3 + 18828N^2 + 6912N - 1296 \quad (809)$$

$$P_{562} = 16N^{11} - 188N^{10} + 263N^9 - 1632N^8 + 3240N^7 - 5097N^6 - 3418N^5 + 740N^4 + 1348N^3 - 1329N^2 - 369N + 378 \quad (810)$$

$$P_{563} = 56N^{11} + 382N^{10} + 982N^9 + 1045N^8 - 326N^7 - 2177N^6 - 1575N^5 + 832N^4 + 743N^3 - 150N^2 - 444N - 136 \quad (811)$$

$$P_{564} = 94N^{11} + 761N^{10} + 2081N^9 + 1940N^8 - 1586N^7 - 5059N^6 + 1989N^5 + 14322N^4 + 17366N^3 + 12660N^2 - 936N - 2160 \quad (812)$$

$$P_{565} = 473N^{11} + 2347N^{10} + 2874N^9 - 2235N^8 - 5646N^7 - 1068N^6 + 634N^5 - 835N^4 + 1593N^3 + 891N^2 + 432N + 108 \quad (813)$$

$$P_{566} = 896N^{11} + 7285N^{10} + 12349N^9 - 37301N^8 - 120535N^7 - 3191N^6 + 258447N^5 + 197745N^4 - 122645N^3 - 171450N^2 + 19008N + 35424 \quad (814)$$

$$P_{567} = 2310N^{11} + 16134N^{10} + 38381N^9 + 22356N^8 - 67974N^7 - 128568N^6 + 6525N^5 + 103266N^4 - 22874N^3 + 17052N^2 + 21600N + 7344 \quad (815)$$

$$P_{568} = 3331N^{11} + 26270N^{10} + 51231N^9 - 57642N^8 - 238659N^7 - 113214N^6 + 13301N^5 - 284390N^4 - 137220N^3 + 669168N^2 + 259632N - 399168 \quad (816)$$

$$P_{569} = 4148N^{11} + 22259N^{10} + 16296N^9 - 96255N^8 - 182904N^7 - 14577N^6 + 182032N^5 + 126841N^4 + 6672N^3 - 13968N^2 + 7776N + 3888 \quad (817)$$

$$P_{570} = -202N^{12} - 1212N^{11} - 1953N^{10} + 1966N^9 + 7125N^8 + 2994N^7 - 4181N^6 - 8280N^5 - 9213N^4 - 868N^3 + 2160N^2 + 432N - 432 \quad (818)$$

$$P_{571} = -100N^{12} - 1085N^{11} - 4256N^{10} - 7025N^9 - 4126N^8 + 1735N^7 + 7416N^6 - 11223N^5 - 71498N^4 - 100698N^3 - 49692N^2 + 9432N + 9936 \quad (819)$$

$$P_{572} = -55N^{12} - 495N^{11} - 1688N^{10} - 2586N^9 - 603N^8 + 3585N^7 + 6022N^6 + 6528N^5 + 4364N^4 + 2496N^3 + 5856N^2 + 4800N + 1344 \quad (820)$$

$$P_{573} = -14N^{12} - 100N^{11} - 19N^{10} + 26N^9 + 541N^8 - 556N^7 - 62N^6 + 14N^5 - 93N^4 + 72N^3 + 163N^2 - 40N - 60 \quad (821)$$

$$P_{574} = 7N^{12} + 20N^{11} + 8N^{10} - 109N^9 - 186N^8 + 234N^7 + 496N^6 + 687N^5 + 1355N^4 + 184N^3 - 488N^2 - 16N + 112 \quad (822)$$

$$P_{575} = 14N^{12} + 174N^{11} + 1019N^{10} + 3349N^9 + 6946N^8 + 12660N^7 + 23003N^6 + 26429N^5 + 6974N^4 - 14616N^3 - 14496N^2 - 6256N - 1056 \quad (823)$$

$$P_{576} = 25N^{12} + 117N^{11} - 32N^{10} - 172N^9 - 23N^8 - 7795N^7 - 15834N^6 + 9242N^5 + 38328N^4 + 23664N^3 - 2304N^2 - 12960N - 4608 \quad (824)$$

$$P_{577} = 43N^{12} - 100N^{11} - 1924N^{10} - 1271N^9 + 11862N^8 + 9003N^7 - 21196N^6 - 19013N^5 + 14107N^4 + 24017N^3 - 14988N^2 - 4428N + 6480 \quad (825)$$

$$P_{578} = 75N^{12} + 652N^{11} - 443N^{10} - 5777N^9 + 5586N^8 + 22584N^7 - 11631N^6 - 45601N^5 + 18305N^4 + 47798N^3 - 42132N^2 - 10152N + 5184 \quad (826)$$

$$P_{579} = 96N^{12} + 409N^{11} - 233N^{10} - 5951N^9 - 10686N^8 + 21813N^7 + 42633N^6 - 33655N^5 - 60658N^4 - 6808N^3 - 20832N^2 - 15984N - 6048 \quad (827)$$

$$P_{580} = 559N^{12} + 1825N^{11} + 14894N^{10} + 7082N^9 - 131685N^8 - 209955N^7 + 210920N^6 + 820088N^5 + 405808N^4 - 635312N^3 - 336768N^2 + 129024N + 96768 \quad (828)$$

$$P_{581} = 848N^{12} + 5271N^{11} + 9132N^{10} - 2123N^9 - 19026N^8 - 3075N^7 + 30280N^6 + 23259N^5 - 4602N^4 + 2684N^3 + 20808N^2 + 2208N - 3456 \quad (829)$$

$$P_{582} = 1136N^{12} + 18501N^{11} + 135921N^{10} + 571962N^9 + 1481748N^8 + 2454501N^7 + 2689333N^6 + 1935348N^5 + 846054N^4 + 386856N^3 + 437616N^2 + 246240N + 54432 \quad (830)$$

$$P_{583} = 1706N^{12} + 10290N^{11} + 16191N^{10} - 13469N^9 - 48873N^8 - 7212N^7 + 63553N^6 + 59895N^5 + 5655N^4 - 20920N^3 - 4320N^2 + 4464N + 3024 \quad (831)$$

$$P_{584} = 7159N^{12} + 43302N^{11} + 82301N^{10} + 16020N^9 - 116257N^8 - 98698N^7 + 17295N^6 - 25720N^5 - 78706N^4 + 38456N^3 + 35936N^2 + 29664N + 7776 \quad (832)$$

$$P_{585} = -777N^{13} - 6846N^{12} - 19922N^{11} - 7090N^{10} + 67924N^9 + 103426N^8 - 5038N^7 - 136214N^6 - 191123N^5 - 110260N^4 + 78808N^3 + 98520N^2 - 5328N - 21600 \quad (833)$$

$$P_{586} = -535N^{13} - 3614N^{12} - 6137N^{11} + 22547N^{10} + 57627N^9 - 16644N^8 + 26215N^7 + 114761N^6 - 162448N^5 - 7034N^4 + 162498N^3 + 57492N^2 - 59400N - 29808 \quad (834)$$

$$P_{587} = -392N^{13} - 2048N^{12} - 7248N^{11} - 1420N^{10} + 36983N^9 + 54705N^8 + 5876N^7 - 82924N^6 - 90399N^5 + 138127N^4 + 192772N^3 - 36672N^2 - 47520N + 15120 \quad (835)$$

$$P_{588} = -172N^{13} - 1091N^{12} - 489N^{11} + 7770N^{10} + 7740N^9 - 11943N^8$$

$$\begin{aligned}
& -24125N^7 - 56548N^6 + 64494N^5 + 168732N^4 - 31032N^3 - 97416N^2 \\
& + 2592N + 23328 \tag{836} \\
P_{589} = & 294N^{13} + 2718N^{12} + 8114N^{11} + 3145N^{10} - 27706N^9 \\
& - 43252N^8 + 4735N^7 + 21551N^6 - 46438N^5 - 49016N^4 + 7181N^3 + 1998N^2 \\
& - 5796N - 1944 \tag{837} \\
P_{590} = & 709N^{13} + 943N^{12} + 411N^{11} + 6029N^{10} - 5923N^9 \\
& - 80745N^8 - 93874N^7 + 173546N^6 + 312405N^5 - 104825N^4 - 238028N^3 \\
& + 75360N^2 + 59616N - 29808 \tag{838} \\
P_{591} = & 1216N^{13} + 7013N^{12} + 7697N^{11} - 23330N^{10} - 88314N^9 \\
& - 43341N^8 + 234716N^7 + 100126N^6 - 272138N^5 + 145814N^4 + 75675N^3 \\
& - 20394N^2 - 47628N - 14904 \tag{839} \\
P_{592} = & 3001N^{13} + 26082N^{12} + 77868N^{11} + 34682N^{10} - 258930N^9 \\
& - 419730N^8 - 53068N^7 + 658734N^6 + 1592601N^5 + 1335568N^4 - 374352N^3 \\
& - 687528N^2 + 47952N + 163296 \tag{840} \\
P_{593} = & -306723N^{14} - 2340885N^{13} - 6607171N^{12} - 6904449N^{11} \\
& + 3987055N^{10} + 16593333N^9 + 13467975N^8 - 1608819N^7 - 8946032N^6 \\
& - 4282236N^5 - 815200N^4 + 587280N^3 - 209664N^2 - 478656N - 145152 \tag{841} \\
P_{594} = & -171287N^{14} - 1211897N^{13} - 3068659N^{12} - 2667841N^{11} \\
& + 1694631N^{10} + 4206249N^9 + 2089679N^8 + 2659013N^7 + 972772N^6 \\
& - 3123356N^5 + 3261648N^4 + 415176N^3 - 2010528N^2 - 1822176N - 476928 \tag{842} \\
P_{595} = & -2895N^{14} - 20361N^{13} - 51007N^{12} - 42853N^{11} + 26203N^{10} \\
& + 52297N^9 - 11709N^8 + 769N^7 + 82896N^6 + 25092N^5 - 26432N^4 - 57760N^3 \\
& - 25056N^2 + 1664 \tag{843} \\
P_{596} = & -17N^{14} - 79N^{13} - 61N^{12} + 615N^{11} + 1747N^{10} - 2965N^9 \\
& - 4979N^8 + 18209N^7 - 11626N^6 - 43220N^5 + 40632N^4 + 29552N^3 - 18784N^2 \\
& - 5568N + 8064 \tag{844} \\
P_{597} = & -16N^{14} - 248N^{13} - 1605N^{12} - 5865N^{11} - 13383N^{10} \\
& - 21272N^9 - 34866N^8 - 72922N^7 - 124252N^6 - 137177N^5 - 102690N^4 \\
& - 60360N^3 - 28240N^2 - 9168N - 1440 \tag{845} \\
P_{598} = & -15N^{14} - 34N^{13} + 63N^{12} + 162N^{11} - 281N^{10} - 912N^9 + 187N^8 + 4578N^7 \\
& + 8700N^6 + 3998N^5 - 3598N^4 - 864N^3 + 2160N^2 + 160N \\
& - 480 \tag{846} \\
P_{599} = & 65N^{14} - 1504N^{13} - 3135N^{12} + 19307N^{11} + 3465N^{10} - 98136N^9 \\
& + 109420N^8 + 97027N^7 - 330420N^6 + 135406N^5 + 262941N^4 \\
& - 271764N^3 - 39312N^2 + 86832N - 11664 \tag{847} \\
P_{600} = & 1711N^{14} + 11977N^{13} + 29375N^{12} + 19397N^{11} - 36003N^{10} - 66801N^9 \\
& - 29659N^8 + 56591N^7 + 168496N^6 + 73084N^5 - 124560N^4 - 50472N^3 \\
& + 18576N^2 + 30240N + 8640 \tag{848} \\
P_{601} = & 3081N^{14} + 36451N^{13} + 160822N^{12} + 265142N^{11} - 191472N^{10} - 1244948N^9 \\
& - 1268974N^8 + 458058N^7 + 1945999N^6 + 1996529N^5 + 616232N^4
\end{aligned}$$

$$\begin{aligned}
& -1244976N^3 - 1177272N^2 + 47088N + 220320 & (849) \\
P_{602} = & -368N^{15} - 496N^{14} - 2334N^{13} - 3335N^{12} + 18115N^{11} + 7032N^{10} - 6186N^9 \\
& +190861N^8 + 72423N^7 - 543466N^6 - 338338N^5 + 327996N^4 + 185840N^3 \\
& -168672N^2 - 35424N + 39744 & (850) \\
P_{603} = & 88N^{15} + 857N^{14} + 6099N^{13} + 5941N^{12} - 21569N^{11} - 38661N^{10} - 8499N^9 \\
& +7159N^8 - 60363N^7 - 9808N^6 + 57620N^5 - 280368N^4 - 138400N^3 \\
& +103200N^2 + 27648N - 24192 & (851) \\
P_{604} = & 203N^{15} + 2177N^{14} + 8464N^{13} + 11458N^{12} - 4055N^{11} - 17152N^{10} - 19631N^9 \\
& -30020N^8 + 33188N^7 + 227867N^6 + 421391N^5 + 272806N^4 - 95256N^3 \\
& -95616N^2 + 12528N + 18144 & (852) \\
P_{605} = & -1300N^{16} - 11671N^{15} - 23280N^{14} + 72661N^{13} + 294991N^{12} + 6582N^{11} \\
& -966464N^{10} - 557054N^9 + 1291890N^8 + 672533N^7 - 1214272N^6 - 94815N^5 \\
& +1410291N^4 + 287604N^3 - 798336N^2 - 241056N + 58320 & (853) \\
P_{606} = & 8N^{16} + 29N^{15} + 26N^{14} - 162N^{13} - 168N^{12} + 1130N^{11} + 695N^{10} - 2837N^9 \\
& -2364N^8 - 8153N^7 - 16755N^6 - 2799N^5 + 5774N^4 - 72N^3 - 2560N^2 + 80N \\
& +480 & (854) \\
P_{607} = & 149N^{16} + 2030N^{15} + 10972N^{14} + 27481N^{13} + 26215N^{12} - 21433N^{11} \\
& -116177N^{10} - 148583N^9 + 245582N^8 + 934607N^7 + 1604033N^6 \\
& +2493562N^5 + 1732746N^4 - 451152N^3 - 502848N^2 + 49248N + 85536 & (855) \\
P_{608} = & 1214N^{16} + 9712N^{15} + 24280N^{14} + 1053N^{13} - 88702N^{12} - 97444N^{11} \\
& +87242N^{10} + 150014N^9 - 43498N^8 - 31956N^7 + 127858N^6 + 63877N^5 \\
& -7306N^4 - 1944N^3 + 16848N^2 + 1296N - 2592 & (856) \\
P_{609} = & 946N^{17} + 7532N^{16} + 9325N^{15} - 65608N^{14} - 173636N^{13} + 141572N^{12} \\
& +759206N^{11} + 131590N^{10} - 1359784N^9 - 740252N^8 + 1131425N^7 \\
& +947158N^6 - 420762N^5 - 502920N^4 + 270144N^3 + 205344N^2 - 82080N \\
& -72576 & (857) \\
P_{610} = & 1711N^{17} + 22243N^{16} + 115213N^{15} + 273811N^{14} + 126835N^{13} - 922871N^{12} \\
& -2451337N^{11} - 2618887N^{10} - 953086N^9 + 204800N^8 + 50168N^7 + 280136N^6 \\
& +502080N^5 + 685152N^4 + 1756800N^3 + 1913472N^2 + 1009152N + 207360 & (858) \\
P_{611} = & 3331N^{17} + 21215N^{16} + 31660N^{15} - 96196N^{14} - 487514N^{13} - 298606N^{12} \\
& +1898000N^{11} + 1099708N^{10} - 4381585N^9 + 2587711N^8 + 7099436N^7 \\
& -10245920N^6 - 3591360N^5 + 14626872N^4 + 6996672N^3 - 2811456N^2 \\
& -1472256N + 217728 & (859) \\
P_{612} = & -1536N^{18} - 11971N^{17} - 25135N^{16} + 108682N^{15} + 571220N^{14} \\
& -186284N^{13} - 3832060N^{12} - 1817070N^{11} + 10172986N^{10} + 8764643N^9 \\
& -14073029N^8 - 15329356N^7 + 17717986N^6 + 8414044N^5 - 22518432N^4 \\
& +2633760N^3 + 7011360N^2 + 1031616N - 870912 & (860) \\
P_{613} = & -377N^{18} - 961N^{17} + 11118N^{16} + 40455N^{15} - 69189N^{14} - 398487N^{13} \\
& +70186N^{12} + 1667114N^{11} + 481053N^{10} - 3152601N^9 - 1445880N^8 \\
& +2408547N^7 + 2408989N^6 - 195499N^5 - 3562764N^4 + 127800N^3
\end{aligned}$$

$$\begin{aligned}
& +2246832N^2 + 94608N - 544320 \tag{861} \\
P_{614} = & 2801N^{18} + 24554N^{17} - 2860N^{16} - 142290N^{15} + 30162N^{14} + 286452N^{13} \\
& -1009366N^{12} - 2154796N^{11} + 2538563N^{10} + 10301106N^9 + 1319154N^8 \\
& -20831058N^7 - 3061678N^6 + 29370560N^5 + 4864952N^4 - 12022176N^3 \\
& +1577952N^2 + 2702592N - 777600 \tag{862} \\
P_{615} = & 7159N^{18} + 64779N^{17} + 161014N^{16} - 186916N^{15} - 1372538N^{14} - 1131058N^{13} \\
& +3030520N^{12} + 4986920N^{11} - 1784857N^{10} - 6856273N^9 - 1718422N^8 \\
& +2267620N^7 - 108620N^6 - 1548464N^5 - 833712N^4 - 281920N^3 - 190272N^2 \\
& +165888N + 103680 \tag{863} \\
P_{616} = & 24535N^{18} + 223137N^{17} + 571468N^{16} - 535672N^{15} - 4455392N^{14} - 3953404N^{13} \\
& +8875142N^{12} + 15762002N^{11} - 5430715N^{10} - 26302661N^9 - 10457122N^8 \\
& +15101398N^7 + 12223476N^6 - 11961968N^5 - 20488992N^4 - 1064736N^3 \\
& +5389632N^2 + 476928N - 870912 \tag{864} \\
P_{617} = & 2310N^{19} + 27684N^{18} + 110213N^{17} + 62581N^{16} - 730907N^{15} - 1801977N^{14} \\
& +261074N^{13} + 5643644N^{12} + 4379456N^{11} - 4397272N^{10} - 2762167N^9 \\
& +2836323N^8 - 8997563N^7 - 14308283N^6 + 2977232N^5 + 8931012N^4 \\
& -1797408N^3 - 4069872N^2 + 461376N + 855360 \tag{865} \\
P_{618} = & -64N^{20} - 5741N^{19} - 9955N^{18} - 64422N^{17} - 2708N^{16} + 645006N^{15} \\
& +483098N^{14} - 1197548N^{13} - 908800N^{12} + 4112583N^{11} + 2097697N^{10} \\
& -20613750N^9 - 4094356N^8 + 29920768N^7 - 21624688N^6 - 27765600N^5 \\
& +9469696N^4 + 9366528N^3 - 6156288N^2 - 1769472N + 1244160 \tag{866} \\
P_{619} = & 1216N^{20} + 19173N^{19} + 121764N^{18} + 379926N^{17} + 472571N^{16} - 429454N^{15} \\
& -1823753N^{14} - 1180304N^{13} + 576898N^{12} + 244797N^{11} + 2069530N^{10} \\
& +13099954N^9 + 35093959N^8 + 47476772N^7 + 20154875N^6 - 11470512N^5 \\
& -5237892N^4 + 5866560N^3 + 3692304N^2 - 933120N - 699840 \tag{867} \\
P_{620} = & -171287N^{24} - 2068332N^{23} - 7720102N^{22} + 2768090N^{21} + 87092589N^{20} \\
& +154633846N^{19} - 244069214N^{18} - 978466140N^{17} - 208517821N^{16} \\
& +2389374224N^{15} + 2200366326N^{14} - 2505722606N^{13} - 4151353097N^{12} \\
& +478650438N^{11} + 2967438446N^{10} + 669674480N^9 - 687107760N^8 \\
& -1428144224N^7 - 1830398016N^6 + 16556928N^5 + 761041152N^4 \\
& +29251584N^3 - 318380544N^2 - 19408896N + 44789760. \tag{868}
\end{aligned}$$

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