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Correlators of heavy–light quark currents in HQET: OPE at three loops

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Abstract

Coefficient functions of the operator product expansion of correlators of HQET heavy–light quark currents are calculated up operators of dimension 4 up to to 3 loops.

1. Heavy–light currents and their correlators in HQET

QCD problems with a single heavy quark Q having momentum $P = Mv + p$ (M is its on-shell mass, $v^2 = 1$) can be described by heavy quark effective theory [1] (HQET, see, e. g., [2, 3, 4]) if its characteristic residual momentum is small ($p \ll M$), and characteristic momenta of light quarks and gluons are also small. QCD operators are expanded in $1/M$, the coefficients are HQET operators of corresponding dimensionalities. For example, QCD heavy–light quark currents at the leading order in $1/M$ are equal to the matching coefficients times the HQET heavy–light currents. These matching coefficients are known at 2 [5, 6] and 3 loops [7]. Anomalous dimensions of all HQET heavy-light currents are the same and known at 2 [8, 9, 10] and 3 loops [11]. Correlators such currents at small distances can be calculated using operator product expansion (OPE); coefficient functions of operators up to dimension 3 are known up to 2 loops [12, 13, 14]. The $\langle G^2 \rangle$ contribution vanishes at 1 loop; the $\langle G^3 \rangle$ one is known at 1 loop [12]. Contributions of quark condensates up to dimension 8

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are known at tree level [12, 15]. Here we calculate the perturbative contribution expanded up to m^4 (m is the light-quark mass) and condensate contributions up to dimension 4 at 3 loops. The perturbative spectral densities of correlators of some QCD heavy–light currents with $m = 0$ in the threshold region at 3 loops were calculated [16]; they are related to the HQET spectral density by the corresponding matching coefficients.

If our heavy quark is b, there are 2 different HQETs: with c quark and without it. The heavy–light HQET currents in these 2 theories are related by the decoupling coefficient, which known up to 3 loops [17]. The HQET current in HQET with c is related to the QCD currents by the matching coefficients. In this paper we shall work in HQET without c quarks. There are $n_l = 3$ dynamic flavors (u, d, s) and the static b quark.

At the leading order in $1/M$ the heavy-quark spin does not interact with gluon field. We may rotate it at will without affecting physics (heavy-quark spin symmetry [18]). We may even switch it off (superflavor symmetry [19]). We shall use the effective theory of a scalar static antiquark. This particle has no antiparticle; its field φ^* contains only annihilation operators. Its coordinate-space free propagator in the v rest frame is $\delta(\vec{x})S_0(x^0)$ where $S_0(t) = -i\theta(t)$. The momentum-space propagator $S_0(p) = 1/(p^0 + i0)$ does not depend on $\vec{p}$. Static-quark lines cannot form loops.

We consider the current

$$j_0 = \varphi_0^* q_0 = Z_j(\alpha_s(\mu)) j(\mu)\,. \tag{1}$$

The correlator of 2 currents in the v rest frame

$$\langle T j_0(x) \bar{\jmath}_0(0) \rangle = \delta(\vec{x}) \Pi_0(x^0) \tag{2}$$

is non-zero only for $x^0 \ge 0$ (the symbol T is superfluous: the product $\bar{\jmath}_0(0) j_0(x) = 0$). The momentum-space correlator

$$\int d^d x \, \langle T j_0(x) \bar{\jmath}_0(0) \rangle e^{ip\cdot x} = \Pi_0(p^0) \tag{3}$$

does not depend on $\vec{p}$. They are related by the 1-dimensional Fourier transform

$$\Pi_0(\omega) = \int_0^\infty dt \, \Pi_0(t) e^{i\omega t}\,, \quad \Pi_0(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \Pi_0(\omega) e^{-i\omega t}\,. \tag{4}$$

The correlator $\Pi_0(\omega)$ has a cut from 0 to $+\infty$, the discontinuity gives the spectral density

$$\rho_0(\omega) = \frac{1}{2\pi} \left[\Pi_0(\omega + i0) - \Pi_0(\omega - i0) \right]\,. \tag{5}$$

The correlator is expressed via the spectral density by the dispersion representation:

$$\Pi_0(\omega) = i \int_0^\infty \frac{d\nu \, \rho_0(\nu)}{\omega - \nu + i0}\,, \quad \Pi_0(t) = \theta(t) \int_0^\infty d\omega \, \rho_0(\omega) e^{-i\omega t}\,. \tag{6}$$

We can analytically continue $\Pi_0(t)$ from $t > 0$ to $t = -i\tau$, $\tau > 0$ and obtain the Euclidean correlator

$$\Pi_0(\tau) = \int_0^\infty d\omega\, \rho_0(\omega) e^{-\omega\tau} \,. \tag{7}$$

The spectral density can be reconstructed from it by the inverse Mellin transform

$$\rho_0(\omega) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\tau\, \Pi_0(\tau) e^{\omega\tau} \,, \tag{8}$$

where a is to the right from all singularities of $\Pi_0(\tau)$.

Borel transform of the correlator $\Pi(\omega)$ is often used in sum rules. In HQET it is defined by

$$\hat{B}_E F(\omega) = \lim_{k\to\infty} \frac{(-\omega)^{k+1}}{k!} \left(\frac{d}{d\omega}\right)^k F(\omega)\bigg|_{\omega=-Ek} \tag{9}$$

It is equivalent to the correlator in imaginary time $\Pi(\tau)$. For example, for the function

$$F(\omega) = \frac{1}{(\nu-\omega-i0)^n} \quad \text{it is} \quad \hat{B}_E F(\omega) = \frac{e^{-\nu/E}}{\Gamma(n) E^{n-1}} \,.$$

The Fourier transform (4) of $F(\omega)$ is

$$F(t) = i\theta(t) \frac{(it)^{n-1}}{\Gamma(n)} e^{-i(\nu-i0)t} \,;$$

its analytical continuation from the half-axis $t > 0$ to the half-axis $t = -i\tau$, $\tau > 0$ is

$$F(\tau) = i \frac{\tau^{n-1}}{\Gamma(n)} e^{-\nu\tau} \,.$$

Therefore,

$$\hat{B}_E \Pi(\omega) = -i \Pi(\tau = 1/E) \,. \tag{10}$$

The static-antiquark propagator in a gluon field is $\delta(\vec{x}_1 - \vec{x}_0) S_0(x_1^0 - x_0^0) \overline{[x_1, x_0]}$, where

$$\overline{[x_1, x_0]} = P \exp\left[-i g_0 \int_{x_0}^{x_1} dx_\mu A_0^\mu(x)\right] \tag{11}$$

is the Wilson line in the antiquark representation, the integral is taken along the straight line from x_0 to x_1. Therefore, the correlator can be written as

$$\Pi_0(t) = \langle q_0(vt) \overline{[vt, 0]} \bar{q}_0(0) \rangle \,. \tag{12}$$

We can consider a more general object [20]

$$F_0(x) = \langle q_0(x) \overline{[x, 0]} \bar{q}_0(0) \rangle \,, \tag{13}$$

where x is not necessarily timelike. The bilocal vacuum average

$$\langle \bar{q}_0(0) [0, x] \Gamma q(x) \rangle = -\operatorname{Tr} \Gamma F_0(x) \,, \tag{14}$$

where $[0,x]$ is the Wilson line in the quark (fundamental) representation, and Γ is a Dirac matrix.

The correlator of the $\overline{\text{MS}}$ renormalized currents $j(\mu)$ still contains ultraviolet (UV) divergences when $t=0$. Subtracting these divergences we obtain the renormalized correlator $\Pi(t;\mu)$. The dispersion representation should contain 3 subtractions:

$$\begin{aligned}
\Pi(\omega;\mu) &= -i\omega^3 \int_0^\infty \frac{d\varepsilon\,\rho(\varepsilon;\mu)}{\varepsilon^3(\varepsilon-\omega-i0)} + \sum_{n=0}^{2} c_n \omega^n\,, \\
\Pi(t;\mu) &= \theta(t) \int_0^\infty d\omega\,\rho(\omega;\mu) e^{-i\omega t} + \sum_{n=0}^{2} c_n i^n \delta^{(n)}(t)\,.
\end{aligned} \tag{15}$$

Divergences of the correlator of renormalized currents are in subtraction terms in (15): in coordinate space they are at $t=0$, in momentum space they are polynomial in ω. More exactly, in dimensional regularization only c_2 contains $1/\varepsilon^n$ divergences, whereas power divergences in $c_{0,1}$ are not seen in this scheme. The renormalized spectral density is simply given by $\rho_0(\omega) = Z_j^2(\alpha_s(\mu))\rho(\omega;\mu)$.

The correlator has 2 Dirac structures

$$\Pi = A + B\rlap/v\,. \tag{16}$$

It is convenient to introduce the currents with definite parities $P=\pm 1$:

$$j_P = \frac{1+P\rlap/v}{2} j\,. \tag{17}$$

Their correlators are

$$P\Pi_P \frac{1+P\rlap/v}{2} \quad \text{where} \quad \Pi_P = P(A+PB) = \frac{P}{4}\,\text{Tr}(1+P\rlap/v)\Pi \tag{18}$$

(we shall see soon why it is convenient to introduce the factor P here).

For sufficiently large $-\omega$ the operator product expansion (OPE) is valid

$$\Pi(\omega;\mu) = \sum_i C_i(\omega;\mu)\langle O_i(\mu)\rangle\,, \tag{19}$$

where O_i are all possible operators. If the q mass is small, we can include operators with powers of $m(\mu)$ in the set O_i and calculate the Wilson coefficients C_i treating q as massless. Then the terms with even-dimensional O_i have Dirac structure $\rlap/v$, and those with odd-dimensional O_i have the structure 1.

Currently we are in the world where the antiquark $\bar{Q}$ has quantum numbers 0^+. Then S-wave $\bar{Q}q$ mesons have $j^P = \frac{1}{2}^+$, and P-wave ones $\frac{1}{2}^-$ and $\frac{3}{2}^-$. The currents $j_\pm$ have quantum numbers of $\frac{1}{2}^\pm$ mesons (currents with quantum numbers of mesons with $j > \frac{1}{2}$ necessarily involve derivatives, we don't consider them). The matrix elements of our currents are

$$\langle 0|j_P(\mu)|M\rangle = F(\mu)u\,, \tag{20}$$

where the meson states are normalized as

$$\langle M,\vec{p}\,'|M,\vec{p}\rangle = (2\pi)^3\delta(\vec{p}\,'-\vec{p}) \tag{21}$$

in the v rest frame, and u is the Dirac wave function of the $\frac{1}{2}^P$ meson M satisfying $\rlap/v u = Pu$ and normalized as $u^+u=1$. The contribution of the meson M to the correlator Π_P and its spectral density ρ_P is

$$\begin{aligned} &\Pi_M(t) = |F|^2 e^{-i\bar{\Lambda}t}\theta(t)\,, \quad \Pi_M(\tau) = |F|^2 e^{-\bar{\Lambda}\tau}\,,\\ &\Pi_M(\omega) = \frac{i|F|^2}{\omega-\bar{\Lambda}+i0}\,, \quad \rho_M(\omega) = |F|^2\delta(\omega-\bar{\Lambda})\,, \end{aligned} \tag{22}$$

where $\bar{\Lambda}$ is the residual energy of this meson. If there are several mesons with given quantum numbers, we get sums of contributions (22); sums become integrals in the continuum spectrum.

Now let's switch on the spin (and parity) $\frac{1}{2}^-$ of the static antiquark $\bar{Q}$ (still at $M=\infty$). The static antiquark is now described by the field $\bar{h}$ satisfying $\bar{h}\rlap/v = -\bar{h}$. The free propagator of this field contains the extra factor $(1-\rlap/v)/2$ as compared to the scalar case. The currents are

$$j_{\Gamma 0} = \bar{h}_0\Gamma q_0\,; \tag{23}$$

in the v rest frame the set of independent Dirac structures Γ is 1, γ^i, $\gamma^{[i}\gamma^{j]}$, $\gamma^{[i}\gamma^j\gamma^{k]}$, where square brackets mean antisymmetrization. Instead of this set, we can use 1, γ_5^{HV}, γ^i, $\gamma_5^{\mathrm{HV}}\gamma^i$, where γ_5^{HV} is the 't Hooft–Veltman γ_5. In HQET renormalized currents with the anticommuting γ_5^{AC} coincide with the corresponding currents with γ_5^{HV}, because their anomalous dimensions are the same [5] (contrary to the QCD case). Therefore, in the following we shall just use γ_5. The correlator of j_1^+ and j_2 is

$$\Pi_{12} = -\operatorname{Tr}\bar{\Gamma}_1\frac{1-\gamma^0}{2}\Gamma_2\Pi\,, \tag{24}$$

where Π is the correlator with the scalar static antiquark, and minus comes from the fermion loop. The Dirac matrices Γ either commute or anticommute with γ^0: $\Gamma\gamma^0 = -P\gamma^0\Gamma$. Both $\Gamma_{1,2}$ must have the same P (otherwise the correlator vanishes), and

$$\Pi_{12} = -\Pi_P\frac{1}{2}\operatorname{Tr}\bar{\Gamma}_1\Gamma_2\,. \tag{25}$$

The same formula works for the spectral densities.

S-wave mesons with light-fields quantum numbers $j^P=\frac{1}{2}^+$ become degenerate doublets 0^-, 1^-; P-wave ones with $j^P=\frac{1}{2}^-$, $\frac{3}{2}^-$ form degenerate doublets 0^+, 1^+ and 1^+, 2^+. The currents with Γ anticommuting with γ^0 (γ_5, γ^i: $P=+1$) have quantum numbers of the S-wave 0^-, 1^- mesons ($j^P=\frac{1}{2}^+$); those with Γ commuting with γ^0 (1, $\gamma_5\gamma^i$: $P=-1$) have quantum numbers of the P-wave 0^+, 1^+ mesons ($j^P=\frac{1}{2}^-$). The spectral density of correlator of

P	Γ_1	Γ_2	Π_{12}
$+1$	γ_5	γ_5	$2\Pi_+$
	γ^i	γ^j	$2\Pi_+\delta^{ij}$
-1	1	1	$2\Pi_-$
	$\gamma_5\gamma^i$	$\gamma_5\gamma^j$	$2\Pi_-\delta^{ij}$

Table 1: Correlators of currents with spin $\frac{1}{2}$ heavy antiquark

the currents with quantum numbers of $0^\mp$ mesons is $2\rho_\pm$, and for $1^\mp$ mesons it is $2\rho_\pm\delta^{ij}$ (Table 1, eq. (25); this is the reason why we introduced the factor P in (18)). A 0^- meson contribution to the spectral density is $|F_{0^-}|^2\delta(\omega-\bar\Lambda)$; for 1^- one it is $|F_{1^-}|^2\delta^{ij}\delta(\omega-\bar\Lambda)$, where

$$\langle 0|\bar h\gamma_5 q|0^-\rangle = F_{0^-}(\mu)\,, \quad \langle 0|\bar h\vec\gamma q|1^-\rangle = F_{1^-}(\mu)\vec e \tag{26}$$

($\vec e$ is the polarization vector of the 1^- meson). Therefore

$$F_{0^-}(\mu) = F_{1^-}(\mu) = \sqrt{2}F(\mu) \tag{27}$$

(this is an example of the heavy-quark spin symmetry). The case of 0^+, 1^+ mesons is similar.

Usually the relativistic normalization of one-particle states is used:

$${}_r\langle M,P'|M,P\rangle_r = (2\pi)^3 2P^0\delta(\vec P'-\vec P) \tag{28}$$

(it becomes meaningless when the meson mass $M\to\infty$, and thus is not usable in HQET; in the meson rest frame $|M\rangle_r = \sqrt{2M}|M\rangle$). The spin-symmetry result (27) can be written in a completely Lorentz-invariant way:

$$\begin{aligned}
&\langle 0|\bar h\Gamma q|M\rangle_r = \sqrt{M}F(\mu)\,\mathrm{Tr}\,\Gamma\mathcal{M}\,, \quad \mathcal{M} = \frac{1+\not{v}}{2}\times\left\{\begin{array}{ll}\gamma_5 & \text{for } 0^- \\ \not{e} & \text{for } 1^-\end{array}\right. , \\
&\not{v}\mathcal{M} = -\mathcal{M}\not{v} = \mathcal{M}\,.
\end{aligned} \tag{29}$$

For example ($P^\mu = Mv^\mu$ is the meson momentum)

$$\begin{aligned}
&\langle 0|\bar h\gamma_5\gamma^\mu q|0^-\rangle_r = \frac{2F(\mu)}{\sqrt{M}}P^\mu\,, \quad \langle 0|\bar h\gamma_5 q|0^-\rangle_r = \frac{2F(\mu)}{\sqrt{M}}M\,, \\
&\langle 0|\bar h\gamma^\mu q|1^-\rangle_r = \frac{2F(\mu)}{\sqrt{M}}Me^\mu\,, \quad \langle 0|\bar h\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]q|1^-\rangle_r = \frac{2F(\mu)}{\sqrt{M}}(e^\mu P^\nu - e^\nu P^\mu)\,.
\end{aligned}$$

Similarly, for 0^+, 1^+ mesons ($j^P = \frac{1}{2}^-$)

$$\mathcal{M} = \frac{1-\not{v}}{2}\times\left\{\begin{array}{ll}1 & \text{for } 0^+ \\ \gamma_5\not{e} & \text{for } 1^+\end{array}\right. , \quad \not{v}\mathcal{M} = \mathcal{M}\not{v} = -\mathcal{M}\,. \tag{30}$$

Of course, phases of $|M\rangle$ states (and hence of $\mathcal{M}$) can be redefined.

The vacuum average (13) for a general (timelike or spacelike) x has 2 Dirac structures

$$F_0(x) = -\frac{1}{4}\left[F_S(x^2) - i\not{x} F_V(x^2)\right]\,,$$
$$F_S(x^2) = \langle \bar{q}(0)[0,x]q(x)\rangle\,,\quad F_V(x^2) = \frac{i}{x^2}\langle \bar{q}(0)[0,x]\not{x} q(x)\rangle\,. \tag{31}$$

In HQET $x = vt$, and the scalar functions $F_{S,V}$ have positive argument $x^2 = t^2$. If x is spacelike, the argument x^2 is negative. When we analytically continue HQET results to $t = -i\tau$, we obtain $F_{S,V}$ of negative argument $-\tau^2$.

2. Perturbative contribution

Perturbative contributions to the correlator are shown in Fig. 1: the one-loop diagram; one of three two-loop ones; and two examples of three-loop diagrams.

We use integration by parts (IBP) to reduce three-loop diagrams to master integrals with the C++ program[1] FIRE6 [24]. Generation of Feynman diagram was done with QGRAF [25] and evaluation of color factors with the FORM [26] package COLOR [27].

There are three non-trivial master integrals. Two of them are known exactly as hypergeometric functions with ε: [28] and [23, 29]; for the last one, only a few terms of the ε expansion are known [16], but this is sufficient for our purpose. This IBP procedure and the master integrals are reviewed in [30].

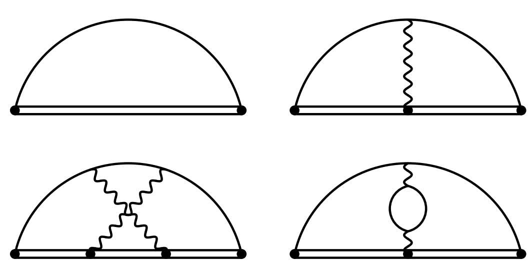

Figure 1: Perturbative contributions to the correlator.

The renormalized perturbative correlator is

$$\Pi_P(\tau;\mu) = \frac{N_c}{\pi^2}\Biggl\{\frac{1}{2\tau^3}\Biggl[1 + C_F\frac{\alpha_s}{4\pi}\Biggl[6L_\tau + 4\Biggl(\frac{\pi^2}{3} + 2\Biggr)\Biggr]$$
$$+ C_F\left(\frac{\alpha_s}{4\pi}\right)^2\Biggl\{C_F\Biggl[18L_\tau^2 + \Biggl(\frac{40}{3}\pi^2 + 43\Biggr)L_\tau - 8\zeta_3 + \frac{8}{45}\pi^4 + \frac{52}{3}\pi^2 + \frac{153}{8}\Biggr]$$

[1]We have also used the Mathematica program LiteRed 1.4 [21, 22] and the REDUCE package Grinder [23] for testing purposes and the identification of the master integrals.

$$+ C_A \Big[22L_\tau^2 + \Big(\frac{76}{9}\pi^2 + 75\Big)L_\tau - 104\zeta_3 - \frac{8}{45}\pi^4 - \frac{5}{27}\pi^2 + \frac{6413}{72}\Big]$$

$$- T_F n_l \Big[8L_\tau^2 + 4\Big(\frac{8}{9}\pi^2 + 7\Big)L_\tau - 32\zeta_3 - \frac{16}{27}\pi^2 + \frac{589}{18}\Big]\Big\}\Big]$$

$$+ P\frac{m}{4\tau^2}\Big[1 + C_F\frac{\alpha_s}{4\pi}\Big[12L_\tau + 4\Big(\frac{\pi^2}{3} + 3\Big)\Big]$$

$$+ C_F\left(\frac{\alpha_s}{4\pi}\right)^2\Big\{C_F\Big[72L_\tau^2 + 2\Big(\frac{32}{3}\pi^2 + 71\Big)L_\tau - 20\zeta_3 + \frac{8}{45}\pi^4 + \frac{52}{3}\pi^2 + \frac{233}{4}\Big]$$

$$+ C_A\Big[44L_\tau^2 + \frac{2}{3}\Big(\frac{38}{3}\pi^2 + 205\Big)L_\tau - 116\zeta_3 - \frac{8}{45}\pi^4 + \frac{31}{27}\pi^2 + \frac{4981}{36}\Big]$$

$$- T_F n_l\Big[16L_\tau^2 + \frac{8}{3}\Big(\frac{4}{3}\pi^2 + 17\Big)L_\tau - 32\zeta_3 - \frac{16}{27}\pi^2 + \frac{401}{9}\Big]\Big\}\Big]$$

$$- \frac{m^2}{8\tau}\Big[1 + 6C_F\frac{\alpha_s}{4\pi}(3L_\tau + 1)$$

$$+ C_F\left(\frac{\alpha_s}{4\pi}\right)^2\Big\{C_F\Big[162L_\tau^2 + \Big(\frac{16}{3}\pi^2 + 109\Big)L_\tau + 40\zeta_3 - \frac{32}{45}\pi^4 - 12\pi^2 + \frac{507}{8}\Big]$$

$$+ C_A\Big[66L_\tau^2 - \Big(\frac{4}{3}\pi^2 - 125\Big)L_\tau - 22\zeta_3 - \frac{4}{15}\pi^4 - \pi^2 + \frac{2789}{24}\Big]$$

$$- T_F n_l\Big[24L_\tau^2 + 36L_\tau + \frac{229}{6}\Big]\Big\}\Big]$$

$$- 2\frac{\sum m_i^2}{\tau}C_F T_F\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\pi^2}{3} - 2\right)$$

$$+ P\frac{m^3}{8}\Big[L_\tau + C_F\frac{\alpha_s}{4\pi}\Big[12L_\tau^2 + 10L_\tau - 2\Big(\frac{2}{3}\pi^2 - 3\Big)\Big]$$

$$+ C_F\left(\frac{\alpha_s}{4\pi}\right)^2\Big\{C_F\Big[96L_\tau^3 + 2\Big(\frac{8}{3}\pi^2 + 59\Big)L_\tau^2 + \Big(16\zeta_3 - \frac{76}{3}\pi^2 + \frac{249}{2}\Big)L_\tau$$

$$- 140\zeta_5 + \frac{64}{3}\pi^2\zeta_3 + 134\zeta_3 - \frac{26}{15}\pi^4 - \frac{40}{3}\pi^2 + \frac{91}{24}\Big]$$

$$+ C_A\Big[\frac{88}{3}L_\tau^3 - \frac{2}{3}(2\pi^2 - 161)L_\tau^2 - \Big(16\zeta_3 + \frac{97}{9}\pi^2 - \frac{889}{6}\Big)L_\tau$$

$$+ 60\zeta_5 - \frac{20}{3}\pi^2\zeta_3 + 89\zeta_3 - \frac{\pi^4}{45} - \frac{277}{27}\pi^2 + \frac{4357}{72}\Big]$$

$$- T_F n_l\Big[\frac{32}{3}L_\tau^3 + \frac{104}{3}L_\tau^2 - \frac{2}{3}\Big(\frac{16}{3}\pi^2 - 71\Big)L_\tau + 32\zeta_3 - \frac{80}{27}\pi^2 + \frac{245}{18}\Big]\Big\}\Big]$$

$$- Pm\Big(\sum m_i^2\Big)C_F T_F\left(\frac{\alpha_s}{4\pi}\right)^2\left(6L_\tau + \frac{\pi^2}{3} - \frac{2}{3}\right)$$

$$- \frac{m^4\tau}{32}\Big[L_\tau - \frac{1}{4} + C_F\frac{\alpha_s}{4\pi}\Big[18L_\tau^2 - \frac{17}{2}L_\tau - \frac{8}{3}\pi^2 + \frac{45}{2}\Big]$$

$$+ C_F\left(\frac{\alpha_s}{4\pi}\right)^2\Big\{C_F\Big[186L_\tau^3 + \Big(\frac{16}{3}\pi^2 - \frac{259}{2}\Big)L_\tau^2 + \Big(16\zeta_3 - \frac{212}{3}\pi^2 + \frac{4155}{8}\Big)L_\tau$$

$$- 420\zeta_5 + 64\pi^2\zeta_3 + 351\zeta_3 - \frac{20}{9}\pi^4 - \frac{113}{9}\pi^2 - \frac{10609}{32}\Big]$$

$$
\begin{aligned}
&+ C_A\left[\frac{154}{3}L_\tau^3 - \left(\frac{4}{3}\pi^2 - \frac{63}{2}\right)L_\tau^2 - \left(16\zeta_3 + \frac{200}{9}\pi^2 - \frac{2603}{8}\right)L_\tau \right.\\
&\quad\left. + 180\zeta_5 - 20\pi^2\zeta_3 + 179\zeta_3 - \frac{\pi^4}{15} + \frac{835}{108}\pi^2 - \frac{64801}{288}\right]\\
&- T_F n_l\left[\frac{56}{3}L^3 + 2L^2 - \left(\frac{64}{9}\pi^2 - \frac{235}{2}\right)L + 64\zeta_3 + \frac{32}{27}\pi^2 - \frac{6137}{72}\right]\Bigg\}\Bigg]\\
&+ m^2\frac{\sum m_i^2}{4}\tau C_F T_F\left(\frac{\alpha_s}{4\pi}\right)^2\left(6L_\tau + \frac{17}{6}\right)\\
&- \frac{\sum m_i^4}{4}\tau C_F T_F\left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(\frac{\pi^2}{3} - \frac{11}{4}\right)L_\tau - 3\zeta_3 - \frac{\pi^2}{2} + \frac{65}{8}\right]\\
&+ \mathcal{O}(m^5, \alpha_s^3)\Bigg\}\,, \qquad (32)
\end{aligned}
$$

where $\alpha_s = \alpha_s^{(n_l)}(\mu)$, $m = m^{(n_l)}(\mu)$ is the mass of the quark q in our current (1), $m_i = m_i^{(n_l)}(\mu)$ are all light-flavor masses (see the last diagram in Fig. 1), and

$$L_\tau = \log\frac{\mu\tau e^{\gamma_E}}{2}\,. \qquad (33)$$

The coefficient functions $C_{m^n}(\mu)$ with $n = 0$, 1, 2 satisfy simple renormalization group (RG) equations (while m^3 mixes with $\bar{q}q$, and m^4 mixes with $m\bar{q}q$ and G^2, Sect. 3). Its solution is

$$
\begin{aligned}
&C_{m^n} \sim \exp\Bigg\{\frac{\alpha_s}{4\pi}(-2\gamma_0 L_\tau + c_1) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[-2\beta_0\gamma_0 L_\tau^2 - 2(\gamma_1 - \beta_0 c_1)L_\tau + c_2\right]\\
&\quad + \mathcal{O}(\alpha_s^3)\Bigg\}\,, \quad \gamma_k = 2\gamma_{jk} - n\gamma_{mk}\,, \quad \gamma_0 = -6C_F(n+1)\,. \qquad (34)
\end{aligned}
$$

Here

$$
\begin{aligned}
\gamma_a(\alpha_s) &= \frac{d\log Z_a}{d\log\mu} = \sum_{n=0}^{\infty}\gamma_{an}\left(\frac{\alpha_s}{4\pi}\right)^{n+1} \qquad (a = j,\, m)\,,\\
\beta(\alpha_s) &= \frac{1}{2}\frac{d\log Z_\alpha}{d\log\mu} = \sum_{n=0}^{\infty}\beta_n\left(\frac{\alpha_s}{4\pi}\right)^{n+1}\,.
\end{aligned}
$$

Our results (32) for $n = 0$, 1, 2 satisfy this condition.

The renormalized spectral density of the OPE terms having dimensionalities ≤ 2 is

$$
\begin{aligned}
&\rho_P^{d\leq 2}(\omega;\mu) = \frac{N_c}{\pi^2}\Bigg\{\frac{\omega^2}{4}\Bigg[1 - C_F\frac{\alpha_s}{4\pi}\left(6L_\omega - \frac{4}{3}\pi^2 - 17\right)\\
&\quad + C_F\left(\frac{\alpha_s}{4\pi}\right)^2\Bigg\{C_F\left[18L_\omega^2 - \left(\frac{40}{3}\pi^2 + 97\right)L_\omega - 8\zeta_3 + \frac{8}{45}\pi^4 + \frac{103}{3}\pi^2 + \frac{1173}{8}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ C_A\left[22L_\omega^2 - \left(\frac{76}{9}\pi^2 + 141\right)L_\omega - 104\zeta_3 - \frac{8}{45}\pi^4 + \frac{238}{27}\pi^2 + \frac{20057}{72}\right] \\
&- T_F n_l\left[8L_\omega^2 - 4\left(\frac{8}{9}\pi^2 + 13\right)L_\omega - 32\zeta_3 + \frac{92}{27}\pi^2 + \frac{1849}{18}\right]\Bigg\}\Bigg] \\
&+ P\frac{m\omega}{4}\Bigg[1 - 4C_F\frac{\alpha_s}{4\pi}\left(3L_\omega - \frac{\pi^2}{3} - 6\right) \\
&+ 4C_F\left(\frac{\alpha_s}{4\pi}\right)^2\Bigg\{C_F\left[18L_\omega^2 - \left(\frac{16}{3}\pi^2 + \frac{143}{2}\right)L_\omega - 5\zeta_3 + \frac{2}{45}\pi^4 + \frac{20}{3}\pi^2 + \frac{1377}{16}\right] \\
&+ C_A\left[11L_\omega^2 - \frac{1}{3}\left(\frac{19}{3}\pi^2 + \frac{337}{2}\right)L_\omega - 29\zeta_3 - \frac{2}{45}\pi^4 + \frac{61}{108}\pi^2 + \frac{13069}{144}\right] \\
&- T_F n_l\left[4L_\omega^2 - \frac{2}{3}\left(\frac{4}{3}\pi^2 + 29\right)L_\omega - 8\zeta_3 + \frac{2}{27}\pi^2 + \frac{1097}{36}\right]\Bigg\}\Bigg] \\
&- \frac{m^2}{8}\Bigg[1 - 6C_F\frac{\alpha_s}{4\pi}(3L_\omega - 1) \\
&+ 2C_F\left(\frac{\alpha_s}{4\pi}\right)^2\Bigg\{C_F\left[81L_\omega^2 - \left(\frac{8}{3}\pi^2 + \frac{109}{2}\right)L_\omega + 20\zeta_3 - \frac{16}{45}\pi^4 - \frac{39}{2}\pi^2 + \frac{507}{16}\right] \\
&+ C_A\left[33L_\omega^2 + \left(\frac{2}{3}\pi^2 - \frac{125}{2}\right)L_\omega - 11\zeta_3 - \frac{2}{15}\pi^4 - 6\pi^2 + \frac{2789}{48}\right] \\
&- T_F n_l\left[12L_\omega^2 - 18L_\omega - 2\pi^2 + \frac{229}{12}\right]\Bigg\}\Bigg] \\
&- \frac{2}{3}\left(\sum m_i^2\right)C_F T_F\left(\frac{\alpha_s}{4\pi}\right)^2(\pi^2 - 6) + \mathcal{O}(\alpha_s^3)\Bigg\}\,, \qquad (35)
\end{aligned}
$$

where

$$
L_\omega = \log\frac{2\omega}{\mu}\,. \qquad (36)
$$

Terms up to two loops agree with [12]; the remaining ones are new. Multiplying the leading m^0 term in the HQET spectral density (35) by the corresponding matching coefficients [5, 6], we reproduce the leading δ^0 terms in the 3-loop QCD spectral densities (10), (14) in [16].

3. Quark and gluon condensates (dimensions 3 and 4)

Some 0-, 1-, and 2-loop diagrams for the quark condensate contribution are shown in Fig. 2. Starting from 2 loops (the last diagram in the figure) contributions proportional to the singlet sum $\sum m_i\langle\bar{q}_i q_i\rangle$ appear. Our result for the coordinate-space correlator[2] is

$$
\Pi_P^q(\tau;\mu) = -P\frac{\langle\bar{q}q\rangle}{4}\Bigg\{1 + 6C_F\frac{\alpha_s}{4\pi}
$$

[2]We have used the well-known method of projectors [31, 32] for computation of various condensate contributions (a similar method was used in [33, 34]).

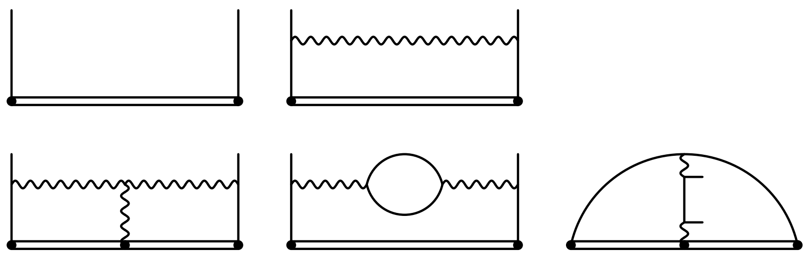

Figure 2: Quark-condensate contributions to the correlator.

$$
\begin{aligned}
&+ C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \Bigg\{ 2C_F \left[4\left(\frac{2}{3}\pi^2 - 1\right) L_\tau - 16\zeta_3 + \frac{10}{3}\pi^2 + 11 \right] \\
&\quad - C_A \left[4\left(\frac{\pi^2}{3} - 7\right) L_\tau - 8\zeta_3 + \pi^2 - \frac{149}{3} \right] - 16 T_F n_l \left(L_\tau + \frac{4}{3} \right) \Bigg\} \\
&+ C_F \left(\frac{\alpha_s}{4\pi}\right)^3 \Bigg\{ C_F^2 \bigg[4\left(18\zeta_3 + \frac{4}{9}\pi^4 + \frac{8}{3}\pi^2 - 35 \right) L_\tau \\
&\qquad + \frac{1600}{3}\zeta_5 - \frac{928}{9}\pi^2\zeta_3 - \frac{140}{3}\zeta_3 + \frac{479}{135}\pi^4 - \frac{8}{9}\pi^2 + 157 \bigg] \\
&\quad + C_F C_A \bigg[\frac{176}{3}\left(\frac{2}{3}\pi^2 - 1\right) L_\tau^2 - 4\left(141\zeta_3 - \frac{4}{45}\pi^4 - \frac{902}{27}\pi^2 - \frac{737}{9} \right) L_\tau \\
&\qquad - \frac{1}{3}\left(1216\zeta_5 - \frac{424}{3}\pi^2\zeta_3 + \frac{23654}{9}\zeta_3 - \frac{3799}{270}\pi^4 - \frac{27122}{81}\pi^2 - \frac{23669}{27} \right) \bigg] \\
&\quad - C_A^2 \bigg[\frac{88}{3}\left(\frac{\pi^2}{3} - 7\right) L_\tau^2 - 4\left(33\zeta_3 + \frac{2}{15}\pi^4 - \frac{164}{27}\pi^2 + \frac{1409}{9} \right) L_\tau \\
&\qquad - 72\zeta_5 + 12\pi^2\zeta_3 - \frac{4856}{27}\zeta_3 + \frac{199}{810}\pi^4 + \frac{4094}{243}\pi^2 - \frac{69583}{81} \bigg] \\
&\quad - 2C_F T_F n_l \bigg[\frac{32}{3}\left(\frac{2}{3}\pi^2 - 1\right) L_\tau^2 - 32\left(3\zeta_3 - \frac{22}{27}\pi^2 - \frac{25}{9} \right) L_\tau \\
&\qquad - \frac{1}{27}\left(5936\zeta_3 - \frac{100}{3}\pi^4 - \frac{3536}{9}\pi^2 - \frac{10253}{3} \right) \bigg] \\
&\quad + 2C_A T_F n_l \bigg[16\left(\frac{\pi^2}{9} - 6\right) L_\tau^2 - 4\left(6\zeta_3 - \frac{32}{27}\pi^2 + \frac{217}{3} \right) L_\tau \\
&\qquad - \frac{1}{27}\left(2258\zeta_3 - \frac{89}{15}\pi^4 - \frac{830}{9}\pi^2 + \frac{27736}{3} \right) \bigg] \\
&\quad + \frac{32}{3}(T_F n_l)^2 \left(4L_\tau^2 + \frac{32}{3}L_\tau + \frac{109}{9} \right) \Bigg\} \Bigg\} \\
&+ \frac{m\langle\bar{q}q\rangle\tau}{16} \Bigg\{ 1 + 2C_F \frac{\alpha_s}{4\pi} \left(3L_\tau - \frac{5}{2} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 4C_F\left(\frac{\alpha_s}{4\pi}\right)^2\Bigg\{C_F\left[\frac{9}{2}L_\tau^2+\left(\frac{4}{3}\pi^2-\frac{35}{4}\right)L_\tau-8\zeta_3+\frac{8}{3}\pi^2-\frac{547}{32}\right]\\
&\quad + C_A\left[\frac{11}{2}L_\tau^2-\frac{1}{3}\left(\pi^2+\frac{61}{4}\right)L_\tau+2\zeta_3-\frac{3}{4}\pi^2+\frac{3415}{96}\right]\\
&\quad - T_F n_l\left(2L_\tau^2-\frac{5}{3}L_\tau+\frac{335}{24}\right)\Bigg\}\\
&+ 4C_F\left(\frac{\alpha_s}{4\pi}\right)^3\Bigg[C_F^2\Bigg[9L_\tau^3+2(4\pi^2-15)L_\tau^2-\left(30\zeta_3-\frac{4}{9}\pi^4-4\pi^2+\frac{1393}{16}\right)L_\tau\\
&\qquad + \frac{1}{3}\left(1450\zeta_5-\frac{712}{3}\pi^2\zeta_3+145\zeta_3+\frac{431}{180}\pi^4+\frac{461}{6}\pi^2-\frac{20141}{32}\right)\Bigg]\\
&\quad + C_F C_A\Bigg[33L_\tau^3+\frac{1}{3}\left(\frac{70}{3}\pi^2-\frac{403}{2}\right)L_\tau^2\\
&\qquad - \left(129\zeta_3-\frac{4}{45}\pi^4-\frac{2551}{54}\pi^2+\frac{6997}{144}\right)L_\tau\\
&\qquad - \frac{1}{3}\left(754\zeta_5-\frac{256}{3}\pi^2\zeta_3+\frac{10715}{9}\zeta_3-\frac{8119}{1080}\pi^4-\frac{16439}{324}\pi^2-\frac{207275}{432}\right)\Bigg]\\
&\quad + C_A^2\Bigg[\frac{242}{9}L_\tau^3-\frac{1}{9}\left(22\pi^2+\frac{365}{2}\right)L_\tau^2+\left(33\zeta_3+\frac{2}{15}\pi^4-\frac{362}{27}\pi^2+\frac{26281}{54}\right)L_\tau\\
&\qquad + \frac{67}{4}\zeta_5-\frac{5}{2}\pi^2\zeta_3+\frac{2951}{27}\zeta_3-\frac{419}{1620}\pi^4-\frac{10103}{3888}\pi^2-\frac{675449}{7776}\Bigg]\\
&\quad - C_F T_F n_l\Bigg[12L_\tau^3+\frac{4}{3}\left(\frac{8}{3}\pi^2-13\right)L_\tau^2-\left(72\zeta_3-\frac{496}{27}\pi^2-\frac{13}{36}\right)L_\tau\\
&\qquad - \frac{1}{27}\left(3058\zeta_3-\frac{223}{15}\pi^4-\frac{1264}{9}\pi^2-\frac{42839}{24}\right)\Bigg]\\
&\quad - C_A T_F n_l\Bigg[\frac{176}{9}L_\tau^3-\frac{2}{9}(4\pi^2+71)L_\tau^2+4\left(9\zeta_3-\frac{34}{27}\pi^2+\frac{2486}{27}\right)L_\tau\\
&\qquad + \frac{1}{27}\left(436\zeta_3-\frac{53}{30}\pi^4-\frac{353}{18}\pi^2-\frac{26521}{36}\right)\Bigg]\\
&\quad + \frac{\left(T_F n_l\right)^2}{3}\left(\frac{32}{3}L_\tau^3-\frac{40}{3}L_\tau^2+\frac{1940}{9}L_\tau+8\zeta_3-\frac{6395}{162}\right)\Bigg]\Bigg\}\\
&+ \left(\sum m_i\langle\bar{q}_i q_i\rangle\right)\frac{\tau}{4}C_F T_F\left(\frac{\alpha_s}{4\pi}\right)^2\Bigg\{\frac{1}{2}\left(\frac{4}{3}\pi^2-11\right)\\
&\quad + \frac{\alpha_s}{4\pi}\Bigg[C_F\left[5\left(\frac{4}{3}\pi^2-11\right)L_\tau+12\zeta_3+\frac{8}{45}\pi^4-\frac{44}{9}\pi^2+\frac{76}{3}\right]\\
&\qquad + \frac{C_A}{3}\left[\left(\frac{100}{3}\pi^2-311\right)L_\tau-256\zeta_3-\frac{7}{15}\pi^4-\frac{211}{9}\pi^2+\frac{3559}{6}\right]\\
&\qquad - 2T_F n_l\left[\frac{4}{3}\left(\frac{4}{3}\pi^2-11\right)L_\tau-16\zeta_3-\frac{44}{27}\pi^2+\frac{313}{9}\right]\Bigg]\Bigg\}, \qquad (37)
\end{aligned}
$$

where $\langle\bar{q}q\rangle$ is renormalized at μ. The terms up to 2 loops in the dimension-3 contribution agree with [12].

Finiteness of the renormalized coefficient function $C_{\bar{q}q}$ provides an independent confirmation of $2\gamma_j - \gamma_{\bar{q}q}$ at 3 loops [11]. This anomalous dimension vanishes at 1 loop; $\gamma_{\bar{q}q} = -\gamma_m$, hence $C_{\bar{q}q}$ has the structure (34) with $n = -1$. We need one more term:

$$C_{\bar{q}q} \sim \exp\left\{ c_1 \frac{\alpha_s}{4\pi} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[-2(\gamma_1 - \beta_0 c_1) L_\tau + c_2\right] \right. \tag{38}$$
$$\left. + \left(\frac{\alpha_s}{4\pi}\right)^3 \left[-4\beta_0(\gamma_1 - \beta_0 c_1) L_\tau^2 - 2(\gamma_2 - 2\beta_0 c_2 - \beta_1 c_1) L_\tau + c_3\right] + \mathcal{O}(\alpha_s^4) \right\},$$

where $\gamma_k = 2\gamma_{jk} + \gamma_{mk}$, $\gamma_0 = 0$. Hence the α_s term contains no L_τ, the α_s^2 one contains L_τ^1, etc.

The dimension-3 operators $O_3 = (m^3, m\sum m_i^2, \bar{q}q)^T$ satisfy the renormalization group equation [35, 36, 37, 38]

$$\frac{dO_3}{d\log\mu} + \gamma_3 O_3 = 0\,, \quad \gamma_3 = \begin{pmatrix} 3\gamma_m & 0 & 0 \\ 0 & 3\gamma_m & 0 \\ \gamma & \gamma' & -\gamma_m \end{pmatrix}, \tag{39}$$
$$\gamma = -\frac{N_c}{4\pi^2}\left\{ 2 + 8 C_F \frac{\alpha_s}{4\pi} \right.$$
$$\left. + C_F\left[C_F(96\zeta_3 - 131) - C_A(48\zeta_3 - 109) - 20 T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\right\},$$
$$\gamma' = 24 \frac{N_c}{\pi^2} C_F T_F \left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\,.$$

Therefore, the coefficient functions $C_3 = (C_{m^3}, C_{m\sum m_i^2}, C_{\bar{q}q})^T$ satisfy the renormalization group equation

$$\frac{dC_3}{d\log\mu} = \frac{\partial C_3}{\partial L_\tau} - 2\beta \frac{\partial C_3}{\partial \log\alpha_s} = (\gamma_3^T - 2\gamma_j) C_3\,. \tag{40}$$

The dimension-4 operators mO_3 satisfy the renormalization group equation similar to (39) but with the anomalous dimension $\gamma_3 + \gamma_m$. Hence we obtain

$$\left[\frac{\partial}{\partial L_\tau} - 2\beta\frac{\partial}{\partial\log\alpha_s} + 2\gamma_j - 3\gamma_m\right] C_{m^3} = \gamma C_{\bar{q}q}\,,$$
$$\left[\frac{\partial}{\partial L_\tau} - 2\beta\frac{\partial}{\partial\log\alpha_s} + 2\gamma_j - 3\gamma_m\right] C_{m\sum m_i^2} = \gamma' C_{\bar{q}q}\,,$$
$$\left[\frac{\partial}{\partial L_\tau} - 2\beta\frac{\partial}{\partial\log\alpha_s} + 2\gamma_j - 4\gamma_m\right] C_{m^4} = \gamma C_{m\bar{q}q}\,,$$
$$\left[\frac{\partial}{\partial L_\tau} - 2\beta\frac{\partial}{\partial\log\alpha_s} + 2\gamma_j - 4\gamma_m\right] C_{m^2\sum m_i^2} = \gamma' C_{m\bar{q}q}\,.$$

Our results (32), (37) satisfy these equations.

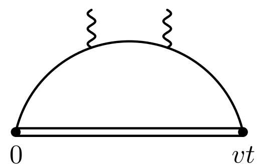

Figure 3: One-loop gluon condensate contribution.

It is well known that the gluon condensate contribution vanishes at 1 loop. In the fixed-point gauge the static quark does not interact with gluons, and the only remaining diagram is shown in Fig. 3. But the G^2 correction to the massless quark propagator $S(x,0)$ vanishes after vacuum averaging [39]. The 2- and 3-loop contributions are

$$\begin{aligned}
\Pi_P^G(\tau;\mu) ={}& \frac{\langle G^2\rangle\tau}{48} T_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{\left[C_F\left(\frac{4}{3}\pi^2-11\right)+\frac{C_A}{2}\left(\frac{4}{3}\pi^2-23\right)\right]\right.\\
&+3\frac{\alpha_s}{4\pi}\left\{C_F^2\left[2\left(\frac{4}{3}\pi^2-11\right)L_\tau+56\zeta_3+\frac{136}{135}\pi^4-\frac{140}{9}\pi^2-\frac{23}{2}\right]\right.\\
&+\frac{C_F C_A}{9}\left[\left(\frac{124}{3}\pi^2-449\right)L_\tau-524\zeta_3-\frac{20}{3}\pi^4+100\pi^2+167\right]\\
&+\frac{C_A^2}{9}\left[11\left(\frac{4}{3}\pi^2-23\right)L_\tau-46\zeta_3+\frac{4}{5}\pi^4+\frac{65}{6}\pi^2-\frac{93}{2}\right]\\
&-\frac{2}{3}C_F T_F n_l\left[\frac{4}{3}\left(\frac{4}{3}\pi^2-11\right)L_\tau-16\zeta_3-\frac{44}{27}\pi^2+\frac{259}{9}\right]\\
&\left.\left.-\frac{1}{3}C_A T_F n_l\left[\frac{4}{3}\left(\frac{4}{3}\pi^2-23\right)L_\tau-16\zeta_3-\frac{20}{27}\pi^2+\frac{253}{9}\right]\right\}\right\},
\end{aligned} \tag{41}$$

where $G^2 = G^a_{\mu\nu}G^{a\mu\nu}$. The anomalous dimension of this operator is [40, 41, 42]

$$\gamma_{G^2} = -2\frac{d\beta}{d\log\alpha_s}\,, \tag{42}$$

and hence the coefficient function must have the structure

$$C_{G^2} \sim \left(\frac{\alpha_s}{4\pi}\right)^2\left\{1+\frac{\alpha_s}{4\pi}\left[2(\beta_0-\gamma_{j0})L_\tau+c\right]+\mathcal{O}(\alpha_s^2)\right\}. \tag{43}$$

Our result (41) satisfies this condition.

The flavor-singlet dimension-4 operators $O_4 = (\sum m_i^4, (\sum m_i^2)^2, \sum m_i\bar{q}_i q_i, G^2)^T$ satisfy the renormalization group equation [35, 36, 37, 38]

$$\frac{dO_4}{d\log\mu}+\gamma_4 O_4 = 0\,, \tag{44}$$

$$\gamma_4 = \begin{pmatrix} 4\gamma_m & 0 & 0 & 0 \\ 0 & 4\gamma_m & 0 & 0 \\ \gamma & \gamma' & 0 & 0 \\ -\frac{d\gamma}{d\log\alpha_s} & -\frac{d\gamma'}{d\log\alpha_s} & 4\frac{d\gamma_m}{d\log\alpha_s} & -2\frac{d\beta}{d\log\alpha_s} \end{pmatrix} .$$

and the corresponding coefficient functions $C_4 = (C_{\sum m_i^4}, C_{(\sum m_i^2)^2}, C_{\sum m_i \bar{q}_i q_i}, C_{G^2})^T$ — the equation

$$\frac{dC_4}{d\log\mu} = (\gamma_4^T - 2\gamma_j)C_4 \, ; \tag{45}$$

hence,

$$\begin{aligned}
&\left[\frac{\partial}{\partial L_\tau} - 2\beta\frac{\partial}{\partial\log\alpha_s} + 2\gamma_j - 4\gamma_m\right] C_{\sum m_i^4} = \gamma C_{\sum m_i \bar{q}_i q_i} - \frac{d\gamma}{d\log\alpha_s} C_{G^2} \, , \\
&\left[\frac{\partial}{\partial L_\tau} - 2\beta\frac{\partial}{\partial\log\alpha_s} + 2\gamma_j - 4\gamma_m\right] C_{(\sum m_i^2)^2} = \gamma' C_{\sum m_i \bar{q}_i q_i} - \frac{d\gamma'}{d\log\alpha_s} C_{G^2} \, , \\
&\left[\frac{\partial}{\partial L_\tau} - 2\beta\frac{\partial}{\partial\log\alpha_s} + 2\gamma_j\right] C_{\sum m_i \bar{q}_i q_i} = 4\frac{d\gamma_m}{d\log\alpha_s} C_{G^2} \, .
\end{aligned}$$

The second equation here is satisfied trivially, because $C_{(\sum m_i^2)^2} = \mathcal{O}(\alpha_s^3)$. Our results (32), (37), (41) satisfy these equations.

4. Higher-dimensional condensates

The tree diagram in Fig. 2 can be written exactly in $x = vt$:

$$\Pi^q(t) = i\theta(t)\langle q(vt)\overline{[vt,0]}\bar{q}(0)\rangle \, . \tag{46}$$

It is expressed via the bilocal quark condensate [43] which has 2 Dirac structures:

$$\langle q(x)\overline{[x,0]}\bar{q}(0)\rangle = -\frac{\langle\bar{q}q\rangle}{4}\left[f_S(x^2) - \frac{i\not{x}}{d} f_V(x^2)\right] . \tag{47}$$

Its expansion in x via local quark condensates is known up to dimension 8 [15]. We use the bases of local condensates [44]

$$\begin{aligned}
&Q^3 = \langle\bar{q}q\rangle \, , \quad Q^5 = i\langle\bar{q}G_{\mu\nu}\sigma^{\mu\nu}q\rangle \, , \quad Q^6 = \langle\bar{q}\not{J}q\rangle \, , \\
&Q_1^7 = \langle\bar{q}G_{\mu\nu}G^{\mu\nu}q\rangle \, , \quad Q_2^7 = i\langle\bar{q}G_{\mu\nu}\tilde{G}^{\mu\nu}\gamma_5 q\rangle \, , \\
&Q_3^7 = \langle\bar{q}G_{\mu\lambda}G^\lambda{}_\nu\sigma^{\mu\nu}q\rangle \, , \quad Q_4^7 = i\langle\bar{q}D_\mu J_\nu\sigma^{\mu\nu}q\rangle \, , \\
&A = i\langle\bar{q}D_\alpha D_\beta D_\gamma D_\delta D_\varepsilon \gamma^{[\alpha}\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^{\varepsilon]}q\rangle \, , \\
&Q_1^8 = i\langle\bar{q}[[G_{\mu\lambda}, G^\lambda{}_\nu]_+, D^\mu]_+\gamma^\nu q\rangle \, , \quad Q_2^8 = -\langle\bar{q}[[G_{\mu\lambda}, \tilde{G}^\lambda{}_\nu], D^\mu]_+\gamma^\nu\gamma_5 q\rangle \, , \\
&Q_3^8 = i\langle\bar{q}[\not{D}G_{\mu\nu}, G^{\mu\nu}]q\rangle \, , \quad Q_4^8 = \langle\bar{q}D^2\not{J}q\rangle \, , \\
&Q_5^8 = i\langle\bar{q}[G_{\mu\nu}, J^\mu]\gamma^\nu q\rangle \, , \quad Q_6^8 = \langle\bar{q}[\tilde{G}_{\mu\nu}, J^\mu]_+\gamma^\nu\gamma_5 q\rangle \, ,
\end{aligned} \tag{48}$$

where $G_{\mu\nu} = gG^a_{\mu\nu}t^a$, $J_\mu = gJ^a_\mu t^a$, $J^a_\mu = D^\nu G^a_{\mu\nu} = g\sum \bar{q}_i\gamma_\mu t^a q_i$; $\sigma_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]}$; operators containing $\tilde{G}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$ and $\gamma_5 = \frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta$ are understood as short notations for the expressions from which both ε tensors are eliminated using $\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta} = -4!\,\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_{\delta]}$. The anomalous condensate A does not vanish in the $\overline{\text{MS}}$ scheme; it is a finite combination of dimension-8 gluon condensates [44].

We obtain the contribution of bilinear quark condensates up to dimension 8 to the correlator at the tree level

$$\begin{aligned}
\Pi^q_P(\tau) = -\frac{1}{4}\Bigg\{ & PQ^3 - \frac{\tau}{d}mQ^3 - P\frac{\tau^2}{2!\,d}\left[\frac{1}{2}Q^5 - m^2Q^3\right] \\
& + \frac{\tau^3}{3!\,d(d+2)}\left[\frac{1}{2}Q^6 + \frac{3}{2}mQ^5 - 3m^3Q^3\right] \\
& + P\frac{\tau^4}{4!\,d(d+2)}\left[3Q^7_1 - \frac{3}{2}Q^7_2 - 3Q^7_3 + Q^7_4 - 2mQ^6 - 3m^2Q^5 + 3m^4Q^3\right] \\
& - \frac{\tau^5}{5!\,d(d+2)(d+4)}\bigg[5A - \frac{5}{2}Q^8_2 + \frac{1}{4}Q^8_3 - \frac{1}{2}Q^8_4 - 3Q^8_5 + 5Q^8_6 \\
& \quad + 5m\left(3Q^7_1 - Q^7_2 - 3Q^7_3 + Q^7_4\right) - 15m^2\left(Q^6 + mQ^5 - m^3Q^3\right)\bigg] \\
& + \mathcal{O}(\tau^6)\Bigg\}\,.
\end{aligned} \tag{49}$$

The terms up to dimension 7 at $m = 0$ agree with [12].

5. Conclusion

The results obtained here can be used for extracting numerical values of F_P (and hence $f_B = f_{B^*}$, f_{B_s}/f_B and similar quantities for 0^+, 1^+ mesons) and $\bar{\Lambda}_P$ (and hence $m_{B_s} - m_B$, $m_{B(0^+)} - m_B$, $m_{B_s(0^+)} - m_{B(0^+)}$, m_b) from HQET sum rules ($1/m_b$ corrections should be calculated separately).

For sufficiently small τ the correlator $\Pi_P(\tau;\mu)$ is given by the truncated OPE series

$$\Pi_P(\tau;\mu) = \int_0^\infty d\omega\,\rho^{d\le 2}_P(\omega;\mu)e^{-\omega\tau} + \Pi^{d\ge 3}_P(\tau;\mu)\,, \tag{50}$$

where the coefficient functions are known as truncated series in α_s. On the other hand, we can represent it as

$$\Pi_P(\tau;\mu) = \int_0^\infty d\omega\,\rho_P(\omega;\mu)e^{-\omega\tau}\,, \tag{51}$$

where the spectral density is given by the ground-state meson contribution (22) plus the continuum of excited states. We can use the rough model of the continuum contribution [45]

$$\rho_P(\omega;\mu) = |F_P(\mu)|^2\delta(\omega - \bar{\Lambda}_P) + \rho^{d\le 2}_P(\omega;\mu)\theta(\omega - \omega_{cP})\,, \tag{52}$$

where ω_{cP} is the effective continuum threshold. Equating these two expressions, we obtain the sum rule

$$|F_P(\mu)|^2 e^{-\bar{\Lambda}_P\tau} = \int_0^{\omega_{cP}} d\omega\, \rho_P^{d\le 2}(\omega;\mu) e^{-\omega\tau} + \Pi_P^{d\ge 3}(\tau;\mu)\,. \tag{53}$$

It is approximately valid at sufficiently large τ, where the continuum contribution is small, and the uncertainty introduced by its rough model is not essential. If there is a window of τ where both conditions are satisfied, we can use this sum rule to extract an approximate value of $F_P(\mu)$.

Differentiating (53) in τ and dividing by (53) we obtain the sum rule for the ground-state residual energy

$$\bar{\Lambda}_P = \frac{\int_0^{\omega_{cP}} d\omega\, \rho_P^{d\le 2}(\omega;\mu)\omega e^{-\omega\tau} - d\Pi_P^{d\ge 3}(\tau;\mu)/d\tau}{\int_0^{\omega_{cP}} d\omega\, \rho_P^{d\le 2}(\omega;\mu) e^{-\omega\tau} + \Pi_P^{d\ge 3}(\tau;\mu)}\,. \tag{54}$$

The continuum thresholds ω_{cP} are tuned in such a way that the resulting $\bar{\Lambda}_P$ do not depend on τ in the region of applicability.

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