## Third order corrections to the semi-leptonic $b \rightarrow c$ and the muon decays

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We compute corrections of order  $\alpha_s^3$  to the decay  $b \to c\ell\bar{\nu}$  taking into account massive charm quarks. In the on-shell scheme large three-loop corrections are found. However, in the kinetic scheme the three-loop corrections are below 1% and thus perturbation theory is well under control. We furthermore provide results for the order  $\alpha_s^3$  corrections to  $b \to u\ell\bar{\nu}$  and the third-order QED corrections to the muon decay which will be important input for reducing the uncertainty of the Fermi coupling constant  $G_F$ .

**Introduction.** The Cabibbo-Kobayashi-Maskawa (CKM) matrix determines the mixing strength in the quark sector and provides furthermore the source for charge-parity (CP) violation in the Standard Model (SM). It is thus of prime importance to determine the parameters of the CKM matrix with highest accuracy. In this Letter we address the elements  $V_{ub}$  and  $V_{cb}$  which are accessible via semi-leptonic *B* meson decays.

At present, the value of  $|V_{cb}|$  from inclusive  $B \to X_c \ell \bar{\nu}$ decays is obtained from global fits [1–3]. The experimental inputs are the semileptonic width and the moments of kinematical distributions measured at Belle [4, 5] and BABAR [6, 7], together with earlier data from CDF [8], CLEO [9] and DELPHI [10]. The most recent determination  $|V_{cb}| = (42.19 \pm 0.78) \times 10^{-3}$  [11] has a relative error of about 1.8%, which is mostly dominated by theoretical uncertainties. In view of the much larger luminosity expected a Belle II in the next years, we are in the need to systematically improve and validate the theoretical predictions for semileptonic *B* decays.

A crucial ingredient for the determination of  $|V_{ub}|$  and  $|V_{cb}|$  is the total semi-leptonic decay rate. With the help of the heavy quark expansion it can be written as a double series in  $\alpha_s$  and  $\Lambda_{\rm QCD}/m_b$ . The  $m_b$ -suppressed corrections are obtained from higher-dimensional operators. In the free-quark approximation, corrections up to  $\mathcal{O}(\alpha_s^2)$ are available [12–19] together with the leading  $\beta_0$  terms at higher orders [20], where  $\beta_0$  is the one-loop coefficient of the QCD beta function. The power corrections of order  $\Lambda^2_{\rm QCD}/m_b^2$  and  $\Lambda^3_{\rm QCD}/m_b^3$  have been computed in [21– 24] to tree-level and in [25–28] to  $\mathcal{O}(\alpha_s)$ . Also  $1/m_b^4$ and  $1/m_b^5$  terms are known, however, only at leading order [29–32]. Note that linear  $1/m_b$  corrections vanish to all orders.

In this Letter we compute the  $\alpha_s^3$  corrections to the leading  $1/m_b$  term of  $\Gamma(B \to X_c \ell \bar{\nu})$ . We incorporate a finite charm quark mass via an expansion in the mass difference  $m_b - m_c$  and show that precise results can be obtained for the physical values of  $m_c$  and  $m_b$ . Our analysis even allows for the limit  $m_c \to 0$  which provides  $\alpha_s^3$  corrections for the decay rate  $\Gamma(B \to X_u \ell \bar{\nu})$ .<sup>1</sup>

A process closely related to  $b \to u \ell \bar{\nu}$  is the muon decay. Its lifetime,  $\tau_{\mu}$ , can be written in the following form

$$\frac{1}{\tau_{\mu}} \equiv \Gamma(\mu^{-} \to e^{-} \nu_{\mu} \bar{\nu}_{e}) = \frac{G_{F}^{2} m_{\mu}^{5}}{192 \pi^{3}} \left(1 + \Delta q\right) \,, \quad (1)$$

where  $G_F$  is the Fermi constant,  $m_{\mu}$  is the muon mass and  $\Delta q$  contains QED and hadronic vacuum polarization corrections (see Ref. [33–35] for details). Note that all weak corrections are absorbed in  $G_F$ . Equation (1) allows for the determination of  $G_F$  if precise measurements of  $\tau_{\mu}$  are combined with accurate QED predictions. We compute for the first time  $\alpha^3$  corrections to  $\Delta q$  by specifying the colour factors of our  $b \rightarrow c \ell \bar{\nu}$  result to QED and taking the limit  $m_c \rightarrow 0$ . This allows for the determination of the third-order coefficient with an accuracy of 15%.

**Calculation.** We apply the optical theorem and consider the forward scattering amplitude of a bottom quark where at leading order the two-loop diagram in Fig. 1(a) has to be considered. It has a neutrino, a lepton and a charm quark as internal particles. The weak interaction is shown as an effective vertex. Our aim is to consider QCD corrections up to third order which adds up to three more loops. Some sample Feynman diagrams are shown in Fig. 1(b-f).

The structure of the Feynman diagrams allows the integration of the massless neutrino-lepton loop which essentially leads to an effective propagator raised to an  $\epsilon$ dependent power, where  $d = 4 - 2\epsilon$  is the space-time dimension. The remaining diagram is at most of four-loop order.

From the technical point of view there are two basic ingredients which are crucial to realize our calculation. First, we perform an expansion in the difference between the bottom and charm quark mass. It has been shown

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<sup>&</sup>lt;sup>1</sup> Note that in our approach one class of diagrams for the  $b \rightarrow u$  transition is missing, namely the one where the charm quark appears as virtual particle in a closed loop. At  $\mathcal{O}(\alpha_s^2)$  these corrections were denoted by  $U_C$  [14, 15].



FIG. 1. Sample Feynman diagrams which contribute to the forward scattering amplitude of a bottom quark at LO (a), NLO (b), NNLO (c) and  $N^3LO$  (d-f). Straight, curly and dashed lines represent quarks, gluons and leptons, respectively. The weak interaction mediated by the W boson is shown as a blob.

in Ref. [19] that the expansion converges quite fast for the physical values of  $m_c$  and  $m_b$ . Second, we apply the so-called method of regions [36, 37] and exploit the similarities to the calculation of the three-loop corrections to the kinetic mass [38].

The method of regions [36, 37] leads to two possible scalings for each loop momentum

- $|k^{\mu}| \sim m_b$  (*h*, hard)
- $|k^{\mu}| \sim \delta \cdot m_b$  (*u*, ultra-soft)

with  $\delta = 1 - m_c/m_b$ . We choose the notion "ultra-soft" for the second scaling to stress the analogy to the calculation of the relation between the pole and the kinetic mass of a heavy quark, see [38, 39]. Note that the momentum which flows through the neutrino-lepton loop,  $\ell$ , has to be ultra-soft since the Feynman diagram has no imaginary part if  $\ell$  is hard.

Let us next consider the remaining (up to three) momentum integrations which can be interpreted as a fourpoint amplitude with forward-scattering kinematics and two external momenta:  $\ell$  and the on-shell momentum  $p^2 = m_b^2$ . This is in close analogy to the scattering amplitude of a heavy quark and an external current considered in Ref. [38]. In fact, the loop momenta can have the following scalings

$\mathcal{O}(\alpha_s)$	h, u
$\mathcal{O}(\alpha_s^2)$	hh, hu, uu
$\mathcal{O}(\alpha_s^3)$	hhh, hhu, huu, uuu

Note that all regions where at least one of the loop momenta scales ultra-soft leads to the same integral families as in Ref. [38, 39]. The pure-hard regions were absent in [38, 39]; they lead to (massive) on-shell integrals.

At this point there is the crucial observation that the integrands in the hard regions do not depend on the loop momentum  $\ell$ . On the other hand, the ultra-soft integrals still depend on  $\ell$ . However, for each individual integral the dependence of the final result on  $\ell$  is of the form

$$(-2p \cdot \ell + 2\delta)^{\alpha} \tag{2}$$

with known exponent  $\alpha$ . This means that it is always possible to perform in a first step the  $\ell$  integration which is of the form

$$\int \mathrm{d}^d \ell \frac{\ell^{\mu_1} \ell^{\mu_2} \cdots}{(-2p \cdot \ell + 2\delta)^{\alpha} (-\ell^2)^{\beta}} \,. \tag{3}$$

A closed formula for such tensor integrals with arbitrary tensor rank and arbitrary exponents  $\alpha$  and  $\beta$  can easily be obtained from the formula provided in Appendix A of Ref. [37]. We thus remain with the loop integrations given in the above table. Similar to Eq. (3) we can integrate all one-loop hard or ultra-soft loops which leaves us with pure hard or pure ultra-soft contributions up to three loops.

A particular challenge of our calculation is the high expansion depth in  $\delta$ . We perform an expansion of all diagrams up to  $\delta^{12}$ . This leads to huge intermediate expressions of the order of 100 GB. Furthermore, for some of the scalar integrals individual propagators are raised to positive and negative powers up to 12, which is a nontrivial task for the reduction to master integrals. For the latter we combine FIRE [40] and LiteRed [41].<sup>2</sup> For the subset of integrals which are needed for the expansion up to  $\delta^{10}$  we also use the stand-alone version of LiteRed [41] as a cross-check. For all regions where at least one of the regions is ultra-soft we can take over the master integrals from [38, 39]. For some of the (complicated) three-loop triple-ultra-soft master integrals higher order  $\epsilon$  terms are needed. The method used for their calculation and the results are given Ref. [39]. All triple-hard master integrals can be found in Ref. [42].

**Results.** We write the total decay rate for the  $b \rightarrow c$  transition in the form

$$\Gamma(B \to X_c \ell \bar{\nu}) = \Gamma_0 \left[ X_0 + C_F \sum_{n \ge 1} \left( \frac{\alpha_s}{\pi} \right)^n X_n \right] + \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}^2}{m_b^2} \right), \qquad (4)$$

with  $C_F = 4/3$ ,  $\Gamma_0 = A_{\rm ew} G_F^2 |V_{cb}|^2 m_b^5 / (192\pi^3)$ ,  $X_0 = 1 - 8\rho^2 - 12\rho^4 \log(\rho^2) + 8\rho^6 - \rho^8$  where  $\rho = m_c/m_b$  and  $\alpha_s \equiv \alpha_s^{(5)}(\mu_s)$  with  $\mu_s$  being the renormalization scale.  $A_{\rm ew} =$ 

 $<sup>^2</sup>$  We thank A. Smirnov for providing us with the private version of FIRE which was crucial for our calculation.



 $\rho = m_c/m_b$ 

0

-50

-100

-150

-200

-250

FIG. 2. The third-order coefficient (see Eq. (4)) as a function of  $\rho = m_c/m_b$  for different expansion depth in  $\delta$ .

1.014 is the leading electroweak correction [43] and  $m_b$ ( $m_c$ ) is the bottom (charm) pole mass. The one- and twoloop results are available from Refs. [12–19]. The main result of our calculation is  $X_3$ . In the following we set all colour factors to their numerical values. Furthermore, we specify the number of massless quarks to 3 and take into account closed charm and bottom loops. For  $\mu = m_b$  we have

$$X_3 = \sum_{n \ge 5} x_{3,n} \delta^n \,, \tag{5}$$

with analytic coefficients  $x_{3,n}$ , which in general depend on  $\log(\delta)$ . For illustration purposes we show explicit results only for the leading term  $x_{3,5}$ . Our result reads

$$C_F x_{3,5} = \frac{533858}{1215} - \frac{20992a_4}{81} + \frac{8744\pi^2\zeta_3}{135} - \frac{6176\zeta_5}{27} - \frac{16376\zeta_3}{135} - \frac{2624l_2^4}{243} + \frac{5344\pi^2l_2^2}{1215} + \frac{179552\pi^2l_2}{405} - \frac{39776\pi^4}{6075} - \frac{1216402\pi^2}{3645},$$
(6)

where  $l_2 = \log(2)$ ,  $a_4 = \text{Li}_4(1/2)$  and  $\zeta_n$  is the Riemann zeta function. Analytic results up to  $\delta^{12}$  can be found in the supplementary material to this Letter [44].

In Fig. 2 we show  $X_3$  as a function of  $\rho = 1 - \delta = m_c/m_b$  where the different curves contain different expansion depths in  $\delta$ . One observes a rapid convergence at the physical point for the  $b \rightarrow c$  decay which amounts to  $\rho \approx 0.3$ . In particular, the curves including terms up to  $\delta^{10}$ ,  $\delta^{11}$  or  $\delta^{12}$  are basically indistinguishable for  $\rho \approx 0.3$  which leads to  $X_3(\rho = 0.28) = -68.4 \pm 0.3$ , where the uncertainty is obtained from the difference of the  $\delta^{11}$  and  $\delta^{12}$  expansion, multiplied by a security factor of five.

For the numerical evaluation it is convenient to cast Eq. (4) in the form

$$\Gamma(B \to X_c \ell \bar{\nu}) = \Gamma_0 X_0 \left[ 1 + \sum_{n \ge 1} \left( \frac{\alpha_s}{\pi} \right)^n Y_n \right]$$

	$Y_1$	$Y_2^{\rm rem}$	$\beta_0 Y_2^{\beta_0}$	$Y_3^{\rm rem}$	$\beta_0^2 Y_3^{\beta_0^2}$
$m_b^{ m OS}, m_c^{ m OS}$	-1.72	3.08	-16.17	48.8	-212.1
$m_b^{ m kin}, m_c^{ m kin}$	-0.94	0.33	-4.08	-5.4	-15.4
$m_b^{\rm kin}, \overline{m}_c(3 { m ~GeV})$	-1.67	-3.39	-3.85	-97.7	69.1
$m_b^{\rm kin}, \overline{m}_c(2 { m ~GeV})$	-1.25	-1.21	-2.43	-68.8	67.9
$\overline{m}_b(\overline{m}_b), \overline{m}_c(3 \text{ GeV})$	3.07	-21.81	35.17	-56.7	119.4

TABLE I. Numerical results for the coefficients  $Y_n$  in Eq. (7) for various renormalization schemes.

$$+ \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{m_b^2}\right),$$
 (7)

with  $\alpha_s \equiv \alpha_s^{(4)}(\mu_s)$  as expansion parameter. In the following we discuss various renormalization schemes for the charm and bottom quark masses, where  $\Gamma_0$  and  $X_0$ are evaluated using the respective numerical values. In Tab. I we provide the corresponding results for the coefficients  $Y_n$ . At two and three-loop orders we split the results into the large- $\beta_0$  contribution and the remaining term

$$Y_{2} = Y_{2}^{\text{rem}} + \beta_{0} Y_{2}^{\beta_{0}} ,$$
  

$$Y_{3} = Y_{3}^{\text{rem}} + \beta_{0}^{2} Y_{3}^{\beta_{0}^{2}} ,$$
(8)

with  $\beta_0 = 11 - 2/3n_l = 9$  where  $n_l = 3$  is the number of massless quarks. Note that the uncertainty of  $Y_3$  due to the expansion in  $\delta$  is of the same order of magnitude as for  $X_3$  discussed above.

For the transition of the on-shell quark masses to the MS scheme we use the three-loop formulae provided in Refs. [45, 46]. Finite- $m_c$  effects in the bottom mass relation are taken from Refs. [47]. The two- and three-loop corrections to the transition from the on-shell to the kinetic scheme are provided in [48] and [38, 39], respectively. Note that the transition to the kinetic scheme also requires the renormalization of the parameters  $\mu_{\pi}^2$  and and  $\rho_D^3$ , which enter the decay rate at order  $1/m_b^2$  and  $1/m_b^3$ , respectively. They receive additive contributions, which enter  $Y_i$  in Eq. (7) [49, 50]. The corresponding corrections up to three-loop order can be found in [39]. Note that we assume a heavy charm quark and thus we have  $(n_l = 3)$ -flavour QCD as starting point for the on-shellkinetic relations. We use the decoupling relation for  $\alpha_s$ up to two-loop order to obtain expressions parameterized in terms of  $\alpha_s^{(4)}$ . For the decoupling scale we use  $\mu_s$ . It has been shown in Ref. [39] that there are no additional charm quark mass effects in the kinetic-on-shell relation. Note that our two-loop results for  $Y_2^{\text{rem}}$  differ from the one of Ref. [2] due to finite charm quark mass effects in the relation between the kinetic and on-shell bottom quark mass and the renormalization of  $\mu_{\pi}^2$  and  $\rho_D^3$  [39]. This leads to a shift of about -0.5% in the leading  $1/m_b$ approximation of the decay rate and thus might have a visible effect on the value of  $|V_{cb}|$ .

For the numerical evaluation of the decay rate we use the input values  $m_b^{OS}=4.7$  GeV,  $m_c^{OS}=1.3$  GeV,

 $m_b^{\rm kin} = 4.526 \text{ GeV}, \ m_c^{\rm kin} = 1.130 \text{ GeV}, \ \overline{m}_b(\overline{m}_b) = 4.163 \text{ GeV}, \ \overline{m}_c(3 \text{ GeV}) = 0.993 \text{ GeV}, \ \overline{m}_c(2 \text{ GeV}) = 1.099 \text{ GeV}, \text{ and } \alpha_s^{(5)}(M_Z) = 0.1179.$  We use RunDec [51] for the running of the  $\overline{\rm MS}$  parameters and the decoupling of heavy particles. For the Wilsonian cutoff in the kinetic scheme we use  $\mu = 1$  GeV both for the bottom and charm quark. For the renormalization scale of  $\alpha_s^{(4)}$ ,  $\mu_s$ , we choose the respective value for the bottom quark mass.

For illustration purpose we provide in Tab. I also results where both masses are defined in the on-shell scheme. It is well known that in this scheme the perturbative series shows a bad convergence behaviour. In fact, we have  $Y_3 \approx -163$  whereas in the schemes where the bottom quark mass is used in the kinetic scheme we have that  $Y_3$  is between -1 and -29. Note, that in the scheme where both quark masses are defined in the  $\overline{\text{MS}}$ scheme the three-loop corrections are more than twice as big which also hints for a worse convergence behaviour. This behaviour clearly shows the advantage of the kinetic scheme which is constructed such that large corrections are resummed into the quark mass value. In fact, all three schemes which involve  $m_b^{\text{kin}}$  demonstrate a good convergence behaviour. Using  $\alpha_s^{(4)}(m_b^{\text{kin}}) = 0.2186$  we obtain for  $\Gamma(B \to X_c \ell \bar{\nu})/\Gamma_0$  in these three schemes

$$\begin{split} m_b^{\rm kin}, m_c^{\rm kin} &: 0.633 \left( 1 - 0.066 - 0.018 - 0.007 \right) \\ &\approx 0.575 \,, \\ m_b^{\rm kin}, \overline{m}_c (3 \text{ GeV}) : 0.700 \left( 1 - 0.116 - 0.035 - 0.010 \right) \\ &\approx 0.587 \,, \\ m_b^{\rm kin}, \overline{m}_c (2 \text{ GeV}) : 0.648 \left( 1 - 0.087 - 0.018 - 0.0003 \right) \\ &\approx 0.580 \,, \end{split}$$

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where the different  $\alpha_s$  orders are displayed separately. We observe that the third-order corrections amount to at most 1% and they are a factor two to three smaller than the corrections of order  $\alpha_s^2$ . A particularly good behaviour is observed for the choice  $\overline{m}_c(2 \text{ GeV})$  where the corrections of order  $\alpha_s^3$  are below the per mille level. Its final result lies between the other two kinetic schemes and deviates from them by about 0.9% and 1.2%, respectively.

In general the large- $\beta_0$  terms provide dominant contributions. However, in all cases the remaining terms are not negligible and often have a different sign. In the kinetic scheme where the charm quark is renormalized in the  $\overline{\text{MS}}$  scheme the remaining contributions are numerically even bigger than the large- $\beta_0$  terms.

It is impressive that the expansion in  $\delta$  shows a good converge behaviour even for  $\delta \to 1$  which corresponds to a massless daughter quark. This allows us to extract the coefficient  $X_3$  for the decay  $b \to u\ell\bar{\nu}$ . A closer look to the  $\delta^{10}$ ,  $\delta^{11}$ , and  $\delta^{12}$  terms in Fig. 2 indicates that the convergence is quite slow for  $\rho \to 0$ . As central value for the three-loop prediction we use our approximation based on the  $\delta^{12}$  term and estimate the uncertainty from the behaviour of the one- and two-loop [52, 53] results for  $\rho = 0$ , where the exact results are known. Incorporating



FIG. 3. The third-order coefficient to  $\Delta q$  introduced in Eq. (1) as a function of  $m_e/m_{\mu}$ .

expansion terms up to order  $\delta^{12}$  we observe a deviation of about 3.5% whereas the  $\delta^{12}$  terms amount to less than 1%, both at one and two loops. At three loops the  $\delta^{12}$ term amounts to about 2%. We thus conservatively estimate the uncertainty to 10% which leads to

$$X_3^u \approx -202 \pm 20$$
. (10)

In this result the contributions with closed charm loops are approximated with  $m_c = 0$ .

In the remaining part of this Letter we specify our results to QED and study the corrections to the muon decay. A comprehensive review of the various correction terms is given in Ref. [34] where  $\Delta q$  in Eq. (1) is parameterized as

$$\Delta q = \sum_{i \ge 0} \Delta q^{(i)} \,. \tag{11}$$

 $\Delta q^{(0)}$  is given by  $X_0 - 1$  (see Eq. (4)) with  $\rho = m_e/m_\mu$ and  $\Delta q^{(1)}$  [33] and  $\Delta q^{(2)}$  [53, 54] are easily obtained after specification of the QCD colour factors to their QED values (see Ref. [34] for analytic results). We introduce  $\Delta q^{(3)} = (\alpha(m_{\mu})/\pi)^3 X_3^{\mu}$ , where  $\alpha(m_{\mu})$  is the fine structure constant in the  $\overline{\rm MS}$  scheme [34]. In Fig. 3 we show the third-order coefficient  $X_3^{\mu}$  for  $0 \leq \rho \leq 0.3$ . At the physical point  $m_e/m_\mu \approx 0.005$  the convergence behaviour is similar to QCD. We estimate  $X_3^{\mu}$  using the same approach as for  $X_3^u$  and examine the one- and twoloop behaviour. Up to an overall factor  $C_F$  the one-loop term is, of course, identical to the  $b \rightarrow u$  transition. Including expansion terms up to  $\delta^{12}$  at two loops leads to a deviation by about 8% from the exact result whereas the  $\delta^{12}$  term itself contributes by about 1%. The threeloop  $\delta^{12}$  amounts to about 2%. Assuming the same relative contribution thus leads to an uncertainty estimate of about 15% and we have

$$\Delta q^{(3)} \approx \left(\frac{\alpha(m_{\mu})}{\pi}\right)^3 (-15.3 \pm 2.3) .$$
 (12)

In Ref. [35] the three-loop corrections were estimated to  $X_3^{\mu} \sim -20$ . With the help of Eq. (1) we obtain

for the  $\alpha^3$  QED contribution to the muon life time  $(-9 \pm 1) \times 10^{-8} \ \mu$ s. This result has to be compared to the current experimental value which is given by  $\tau_{\mu} = 2.1969811 \pm 0.0000022 \ \mu$ s [55]. The new correction terms are almost two orders of magnitude smaller than the experimental uncertainty. Thus, an updated value of  $G_F$  can only be extracted once the latter has been improved.

**Conclusions.** In this Letter we have computed threeloop corrections of order  $\alpha_s^3$  to the total decay rate  $\Gamma(B \to X_c l \bar{\nu})$  including finite charm quark mass effects. We perform an expansion around the equal-mass case and demonstrate that a good convergence at the physical point is observed after taking into account eight expansion terms. Our result is one of the very few third-order results to physical quantities available to date involving two different mass scales.

We can extend our considerations to the case of a massless charm quark and thus obtain corrections of order  $\alpha_s^3$ to  $\Gamma(B \to X_u \ell \bar{\nu})$ , although with a larger uncertainty of about 10%. After specifying our findings to QED we

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furthermore obtain predictions for the third-order corrections to the muon decay. Here we estimate the uncertainty to 15%.

The decay rate  $\Gamma(B \to X_c \ell \bar{\nu})$  is an important ingredient for the determination of the CKM matrix element  $|V_{cb}|$ . However, a detailed analysis (see, e.g., Ref. [2]) also requires the knowlegde of moments of kinematic distributions. The method described in this Letter can also be applied to the calculation of such moments at order  $\alpha_s^3$ , although at the cost of significantly increased computer resources.

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