

Reconstructing Effective Lagrangians Embedding Residual Family Symmetries

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Abstract

We consider effective Lagrangians which, after electroweak- and family-symmetry breaking, yield fermionic mass matrices and/or other flavoured couplings exhibiting residual family symmetries (RFS). Thinking from the bottom up, these RFS intimately link ultra-violet (UV) Beyond-the-Standard Model (BSM) physics to infrared flavour phenomenology without direct reference to any (potentially unfalsifiable) UV dynamics. While this discussion is typically performed at the level of RFS group generators and the UV flavour groups they can close, we now also focus on the RFS-implied shape of the low-energy mass/coupling matrices. We then show how this information can be used to algorithmically guide the reconstruction of an effective Lagrangian, thereby forming top-down models realizing the typical bottom-up phenomenological conclusions. As a first application we take results from scans of finite groups capable of controlling (through their RFS) CKM or PMNS mixing within the SM alone. We then extend this to recently studied scenarios where RFS also control special patterns of leptoquark couplings, thus providing proof-in-principle completions for such ‘Simplified Models of Flavourful Leptoquarks.’

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1 Introduction

The unexplained quantization of 20-22 free and physical parameters associated to the masses, mixings, and CP-violating phases of the Standard Model’s (SM) flavour sector (the so-called *Flavour Puzzle*) represents an open challenge for theoretical constructions Beyond the SM (BSM). While these parameters are technically natural, their appearance in the quark sector is associated to an explicit breaking of the $U(3)^5$ global flavour symmetry otherwise present in the SM Lagrangian [1], while the observation of neutrino masses is already a definitive new physics phenomenon. Furthermore, the actual values of fermionic masses and mixings exhibit tantalizing hierarchies, including dramatically different patterns between quark and lepton sectors. These observations beg for a dynamical origin for flavour, and countless BSM models based on family symmetries have been devised to that end, with some even attempting explanations for the presence of the otherwise arbitrary flavour index ($i = 1, 2, 3$) in the first place.

However, the model space is underdetermined — multiple models based on different symmetries can predict the same phenomenology, and often models based on the same family symmetry can yield different infrared (IR) predictions when (unfalsifiable) tweaks to ultraviolet (UV) Lagrangian parameters are made. Indeed, it may be impossible to determine a

true theory of flavour in the absence of any convincing observation of new physics that distinguishes SM fermion generations, especially since reliable experimental constraints already exist for all but the leptonic (Dirac) CP-violating phase, absolute neutrino masses, and additional parameters depending on whether neutrinos are Majorana particles. Therefore most model predictions should actually be considered ‘post-dictions.’

One might then pursue a formalism for describing BSM flavour in more model-independent ways, focusing only on connecting patterns of family-symmetry breaking (which can themselves be generically motivated, perhaps in stringy theories — see [2, 3], e.g.) to the relevant IR phenomenology, and not on unfalsifiable Lagrangians based on new heavy states or dynamics that may be associated to that symmetry breaking. Residual Family Symmetries (RFS) provide just such a formalism, as they promote accidental Abelian symmetries of the SM mass sector to the residual subgroups of a UV flavour symmetry \mathcal{G}_F . In bases where the physical mixing parameters appear in the SM Yukawa Lagrangian (any basis other than the mass-eigenstate basis), one then notes that the Abelian generators associated to RFS are themselves functions of the physical mixing parameters. Closing flavour groups with these generators then provides the desired, model-independent link between UV symmetries and IR mixing phenomenology, and multiple analytic and computational studies have been performed to uncover viable \mathcal{G}_F [4–29].

Of course, if a particular ‘simplified model’ (or class of simplified models) based on the RFS formalism is singled out due to new measurements in the flavour sector, a more complete description of the physics will be desired. In this paper we provide a method to (re)construct effective Lagrangians that recover the symmetry breaking distilled in RFS scenarios. That is, we show how to construct a top-down model from a bottom-up phenomenological observation/conclusion. We do so by focusing on the intimate link between RFS generators and the implied shape of an RFS-invariant mass/coupling matrix. After all, RFS are symmetries of mass matrices and not the full SM Lagrangian and so, up to possible ambiguities associated to the group-theoretical properties of RFS generators, a specific symmetry-breaking pattern from the UV \mathcal{G}_F to a given RFS implies a specific IR mass/coupling matrix. This shape then hints at relevant multiplet charge assignments under the parent \mathcal{G}_F , which when combined with RFS-implied vacuum expectation values (VEV) for family-symmetry breaking scalar flavons, can be used to algorithmically construct an effective Lagrangian. We first apply this method to models addressing SM mixing structures alone, i.e. the U_{PMNS} or U_{CKM} matrices, and then also to a class of ‘Simplified Models of Flavourful Leptoquarks’ developed in [30, 31]. These models include a new Yukawa-like coupling between the leptoquark and SM quark and lepton doublets which is, in addition to the SM mixing, controlled by RFS. They therefore generate rich, flavour-dependent phenomenology at the Large Hadron Collider (LHC) and other precision experiments which can be used to probe their predictions.

The paper develops as follows: in Section 2 we give a pedagogical review of the RFS formalism, making explicit the intimate connection between RFS generators and implied mass shapes, while also describing bottom-up techniques to close UV flavour groups. We then discuss how to take those results and build an effective UV Lagrangian. In Section 3 we apply this recipe to models reproducing U_{PMNS} or U_{CKM} before moving to leptoquark applications in Section 4. We conclude in Section 5 and give relevant information for the finite groups we employ in Appendix A.

2 RFS: Bottom-Up Formalism for Top-Down Models

The core assumption of the RFS paradigm is that a parent flavour symmetry $\mathcal{G}_{\mathcal{F}}$ is broken in such a way that, after subsequent EWSB, RFS mediated by subgroups of $\mathcal{G}_{\mathcal{F}}$ are preserved in some or all of the SM mass matrices, or indeed any other term controlled by the original flavour symmetry. For example, a natural symmetry breaking pattern through intermediate groups controlling the lepton and quark sectors of the SM is schematically illustrated by

$$\mathcal{G}_{\mathcal{F}} \rightarrow \begin{cases} \mathcal{G}_{\mathcal{L}} \rightarrow \begin{cases} \mathcal{G}_{\nu} \\ \mathcal{G}_l \end{cases} \\ \mathcal{G}_{\mathcal{Q}} \rightarrow \begin{cases} \mathcal{G}_u \\ \mathcal{G}_d \end{cases} \end{cases} \quad (1)$$

Of course other patterns beyond (1), perhaps without the intermediate $\mathcal{G}_{\mathcal{L},\mathcal{Q}}$ or which only address either the quark or lepton sector individually, are also conceivable. $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$ can in principle be Abelian or non-Abelian, continuous or discrete, although for the remainder of this paper we will assume that $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$ are non-Abelian, such that irreducible multiplets of dimension greater than one can be arranged in flavour space. Furthermore, we will only work with non-Abelian discrete symmetries (NADS) when constructing explicit models in Sections 3-4, although our general approach and analysis in this section is equally applicable to non-Abelian continuous flavour groups as well. Finally, the RFS \mathcal{G}_a with $a \in \{u, d, l, \nu\}$ must be Abelian, and in particular Abelian cyclic groups of order n ,

$$\mathcal{G}_a \cong \mathbb{Z}_a^{n_a}, \quad (2)$$

when $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$ are themselves discrete. Discrete product groups of the form $\mathcal{G}_a \sim \mathbb{Z} \times \mathbb{Z} \times \dots$ are also possible.

2.1 The Infrared Lagrangian

To review how the RFS chain in (1) is naturally motivated, we follow prior discussions (see e.g. [4,9]) and first write down the SM Yukawa Lagrangian, after EWSB, in the mass-eigenstate basis:

$$\mathcal{L}_{mass}^{SM} \supset \frac{1}{2} \bar{\nu}_L^c m_\nu \nu_L + \bar{E}_R m_l l_L + \bar{d}_R m_d d_L + \bar{u}_R m_u u_L + \text{h.c.}, \quad (3)$$

where m_a are diagonal matrices of mass eigenvalues and where we have assumed a Majorana neutrino mass term to illustrate our point, although our RFS approach applies straightforwardly to a Dirac mass $\propto \bar{\nu}_R m_\nu \nu_L$ as well. From (3) we observe that the Lagrangian is invariant under Abelian transformations on the fermion fields:¹

$$\begin{aligned} \nu_L &\rightarrow T_{\nu_i} \nu_L, & \text{with } T_{\nu_1} &= \text{diag}(1, -1, -1) & \text{and } T_{\nu_2} &= \text{diag}(-1, 1, -1), \\ f &\rightarrow T_f f, & \text{with } T_f &= \text{diag}(e^{i\alpha_f}, e^{i\beta_f}, e^{i\gamma_f}) & \text{for } f &\in \{E_R, l_L, d_R, d_L, u_R, u_L\}. \end{aligned} \quad (4)$$

¹Other basis choices can be made for the Klein generators T_{ν_i} .

Hence we promote these accidental symmetries to RFS, and note that, for the case of Majorana neutrinos, the (maximal) RFS generated by T_{ν_i} is a Klein four-group [4],

$$\mathcal{G}_\nu \cong \mathbb{Z}_\nu^2 \times \mathbb{Z}_\nu^2. \quad (5)$$

Similarly, the generators T_f simply represent re-phasing freedoms of the three fermion generations in each family's Dirac mass term,

$$\mathcal{G}_f \cong U(1)^3, \quad (6)$$

and of course a Dirac neutrino mass term would be generically invariant under (6) instead of (5). When (2) is realized the otherwise continuous phases in T_f are quantized as

$$\{\alpha, \beta, \gamma\}_f \stackrel{!}{=} \frac{2\pi}{m} \{a, b, c\}_f. \quad (7)$$

Finally, from (3) one also finds that $T_{f_L} \stackrel{!}{=} T_{f_R}$ for the terms to be invariant. Of course, left- and right-chiral fermions can be charged differently in the complete flavour theory invariant under $\mathcal{G}_{\mathcal{F}, \mathcal{L}, \mathcal{Q}}$.

However, (3)-(4) tell us nothing about the physical predictions associated to the family symmetry breaking in (1). It is only when we rotate to a basis where the Yukawa terms contain information about fermionic mixing that the RFS is useful as a bottom-up tool. Take the standard 'flavour basis' of the SM, where charged-current (CC) interactions are diagonal, as an example. Here (3) is transformed to

$$\mathcal{L}_{flav}^{SM} \supset \frac{1}{2} \bar{\nu}_L^c \underbrace{U_\nu^* m_\nu U_\nu^\dagger}_{m_{\nu U}} \nu_L + \bar{E}_R \underbrace{U_E m_l U_l^\dagger}_{m_{lU}} l_L + \bar{d}_R \underbrace{U_D m_d U_d^\dagger}_{m_{dU}} d_L + \bar{u}_R \underbrace{U_U m_u U_u^\dagger}_{m_{uU}} u_L + \text{h.c.}, \quad (8)$$

where the left-handed unitary matrices U have physical effects in the CC through the presence of the CKM and PMNS overlap matrices:

$$U_{CKM} \equiv U_u^\dagger U_d, \quad U_{PMNS} \equiv U_l^\dagger U_\nu. \quad (9)$$

Both $U_{CKM, PMNS}$ are 3×3 matrices in flavour space and are parameterized by three mixing angles $\theta_{12, 23, 13}^{q,l}$ and one Dirac CP-violating phase $\delta^{q,l}$. If neutrinos are Majorana particles, the PMNS also encodes two additional phases $\alpha_{1,2}$. Hence it is clear that the redefined mass matrices m_{aU} (where a denotes all fermions) in (8) are themselves 3×3 matrices in flavour space, and are of course related to the SM Yukawa couplings Y_a through the Higgs VEV v ,

$$m_{aU} \equiv \frac{v}{\sqrt{2}} Y_a. \quad (10)$$

Obviously (10) does not hold for the Majorana neutrino mass term written explicitly above.

Let us now examine the RFS of (8). Here one observes that the Lagrangian is invariant under transformations of the form

$$a \rightarrow T_{aU} a, \quad \text{with} \quad T_{aU} = U_a T_a U_a^\dagger, \quad (11)$$

as opposed to those of (4). Indeed, the RFS generator T_{aU} now knows about the physical mixing matrices U_a , which means that any parent group $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$ with subgroup \mathcal{G}_a generated by T_{aU} can be connected to a physical mixing prediction embedded in U_a . In this way the RFS intimately links the IR phenomenology to the UV symmetry without reference to any of the dynamics associated to realizing (1). RFS therefore provide a powerful, bottom-up means of understanding observed patterns of flavour mixing in a rather model-independent way, as the only assumption made thus far is that the accidental flavour symmetries of the SM mass sector encoded in (4) are in fact the global RFS of a complete flavour theory broken as in (1) or its analogues.² In this way the RFS formalism defines a set of ‘simplified flavour models,’ which can easily be extended to BSM constructions as well — see [34] for a recent application of RFS to the Yukawa sector of multi-Higgs doublet models, where they were shown to be capable of controlling dangerous flavour-changing neutral currents alongside of fermionic mixing, and [30,31] where it is demonstrated that they can also structure the flavour patterns of leptoquark couplings (we also address some of these models in Section 4 below).

However, (8) is but one of an infinite number of bases that the Yukawa sector can be written in. In the presence of BSM couplings that introduce new (physical) mixings, one may want to work in a different one in order to preserve diagonal CC (see the ‘leptoflavour basis’ discussed in Section 4). Or one may be motivated to change basis due to the ease of use of certain (basis-dependent) group product rules. Regardless, the trend as regards the associated RFS symmetry transformation is trivially clear; a rotation on a mass-eigenstate field a with unitary matrix V_a^\dagger equivalently implies a basis-change on the corresponding RFS mass-basis generator T_a through the same matrix:

$$a \rightarrow V_a^\dagger a \iff T_a \rightarrow V_a T_a V_a^\dagger. \quad (12)$$

The statement holds vice versa as well, since otherwise T_a would no longer be an RFS generator, as it would not leave the associated mass/Yukawa term invariant. In Section 2.3 we will apply the logically equivalent statement to (12) to study the RFS-invariant mass matrices themselves, in an effort to guide the reconstruction of an effective Lagrangian with manifest $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$.

Before doing so, it is important to address a couple of subtleties in the approach, for clarity. First, the RFS are not symmetries of the full IR Lagrangian. CC interactions do not respect them without additional assumptions relating RFS within the quark and/or lepton sector. This is realized naturally in most flavour models, however, since the breaking of $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$ is typically only communicated to the Yukawa sector, perhaps through scalar flavons developing VEV. This is the approach we will take in what follows, although it is worth noting that family-symmetry breaking can also occur through other mechanisms, e.g. orbifold compactifications.

And secondly, we recall that a bottom-up RFS analysis alone cannot recover the exact mixing prediction associated to the model sketched by (1) unless \mathcal{G}_a distinguishes all three fermion generations, i.e. the associated RFS generator(s) T_a needs to have three distinct eigenvalues (or multiple T_{a_i} need to be present when T_a has fewer than three distinct eigenvalues). This

²The RFS formalism therefore characterizes a class of flavour models, sometimes referred to as ‘direct’ or ‘semi-direct’ in the literature [32], but it is of course not entirely generic. It is plausible that (4) instead represent truly accidental symmetries, and that T_a do not generate the subgroups of $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$. These types of models are sometimes called ‘indirect’ — see [33] for a successful GUT-inspired example.

point becomes clear in the following flavour-basis equality:

$$T_{aU} = U_a T_a^{ii=jj} U_a^\dagger = U_a R_a^{ij} T_a^{ii=jj} R_a^{ji*} U_a^\dagger, \quad \text{with} \quad R^{ij} \equiv \begin{pmatrix} \cos \theta_{ij} & \sin \theta_{ij} e^{-i\delta_{ij}} \\ -\sin \theta_{ij} e^{i\delta_{ij}} & \cos \theta_{ij} \end{pmatrix}. \quad (13)$$

That is, the RFS generator cannot distinguish between the mixing matrix U_a and $U_a \cdot R_a$, with the latter having free parameters in the degenerate (i, j) sector of T_a . In complete models these free parameters can either be fit to data or quantized as a result of other mechanisms, like further auxiliary or accidental symmetries of the Lagrangian. We will discuss the top-down implications of (13) in upcoming sections.

2.2 Closing Ultraviolet Flavour Groups

Given (12), one must then have a procedure for recovering the associated parent groups $\mathcal{G}_{\mathcal{F}, \mathcal{L}, \mathcal{Q}}$, as the Abelian \mathcal{G}_a alone are insufficient to model patterns of physical mixing. Many groups have attempted this by performing either analytic or computational studies of the classes of $\mathcal{G}_{\mathcal{F}, \mathcal{L}, \mathcal{Q}}$ that can break to desired subgroups \mathcal{G}_a , given that specific (phenomenologically viable) shapes for U_a must be achieved in a realistic model. On the other hand, the **GAP** language for computational finite algebra [35, 36] has been indispensable when searching for NADS with automated techniques, as it has a large library of small groups catalogued along with vast amounts of associated group-theoretical information (conjugacy classes, order, irreducible representations, etc.).

In what follows we will use the bottom-up approach to ‘reconstructing’ NADS first discussed in [19, 22], but recently applied to a class of BSM leptoquark models in [30, 31]. Here one assumes that the RFS generators form the complete generating set for $\mathcal{G}_{\mathcal{F}, \mathcal{L}, \mathcal{Q}}$, such that the latter are recovered upon using **GAP** to close all elements of the former:

$$\mathcal{G}_{\mathcal{F}} \cong \{\hat{T}_d, \hat{T}_l, \hat{T}_u, \hat{T}_\nu\}, \quad (14)$$

$$\mathcal{G}_{\mathcal{F}} \cong \underbrace{\{\hat{T}_d, \hat{T}_u\}}_{\mathcal{G}_{\mathcal{Q}}} \times \underbrace{\{\hat{T}_l, \hat{T}_\nu\}}_{\mathcal{G}_{\mathcal{L}}}, \quad (15)$$

$$\mathcal{G}_{\mathcal{L}} \cong \{\hat{T}_l, \hat{T}_\nu\}, \quad (16)$$

$$\mathcal{G}_{\mathcal{Q}} \cong \{\hat{T}_u, \hat{T}_d\}, \quad (17)$$

where (14) reconstructs a parent group generated by all family sectors, (15) forms a direct product parent group of lepton and quark symmetries, and (16)-(17) assume that the NADS only controls either lepton or quark mixing, but not both. Other scenarios could also be envisaged, e.g. one where $\mathcal{G}_{\mathcal{L}, \mathcal{Q}}$ are formed as in (16)-(17), but where $\mathcal{G}_{\mathcal{F}}$ is not their direct product group as in (15), but instead any larger group containing $\mathcal{G}_{\mathcal{L}, \mathcal{Q}}$. Regardless, the hatted (\hat{T}) notation in (14)-(17) simply indicates any basis where the generators know about physical mixing parameters,

$$\hat{T}_a \equiv \hat{T}_a(\theta_{ij}^a, \delta_{ij}^a, \dots). \quad (18)$$

Of course, when searching for NADS in the bottom-up approach one must also apply a discretization scheme to all of these mixing parameters, and details on this procedure and other cuts made regarding group order, etc., can be found on case-by-case bases in [19, 22, 31].

However, it does not matter how one finds the parent symmetry with associated RFS for our present purposes. Any procedure is appropriate, as long as all relevant information about the RFS can be extracted, which we now discuss in detail.

2.3 Guided Reconstruction for Effective Lagrangians

As demonstrated above, the implied shape of the generators \hat{T}_a given an RFS-invariant Lagrangian is the information required for connecting the IR \mathcal{G}_a to the UV $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$. But is there a systematic way of using the recovered non-Abelian parent group to build an effective Lagrangian \mathcal{L}_Y (a model, that is) that exhibits, upon family- and electroweak- symmetry breaking, the simplified construction of (1)? A straightforward approach³ is based on the implied RFS invariances of (a) the IR mass matrix and (b) new scalar flavons whose VEV implement the breaking patterns of (1).

Concerning (a), we note that a generic rotation V_a^\dagger on the mass-eigenstate terms in (3) yields new mass matrices of the form

$$\hat{m}_a \equiv V_{aR} m_a V_{aL}^\dagger \iff \hat{T}_a^\dagger \hat{m}_a^\dagger \hat{m}_a \hat{T}_a = \hat{m}_a^\dagger \hat{m}_a = V_{aL} m_a^\dagger m_a V_{aL}^\dagger, \quad (19)$$

where we observed that, by construction, the Hermitian combination of this term is invariant under \hat{T}_a given in (12). Hence, as m_a can always be written as a diagonal matrix of mass eigenvalues, the RFS-invariant quantity $\hat{m}_a^\dagger \hat{m}_a$ can always be written out in model space, once the rotations V_a are specified. In the class of simplified models we have reviewed above, V_a can always be extracted from the (known) IR phenomenology that is predicted. In this way (19) provides the rubric for completing the simplified model, as a \mathcal{L}_Y that reproduces it will, by construction, embed the desired RFS.

Then flavons (point (b)) provide a candidate mechanism for breaking $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$ down to the desired RFS-invariant mass shapes. The \hat{T}_a invariant mass matrices are then obtained, after the flavon expands around its VEV, from Lagrangian terms of the form

$$\mathcal{L}_Y \supset \frac{\hat{y}_a}{\Lambda} [\bar{A}_R \phi_a H A_L]_{\mathbf{1}}, \quad (20)$$

where A_L denotes the associated $SU(2)_L$ doublet for the family sector, A_R is the $SU(2)_L$ singlet, H is a Higgs doublet, and \hat{y}_a is the effective coupling suppressed by the new physics scale Λ integrated out of the effective operator. To enforce \hat{T}_a invariant masses, we can use the following condition regarding the VEV direction of the flavon field [4]

$$\hat{T}_a^\dagger \langle \phi_a \rangle = \hat{e}_a^* \langle \phi_a \rangle \iff \hat{T}_a \langle \phi_a \rangle = \hat{e}_a \langle \phi_a \rangle \quad \text{with} \quad \hat{e}_a^* \cdot \hat{e}_a = 1, \quad (21)$$

where \hat{e}_a is a (scalar) eigenvalue, and the \Leftrightarrow is due to the fact that T_a is assumed without loss of generality (since we work with finite groups) to be unitary. From the group theory perspective, it is of course obvious that $\langle \phi_a \rangle$ (or indeed $\langle \phi_a \rangle \cdot \langle \phi_a \rangle$) should preserve $\mathcal{G}_a \cong \hat{T}_a$ when (21) holds. Therefore (21) provides a condition on operators of the form (20) which yield mass matrices of the form in (19), and thereby realize the simplified construction in (1).

³We do not claim that this is the unique prescription, but rather a simple and economical one.

Summary of Effective Lagrangian Reconstruction

In summary, once a flavour group $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$ is determined with RFS generated by $\{\hat{T}_a\}$, which are assumed to know about the physical mixings the RFS mechanism (and therefore the reconstructed model) controls, the following procedure can be followed to yield \mathcal{L}_Y :

1. Write out the explicit representations of $\{\hat{T}_a\}$ in a basis amenable to manipulating the group product rules of $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$. This is typically determined by identifying $\{\hat{T}_a\}$ with specific group elements in a given irreducible representation. If this basis differs from the one in which $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$ was originally recovered, take account of the additional transformations in the IR Lagrangian. This determines the *model basis*.
2. For each family sector a with an active RFS, write down a new flavon ϕ_a whose VEV respects (21) in the model basis, hence deriving the model-space orientation of $\langle\phi_a\rangle$. The flavon ϕ_a is taken to be charged under the irreducible representation that \hat{T}_a was identified with in Step 1.
3. Derive the expected form of the model-basis mass/coupling matrix in each family sector, which is given by (19) for a generic set of model-basis transformation(s) V_a away from the mass-eigenstate basis, and form the Hermitian combination which preserves the information from the physical mixings in the theory.
4. For each family sector a with an active RFS, create an effective Yukawa-like operator with ϕ_a and build invariants of the form in (20), or a similar invariant of the form $\propto [L_L L_L \phi_\nu \phi_\nu]_{\mathbf{1}}$ for Majorana neutrinos. Multiple such invariants may be required in a given family sector, depending on the kinds of irreducible representations implied in Steps 1-3. The goal is to recover the RFS-invariant mass/coupling shapes from Step 3, with a one-to-one mapping between physical and model parameters, and of course the shapes of $\{\hat{T}_a\}$ and $\hat{m}_a^\dagger \hat{m}_a$ already hint at appropriate generations charges under $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$.
5. Construct the Hermitian Yukawa coupling $\hat{Y}_a^\dagger \hat{Y}_a$ from (20),⁴ such that a comparison with the quantities in Step 3 can be made. Map the model parameters (e.g. $\{\hat{y}_{ai}, v, \dots\}$) to physical parameters (e.g. $\{m_{ai}, \theta_{ij}^a, \dots\}$). If this mapping is not one-to-one, the model may appear to require some fine-tuning of parameters, although we will show that this could be a misleading conclusion if the expected RFS-invariant mass matrices have not been generalized with the free parameters permitted through the relationships in (13). Also be sure to check that the implied mass eigenvalues are physical. If not, additional operators may need to be added.

If Steps 1-5 are successful, the resulting model will exhibit the RFS symmetry-breaking patterns and desired phenomenology embedded in the original simplified models, thereby providing an EFT completion.

⁴It is often preferred to instead construct terms in the LR basis, with operators $\propto \bar{A}_L \phi_a H A_R$. In this case, simply identify the predicted Yukawa coupling from this term as \hat{Y}_a^\dagger , and then proceed to build $\hat{Y}_a^\dagger \hat{Y}_a$. We will do this in some of the models below. Also, it is obvious that Majorana neutrinos do not require the construction of the Hermitian object $\hat{m}_\nu^\dagger \hat{m}_\nu$.

3 Application to Models of SM Flavour

In this section we apply the strategy outlined in Section 2.3 to flavour models reproducing SM mixing matrices, i.e. the PMNS or CKM matrices defined in (9).

3.1 A_4 Altarelli-Feruglio Model for U_{PMNS}

As a first application of the algorithm described in Section 2, we now show how the famous Altarelli-Feruglio model of leptonic flavour [37, 38] can be reconstructed with only minimal knowledge of its low-energy predictions. In particular, its IR phenomenology is characterized by the breaking of the tetrahedral A_4 group to \mathbb{Z}_3 and \mathbb{Z}_2 RFS in the charged lepton and neutrino sectors,

$$\mathcal{G}_{\mathcal{L}} \cong A_4 \rightarrow \begin{cases} \mathcal{G}_\nu \cong \mathbb{Z}_2 \\ \mathcal{G}_l \cong \mathbb{Z}_3, \end{cases} \quad (22)$$

which are respectively generated by

$$T_\nu = \text{diag}(-1, 1, -1), \quad T_l = \text{diag}(1, \omega^2, \omega), \quad \omega \equiv e^{2\pi i/3}. \quad (23)$$

The model assumes Majorana neutrino masses, and its LO mixing prediction is the TBM matrix defined in (76),⁵

$$U_{PMNS} \equiv \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}, \quad (24)$$

which is realized in a special flavour basis, where $U_l = \mathbb{1}$ and $U_\nu \equiv U_{PMNS}$.⁶ Given knowledge of (23)-(24), we now have all of the information necessary to apply our reconstruction algorithm.

We first use these equations to infer that, in the model basis we will construct our effective Lagrangian, the relevant generators are given by

$$T_{\nu U} = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T_{lU} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad (25)$$

which we immediately identify as triplet $\mathbf{3}$ representations from the A_4 review in [37], and which we can use to solve for flavon VEV in each family sector,

$$T_{\nu U, lU} \langle \phi_{\nu, l} \rangle = \langle \phi_{\nu, l} \rangle \implies \langle \phi_\nu \rangle = \begin{pmatrix} v_\nu \\ v_\nu \\ v_\nu \end{pmatrix}, \quad \langle \phi_l \rangle = \begin{pmatrix} v_l \\ 0 \\ 0 \end{pmatrix}, \quad (26)$$

⁵In this particular example, our convention for the PMNS mixing is different than that adopted later in the paper, cf. (76), in order to better reproduce those of [38].

⁶Note that one is always free to move to this basis, and so knowledge of this fact about the Altarelli-Feruglio model does not bias our approach in what follows.

and to conclude that the model-basis mass matrices invariant under them are characterized by

$$m_{\nu U} \stackrel{!}{=} \frac{1}{3} \begin{pmatrix} (2m_{\nu_1} + m_{\nu_2}) & (m_{\nu_2} - m_{\nu_1}) & (m_{\nu_2} - m_{\nu_1}) \\ (m_{\nu_2} - m_{\nu_1}) & \frac{1}{2}(3m_{\nu_3} + 2m_{\nu_2} + m_{\nu_1}) & \frac{1}{2}(m_{\nu_1} + 2m_{\nu_2} - 3m_{\nu_3}) \\ (m_{\nu_2} - m_{\nu_1}) & \frac{1}{2}(m_{\nu_1} + 2m_{\nu_2} - 3m_{\nu_3}) & \frac{1}{2}(3m_{\nu_3} + 2m_{\nu_2} + m_{\nu_1}) \end{pmatrix}, \quad (27)$$

$$m_{lU}^\dagger m_{lU} \stackrel{!}{=} \begin{pmatrix} m_e^2 & 0 & 0 \\ 0 & m_\mu^2 & 0 \\ 0 & 0 & m_\tau^2 \end{pmatrix}, \quad (28)$$

which we can further use to infer charge assignments under A_4 for SM fields and the new flavons in (26). We recall that A_4 is $\mathcal{O}(12)$ and has four irreducible representations: a triplet $\mathbf{3}$ and three singlets $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$, with $\mathbf{1}$ denoting the trivial representation. While A_4 is an exceptionally well studied finite group, we repeat the relevant product rules in this basis for completeness in Appendix A.1, where it is clear that in order to build up non-trivial Yukawa matrices, the SM $SU(2)$ doublet L_L and corresponding flavons (from (26)) will need to be assigned to the triplet representation:

$$L_L \sim \mathbf{3}, \quad \phi_\nu \sim \mathbf{3}, \quad \phi_1 \sim \mathbf{3}. \quad (29)$$

Given this, we then consider the charged lepton mass term and observe from (25) and (28) that SM generations do not ‘talk’ to one another through the A_4 symmetry, and so we assign a different singlet to each RH SM field:

$$e_R \sim \mathbf{1}, \quad \mu_R \sim \mathbf{1}'', \quad \tau_R \sim \mathbf{1}', \quad (30)$$

and hence combinations of $[\phi_l L_L]$ as needed in (20) must themselves form one-dimensional representations of A_4 . Noting this, one quickly deduces the LO effective Yukawa Lagrangian for this sector:

$$\mathcal{L}_l \supset a_e e_R [\phi_l L_L]_{\mathbf{1}} + b_e \mu_R [\phi_l L_L]_{\mathbf{1}''} + c_e \tau_R [\phi_l L_L]_{\mathbf{1}'} + \text{h.c.} + \dots, \quad (31)$$

where the $[\dots]_{\mathbf{1}}$ notation indicates that the bracketed fields contract to the indicated singlet under the A_4 product rules given in (127). Each individual term in (31) is then an A_4 and SM gauge singlet, once contracted with the corresponding RH isospin singlets. The additional terms implied in (31) correspond to higher-order operators in the OPE allowed by successive flavon and SM field insertions, given their associated symmetry assignments. We will discuss these below, along with additional symmetries irrelevant to the RFS formalism. Regardless, one immediately finds that (31) generates the desired mass matrix from (28), with the relations between masses and Lagrangian parameters easily found to be

$$m_e = a_e v \frac{v_l}{\Lambda}, \quad m_\mu = b_e v \frac{v_l}{\Lambda}, \quad m_\tau = c_e v \frac{v_l}{\Lambda}, \quad (32)$$

with v the Higgs VEV realizing EWSB.

	L_L	e_R	μ_R	τ_R	ϕ_ν	ϕ_l	ξ
A_4	3	1	1''	1'	3	3	1

Table 1: Relevant field and A_4 symmetry content from [38].

Moving now to the neutrino masses, the $m_{\nu U}$ implied in (27) has non-trivial structure in all matrix sectors, a fact concurrent with 1) our observation that ϕ_ν and L_L should be charged as A_4 triplets, and 2) the fact the Altarelli-Feruglio Model predicts a Majorana neutrino mass matrix, which is itself implied (or at least consistent with) the \mathbb{Z}_2 neutrino RFS. In the low-energy EFT, a Majorana neutrino mass is necessarily $\propto L_L L_L$. We therefore conclude that an operator of the form⁷

$$\mathcal{L}_\nu \supset a_\nu [\phi_\nu L_L L_L]_{\mathbf{1}} + \text{h.c.} + \dots, \quad (33)$$

should be included in the Lagrangian. This term generates a contribution to $m_{\nu U}$,

$$m_{\nu U} \simeq \frac{v^2}{\Lambda} \begin{pmatrix} 2B/3 & -B/3 & -B/3 \\ -B/3 & 2B/3 & -B/3 \\ -B/3 & -B/3 & 2B/3 \end{pmatrix}, \quad B \equiv 2a_\nu \frac{v_\nu}{\Lambda}, \quad (34)$$

which, while invariant under (25) (as it must be by construction), fails to realize the required neutrino phenomenology, as it has only two distinct eigenvalues: 0 and B . In other words, it cannot map to the generic, RFS-invariant form in (27) that we have deduced, in the absence of (unphysical) assumptions about the mass eigenvalues embedded in it. The obvious solution is to introduce a further flavon ξ whose VEV $\langle \xi \rangle = u$ does not break \mathcal{G}_ν and which can couple to the $L_L L_L$ bilinear. To that end we introduce ξ as an A_4 singlet, which adds an additional contribution to (33)-(34),

$$\mathcal{L}_\nu \supset a_\nu [\phi_\nu L_L L_L]_{\mathbf{1}} + b_\nu \xi [L_L L_L]_{\mathbf{1}} \implies m_{\nu U} \simeq \frac{v^2}{\Lambda} \begin{pmatrix} A + 2B/3 & -B/3 & -B/3 \\ -B/3 & 2B/3 & A - B/3 \\ -B/3 & A - B/3 & 2B/3 \end{pmatrix}, \quad A \equiv 2b_\nu \frac{u}{\Lambda}. \quad (35)$$

This matrix is still invariant under (25), is diagonalized by U_{TBM} , and has mass eigenvalues given by

$$m_{\nu_1} = \frac{v^2}{\Lambda} (A + B), \quad m_{\nu_2} = \frac{v^2}{\Lambda} A, \quad m_{\nu_3} = \frac{v^2}{\Lambda} (B - A), \quad (36)$$

which is fully consistent with the matrix form in (27).

In conclusion, with the knowledge of the parent flavour symmetry A_4 , the neutrino and charged lepton RFS in (23), and the PMNS mixing prediction given in (24), we have easily inferred the field and symmetry content in Table 1 and the following LO effective Yukawa Lagrangian:

$$\mathcal{L}_Y \supset a_e e_R [\phi_l L_L]_{\mathbf{1}} + b_e \mu_R [\phi_l L_L]_{\mathbf{1}'} + c_e \tau_R [\phi_l L_L]_{\mathbf{1}''} + a_\nu [\phi_\nu L_L L_L]_{\mathbf{1}} + b_\nu \xi [L_L L_L]_{\mathbf{1}} + \text{h.c.} + \dots \quad (37)$$

⁷Note that a see-saw realization of this IR Majorana term is also possible — see [38].

This is to be compared to eq(12) in [38], where it is found to be equivalent to the non-SUSY version of the Altarelli-Feruglio Lagrangian — we have ‘reconstructed’ this model from the bottom up.

However, as is well known, the complete model of [38] is more involved than just its LO Yukawa terms. Furthermore, we made choices in the above discussion that, a priori, may seem ad-hoc. We will now discuss some of these subtleties for this particular model, as well as their broader implications for our generic approach, although in forthcoming models we will typically leave these discussions implicit, unless they become particularly relevant for the physics at hand.

Mass and Mixing Prediction Ambiguities

We have observed in the preceding section that knowledge of the IR RFS and mixing prediction is not guaranteed to tell us *everything* required to build the LO terms in the EFT. For one, as became clear between (34)-(35), the RFS has no control over the quantization of the mass eigenvalues, but only the mixing associated to them.⁸ As we saw, (34) exhibits the required \mathbb{Z}_2 invariance, but does not map to the (more) generic RFS-invariant form in (27) without imposing,

$$m_{\nu_1} \rightarrow m_{\nu_3}, \quad m_{\nu_2} \rightarrow 0, \quad (38)$$

which is unphysical (there are two non-zero mass splittings measured for low-energy neutrinos). This motivated the introduction of the singlet ξ whose VEV also breaks $\mathcal{G}_{\mathcal{L}}$, but does *not* break \mathcal{G}_ν , and so does not upset the TBM prediction for the PMNS. In general, this is a good strategy when reconstructing a given Majorana-neutrino-sector Lagrangian for which one does not have a literature reference, as we did here — operators $\propto [L_L L_L]_1$ will always preserve a given RFS if augmented only by a scalar singlet.

And secondly, in the absence of a reference Lagrangian, one can also reconstruct a mass term associated to a different mixing prediction, when degenerate eigenvalues exist in RFS generators. This was highlighted explicitly for the mixing in (13), but of course this also has implications for the associated RFS-invariant mass matrix. In the Altarelli-Feruglio case, we observe that T_l has three distinct eigenvalues, and so U_l is uniquely predicted as the identity matrix. However, as only one \mathbb{Z}_2 is explicitly preserved in the neutrino sector, the truly generic RFS-invariant mass matrix is given by

$$\begin{aligned} m_{\nu U}^{11} &= \frac{1}{3} (m_{\nu_2} + 2m_{\nu_1} \cos^2 \theta_{13} + 2m_{\nu_3} e^{2i\delta_{13}} \sin^2 \theta_{13}), \\ m_{\nu U}^{12} &= \frac{1}{3} (m_{\nu_2} - m_{\nu_1} \cos^2 \theta_{13} + \sqrt{3} e^{-i\delta_{13}} (m_{\nu_1} - e^{2i\delta_{13}} m_{\nu_3}) \cos \theta_{13} \sin \theta_{13} - m_{\nu_3} \sin^2 \theta_{13} e^{2i\delta_{13}}), \\ m_{\nu U}^{13} &= \frac{1}{3} (m_{\nu_2} - m_{\nu_1} \cos^2 \theta_{13} - \sqrt{3} e^{-i\delta_{13}} (m_{\nu_1} - e^{2i\delta_{13}} m_{\nu_3}) \cos \theta_{13} \sin \theta_{13} - m_{\nu_3} \sin^2 \theta_{13} e^{2i\delta_{13}}), \\ m_{\nu U}^{22} &= \frac{1}{18} \left(6m_{\nu_2} + m_{\nu_1} \left(\sqrt{3} \cos \theta_{13} - 3e^{-i\delta_{13}} \sin \theta_{13} \right)^2 + m_{\nu_3} \left(3 \cos \theta_{13} + \sqrt{3} e^{i\delta_{13}} \sin \theta_{13} \right)^2 \right), \\ m_{\nu U}^{23} &= \frac{1}{6} (2m_{\nu_2} + (m_{\nu_1} - 3m_{\nu_3}) \cos^2 \theta_{13} + e^{-2i\delta_{13}} (-3m_{\nu_1} + e^{4i\delta_{13}} m_{\nu_3}) \sin^2 \theta_{13}), \end{aligned} \quad (39)$$

⁸This is a well known fact about RFS that are only active in SM mass terms.

$$m_{\nu U}^{33} = \frac{1}{18} \left(6m_{\nu_2} + e^{-2i\delta_{13}} m_{\nu_1} \left(\sqrt{3} e^{i\delta_{13}} \cos \theta_{13} + 3 \sin \theta_{13} \right)^2 + m_{\nu_3} \left(-3 \cos \theta_{13} + \sqrt{3} e^{i\delta_{13}} \sin \theta_{13} \right)^2 \right),$$

with θ_{13} and δ_{13} defined as in (13). This complex symmetric matrix is diagonalized by U_{TBM-13}^T . $m_{\nu U} \cdot U_{TBM-13} = m_\nu$, with

$$U_{TBM-13} = \begin{pmatrix} \sqrt{2/3} \cos \theta_{13} & 1/\sqrt{3} & \sqrt{2/3} e^{-i\delta_{13}} \sin \theta_{13} \\ -\frac{\cos \theta_{13}}{\sqrt{6}} + \frac{e^{i\delta_{13}} \sin \theta_{13}}{\sqrt{2}} & 1/\sqrt{3} & -\frac{\cos \theta_{13}}{\sqrt{2}} - \frac{e^{-i\delta_{13}} \sin \theta_{13}}{\sqrt{6}} \\ -\frac{\cos \theta_{13}}{\sqrt{2}} - \frac{e^{-i\delta_{13}} \sin \theta_{13}}{\sqrt{6}} & 1/\sqrt{3} & \frac{\cos \theta_{13}}{\sqrt{2}} - \frac{e^{-i\delta_{13}} \sin \theta_{13}}{\sqrt{6}} \end{pmatrix}, \quad (40)$$

from which θ_{13} and δ_{13} can be fit to experimental data, yielding a phenomenologically successful PMNS matrix. The point here is that knowledge of the RFS alone does not nail down the mixing or mass matrix prediction that a top-down EFT can yield, when the generators of said RFS do not distinguish all three generations. That the Altarelli-Feruglio model predicts (24) and not (40) at leading order is due to the accidental invariance of the $m_{\nu U}$ in (35) under the $\mu - \tau$ operator⁹

$$T_{\mu-\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (41)$$

which (39) does not respect. This invariance is not associated to the RFS, as $\mathbb{Z}_2 \times \mathbb{Z}_2^{\mu\tau}$ is not a subgroup of A_4 . Rather, the accidental invariance of (35) under (41) is due to the absence of additional operators in (37), which is a consequence of additional symmetries unrelated to the RFS, which we will now discuss.

Further Symmetries and Fields

The astute reader will notice that (37) does not contain the most generic set of operators invariant under the A_4 flavour symmetry and SM gauge symmetries. For example, a term of the form (briefly restoring Higgs fields to maintain clarity)

$$\mathcal{O} \sim L_L L_L H H \quad (42)$$

is also allowed, as are the four operators corresponding to (37) but with $\phi_\nu \leftrightarrow \phi_l$. Indeed, these additional contributions to the LO Lagrangian are forbidden by a \mathbb{Z}_3 *shaping* symmetry,

$$\{e, \mu, \tau\}_R \sim \omega^2, \quad \{H, \phi_l\} \sim 1, \quad \{L_L, \xi, \phi_\nu\} \sim \omega, \quad (43)$$

which limits contact interactions between certain fields. As the RFS have nothing to do with these shaping symmetries, in what follows we will simply assume that either they are not needed or, more commonly, that they can always be found such that only desired operators in the $1/\Lambda$ OPE are recovered.

We also ignored all of the dynamics required to obtain the VEV derived in (26). As mentioned in the introduction, flavon VEV can be realized via the minimization of an appropriate

⁹Note that, using the basis choice for the Klein RFS generating set in (4), one would instead derive the matrix $-T_{\mu-\tau}$ from the bottom-up. which still leaves $m_{\nu U}$ invariant but corresponds to a different \hat{e}_ν .

scalar potential. SUSY is assumed for the Altarelli-Feruglio model, such that (37) is understood as one part of the overall superpotential, whilst yet another flavon $\tilde{\xi}$ that breaks A_4 is introduced alongside additional ‘driving’ superfields ϕ_0^l, ϕ_0^{ν} and ξ_0 . All fields are then further charged under a traditional R symmetry $U(1)_R$ that distinguishes matter, symmetry-breaking, and driving/alignment fields. We again ignore all such discussion in upcoming models, as we simply assume that the required VEV alignment can be achieved.

Finally, we mentioned that the RFS do not constrain mass eigenvalues. That means that mass hierarchies must be understood with some other mechanism. In the case of [38], this is achieved with an additional Froggatt-Nielsen [39] $U(1)_{FN}$, under which the τ_R, μ_R , and e_R generations are assigned 0, q and $2q$, and additional flavons θ are introduced whose VEV create hierarchical mass suppressions $\sim \lambda \equiv \langle \theta \rangle / \Lambda$: $c_e \approx \mathcal{O}(1)$, $b_e \approx \mathcal{O}(\lambda^q)$, and $a_e \approx \mathcal{O}(\lambda^{2q})$. Again, such symmetries can always be imposed in addition to the core flavour symmetries yielding the RFS of interest. We therefore do not mention them further in what follows.

Higher-order Operators

We have only reconstructed the LO Yukawa Lagrangian in the $1/\Lambda$ OPE. Higher-order terms associated to more SM or flavon field insertions (but which are still invariant under *all* assumed symmetries) can of course be found, and these will generate small corrections to the phenomenological conclusions of the LO Lagrangian. In the Altarelli-Feruglio model, the leading such terms are given by

$$\mathcal{O} \sim [e_R L_L \phi_l \phi_l]_{\mathbf{1}}, \quad \mathcal{O} \sim [\mu_R L_L \phi_l \phi_l]_{\mathbf{1}}, \quad \mathcal{O} \sim [\tau_R L_L \phi_l \phi_l]_{\mathbf{1}}, \quad (44)$$

for the charged leptons, and

$$\mathcal{O} \sim [\phi_l \phi_\nu]_{\mathbf{1}'} [L_L L_L]_{\mathbf{1}''}, \quad \mathcal{O} \sim [\phi_l \phi_\nu]_{\mathbf{1}''} [L_L L_L]_{\mathbf{1}'}, \quad \mathcal{O} \sim \xi [\phi_l L_L L_L]_{\mathbf{1}}, \quad (45)$$

for the neutrinos. These operators will add small corrections to the predictions for the associated mixing matrices, which can bring them closer to experiment. However, in general, they may softly break the RFS preserved at LO,¹⁰ and so studying them in generality is again beyond our scope in what follows.

3.2 $(\mathbb{Z}_{14} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ Model for U_{CKM}

We now consider a simple model based on the finite group $(\mathbb{Z}_{14} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ that makes predictions for CKM quark mixing. The RFS symmetry-breaking pattern to the down and up quark sectors is illustrated by

$$\mathcal{G}_Q \cong (\mathbb{Z}_{14} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \rightarrow \begin{cases} \mathcal{G}_u \cong \mathbb{Z}_2 \\ \mathcal{G}_d \cong \mathbb{Z}_2 \end{cases}, \quad (46)$$

with $\mathcal{G}_{u,d}$ generated by

$$T_u = \text{diag}(1, -1, -1), \quad T_d = \text{diag}(1, -1, -1). \quad (47)$$

¹⁰Combinations of flavons generally give rise to different directions in flavour space, so-called effective alignments [40, 41].

The model predicts the LO CKM mixing prediction to be of the Cabibbo form,

$$U_{CKM} = \begin{pmatrix} \cos \frac{\pi}{14} & \sin \frac{\pi}{14} & 0 \\ -\sin \frac{\pi}{14} & \cos \frac{\pi}{14} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (48)$$

which, while insufficient to fully reproduce the known three-dimensional structure of the CKM, does capture the dominant mixing in the $(1, 2)$ sector, i.e. the Cabibbo angle θ_C . Further corrections are highly suppressed, $\propto \mathcal{O}(\theta_C^2, \theta_C^3)$, and will be briefly mentioned below.

As in Section 3.1, we can immediately construct the flavour-basis generators, under the (common) assumption that the down quarks are already diagonal, such that the entirety of the CKM mixing is encoded in the up sector. We immediately find

$$T_{uU} = \begin{pmatrix} \cos \frac{\pi}{7} & \sin \frac{\pi}{7} & 0 \\ \sin \frac{\pi}{7} & -\cos \frac{\pi}{7} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_{dU} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (49)$$

In principle, we can use (49) to proceed with the algorithm as described in Section 2. However, in what follows we find it convenient to work in a different basis where the non-trivial entries of T_{uU} are in the $(2, 3)$ sector.¹¹ To that end, we consider the following unitary transformation on the weak-eigenstate generators:

$$P = \frac{1}{2} \begin{pmatrix} 1 & 1 + \frac{i}{\sqrt{2}} & 1 - \frac{i}{\sqrt{2}} \\ -1 & 1 - \frac{i}{\sqrt{2}} & 1 + \frac{i}{\sqrt{2}} \\ \sqrt{2} & -i & i \end{pmatrix}. \quad (50)$$

Applying P with $T''_{u,d} = P^\dagger T_{uU,dU} P$ we get the following expressions for the RFS-generators in the model basis:¹²

$$T''_u = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & e^{i\pi/7} \\ 0 & e^{-i\pi/7} & 0 \end{pmatrix}, \quad T''_d = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (51)$$

We now want to identify these matrices with $(\mathbb{Z}_{14} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ generating elements in certain irreducible representations. However, we have not found sources available that catalogue the properties of this group. For this reason we derived the relevant product rules and group information ourselves, and have provided them in Appendix A.2. There we see that $T''_{u,d}$ can be easily expressed in terms of the group generators \mathbf{a} , \mathbf{b} and \mathbf{c} :

$$T''_d = \mathbf{bab}, \quad T''_u = \mathbf{ac} \quad \text{for} \quad \mathbf{1}_{--} + \mathbf{2}_{\mathbf{3}+-}. \quad (52)$$

¹¹We also attempted a construction with this symmetry that treated leptons as well as quarks, which we reference at the end of Section 4.2, and where the current basis was required.

¹²In what follows, we will always use double-primed ($\prime\prime$) notation when constructing objects in the basis we intend to build the model, unless that basis has already been given a specific label (as with the flavour basis in Section 3.1).

	$Q_L''^1$	$Q_L''^{23}$	$u_R''^1$	$u_R''^2$	$u_R''^3$	$d_R''^1$	$d_R''^2$	$d_R''^3$	ϕ_u	ϕ_d
$(\mathbb{Z}_{14} \times \mathbb{Z}_2) \times \mathbb{Z}_2$	$\mathbf{1}_{--}$	$\mathbf{2}_{3+-}$	$\mathbf{1}_{++}$	$\mathbf{1}_{-+}$	$\mathbf{1}_{--}$	$\mathbf{1}_{++}$	$\mathbf{1}_{-+}$	$\mathbf{1}_{--}$	$\mathbf{2}_{3+-}$	$\mathbf{2}_{3+-}$

Table 2: Representations of the quarks under $(\mathbb{Z}_{14} \times \mathbb{Z}_2) \times \mathbb{Z}_2$.

Critically, we observe that (51)-(52) indicate that $\mathbf{2}_{3+-}$ is the appropriate $(\mathbb{Z}_{14} \times \mathbb{Z}_2) \times \mathbb{Z}_2$ charge for the two flavons $\phi_{u,d}$ that we introduce according to the algorithm in Section 2, and we can use (51) to work out the expressions for these doublet VEV, finding

$$\langle \phi_u \rangle = v_u \begin{pmatrix} 1 \\ -e^{-i\pi/7} \end{pmatrix}, \quad \langle \phi_d \rangle = v_d \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (53)$$

Finally, one derives that in the absence of mixing ambiguities, the model-basis mass matrices are given by

$$m_u''^\dagger m_u'' \stackrel{!}{=} \begin{pmatrix} m_{u_3}^2 & 0 & 0 \\ 0 & \frac{1}{2}(m_{u_1}^2 + m_{u_2}^2) & \frac{1}{2}e^{i\pi/7}(m_{u_1}^2 - m_{u_2}^2) \\ 0 & \frac{1}{2}e^{-i\pi/7}(m_{u_1}^2 - m_{u_2}^2) & \frac{1}{2}(m_{u_1}^2 + m_{u_2}^2) \end{pmatrix},$$

$$m_d''^\dagger m_d'' \stackrel{!}{=} \begin{pmatrix} m_{d_3}^2 & 0 & 0 \\ 0 & \frac{1}{2}(m_{d_1}^2 + m_{d_2}^2) & \frac{1}{2}(m_{d_1}^2 - m_{d_2}^2) \\ 0 & \frac{1}{2}(m_{d_1}^2 - m_{d_2}^2) & \frac{1}{2}(m_{d_1}^2 + m_{d_2}^2) \end{pmatrix}, \quad (54)$$

where m_{A_i} are the associated mass eigenvalues, and where we have used dagger combinations for the charged fermions to remove the dependence on RH transformations. Of course, in deriving (54), we have been careful to keep track of the additional basis change implied by operating with P in (50).

The results in (51)-(54) strongly indicate that the second and third generations of LH quarks should transform as a $(\mathbb{Z}_{14} \times \mathbb{Z}_2) \times \mathbb{Z}_2$ doublet, while the first generation transforms as a non-trivial singlet. Similarly, the second and third generations of RH up and down quarks should transform as a non-trivial singlet, while the first generation of both families transforms trivially. Furthermore, (52) already indicated that the flavons $\phi_{d,u}$ associated to these sectors should transform as a $\mathbf{2}_{3+-}$, a fact that helped us derive (53). This information is summarized in Table 2.

Assuming a shaping symmetry to prevent ϕ_i from coupling to undesirable sectors, one can straightforwardly build up the Yukawa sector for the quarks in the model basis using Table 2,

$$\mathcal{L}_Y \supset a_u \bar{Q}_L''^1 u_R''^3 + b_u \left[\bar{Q}_L''^{23} \phi_u \right]_{\mathbf{1}_{++}} u_R''^1 + c_u \left[\bar{Q}_L''^{23} \phi_u \right]_{\mathbf{1}_{-+}} u_R''^2$$

$$+ a_d \bar{Q}_L''^1 d_R''^3 + b_d \left[\bar{Q}_L''^{23} \phi_d \right]_{\mathbf{1}_{++}} d_R''^1 + c_d \left[\bar{Q}_L''^{23} \phi_d \right]_{\mathbf{1}_{-+}} d_R''^2, \quad (55)$$

where we have omitted Higgs fields and scale suppressions. Using the vevs from (53) and

product rules from Appendix A.2, we get the following Yukawa matrices:

$$Y_u''^\dagger = v_u \begin{pmatrix} 0 & 0 & a_u/v_u \\ b_u & -c_u & 0 \\ -b_u e^{-i\pi/7} & -c_u e^{-i\pi/7} & 0 \end{pmatrix}, \quad Y_d''^\dagger = v_d \begin{pmatrix} 0 & 0 & a_d/v_d \\ b_d & -c_d & 0 \\ -b_d & -c_d & 0 \end{pmatrix}. \quad (56)$$

Assembling these into their Hermitian combinations, one immediately finds

$$Y_u''^\dagger Y_u'' = v_u^2 \begin{pmatrix} |a_u|^2/v_u^2 & 0 & 0 \\ 0 & |b_u|^2 + |c_u|^2 & e^{i\pi/7}(|c_u|^2 - |b_u|^2) \\ 0 & -e^{-i\pi/7}(|b_u|^2 - |c_u|^2) & |b_u|^2 + |c_u|^2 \end{pmatrix},$$

$$Y_d''^\dagger Y_d'' = v_d^2 \begin{pmatrix} |a_d|^2/v_d^2 & 0 & 0 \\ 0 & |b_d|^2 + |c_d|^2 & |b_d|^2 - |c_d|^2 \\ 0 & |b_d|^2 - |c_d|^2 & |b_d|^2 + |c_d|^2 \end{pmatrix}, \quad (57)$$

which directly map to (54) with

$$|a_u|^2 \leftrightarrow |m_{u3}|^2, \quad |b_u|^2 \leftrightarrow |m_{u2}|^2, \quad |c_u|^2 \leftrightarrow |m_{u1}|^2,$$

and analogous relations for the mapping of Y_d , up to VEV factors and multiplicative constants. We have therefore reconstructed a successful top-down model, as it exhibits the required symmetry breaking in (46) and recovers the CKM mixing prediction in (48).

4 Application to Models of Flavourful Leptoquarks

As a particularly relevant extension of the field content of the SM, we now apply our algorithm to a class of flavoured leptoquark models defined in [30, 31], which we will briefly review for completeness. Leptoquarks have become popular in the recent literature due to their ability to resolve (potential) anomalies in heavy meson decay observables like \mathcal{R}_{K^*} observed by LHCb [42, 43]. Here we allow the SM to be augmented by one of the following bosons, denoted the ‘scalar triplet,’ ‘vector singlet,’ and ‘vector triplet,’ whose charge assignments under the SM gauge group are respectively given by (in the form $\mathcal{G}_{SM} \equiv SU(3)_C \times SU(2)_L \times U(1)_Y$),

$$\Delta_3 \sim (\bar{\mathbf{3}}, \mathbf{3}, 1/3), \quad \Delta_1^\mu \sim (\mathbf{3}, \mathbf{1}, 2/3), \quad \Delta_3^\mu \sim (\mathbf{3}, \mathbf{3}, 2/3). \quad (58)$$

These leptoquarks are easily motivated in the UV by Grand Unified constructions, or in models with new gauge interactions (see e.g. [44, 45]), and all can successfully account for $\mathcal{R}_{K^{(*)}} < 1$ [46]. The SM-gauge invariant operators they source are given by

$$\begin{aligned} \Delta_3 : \quad & \mathcal{L} \supset y_{3,ij}^{LL} \bar{Q}_L^{C i,a} \epsilon^{ab} (\tau^k \Delta_3^k)^{bc} L_L^{j,c} + z_{3,ij}^{LL} \bar{Q}_L^{C i,a} \epsilon^{ab} ((\tau^k \Delta_3^k)^\dagger)^{bc} Q_L^{j,c} + \text{h.c.} \\ \Delta_1^\mu : \quad & \mathcal{L} \supset x_{1,ij}^{LL} \bar{Q}_L^{i,a} \gamma^\mu \Delta_{1,\mu} L_L^{j,a} + x_{1,ij}^{RR} \bar{d}_R^i \gamma^\mu \Delta_{1,\mu} E_R^j + x_{1,ij}^{\overline{RR}} \bar{u}_R^i \gamma^\mu \Delta_{1,\mu} \nu_R^j + \text{h.c.} \\ \Delta_3^\mu : \quad & \mathcal{L} \supset x_{3,ij}^{LL} \bar{Q}_L^{i,a} \gamma^\mu (\tau^k \Delta_{3,\mu}^k)^{ab} L_L^{j,b} + \text{h.c.} \end{aligned} \quad (59)$$

with $\{i, j\}$ denoting flavour indices, $\{a, b\}$ denoting SU(2) indices, and $k = 1, 2, 3$ for the Pauli matrices.¹³ Following [47] and redefining the components of the scalar triplet state according to,¹⁴

$$\Delta_3^{4/3} = (\Delta_3^1 - i\Delta_3^2) / \sqrt{2}, \quad \Delta_3^{-2/3} = (\Delta_3^1 + i\Delta_3^2) / \sqrt{2}, \quad \Delta_3^{1/3} = \Delta_3^3, \quad (60)$$

contracting the SU(2) indices of (59), and ignoring the diquark operator of Δ_3 (for simplicity, although RFS can control it— see [30]), one then finds that the Yukawa/mass sector of the SM is enhanced to

$$\begin{aligned} \mathcal{L}_{mass} \supset & \frac{1}{2} \bar{\nu}_L^c m_\nu \nu_L + \bar{E}_R m_l l_L + \bar{d}_R m_d d_L + \bar{u}_R m_u u_L \\ & + \bar{d}_L^C \lambda_{dl} l_L \Delta_3^{4/3} + \bar{d}_L^C \lambda_{d\nu} \nu_L \Delta_3^{1/3} + \bar{u}_L^C \lambda_{ul} l_L \Delta_3^{1/3} + \bar{u}_L^C \lambda_{u\nu} \nu_L \Delta_3^{-2/3} \\ & + \text{h.c.} \end{aligned} \quad (61)$$

with the novel leptoquark couplings λ_{QL} normalized to the first term, λ_{dl} , which in the mass-eigenstate basis of the SM fermions is generically parameterized by

$$-\sqrt{2} (U_d^T y_3^{LL} U_l) \equiv \lambda_{dl} = \begin{pmatrix} \lambda_{de} & \lambda_{d\mu} & \lambda_{d\tau} \\ \lambda_{se} & \lambda_{s\mu} & \lambda_{s\tau} \\ \lambda_{be} & \lambda_{b\mu} & \lambda_{b\tau} \end{pmatrix}. \quad (62)$$

The other couplings in (61) are related to λ_{dl} via SU(2) relations, and are given by

$$\lambda_{d\nu} = \frac{1}{\sqrt{2}} \lambda_{dl} U_{PMNS}, \quad \lambda_{ul} = \frac{1}{\sqrt{2}} U_{CKM}^* \lambda_{dl}, \quad \lambda_{u\nu} = -U_{CKM}^* \lambda_{dl} U_{PMNS}. \quad (63)$$

Given these new flavoured couplings, we defined multiple classes of simplified models based on the RFS formalism in [30,31]. In particular, we assumed that the natural RFS of the SM (cf. (4)) also hold in the new leptoquark terms of (61). This allowed us to constrain the λ_{QL} couplings via RFS invariances of the form

$$\exists \{Q, L\}, \quad T_Q^{(T, \dagger)} \lambda_{QL} T_L \stackrel{!}{=} \lambda_{QL}, \quad (64)$$

with Q, L respectively representing arbitrary quark (d, u) and lepton (l, ν) families, and the transposed ‘ T ’ (daggered ‘ \dagger ’) T_Q corresponding to scalar (vector) leptoquark(s). Critically, different phase relationships amongst RFS generators $T_{Q,L}$ correspond to different textures in λ_{QL} , and the extent to which free parameters remain in (62)-(63) is a function of the amount of symmetry present in any given term. Precision flavour data from (e.g.) $B - \bar{B}$ mixing, lepton-flavour-violating (LFV) observables like $\mu \rightarrow e\gamma$, and the anomalous \mathcal{R} ratios also constrain the viable textures (and hence also the phenomenologically viable RFS relationships) in (62).

For example, in [30] we insisted that RFS hold in *all* lepton and quark sectors of the SM and leptoquark couplings, and this led to only $\mathcal{O}(10)$ viable textures for λ_{dl} with only a

¹³The physics of leptoquarks is thoroughly reviewed in [47].

¹⁴We will write the following equations explicitly for the scalar triplet, although analogous expressions are easily derived for the other two leptoquark states of (58). Superscripts on the LHS denote electric charges.

single real parameter, once all experimental and symmetry constraints were made.¹⁵ Then, in [31], we relaxed the symmetry assumptions and enforced an RFS invariance in some or all of the SM mass terms, but *only* in the $d - l$ leptoquark coupling, where either or both $T_{d,l}$ were allowed to act; in this symmetry environment, invariance in $\lambda_{d\nu,ul,u\nu}$ is inherited via SU(2) relationships as in (63). These two types of simplified models were labeled ‘SE1’ and ‘SE2’ respectively, with the former likely requiring more intricate model building to account for the fact the RFS distinguishes members of SU(2) doublets in individual leptoquark terms after EWSB. The SE2 models, on the other hand, represent highly natural relaxations of the SE1 constructions, and can easily be realized with single flavon EFTs as per our algorithm in Section 2, which we will now show.

The Leptoflavour Basis

Viable NADS that can realize the SE2 symmetry predictions for $U_{PMNS,CKM}$ and λ_{QL} must be uncovered in order to apply our algorithm (cf. Section 2.2), and to that end a **GAP** scan was performed in [31]. We performed that scan in the ‘leptoflavour basis’, where information about all relevant physical mixings could be extracted, and which we now review.

Recall that, in the mass-eigenstate basis, the RFS generators (and therefore $\mathcal{G}_{\mathcal{F},\mathcal{L},\mathcal{Q}}$) do not know about $U_{PMNS,CKM}$, even in the case of only SM field content. However, we can go to a special form of the ‘flavour basis’ (cf. (8)) by doing no change of basis (or trivial change of basis with the identity matrix) on the LH d , l , and a change of basis via the CKM for the LH u and the PMNS for the LH ν . With leptoquarks present, we simultaneously want to encode the additional information present in λ_{QL} together with the SM mixing matrices, so we choose the leptoflavour basis to be the one where the SM mixing is all in the mixing matrices, and therefore the charged current should be diagonal — similarly to what we have in the flavour basis — but additionally, we choose to have λ_{dl} diagonal. To diagonalize λ_{dl} , we require a further non-trivial change of basis in the LH d , l , which must be cancelled with rotations in the u and ν respectively, to have the CC diagonal. So, starting from the mass basis (where λ_{dl} is defined as in (62)), d changes by Λ_d , l by Λ_l and u changes by both CKM and by Λ_d (canceling the presence of Λ_d in the CC and making it diagonal), while ν changes by both PMNS and by Λ_l (canceling the presence of the Λ_l in the CC, and making it diagonal). Finally, the case without leptoquarks appears correctly as the limit with $\Lambda_d = \Lambda_l = 1$.

To see this explicitly, we apply the following ‘leptoflavour’ basis transformations on the mass eigenstates:

$$\begin{aligned} l_L &\rightarrow \Lambda_l^\dagger l'_L, & d_L &\rightarrow \Lambda_d^\dagger d'_L, & \nu_L &\rightarrow U_{PMNS}^\dagger \Lambda_l^\dagger \nu'_L, & u_L &\rightarrow U_{CKM} \Lambda_d^\dagger u'_L, \\ E_R &\rightarrow \Lambda_E^\dagger E'_R, & d_R &\rightarrow \Lambda_D^\dagger d'_R, & \nu_R &\rightarrow \Lambda_R^\dagger \nu'_R, & u_R &\rightarrow \Lambda_U^\dagger u'_R. \end{aligned} \quad (65)$$

This yielded the leptoflavour basis Lagrangian,

$$\mathcal{L} \supset \frac{g}{\sqrt{2}} \bar{l}'_L \gamma^\mu \nu'_L W_\mu^- + \frac{g}{\sqrt{2}} \bar{d}'_L \gamma^\mu u'_L W_\mu^-$$

¹⁵We also insisted that at least two generations be distinguished by the eigenvalues of $T_{Q,L}$, so that the action of the RFS was not trivial in a given family sector.

Basis-Dependent RFS Quantities			
Basis	Transform $\nu_L \rightarrow$	Mass Matrix	RFS Generator
Mass	ν_L	m_ν	T_ν
Flavour	$U_\nu^\dagger \nu_L$	$U_\nu^* m_\nu U_\nu^\dagger$	$U_\nu T_\nu U_\nu^\dagger$
Leptoflavour	$U_{PMNS}^\dagger \Lambda_l^\dagger \nu'_L$	$\Lambda_l^* U_{PMNS}^* m_\nu U_{PMNS}^\dagger \Lambda_l^\dagger$	$\Lambda_l U_{PMNS} T_\nu U_{PMNS}^\dagger \Lambda_l^\dagger$
Model	$U_{PMNS}^\dagger \Lambda_l^\dagger P \nu''_L$	$P^T \Lambda_l^* U_{PMNS}^* m_\nu U_{PMNS}^\dagger \Lambda_l^\dagger P$	$P^\dagger \Lambda_l U_{PMNS} T_\nu U_{PMNS}^\dagger \Lambda_l^\dagger P$

Table 3: The RFS generators associated with a given basis change away from the original mass-eigenstate ν_L .

$$\begin{aligned}
& + \frac{1}{2} \bar{\nu}'_L{}^c \Lambda_l^* U_{PMNS}^* m_\nu U_{PMNS}^\dagger \Lambda_l^\dagger \nu'_L + \bar{E}'_R \Lambda_E m_l \Lambda_l^\dagger l'_L + \bar{d}'_R \Lambda_D m_d \Lambda_d^\dagger d'_L + \bar{u}'_R \Lambda_U m_u U_{CKM} \Lambda_d^\dagger u'_L \\
& + \frac{1}{\sqrt{2}} \bar{d}'_L{}^c \Lambda_d^* \lambda_{dl} \Lambda_l^\dagger \nu'_L \Delta_3^{1/3} + \bar{d}'_L{}^c \Lambda_d^* \lambda_{dl} \Lambda_l^\dagger l'_L \Delta_3^{4/3} + \bar{u}'_L{}^c \Lambda_d^* \lambda_{dl} \Lambda_l^\dagger \nu'_L \Delta_3^{-2/3} + \frac{1}{\sqrt{2}} \bar{u}'_L{}^c \Lambda_d^* \lambda_{dl} \Lambda_l^\dagger l'_L \Delta_3^{1/3} \\
& + \text{h.c.}, \tag{66}
\end{aligned}$$

which is invariant under the following LH RFS generators:

$$T'_l = \Lambda_l T_l \Lambda_l^\dagger, \quad T'_\nu = \Lambda_l U_{PMNS} T_\nu U_{PMNS}^\dagger \Lambda_l^\dagger, \quad T'_d = \Lambda_d T_d \Lambda_d^\dagger, \quad T'_u = \Lambda_d U_{CKM}^\dagger T_u U_{CKM} \Lambda_d^\dagger, \tag{67}$$

and the following RH RFS generators:

$$T'_E = \Lambda_E T_E \Lambda_E^\dagger, \quad T'_R = \Lambda_R T_\nu \Lambda_R^\dagger, \quad T'_D = \Lambda_D T_d \Lambda_D^\dagger, \quad T'_U = \Lambda_U T_u \Lambda_U^\dagger, \tag{68}$$

where T'_R holds only in the case of Dirac neutrinos. As mentioned above, in the limit where $\Lambda_{l,d} \rightarrow \mathbb{1}$, (67) reduces to the SM-only flavour-basis generating set! Given the leptoflavour basis, bottom-up scans as described in Section 2.2 were then performed in [31] with (67), so that parent family groups were closed according to (14)-(17). As it turns out, many NADS were discovered, including members of the popular S_N , A_N , $\Delta(3N^2)$, $\Delta(6N^2)$, $\Sigma(3N^2)$, $\Sigma(3N^3)$, and D_N finite group series.

As a final preparation for the reconstruction of the RFS-invariant Lagrangian in these extended leptokuark scenarios, we allow for the possibility that an additional basis change will be amenable for manipulating the group product rules of the NADS discovered in [31]. Hence we rotate via a generic matrix P (which can be set to the identity matrix in the event it is unnecessary),

$$a' \rightarrow P a'', \tag{69}$$

and so the effective mass terms are now given by

$$\mathcal{L} \supset \frac{1}{2} \bar{\nu}''_L{}^c \underbrace{P^T \Lambda_l^* U_{PMNS}^* m_\nu U_{PMNS}^\dagger \Lambda_l^\dagger P}_{m''_\nu} \nu''_L + \bar{u}''_R \underbrace{P^\dagger \Lambda_U m_u U_{CKM} \Lambda_d^\dagger P}_{m''_u} u''_L$$

$$+ \bar{E}_R'' \underbrace{P^\dagger \Lambda_E m_l \Lambda_l^\dagger P}_{m_l''} l_L'' + \bar{d}_R'' \underbrace{P^\dagger \Lambda_D m_d \Lambda_d^\dagger P}_{m_d''} d_L'', \quad (70)$$

where we have assumed that A_R also transforms with P . The mass matrices in this basis are labeled by m'' , and are clearly non-diagonal. The remaining leptoquark terms of (61) are similarly given in this basis by

$$\begin{aligned} \mathcal{L} \supset & \frac{1}{\sqrt{2}} \bar{d}_L'' \underbrace{P^T \Lambda_d^* \lambda_{dl} \Lambda_l^\dagger P}_{\chi_{d\nu}''} \nu_L'' \Delta_3^{1/3} + \bar{d}_L'' \underbrace{P^T \Lambda_d^* \lambda_{dl} \Lambda_l^\dagger P}_{\chi_{di}''} l_L'' \Delta_3^{4/3} \\ & + \bar{u}_L'' \underbrace{P^T \Lambda_d^* \lambda_{dl} \Lambda_l^\dagger P}_{\chi_{u\nu}''} \nu_L'' \Delta_3^{-2/3} + \frac{1}{\sqrt{2}} \bar{u}_L'' \underbrace{P^T \Lambda_d^* \lambda_{dl} \Lambda_l^\dagger P}_{\chi_{ui}''} l_L'' \Delta_3^{1/3}. \end{aligned} \quad (71)$$

These operators already reveal the natural form of their respective RFS generators. A summary of all basis changes on (e.g.) the neutrino field and associated changes in m_ν and T_ν are given in Table 3, tracking all the way from the mass-eigenstate basis to the model basis of (70)-(71).

Hence, given a specific NADS, its RFS, and the associated predictions for $U_{CKM,PMNS}$ and λ_{dl} , one can use (70)-(71) to reconstruct the UV EFT. We will now consider two such models, one based on the $\Delta(96)$ group and one based on the D_{15} member of the Dihedral series D_N . All of the relevant bottom-up information for these groups is given in Table 4, and the parameters $x_{e,\mu}$ are defined in the following textures:

$$\lambda_{dl}^{[e0]} = \lambda_{be} \begin{pmatrix} 0 & 0 & 0 \\ x_e & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_{dl}^{[\mu 0]} = \lambda_{b\mu} \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_\mu & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{with } x_X = \frac{\lambda_{sX}}{\lambda_{bX}}, \quad (72)$$

which are the consequence of special relationships amongst RFS generators — $\lambda_{dl}^{[e0]}$ corresponds to $-\alpha_l = \beta_d = \gamma_d$ while $\lambda_{dl}^{[\mu 0]}$ corresponds to $-\beta_l = \beta_d = \gamma_d$.¹⁶ By construction these couplings are diagonalized with $\Lambda_{d,l}$ in the combination $\Lambda_d^* \lambda_{dl} \Lambda_l^\dagger$, with

$$\Lambda_d(x_{e/\mu}) = \begin{pmatrix} 0 & \frac{x_{e,\mu}}{\sqrt{1+x_{e,\mu}^2}} & \frac{1}{\sqrt{1+x_{e,\mu}^2}} \\ 0 & -\frac{1}{\sqrt{1+x_{e,\mu}^2}} & \frac{1}{\sqrt{1+x_{e,\mu}^2}} \\ 1 & 0 & 0 \end{pmatrix}, \quad \Lambda_l^e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_l^\mu = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (73)$$

The quark matrix Λ_d in (73) left-diagonalizes both patterns in (72), whereas Λ_l^e right-diagonalizes $\lambda_{dl}^{[e0]}$ and Λ_l^μ right-diagonalizes $\lambda_{dl}^{[\mu 0]}$.

On the other hand, the parameters $t_{\theta_{\mu\tau}} \equiv \tan \theta_{\mu\tau}$ and θ_C are defined in the following LO PMNS and CKM textures:

$$U_{PMNS} \simeq U_{\mu\tau} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \cos \theta_{\mu\tau} & \sqrt{2} \sin \theta_{\mu\tau} & 0 \\ -\sin \theta_{\mu\tau} & \cos \theta_{\mu\tau} & 1 \\ \sin \theta_{\mu\tau} & -\cos \theta_{\mu\tau} & 1 \end{pmatrix} + \mathcal{O}(\theta_{13}^l), \quad (74)$$

¹⁶These are the relationships for the scalar triplet, while for the vector singlet and triplet the minus signs do not appear. See [31] for details, where other couplings are also controlled.

$$U_{CKM} \simeq U_C \equiv \begin{pmatrix} \cos \theta_C & \sin \theta_C & 0 \\ -\sin \theta_C & \cos \theta_C & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\theta_C^2, \theta_C^3), \quad (75)$$

which were specified as the SM mixing to be recovered in [31].¹⁷ While the exact forms of (74) and (75) are excluded by current global fits to experiment [48], they nevertheless provide excellent approximations to the data. The $\mu - \tau$ symmetric matrix in (74), for example, reproduces global fits to the PMNS matrix up to corrections on the order of the smallest ‘reactor angle,’ θ_{13}^l , and its free parameter $\theta_{\mu\tau}$ can be fit to many well-studied textures like the tri-bimaximal [49], bi-maximal [50], golden ratio [51, 52], and hexagonal forms [53, 54]:

$$U_{\mu\tau}(\theta_{\mu\tau}) \rightarrow \begin{cases} U_{TBM} & \Leftrightarrow \tan \theta_{\mu\tau} = \frac{1}{\sqrt{2}} \\ U_{BM} & \Leftrightarrow \tan \theta_{\mu\tau} = 1 \text{ or } \theta_{\mu\tau} = \frac{\pi}{4} \\ U_{GR_1} & \Leftrightarrow \tan \theta_{\mu\tau} = \frac{2}{(1+\sqrt{5})} \\ U_{GR_2} & \Leftrightarrow \theta_{\mu\tau} = \frac{\pi}{5} \\ U_{HM} & \Leftrightarrow \tan \theta_{\mu\tau} = \frac{1}{\sqrt{3}} \text{ or } \theta_{\mu\tau} = \frac{\pi}{6}. \end{cases} \quad (76)$$

Furthermore, corrections to (74) can naturally be realized by higher-order terms in the OPE beyond (20), which can softly break the RFS embedded in the LO contribution, or also through renormalization group flow (RGE) [55–62] between the scale at which $\mathcal{G}_{\mathcal{F}}$ is broken and the IR, where global fits are performed.

Similarly, (75) provides an excellent description of the dominant Cabibbo mixing of the CKM matrix. Unlike the PMNS, the CKM is extremely hierarchical, with mixings in the (2,3) and (1,3) sectors suppressed by one to two orders of magnitude with respect to the Cabibbo sector. This suppression again hints at further contributions to (75) from higher-order terms in (20) and/or RGE corrections.

Of course, precisely calculating the corrections expected to (74)-(75) depends on the complete UV flavour model, including not only the full field and symmetry content, but also the presence or lack thereof of supersymmetry. Specifying this is well beyond the scope of our present paper, and so we consider (74)-(75) sufficiently accurate to develop our approach to reconstructing effective Lagrangians from RFS.

4.1 $\Delta(96)$ Model for U_{PMNS} and Leptoquarks

As a first example incorporating leptoquarks we construct a $\Delta(96)$ flavour model from the scan results in [31], which are repeated in the first row of Table 4. This model predicts tri-bimaximal mixing $U_{PMNS} = U_{\mu\tau}(\arctan 1/\sqrt{2})$ and the electron isolation pattern $\lambda_{dl}^{[e0]}$ for the leptoquark coupling through its symmetry breaking to RFS, illustrated in

$$\mathcal{G}_{\mathcal{L}} \cong \Delta(96) \rightarrow \begin{cases} \mathcal{G}_\nu \cong \mathbb{Z}_4 \\ \mathcal{G}_l \cong \mathbb{Z}_3. \end{cases} \quad (77)$$

¹⁷After all, ours is a bottom-up approach, and therefore closing finite groups via (14)-(17) requires that some textures for $U_{PMNS,CKM}$ are fed to the algorithm.

Finite Groups, RFS, and Mixing Predictions						
$\{x_{e/\mu}, t_{\theta_{\mu\tau}}, \theta_C\}$	T_d^{ii}	T_l^{ii}	T_u^{ii}	T_ν^{ii}	GAP-ID	$\mathcal{G}_{\mathcal{F}}$
$\{\text{N.A.}, \frac{1}{\sqrt{2}}, \text{N.A.}\}$	N.A.	$[1, \omega_3, \omega_3^2]$	N.A.	$[\omega_4, 1, -\omega_4]$	[96, 64]	$\Delta(96)$
$\{1, \frac{1}{\sqrt{3}}, \frac{\pi}{15}\}$	[1,-1,-1]	[1,-1,-1]	[1,-1,-1]	[-1,1,-1]	[30,3]	D_{15}
$\{1, 1, \frac{\pi}{14}\}$	[1,-1,-1]	[1,-1,-1]	[1,-1,-1]	$[\omega_4, -\omega_4, 1]$	[56,7]	$(\mathbb{Z}_{14} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$

Table 4: The scan results from [31] that we will use to reconstruct models in Sections 4.1-4.2. Note that we have relabeled $D_{30} \rightarrow D_{15}$ from [31] in order to reflect the conventions of [63].

As evident from the fact that T_ν generates $\mathcal{G}_\nu \cong \mathbb{Z}_4$, this model features Dirac rather than Majorana neutrino masses. In Table 4 one can read off the specific charges of the mass-eigenstate RFS generators T_l and T_ν . We note their basis-independent Traces are respectively 0 and 1, which will soon help us identify the conjugacy class to which they belong within $\Delta(96)$.

Table 4 gives all the information required to move to the leptoflavour basis, where the RFS generators take the forms

$$T_l' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix}, \quad T_\nu' = \frac{1}{3} \begin{pmatrix} 1 + 2i & -1 + i & 1 - i \\ -1 + i & 1 - i & -1 - 2i \\ 1 - i & -1 - 2i & 1 - i \end{pmatrix}, \quad (78)$$

where $\omega_4 = e^{i2\pi/4} = i$. We want to identify these generators with group elements of $\Delta(96)$, and to do so we use the catalogue in [65], repeating the relevant product rules in Appendix A.3. As before, we find it convenient to perform a P transformation on the leptoflavour basis, so that we go to a basis where the combination of $\Delta(96)$ generators $\mathbf{a}_{\mathbf{3}_1}^2, \mathbf{c}_{\mathbf{3}_1}, \mathbf{d}_{\mathbf{3}_1}$ for the $\mathbf{3}_1$ representation, found in [65], is diagonal. We note that we use the same naming of the generators as in [65], only differentiating them with the boldface to further avoid confusion with our naming for the coefficients (in this and in other sections). The P matrix we use is

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (79)$$

which leads to

$$T_l'' = \begin{pmatrix} \omega_3 & 0 & 0 \\ 0 & \omega_3^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_\nu'' = \frac{1}{3} \begin{pmatrix} 1 - i & 1 + 2i & 1 - i \\ 1 + 2i & 1 - i & 1 - i \\ 1 - i & 1 - i & 1 + 2i \end{pmatrix}. \quad (80)$$

With this change of basis, we are able to match T_l'' with the diagonal $\mathbf{a}^2\mathbf{cd}$ element of the $\bar{\mathbf{3}}_1$ representation of $\Delta(96)$, the conjugate representation to $\mathbf{3}_1$, which as expected has zero trace and lies within conjugacy class \mathcal{C}_6 [65]. According to the character of T_ν'' , it could be within

	\bar{L}''	$(E_R''^1, E_R''^2)$	$E_R''^3$	$(\nu_R''^1, \nu_R''^2)$	$\nu_R''^3$	ϕ_ν	$\phi_{\nu 2}$	ϕ_l	$\phi_{l 2}$
$\Delta(96)$	$\mathbf{3}_1$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\bar{\mathbf{3}}_1$	$\bar{\mathbf{3}}_1$	$\bar{\mathbf{3}}_1$	$\bar{\mathbf{3}}_1'$

Table 5: Relevant field and $\Delta(96)$ symmetry content.

\mathcal{C}_5 or \mathcal{C}_9 in the same $\bar{\mathbf{3}}_1$ representation, and indeed we found it to match the element $\mathbf{a}^2\mathbf{bc}^2\mathbf{d}^3$. With this information we introduce two flavons $\phi_{l,\nu}$, for which we use (80) to derive candidate VEV in the triplet directions

$$\langle \phi_l \rangle = \begin{pmatrix} 0 \\ 0 \\ v_l \end{pmatrix}, \quad \langle \phi_\nu \rangle = \begin{pmatrix} v_\nu \\ v_\nu \\ v_\nu \end{pmatrix}, \quad (81)$$

which are invariant under T_l'' for $\bar{\mathbf{3}}_1$ (and for $\bar{\mathbf{3}}_1'$) and T_ν'' for $\bar{\mathbf{3}}_1$, respectively. Finally, the RFS-invariant mass combinations in this basis are given by

$$m_\nu''^\dagger m_\nu'' \stackrel{!}{=} \frac{1}{3} \begin{pmatrix} \frac{1}{2}(m_{\nu_1}^2 + 2m_{\nu_2}^2 + 3m_{\nu_3}^2) & \frac{1}{2}(m_{\nu_1}^2 + 2m_{\nu_2}^2 - 3m_{\nu_3}^2) & (m_{\nu_2}^2 - m_{\nu_1}^2) \\ \frac{1}{2}(m_{\nu_1}^2 + 2m_{\nu_2}^2 - 3m_{\nu_3}^2) & \frac{1}{2}(m_{\nu_1}^2 + 2m_{\nu_2}^2 + 3m_{\nu_3}^2) & (m_{\nu_2}^2 - m_{\nu_1}^2) \\ (m_{\nu_2}^2 - m_{\nu_1}^2) & (m_{\nu_2}^2 - m_{\nu_1}^2) & (2m_{\nu_1}^2 - m_{\nu_2}^2) \end{pmatrix},$$

$$m_l''^\dagger m_l'' \stackrel{!}{=} \begin{pmatrix} m_{l_2}^2 & 0 & 0 \\ 0 & m_{l_3}^2 & 0 \\ 0 & 0 & m_{l_1}^2 \end{pmatrix}, \quad (82)$$

which we will build below. Note that there are no mixing ambiguities associated to these matrices.

The Lepton Sector

We now build the model by assigning L_L as a $\bar{\mathbf{3}}_1$, and the RH leptons as combinations of a $\Delta(96)$ trivial singlet and a doublet, which we designate as $\mathbf{1} + \mathbf{2}$ e.g. $E_R''^3 \sim \mathbf{1}$, $E_R''^{12} \equiv (E_R''^1, E_R''^2) \sim \mathbf{2}$ and similarly for $\nu_R''^3 \sim \mathbf{1}$ and $\nu_R''^{12} \sim \mathbf{2}$. In this case the $\Delta(96)$ invariant Yukawa terms for charged leptons and for neutrinos are very similar, of the type

$$\mathcal{L}_Y \supset a_{\nu,e} \left[\bar{L}_L'' \phi_f \right]_1 f_R''^3 + b_{\nu,e} \left[\bar{L}_L'' \phi_f \right]_2 f_R''^{12}, \quad (83)$$

where f stands for either charged leptons or neutrinos and the Higgs field is omitted for simplicity. For the neutrino sector we find that the $(1, 1, 1)$ direction gives rise to

$$Y_\nu''^\dagger = a_\nu v_\nu \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + b_\nu v_\nu \begin{pmatrix} \omega_3 & 1 & 0 \\ \omega_3 & 1 & 0 \\ \omega_3 & 1 & 0 \end{pmatrix}, \quad (84)$$

which combines into

$$Y_\nu''^\dagger Y_\nu'' = (2|b_\nu|^2 + |a_\nu|^2) v_\nu^2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (85)$$

A simplified version of this could be obtained through a shaping symmetry removing the coupling either to ν_R^{12} ($b_\nu = 0$) or to ν_R^3 ($a_\nu = 0$). This matrix embeds the correct PMNS matrix predicted by the RFS framework, but with two massless neutrinos. Given that the charged lepton invariants are very similar, we can quickly construct the respective Yukawa matrix for this sector as well:

$$Y_l''^\dagger = a_e v_l \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b_e v_l \begin{pmatrix} 0 & 1 & 0 \\ \omega_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (86)$$

leading to

$$Y_l''^\dagger Y_l'' = v_l^2 \begin{pmatrix} |b_e|^2 & 0 & 0 \\ 0 & |b_e|^2 & 0 \\ 0 & 0 & |a_e|^2 \end{pmatrix}. \quad (87)$$

Hence the invariant operators give rise to a diagonal Yukawa coupling but with two degenerate charged lepton masses, which is clearly unphysical.

In order to make the model realistic, we first note that the directions $\langle \phi_\nu \rangle \sim \{(1, 1, 1), (-1, 1, 0), (-1, -1, 2)\}^T$ are the eigenvectors of T_ν'' with eigenvalues $\hat{e}_\nu = \{1, -i, i\}$, respectively. While we initially selected the first eigensystem in (81) with $\hat{e}_\nu = 1$, according to (21) we are free to choose any of them. Taking either $(-1, 1, 0)$ or $(-1, -1, 2)$ for an additional flavon $\phi_{\nu 2}$'s orientation allows one to generate further non-zero masses in m_ν'' .

At the same time, for the charged leptons, it is possible to break the mass degeneracy by having an additional triplet flavon ϕ_{l2} in the $\bar{\mathbf{3}}_1$ representation, aligned in the same direction as ϕ_l . In summary, with the invariant terms

$$\begin{aligned} \mathcal{L}_Y \supset & a_e \left[\bar{L}_L'' \phi_l \right]_1 E_R''^3 + b_e \left[\bar{L}_L'' \phi_l \right]_2 E_R''^{12} + c_e \left[\bar{L}_L'' \phi_{l2} \right]_2 E_R''^{12} \\ & + a_\nu \left[\bar{L}_L'' \phi_\nu \right]_2 \nu_R''^{12} + b_\nu \left[\bar{L}_L'' \phi_{\nu 2} \right]_1 \nu_R''^3, \end{aligned} \quad (88)$$

the degeneracy of the eigenvalues is lifted as $[\bar{L}_L \phi_l]_2 \propto (\bar{L}_L^1, \omega_3 \bar{L}_L^2)$ whereas $[\bar{L}_L \phi_{l2}]_2 \propto (\bar{L}_L^1, -\omega_3 \bar{L}_L^2)$. Explicitly, we aim for a normal mass hierarchy by picking $(-1, 1, 0)$ as the additional direction, with a shaping symmetry which distinguishes the neutrino flavons such that each only couples to one of the right-handed neutrino fields. Taking $v_\nu = v_{\nu 2}$, the Yukawa term in the neutrino sector then corresponds to

$$Y_\nu''^\dagger = a_\nu v_\nu \begin{pmatrix} \omega_3 & 1 & 0 \\ \omega_3 & 1 & 0 \\ \omega_3 & 1 & 0 \end{pmatrix} + b_\nu v_\nu \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and therefore we have

$$Y_\nu''^\dagger Y_\nu'' = v_\nu^2 \begin{pmatrix} 2a_\nu^2 + b_\nu^2 & 2a_\nu^2 - b_\nu^2 & 2a_\nu^2 \\ 2a_\nu^2 - b_\nu^2 & 2a_\nu^2 + b_\nu^2 & 2a_\nu^2 \\ 2a_\nu^2 & 2a_\nu^2 & 2a_\nu^2 \end{pmatrix}, \quad Y_l''^\dagger Y_l'' = v_l^2 \begin{pmatrix} |b_e - c_e|^2 & 0 & 0 \\ 0 & |b_e + c_e|^2 & 0 \\ 0 & 0 & |a_e|^2 \end{pmatrix},$$

with a_ν from the contraction of $(1, 1, 1)$ with the $\Delta(96)$ doublet and b_ν from the contraction of $(-1, 1, 0)$ with the singlet right-handed neutrino, respectively. These map to (82) with

$$\begin{aligned} m_{l_2}^2 &\leftrightarrow |b_e - c_e|^2, & m_{l_3}^2 &\leftrightarrow |b_e + c_e|^2, & m_{l_1}^2 &\leftrightarrow |a_e|^2, \\ m_{\nu_2}^2 &\leftrightarrow |a_\nu|^2, & m_{\nu_3}^2 &\leftrightarrow |b_\nu|^2, & m_{\nu_1}^2 &= 0, \end{aligned} \quad (89)$$

again up to constant prefactors and VEV, thereby realizing the desired shapes.

The Leptoquark Sector

As seen in Table 4, from the bottom-up perspective of the scans in [31], one does not have control over the coupling x_e in (72) when only lepton symmetries are active. The T_l symmetry controls the overall shape of the term (electron isolation), but not the quantization of the ratio of $\lambda_{se}/\lambda_{be}$. This can be seen practically by observing that $\Delta(96)$ is generated by $\Delta(96) \cong \{T_l', T_\nu'\}$, and neither of these RFS generators knows about x_e . Hence, one derives that in the model basis the generic RFS-invariant leptoquark coupling is given by

$$\lambda_{dl}''^\dagger \lambda_{dl}'' \stackrel{!}{=} (1 + |x_e|^2) |\lambda_{be}|^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (90)$$

where we observe that, thanks to the Hermitian combination we have constructed, the appearance of x_e in this relationship is not due to the mixing matrix Λ_d , but instead the mass-eigenstate isolation pattern itself (cf. (72)).

This term can now be easily built using one of the charged lepton flavons, taking the leptoquark field to transform either as $\mathbf{1}$ (selects ϕ_l) or $\mathbf{1}'$ (selects ϕ_{l2}). For simplicity we consider the trivial singlet option:

$$\mathcal{L} \supset a_\Delta^i \bar{Q}_L''^i [L_L'' \phi_{l1}]_1 \Delta. \quad (91)$$

As in the A_4 models described in [64], contracting $[L_L \phi_{l1}]_1$ gives one of the lepton isolation cases ensuring the leptoquark couples only to one lepton flavour. In this case the VEV in the model building basis is $(0, 0, 1)$, leading to

$$\lambda_{dl}'' = a_\Delta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (92)$$

This is written in an unknown quark basis where the third row corresponds to the specific combination of the three components of \bar{Q}_L'' and a_Δ which is the appropriate function of the

three a_{Δ}^i coefficients. In the mass-eigenstate basis, the model yields the electron isolation pattern as expected from the results in our previous paper [31]. To be more precise, we can sum over the uncertainty of the quark sector that we are not controlling with the symmetry. As λ_{dl}'' only has entries in the third column, the resulting $\lambda_{dl}''^\dagger \lambda_{dl}''$ combination only has a non-zero (3,3) entry proportional to the modulus of the third column vector, therefore the model is indeed predicting the structure in (90).

4.2 D_{15} Model for U_{CKM} , U_{PMNS} , and Leptoquarks

We now consider a D_{15} model¹⁸ that makes predictions for both CKM and PMNS mixing alongside of the ratio of leptoquark couplings denoted by x_μ . The scan result from [31] is repeated in Table 4, whose first column reveals that a hexagonal PMNS matrix U_{HM} is predicted alongside of Cabibbo mixing with $\theta_C = \pi/15$ for the CKM matrix, while the second through fourth columns reveal the following symmetry-breaking pattern:

$$\mathcal{G}_{\mathcal{F}} \cong D_{15} \rightarrow \mathcal{G}_{\nu,l,u,d} \cong \mathbb{Z}_2. \quad (93)$$

From Table 4 we can immediately construct the leptoflavour-basis RFS generators with (67), finding that the neutrino and up-quark matrix have non-trivial structure in all three matrix sectors.

As above, we attempt to find a basis within which its easy to manipulate the relevant D_{15} group product rules. To that end, we consider the following unitary transformation that block diagonalizes the leptoflavour-basis generators:

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}. \quad (94)$$

Applying P with $T_f'' = P^\dagger T_f' P$ we get the following expressions for the RFS-generators in the model basis:

$$T_\nu'' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad T_u'' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos \frac{2\pi}{15} & \sin \frac{2\pi}{15} \\ 0 & \sin \frac{2\pi}{15} & -\cos \frac{2\pi}{15} \end{pmatrix}, \quad T_{l,d}'' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (95)$$

We now want to identify these matrices with D_{15} generating elements in certain irreducible representations, and multiple sources are available that catalogue properties of the dihedral series D_N . We have distilled the relevant information specific to D_{15} in Appendix A.4, from which we see that $T_{u,d,l,\nu}''$ can be easily expressed in terms of the group elements \mathbf{a} and \mathbf{b} for the combination $\mathbf{1}_- + \mathbf{2}_1$ and $\mathbf{1}_- + \mathbf{2}_5$:

$$\begin{aligned} T_d'' = \mathbf{b}, \quad T_u'' = \mathbf{ab} & \quad \text{for} \quad \mathbf{1}_- + \mathbf{2}_1, \\ T_l'' = \mathbf{b}, \quad T_\nu'' = \mathbf{ab} & \quad \text{for} \quad \mathbf{1}_- + \mathbf{2}_5. \end{aligned} \quad (96)$$

¹⁸Dihedral groups and their double-valued cousins have been favored in the model-building community for some time, see e.g. [66] for an early example of the latter. In fact, the quark sector of the current model we consider can also be mapped to a scan result from [22].

Here $\mathbf{1}_-$ is the lone non-trivial D_{15} singlet, and $\mathbf{2}_{1,5}$ denote two of the seven doublets of the group. Critically, we observe that (95)-(96) indicate that $\mathbf{2}_1$ ($\mathbf{2}_5$) is the appropriate D_{15} charge for the two flavons $\phi_{u,d}$ ($\phi_{l,\nu}$) that we introduce according to the algorithm in Section 2, and we can use (95) to work out the expressions for these doublet VEV, finding

$$\langle \phi_\nu \rangle = v_\nu \left(1, \sqrt{3} \right)^T, \quad \langle \phi_u \rangle = v_u \left(\cos \frac{\pi}{15}, \sin \frac{\pi}{15} \right)^T, \quad \langle \phi_{d,l} \rangle = v_{d,l} (1, 0)^T. \quad (97)$$

Finally, one derives that in the absence of mixing ambiguities, the model-basis mass matrices are given by

$$\begin{aligned} m_\nu'' &\stackrel{!}{=} \begin{pmatrix} m_{\nu_3} & 0 & 0 \\ 0 & \frac{1}{4}(3m_{\nu_1} + m_{\nu_2}) & \frac{\sqrt{3}}{4}(m_{\nu_2} - m_{\nu_1}) \\ 0 & \frac{\sqrt{3}}{4}(m_{\nu_2} - m_{\nu_1}) & \frac{1}{4}(m_{\nu_1} + 3m_{\nu_2}) \end{pmatrix}, \\ m_l''^\dagger m_l'' &\stackrel{!}{=} \begin{pmatrix} \frac{1}{2}(m_{l_2}^2 + m_{l_3}^2) & 0 & \frac{1}{2}(m_{l_2} - m_{l_3})(m_{l_2} + m_{l_3}) \\ 0 & m_{l_1}^2 & 0 \\ \frac{1}{2}(m_{l_2} - m_{l_3})(m_{l_2} + m_{l_3}) & 0 & \frac{1}{2}(m_{l_2}^2 + m_{l_3}^2) \end{pmatrix}, \\ m_u''^\dagger m_u'' &\stackrel{!}{=} \begin{pmatrix} m_{u_3}^2 & 0 & 0 \\ 0 & m_{u_1}^2 \cos^2 \frac{\pi}{15} + m_{u_2}^2 \sin^2 \frac{\pi}{15} & \frac{1}{2}(m_{u_1} - m_{u_2})(m_{u_1} + m_{u_2}) \sin \frac{2\pi}{15} \\ 0 & \frac{1}{2}(m_{u_1} - m_{u_2})(m_{u_1} + m_{u_2}) \sin \frac{2\pi}{15} & m_{u_2}^2 \cos^2 \frac{\pi}{15} + m_{u_1}^2 \sin^2 \frac{\pi}{15} \end{pmatrix}, \\ m_d''^\dagger m_d'' &\stackrel{!}{=} \begin{pmatrix} m_{d_3}^2 & 0 & 0 \\ 0 & m_{d_1}^2 & 0 \\ 0 & 0 & m_{d_2}^2 \end{pmatrix}, \end{aligned} \quad (98)$$

where m_{A_i} are the associated mass eigenvalues, and where we have used dagger combinations for the charged fermions to remove the dependence on RH transformations. However, unlike the $\Delta(96)$ model of Section 4.1, we see from Table 4 that all of the RFS generators have degenerate eigenvalues, and hence there are again freedoms in the associated mass and mixing matrices thanks to (13). We will discuss these when they become relevant below.

The Quark Sector

The results in (95)-(98) strongly indicate that the second and third generations of LH quarks should transform as a D_{15} doublet, while the first generation of up quarks transforms as a non-trivial singlet. Similarly, the first and third generations of RH up and down quarks should transform as a non-trivial singlet, while the second generation of both families transforms trivially. Furthermore, (96) indicates that the flavons $\phi_{d,u}$ associated to these sectors should transform as a $\mathbf{2}_1$ under D_{15} , a fact that helped us derive (97). This information is summarized in Table 6.

With an appropriate shaping symmetry preventing ϕ_a from coupling to undesirable sec-

	Q''^1	Q''^{23}	u''^1	u''^2	u''^3	d''^1	d''^2	d''^3	ϕ_u	ϕ_d
D_{15}	$\mathbf{1}_-$	$\mathbf{2}_1$	$\mathbf{1}_-$	$\mathbf{1}$	$\mathbf{1}_-$	$\mathbf{1}_-$	$\mathbf{1}$	$\mathbf{1}_-$	$\mathbf{2}_1$	$\mathbf{2}_1$

Table 6: Representations of the quarks under D_{15}

tors,¹⁹ one can quickly obtain the model-basis Yukawa sector for the quarks using Table 6,

$$\begin{aligned} \mathcal{L}_Y \supset & a_u \bar{Q}_L''^1 u_R''^1 + b_u \left[\bar{Q}_L''^{23} \phi_u \right]_{\mathbf{1}} u_R''^2 + c_u \left[\bar{Q}_L''^{23} \phi_u \right]_{\mathbf{1}_-} u_R''^3 \\ & + a_d \bar{Q}_L''^1 d_R''^1 + b_d \left[\bar{Q}_L''^{23} \phi_d \right]_{\mathbf{1}} d_R''^2 + c_d \left[\bar{Q}_L''^{23} \phi_d \right]_{\mathbf{1}_-} d_R''^3, \end{aligned} \quad (99)$$

where Higgs fields and scale suppressions are again omitted. Using the VEV from (97) and product rules from Appendix A.4, we get the following Yukawa matrices

$$Y_u''^\dagger = v_u \begin{pmatrix} a_u/v_u & 0 & 0 \\ 0 & b_u \cos \frac{\pi}{15} & -c_u \sin \frac{\pi}{15} \\ 0 & b_u \sin \frac{\pi}{15} & c_u \cos \frac{\pi}{15} \end{pmatrix}, \quad Y_d''^\dagger = v_d \begin{pmatrix} a_d/v_d & 0 & 0 \\ 0 & b_d & 0 \\ 0 & 0 & c_d \end{pmatrix}. \quad (100)$$

Assembling these into their Hermitian combinations, one arrives at

$$\begin{aligned} Y_u''^\dagger Y_u'' &= v_u^2 \begin{pmatrix} |a_u|^2/v_u^2 & 0 & 0 \\ 0 & |b_u|^2 \cos^2 \frac{\pi}{15} + |c_u|^2 \sin^2 \frac{\pi}{15} & \frac{1}{2} (|b_u|^2 - |c_u|^2) \sin \frac{2\pi}{15} \\ 0 & \frac{1}{2} (|b_u|^2 - |c_u|^2) \sin \frac{2\pi}{15} & |c_u|^2 \cos^2 \frac{\pi}{15} + |b_u|^2 \sin^2 \frac{\pi}{15} \end{pmatrix}, \\ Y_d''^\dagger Y_d'' &= v_d^2 \begin{pmatrix} |a_d|^2/v_d^2 & 0 & 0 \\ 0 & |b_d|^2 & 0 \\ 0 & 0 & |c_d|^2 \end{pmatrix}, \end{aligned} \quad (101)$$

which maps, up to prefactors and VEV, to (98) with

$$|a_u|^2 \leftrightarrow |m_{u_3}|^2, \quad |b_u|^2 \leftrightarrow |m_{u_1}|^2, \quad |c_u|^2 \leftrightarrow |m_{u_2}|^2,$$

and analogous relations for the mapping of Y_d .

The Lepton Sector

Similarly, the matrices in (95)-(98) suggest that the second and third LH generations of $SU(2)$ doublet leptons transform as a $\mathbf{2}_5$ D_{15} doublet, along with the associated flavons $\phi_{\nu,l}$. The $L_L''^1$ and first and third generations of E_R'' are to be charged as non-trivial singlets, while $E_R''^2$ transforms trivially. Using this information, assembled in Table 7, one reconstructs the LO

¹⁹However, we will soon see that off-diagonal entries in $Y_d''^\dagger$ will become desirable once we begin to discuss the leptokuark sector below, and therefore the implied shaping symmetry present in (99) will be modified to allow such additional operators.

	L''^1	L''^{23}	$E_R''^1$	$E_R''^2$	$E_R''^3$	ϕ_ν	ϕ_l
D_{15}	$\mathbf{1}_-$	$\mathbf{2}_5$	$\mathbf{1}_-$	$\mathbf{1}$	$\mathbf{1}_-$	$\mathbf{2}_5$	$\mathbf{2}_5$

Table 7: Representations of the leptons under D_{15}

Lagrangian as

$$\begin{aligned} \mathcal{L}_Y \supset & a_e \bar{L}_L''^1 E_R''^1 + b_e \left[\bar{L}_L''^{23} \phi_l \right]_{\mathbf{1}} E_R''^2 + c_e \left[\bar{L}_L''^{23} \phi_l \right]_{\mathbf{1}_-} E_R''^3 + d_e \bar{L}_L''^1 E_R^3 + \epsilon_e \left[\bar{L}_L''^{23} \phi_l \right]_{\mathbf{1}_-} E_R''^1 \\ & + a_\nu \bar{L}_L^{c''1} L_L''^1 + b_\nu \left[\bar{L}_L^{c''23} \phi_\nu \right]_{\mathbf{1}} \left[\bar{L}_L''^{23} \phi_\nu \right]_{\mathbf{1}} + c_\nu \left[\bar{L}_L^{c''23} \phi_\nu \right]_{\mathbf{1}_-} \left[\bar{L}_L''^{23} \phi_\nu \right]_{\mathbf{1}_-}. \end{aligned} \quad (102)$$

We quickly derive the following terms for neutrino masses and charged lepton Yukawas

$$m_\nu'' = v_\nu^2 \begin{pmatrix} a_\nu/v_\nu^2 & 0 & 0 \\ 0 & b_\nu + 3c_\nu & \sqrt{3}(b_\nu - c_\nu) \\ 0 & \sqrt{3}(b_\nu - c_\nu) & 3b_\nu + c_\nu \end{pmatrix}, \quad Y_l''^\dagger = v_l \begin{pmatrix} a_e/v_l & 0 & d_e/v_l \\ 0 & b_e & 0 \\ \epsilon_e & 0 & c_e \end{pmatrix}, \quad (103)$$

which gives

$$Y_l''^\dagger Y_l'' = v_l^2 \begin{pmatrix} (|a_e|^2 + |d_e|^2)/v_l^2 & 0 & (d_e c_e^* + \epsilon_e^* a_e)/v_l \\ 0 & |b_e|^2 & 0 \\ (\epsilon_e a_e^* + d_e^* c_e)/v_l & 0 & |c_e|^2 + |\epsilon_e|^2 \end{pmatrix}. \quad (104)$$

While the neutrino mass maps directly to (98), the charged lepton term apparently does not without additional fine tuning of the parameters. For example, setting $d_e = \epsilon_e$ and $c_e = a_e$, one can recover the corresponding term in (98), and only then will a diagonal matrix of eigenvalues be returned for $Y_l^\dagger Y_l$ upon (un)rotating (104) to the mass-eigenstate basis with Λ_l and P . However, we simultaneously observe that the coupling $Y_l''^\dagger Y_l''$ still respects the required RFS invariance,

$$T_l''^\dagger Y_l''^\dagger Y_l'' T_l'' = Y_l''^\dagger Y_l''. \quad (105)$$

Are these claims contradictory? After all, we argued that successfully mapping to (98) is a sufficient condition for ensuring that the EFT yields the desired IR phenomenology and RFS symmetry-breaking patterns, and while it appears that (104) cannot do so without unappealing assumptions, the RFS invariance still holds in (105).

The solution to this puzzle resides in the fact that, as in the A_4 Altarelli-Feruglio case, (98) does *not*, in fact, represent the most generic set of RFS-invariant mass matrices, and we will now show that the apparent fine-tuning required in mapping (104) to (98) can be understood as a top-down manifestation of (13).

We begin by recalling that the T_l'' generator cannot distinguish between Λ_l and $\Lambda_l \cdot R_{23}$,

$$T_l'' = \tilde{T}_l = P^\dagger \Lambda_l R_{23} T_l R_{23}^\dagger \Lambda_l^\dagger P, \quad (106)$$

where our tilde notation indicates that this can be understood a basis change on the charged lepton field, such that (starting from the mass-eigenstate basis), we are now operating with

$$l_L \rightarrow R_{23}^\dagger \Lambda_l^\dagger P \tilde{l}_L, \quad (107)$$

and so the most generic mass matrix invariant under the $\tilde{T}_l = T_l''$ generator that our algorithm to find D_{15} knows about is given by

$$\tilde{m}_l^\dagger \tilde{m}_l = P^\dagger \Lambda_L R_{23} m_l^\dagger m_l R_{23}^\dagger \Lambda_l^\dagger P \iff \tilde{T}_l^\dagger \tilde{m}_l^\dagger \tilde{m}_l \tilde{T}_l = \tilde{m}_l^\dagger \tilde{m}_l = T_l''^\dagger \tilde{m}_l^\dagger \tilde{m}_l T_l'' \quad (108)$$

with the the mass matrix elements given by

$$\begin{aligned} \left(\tilde{m}_l^\dagger \tilde{m}_l\right)^{11} &= \frac{1}{2} \left(m_{l_2}^2 + m_{l_3}^2 + (m_{l_3}^2 - m_{l_2}^2) \cos \delta_{23} \sin 2\theta_{23}\right), \\ \left(\tilde{m}_l^\dagger \tilde{m}_l\right)^{13} &= \frac{1}{2} \left(m_{l_2}^2 - m_{l_3}^2\right) \left(\cos 2\theta_{23} - i \sin 2\theta_{23} \sin \delta_{23}\right), \\ \left(\tilde{m}_l^\dagger \tilde{m}_l\right)^{22} &= m_{l_1}^2, \\ \left(\tilde{m}_l^\dagger \tilde{m}_l\right)^{31} &= \frac{1}{2} \left(m_{l_2}^2 - m_{l_3}^2\right) \left(\cos 2\theta_{23} + i \sin 2\theta_{23} \sin \delta_{23}\right), \\ \left(\tilde{m}_l^\dagger \tilde{m}_l\right)^{33} &= \frac{1}{2} \left(m_{l_2}^2 + m_{l_3}^2 + (m_{l_2}^2 - m_{l_3}^2) \cos \delta_{23} \sin 2\theta_{23}\right), \\ \left(\tilde{m}_l^\dagger \tilde{m}_l\right)^{12} &= \left(\tilde{m}_l^\dagger \tilde{m}_l\right)^{21} = \left(\tilde{m}_l^\dagger \tilde{m}_l\right)^{23} = \left(\tilde{m}_l^\dagger \tilde{m}_l\right)^{32} = 0, \end{aligned} \quad (109)$$

and where $\{\delta, \theta\}_{23}$ denote the free parameters our formalism has no control over.

However, (107) also implies that a diagonal charged current in this basis (we have not applied a change on any other field) implies a modified PMNS matrix in the (physical) mass-eigenstate basis:

$$\tilde{l}_L \gamma_\mu \tilde{\nu}_L W_\mu^- \underset{\text{mass basis}}{\longleftrightarrow} \tilde{l}_L \gamma_\mu \underbrace{R_{23}^\dagger U_{HM}}_{U_{PMNS}} \nu_L W_\mu^-. \quad (110)$$

This is consistent with the well-known fact that, in a more generic flavour basis, a degeneracy in T_l should translate to a free parameter in U_l and therefore also U_{PMNS} .

Now, one might be tempted to conclude that we should also translate ν_L with a compensating factor of R_{23} , so that the definition of the PMNS is preserved a la

$$\nu_L \rightarrow U_{HM}^\dagger R_{23}^\dagger \Lambda_l^\dagger P \tilde{\nu}_L \implies \tilde{l}_L \gamma_\mu \tilde{\nu}_L W_\mu^- \underset{\text{mass basis}}{\longleftrightarrow} \tilde{l}_L \gamma_\mu \underbrace{U_{HM}}_{U_{PMNS}} \nu_L W_\mu^-. \quad (111)$$

However, this corresponds to a generic neutrino mass matrix given by

$$\tilde{m}_\nu = P^T \Lambda_l^* R_{23}^* U_{HM}^* m_\nu U_{HM}^\dagger R_{23}^\dagger \Lambda_l^\dagger P, \quad (112)$$

which is left invariant under

$$\tilde{T}_\nu = P^\dagger \Lambda_l R_{23} U_{HM} T_\nu U_{HM}^\dagger R_{23}^\dagger \Lambda_l^\dagger P \neq T_\nu''. \quad (113)$$

That is, the neutrino generator *knows* about this basis change, which differs from our original observation that D_{15} doesn't know the difference between T_l'' and \tilde{T}_l . Indeed, we have checked that (at least at certain values of δ_{23} and θ_{23}) the group generated by $\mathcal{G}_F \cong \{\tilde{T}_l, \tilde{T}_\nu, \tilde{T}_u, \tilde{T}_d\}$

is *not* D_{15} , and may not even be finite! So making this compensating change in (111) is inconsistent with the starting point of our analysis, and cannot be done.

In conclusion, moving to the tilde basis requires no more work from the model building side, but implies a generalized RFS-invariant charged lepton mass matrix, and therefore a generalized prediction for the physical PMNS matrix given by $U_{PMNS} = R_{23}^\dagger \cdot U_{HM}$. The parameters of R_{23} are therefore functions of the (unspecified) coupling strengths of the EFT operators, since when solving the system of equations implied by mapping (109) to (104), one easily sees that

$$\begin{aligned} m_{l_1}^2 &\leftrightarrow |b_e|^2, \\ m_{l_2}^2 &\leftrightarrow (|d_e|^2 + |\epsilon_e|^2 + |a_e|^2 + |c_e|^2 + (|\epsilon_e|^2 + |c_e|^2 - |a_e|^2 - |d_e|^2) \csc 2\theta_{23} \sec \delta_{23}), \\ m_{l_3}^2 &\leftrightarrow (|d_e|^2 + |\epsilon_e|^2 + |a_e|^2 + |c_e|^2 + (|d_e|^2 + |a_e|^2 - |c_e|^2 - |\epsilon_e|^2) \csc 2\theta_{23} \sec \delta_{23}), \end{aligned} \quad (114)$$

up to VEV and prefactors, with

$$\begin{aligned} \delta_{23} &= \arctan \left(\operatorname{Im} \left[\frac{2(d_e c_e^* + \epsilon_e^* a_e)}{|a_e|^2 + |d_e|^2 - |c_e|^2 - |\epsilon_e|^2} \right] \right), \\ 2\theta_{23} &= \arctan \left(-\operatorname{Re} \left[\frac{2(d_e c_e^* + \epsilon_e^* a_e) \cos \delta_{23}}{|a_e|^2 + |d_e|^2 - |c_e|^2 - |\epsilon_e|^2} \right]^{-1} \right). \end{aligned} \quad (115)$$

Hence, while no fine-tuning is required in achieving this map, the model's prediction for U_{PMNS} is ambiguous up to the quantization of the couplings $\{a_e, \dots, \epsilon_e\}$, which may result from a higher UV symmetry (e.g. a GUT) that relates the otherwise independent operators of the EFT. Such an attempt is obviously well beyond our scope in this paper. However, thinking from a more phenomenological perspective, one can instead fit the parameters $\{\theta_{23}, \delta_{23}\}$ to available experimental data for the PMNS matrix, which then implies relationships amongst the top-down model's parameters, according to (114)-(115). Regardless, we see clearly that the ignorance of R_{23} in the *bottom-up* RFS generation of $\mathcal{G}_{\mathcal{F}}$ has consistently manifested itself in a certain lack of predictivity in the *top-down* EFT.

The Leptoquark Sector

The naive RFS-invariant leptoquark coupling for the $d - l$ operator expected in the model basis is given by

$$\lambda_{dl}'' \stackrel{!}{=} \frac{\lambda_{b\mu}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad (116)$$

This can be achieved using the following Lagrangian

$$\mathcal{L} \supset a_\Delta \bar{Q}_L''^1 L_L''^1 + b_\Delta \bar{Q}_L''^1 \left[L_L''^{23} \phi_l \right]_{\mathbf{1}_-} + c_\Delta \left[\bar{Q}_L''^{23} \phi_d \right]_{\mathbf{1}_-} \left[L_L''^{23} \phi_l \right]_{\mathbf{1}_-} + d_\Delta \left[\bar{Q}_L''^{23} \phi_d \right]_{\mathbf{1}_-} L_L''^1, \quad (117)$$

but again only with an additional tuning of the parameters,

$$a_\Delta = b_\Delta = c_\Delta = d_\Delta \implies \lambda_{dl}'' = \begin{pmatrix} a_\Delta & 0 & a_\Delta \\ 0 & 0 & 0 \\ a_\Delta & 0 & a_\Delta \end{pmatrix}. \quad (118)$$

However, following the above discussion for charged leptons, (116) is modified when we consider the R_{23} free rotation yielding (109), and results in the matrix

$$\tilde{\lambda}_{dl} \stackrel{!}{=} \frac{\lambda_{b\mu}}{\sqrt{2}} \begin{pmatrix} \mathcal{E}_{--}^l & 0 & \mathcal{E}_{+-}^l \\ 0 & 0 & 0 \\ \mathcal{E}_{--}^l & 0 & \mathcal{E}_{+-}^l \end{pmatrix}, \quad \mathcal{E}_{\pm\pm}^l \equiv \left(\cos \theta_{23}^l \pm e^{\pm i\delta_{23}^l} \sin \theta_{23}^l \right), \quad (119)$$

whose first and third columns, corresponding to lepton generations, are now distinguished.²⁰ Hence the need to fine-tune parameters between them disappears, although the otherwise independent couplings $\{a_\Delta, b_\Delta, c_\Delta, d_\Delta\}$ and $\{a_e, b_e, c_e, d_e\}$ are linked through (115) — they need to be simultaneously fit to functions of the same physical PMNS parameters, and hence are quite correlated.

Continuing, we also now note that the symmetry structure exposed in [31] (cf. Table 4) also permits a rotation in the (2,3) sector of T_d , and the conversation above as regards the corresponding basis change on the charged-lepton field can be had equally for the down quark, resulting in a further modified (119) which also distinguishes rows (quark generations),

$$\tilde{\tilde{\lambda}}_{dl} \stackrel{!}{=} \frac{\lambda_{b\mu}}{\sqrt{2}} \begin{pmatrix} \mathcal{E}_{--}^d & \mathcal{E}_{--}^l & 0 & \mathcal{E}_{--}^d & \mathcal{E}_{+-}^l \\ 0 & 0 & 0 & 0 & 0 \\ \mathcal{E}_{++}^d & \mathcal{E}_{--}^l & 0 & \mathcal{E}_{++}^d & \mathcal{E}_{+-}^l \end{pmatrix}, \quad (120)$$

with $\mathcal{E}_{\pm\pm}^d$ defined analogously to $\mathcal{E}_{\pm\pm}^l$. Then no fine-tuning of the model's couplings in (117) will be required since the mapping

$$a_\Delta \leftrightarrow \lambda_{b\mu} \mathcal{E}_{--}^d \mathcal{E}_{--}^l, \quad b_\Delta \leftrightarrow \lambda_{b\mu} \mathcal{E}_{--}^d \mathcal{E}_{+-}^l, \quad c_\Delta \leftrightarrow \lambda_{b\mu} \mathcal{E}_{++}^d \mathcal{E}_{--}^l, \quad d_\Delta \leftrightarrow \lambda_{b\mu} \mathcal{E}_{++}^d \mathcal{E}_{+-}^l \quad (121)$$

is achieved, although again at the expense of the phenomenology not being uniquely nailed down. But we must also then account for the fact that the R_{23} rotation in the quark sector simultaneously changes the prediction for the associated RFS-invariant mass matrix to

$$\tilde{\tilde{m}}_d^\dagger \tilde{\tilde{m}}_d \stackrel{!}{=} \begin{pmatrix} m_{d_3}^2 \cos^2 \theta_{23}^q + m_{d_2}^2 \sin^2 \theta_{23}^q & 0 & -e^{i\delta_{23}^q} (m_{d_2}^2 - m_{d_3}^2) \cos \theta_{23}^q \sin \theta_{23}^q \\ 0 & m_{d_1}^2 & 0 \\ e^{-i\delta_{23}^q} (m_{d_3}^2 - m_{d_2}^2) \cos \theta_{23}^q \sin \theta_{23}^q & 0 & m_{d_2}^2 \cos^2 \theta_{23}^q + m_{d_3}^2 \sin^2 \theta_{23}^q \end{pmatrix}, \quad (122)$$

which further corresponds to an altered CKM prediction,

$$U_{CKM} \rightarrow \begin{pmatrix} \cos \frac{\pi}{15} & \cos \theta_{23}^q \sin \frac{\pi}{15} & e^{-i\delta_{23}^q} \sin \theta_{23}^q \sin \frac{\pi}{15} \\ -\sin \frac{\pi}{15} & \cos \theta_{23}^q \cos \frac{\pi}{15} & e^{-i\delta_{23}^q} \sin \theta_{23}^q \cos \frac{\pi}{15} \\ 0 & -e^{i\delta_{23}^q} \sin \theta_{23}^q & \cos \theta_{23}^q \end{pmatrix}. \quad (123)$$

These changes appear problematic at first sight, since we have already successfully achieved the desired mapping from (101) to (98), i.e. the CKM prediction with no mixing ambiguity. Indeed, the Lagrangian in (99) does not have operators that can map to the additional

²⁰ Note that in the definition of $\mathcal{E}_{\pm\pm}^l$, the two \pm labels correspond to the first and second \pm appearing on the RHS, respectively, while the l, d superscript denotes parameters from the lepton and down quark sectors.

contributions in the $(1, 3)$ sector of (122). However, we recall that the implicit assumption in building (99) (and all other effective \mathcal{L}_Y in this paper) is that unspecified shaping symmetries forbid undesirable operators from contributing to the Yukawa. So, once (122) is the appropriate term to be recovered in the Lagrangian of (99), one can then assume a different shaping symmetry, such that further operators contribute in a way that allows for a one-to-one mapping. For example, one can easily obtain the modified quark Lagrangian as the one in (99) with additional down quark terms which are in fact invariant under D_{15} (given that $d_R''^1$ and $d_R''^3$ are not in fact distinguished by D_{15}). The down sector would then be

$$\mathcal{L}_Y \supset a_d \bar{Q}_L''^1 d_R''^1 + b_d \left[\bar{Q}_L''^{23} \phi_d \right]_{\mathbf{1}} d_R''^2 + c_d \left[\bar{Q}_L''^{23} \phi_d \right]_{\mathbf{1}_-} d_R''^3 + a'_d \bar{Q}_L''^1 d_R''^3 + c'_d \left[\bar{Q}_L''^{23} \phi_d \right]_{\mathbf{1}_-} d_R''^1, \quad (124)$$

where the a'_d and c'_d terms are clearly the terms with the unprimed couplings after undergoing a swap of $d_R''^1$ and $d_R''^3$. It is simple to see they create entries in the mass matrix that will allow a successful map to (122):

$$\tilde{Y}_d^\dagger = v_d \begin{pmatrix} a_d/v_d & 0 & a'_d/v_d \\ 0 & b_d & 0 \\ c'_d & 0 & c_d \end{pmatrix}. \quad (125)$$

In this scenario one loses some predictivity over the CKM mixing, since (123) leaves $\{\theta_{23}^q, \delta_{23}^q\}$ unquantized — they will become functions of the free operator couplings in a manner analogous to (115). On the other hand, the fine-tuning issue in λ_{dl} is resolved, and (123) anyway better approximates global fits to the experimental CKM matrix than the original Cabibbo form we predicted. Hence θ_{23}^q can be fit to the data, which then leads to more precise EFT predictions in the (unmeasured) leptoquark coupling of (120).

Further Comments on the Appearance of Free Parameters in Effective Models

We have seen that the simple equivalence evident in (13) can be important when building \mathcal{L}_Y realizing family-symmetry breaking of the form $\mathcal{G}_{\mathcal{F}, \mathcal{L}, \mathcal{Q}} \rightarrow \mathcal{G}_a \cong \{\hat{T}_a\}$. While this phenomenological ambiguity was discussed from a bottom-up perspective in [31] and multiple prior references from other authors, its consequences from a top-down model-building perspective have, to our knowledge, not been appreciated. We now see that, in the absence of a proper accounting of (13), the implied RFS-invariant mass/coupling shapes are unnecessarily restrictive, possibly leading to the erroneous conclusion that fine-tunings of model parameters are required. Upon considering the full implications of (13) on these shapes, these fine-tunings are resolved in favor of one-to-one mappings between model and physical parameters, albeit at the expense of the EFT's predictivity. In short, the bottom-up mathematical ambiguity of (13) can consistently manifest itself as a top-down phenomenological ambiguity in a given model's IR mass and mixing spectrum.

However, our D_{15} analysis still leaves some questions unanswered. For example, why was no tuning required in the neutrino or up quark mass matrices, where we also only attempted a map to the naive RFS-invariant mass matrices, but where Table 4 clearly indicates that free parameters can be introduced into these sectors as well? While it is beyond our present scope to answer this question conclusively, we suspect that the answer lies in the group product rules at hand, which as a function of the group closure will (at least in the basis we consider)

likely be driven by the CKM and PMNS structures entirely embedded in the up and neutrino sectors. After all, D_{15} is only armed with a handful of doublets from which we can form invariants according to (20), whereas a larger group that contains, e.g., triplet representations might allow a broader and more diverse set of invariants from which we can form the naive RFS-invariant shapes of (98). This suspicion is at least consistent with the fact that, while studying D_{15} , we also attempted to build the final model presented in Table 4, based on the same symmetry $(Z_{14} \times Z_2) \rtimes Z_2$ that we used for the CKM prediction in Section 3.2. There we again found that no tuning was required until we tried to model $m_l''^\dagger m_l''$ and the subsequent leptoquark coupling λ_{dl}'' , where the need became apparent exactly as in D_{15} . As conjectured, $(Z_{14} \times Z_2) \rtimes Z_2$ also only has doublet and singlet irreducible representations. Unfortunately though, we did not find a candidate symmetry in [31] that allows us to test to this hypothesis, and so it may be interesting to perform a similar group theory scan while allowing for larger finite groups to pass the self-imposed cuts. On the other hand, the simultaneous introduction of the inherently 2D Cabibbo form of (75) alongside the inherently 3D $\mu - \tau$ symmetric form of (74) into the scans may inevitably lead to similar results as those in [31]. We will leave the resolution of these questions to future study — our method, as demonstrated in Sections 2-4, works regardless of their conclusions.

5 Summary and Outlook

We have shown how to use the RFS of the Yukawa sector of an IR Lagrangian, i.e. one where electroweak- and family-symmetry breaking has already occurred, to systematically reconstruct a UV effective Lagrangian that respects SM gauge symmetries and non-Abelian flavour symmetries $\mathcal{G}_{\mathcal{F}}$ which contain such RFS as subgroups. Our method is thus complimentary to prior scans of family groups performed in order to identify phenomenologically viable $\mathcal{G}_{\mathcal{F}}$ and symmetry-breaking patterns *without* specifying concrete UV Lagrangians — that is, we can use bottom-up, model-independent information to algorithmically construct top-down models with an explicit field and symmetry content. We have shown four such examples, two where only SM fermionic mixing (CKM or PMNS matrices) is controlled, and two where SM mixing matrices and flavoured leptoquark couplings are structured with the RFS. We thus provide ‘proof-in-principle’ routes to UV completions for the simplified models outlined in [30,31]. Our study has also helped to clarify commentary in prior literature as regards the role of eigenvalue degeneracies in RFS generators and associated mixing ambiguities in top-down flavour models.

Furthermore, leptoquark extensions of the SM represent but one of many BSM scenarios with non-trivial flavour structure that can be studied within the RFS paradigm, which bypasses potentially unfalsifiable aspects of model building and offers a mechanism for identifying classes of simplified models and their phenomenological implications. Our results in this paper indicate that analogous, model-independent RFS applications to (e.g.) multi-Higgs-doublet models (cf. [34]) or softly-broken SUSY can also be readily ‘completed’ if deemed necessary by a particular experimental signature, and can therefore be confidently studied in the meantime without reference to UV dynamics. We leave these possible extensions to future work.

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A Details on Representation Theory

In this Appendix we provide the relevant group product rules and associated Clebsch-Gordon coefficient structure for the finite groups employed in the sample models of the main paper, and in the bases in which they are built.²¹

A.1 A_4

For two triplets generically parameterized by $x_i = (x_1, x_2, x_3)$ and $y_i = (y_1, y_2, y_3)$, the product rule between them gives

$$\mathbf{3} \times \mathbf{3} \sim \mathbf{1} + \mathbf{1}' + \mathbf{1}'' + \mathbf{3}_S + \mathbf{3}_A . \quad (126)$$

In the basis of the generators given in (25), the singlet Clebsch-Gordan structure is then given by

$$\begin{aligned} \mathbf{1} &\sim (x_1y_1 + x_2y_3 + x_3y_2) , \\ \mathbf{1}' &\sim (x_3y_3 + x_1y_2 + x_2y_1) , \\ \mathbf{1}'' &\sim (x_2y_2 + x_1y_3 + x_3y_1) , \end{aligned} \quad (127)$$

whilst the symmetric (S) and anti-symmetric (A) triplet combinations are given by

$$\begin{aligned} \mathbf{3}_S &\sim \frac{1}{3} (2x_1y_1 - x_2y_3 - x_3y_2, 2x_3y_3 - x_1y_2 - x_2y_1, 2x_2y_2 - x_1y_3 - x_3y_1) , \\ \mathbf{3}_A &\sim \frac{1}{2} (x_2y_3 - x_3y_2, x_1y_2 - x_2y_1, x_1y_3 - x_3y_1) . \end{aligned} \quad (128)$$

Finally, the singlet product rules are found to be:

$$\mathbf{1}' \times \mathbf{1}' \sim \mathbf{1}'' , \quad \mathbf{1}' \times \mathbf{1}'' \sim \mathbf{1} , \quad \mathbf{1}'' \times \mathbf{1}'' \sim \mathbf{1}' . \quad (129)$$

²¹See e.g. [63] for an exhaustive catalogue of finite group series and their properties.

A.2 $(\mathbb{Z}_{14} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$

We have not found the associated product rules and Clebsch-Gordan factors for this group in the literature, and hence have derived them for ourselves. As such, we provide a touch more information here than for other groups in this Appendix.

The relevant group information we require for $(\mathbb{Z}_{14} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ can be accessed via the **GAP** package, using its identification number [56, 7]. However, **GAP** is using a four-generator basis while the minimal generating set, that we will work with, only has three generators. Making this conversion, we find that the full list of irreducible representations and associated generators is given by

$$\begin{aligned}
\mathbf{1}_{++} : \quad & \mathbf{a} = 1, \quad \mathbf{b} = 1, \quad \mathbf{c} = 1, \\
\mathbf{1}_{+-} : \quad & \mathbf{a} = 1, \quad \mathbf{b} = -1, \quad \mathbf{c} = 1, \\
\mathbf{1}_{-+} : \quad & \mathbf{a} = -1, \quad \mathbf{b} = 1, \quad \mathbf{c} = 1, \\
\mathbf{1}_{--} : \quad & \mathbf{a} = -1, \quad \mathbf{b} = -1, \quad \mathbf{c} = 1, \\
\mathbf{2}_0 : \quad & \mathbf{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\mathbf{2}_{n++} : \quad & \mathbf{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \omega_7^n & 0 \\ 0 & \omega_7^{-n} \end{pmatrix}, \\
\mathbf{2}_{n+-} : \quad & \mathbf{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \omega_7^n & 0 \\ 0 & \omega_7^{-n} \end{pmatrix}, \\
\mathbf{2}_{n-+} : \quad & \mathbf{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \omega_7^n & 0 \\ 0 & \omega_7^{-n} \end{pmatrix}, \\
\mathbf{2}_{n--} : \quad & \mathbf{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \omega_7^n & 0 \\ 0 & \omega_7^{-n} \end{pmatrix},
\end{aligned} \tag{130}$$

where $n = 1, 2, 3$ and $\omega_7 = e^{\frac{2i\pi}{14}}$.

As can be seen, representations are not always real. Taking the generic doublet to be $\sim (x_1, x_2)$, conjugate representations can be expressed in terms of the original ones following

$$\bar{\mathbf{2}}_{\mathbf{k}\rho\sigma} \sim (\bar{x}_2, \bar{x}_1) \sim \mathbf{2}_{\mathbf{k}\sigma\rho}, \tag{131}$$

and similarly for $\bar{\mathbf{2}}_0$, while all the singlets are real.

The product rules for singlets can be obtained trivially by noting the action of the generators \mathbf{a} , \mathbf{b} , which is indicated by the first and second subscript respectively:

$$\begin{aligned}
\mathbf{1}_{\pm\pm} \times \mathbf{1}_{\pm\pm} & \sim \mathbf{1}_{++} \\
\mathbf{1}_{\pm\mp} \times \mathbf{1}_{\pm\pm} & \sim \mathbf{1}_{+-}
\end{aligned}$$

$$\begin{aligned}
\mathbf{1}_{\mp\pm} \times \mathbf{1}_{\pm\pm} &\sim \mathbf{1}_{-+} \\
\mathbf{1}_{\pm\mp} \times \mathbf{1}_{\mp\pm} &\sim \mathbf{1}_{--} .
\end{aligned} \tag{132}$$

The full list of product rules can be obtained by explicitly checking the transformation properties. Assuming the first doublet is given by (x_1, x_2) and the second one by (y_1, y_2) , we obtain the following product rules:

$$\begin{aligned}
\mathbf{2}_0 \times \mathbf{2}_0 &\sim [x_1y_1 + x_2y_2]_{\mathbf{1}_{++}} + [x_2y_2 - x_1y_1]_{\mathbf{1}_{--}} + [x_1y_2 + x_2y_1]_{\mathbf{1}_{+-}} + [x_2y_1 - x_1y_2]_{\mathbf{1}_{-+}} , \\
\mathbf{2}_{n++} \times \mathbf{2}_0 &\sim \begin{bmatrix} x_1y_1 \\ x_2y_2 \end{bmatrix}_{\mathbf{2}_{n+-}} + \begin{bmatrix} x_1y_2 \\ x_2y_1 \end{bmatrix}_{\mathbf{2}_{n-+}} , \\
\mathbf{2}_{n++} \times \mathbf{2}_{n++} &\sim [x_1y_2 + x_2y_1]_{\mathbf{1}_{++}} + [x_2y_1 - x_1y_2]_{\mathbf{1}_{--}} + \mathbf{2}_{k++} , \\
\mathbf{2}_{n++} \times \mathbf{2}_{n+-} &\sim \begin{bmatrix} x_2y_1 \\ x_1y_2 \end{bmatrix}_{\mathbf{2}_0} + \mathbf{2}_{k+-} , \\
\mathbf{2}_{n++} \times \mathbf{2}_{n-+} &\sim \begin{bmatrix} x_1y_2 \\ x_2y_1 \end{bmatrix}_{\mathbf{2}_0} + \mathbf{2}_{k-+} , \\
\mathbf{2}_{n++} \times \mathbf{2}_{n--} &\sim [x_2y_1 + x_1y_2]_{\mathbf{1}_{+-}} + [x_2y_1 - x_1y_2]_{\mathbf{1}_{-+}} + \mathbf{2}_{k--} , \\
\mathbf{2}_{n+-} \times \mathbf{2}_0 &\sim \begin{bmatrix} x_1y_1 \\ x_2y_2 \end{bmatrix}_{\mathbf{2}_{n++}} + \begin{bmatrix} x_1y_2 \\ x_2y_1 \end{bmatrix}_{\mathbf{2}_{n--}} , \\
\mathbf{2}_{n+-} \times \mathbf{2}_{n+-} &\sim [x_2y_1 + x_1y_2]_{\mathbf{1}_{+-}} + [x_2y_1 - x_1y_2]_{\mathbf{1}_{-+}} + \mathbf{2}_{k++} , \\
\mathbf{2}_{n+-} \times \mathbf{2}_{n-+} &\sim [x_2y_1 + x_1y_2]_{\mathbf{1}_{++}} + [x_2y_1 - x_1y_2]_{\mathbf{1}_{--}} + \mathbf{2}_{k--} , \\
\mathbf{2}_{n+-} \times \mathbf{2}_{n--} &\sim \begin{bmatrix} x_2y_1 \\ x_1y_2 \end{bmatrix}_{\mathbf{2}_0} + \mathbf{2}_{k-+} , \\
\mathbf{2}_{n-+} \times \mathbf{2}_0 &\sim \begin{bmatrix} x_1y_2 \\ x_2y_1 \end{bmatrix}_{\mathbf{2}_{n++}} + \begin{bmatrix} x_1y_1 \\ x_2y_2 \end{bmatrix}_{\mathbf{2}_{n--}} , \\
\mathbf{2}_{n-+} \times \mathbf{2}_{n-+} &\sim [x_2y_1 + x_1y_2]_{\mathbf{1}_{+-}} + [x_2y_1 - x_1y_2]_{\mathbf{1}_{-+}} + \mathbf{2}_{k++} , \\
\mathbf{2}_{n-+} \times \mathbf{2}_{n--} &\sim \begin{bmatrix} x_1y_2 \\ x_2y_1 \end{bmatrix}_{\mathbf{2}_0} + \mathbf{2}_{k+-} , \\
\mathbf{2}_{n--} \times \mathbf{2}_0 &\sim \begin{bmatrix} x_1y_1 \\ x_2y_2 \end{bmatrix}_{\mathbf{2}_{n-+}} + \begin{bmatrix} x_1y_2 \\ x_2y_1 \end{bmatrix}_{\mathbf{2}_{n+-}} , \\
\mathbf{2}_{n--} \times \mathbf{2}_{n--} &\sim [x_2y_1 + x_1y_2]_{\mathbf{1}_{++}} + [x_2y_1 - x_1y_2]_{\mathbf{1}_{--}} + \mathbf{2}_{k++} ,
\end{aligned} \tag{133}$$

where

$$\mathbf{2}_{\mathbf{k}\rho\sigma} \sim \begin{cases} \begin{bmatrix} x_1 y_1 \\ x_2 y_2 \end{bmatrix}_{\mathbf{2}_{\mathbf{2}\rho\sigma}} & \text{for } n = 1 \\ \begin{bmatrix} x_2 y_2 \\ x_1 y_1 \end{bmatrix}_{\mathbf{2}_{\mathbf{3}\sigma\rho}} & \text{for } n = 2 \\ \begin{bmatrix} x_2 y_2 \\ x_1 y_1 \end{bmatrix}_{\mathbf{2}_{\mathbf{1}\sigma\rho}} & \text{for } n = 3 \end{cases} . \quad (134)$$

While we expanded the above doublet product rules in (133) for simplicity and utility, we will express the other (less useful) doublet product rules in the following compacted formula:

$$\mathbf{2}_{\mathbf{n}\rho\sigma} \times \mathbf{2}_{\mathbf{n}'\rho'\sigma'} \sim \mathbf{2}_{\mathbf{X}} + \mathbf{2}_{\mathbf{Y}} , \quad (135)$$

with $n \neq n'$ and

$$\mathbf{2}_{\mathbf{X}} \sim \begin{cases} \begin{bmatrix} x_1 y_1 \\ x_2 y_2 \end{bmatrix}_{\mathbf{2}_{(\mathbf{n}+\mathbf{n}')(\rho\rho')(\sigma\sigma')}} & \text{for } n + n' < 7 - (n + n') \\ \begin{bmatrix} x_2 y_2 \\ x_1 y_1 \end{bmatrix}_{\mathbf{2}_{(\mathbf{7}-\mathbf{n}-\mathbf{n}')(\sigma\sigma')(\rho\rho')}} & \text{for } n + n' > 7 - (n + n') \end{cases} , \quad (136)$$

$$\mathbf{2}_{\mathbf{Y}} \sim \begin{cases} \begin{bmatrix} x_1 y_2 \\ x_2 y_1 \end{bmatrix}_{\mathbf{2}_{(\mathbf{n}-\mathbf{n}')(\rho\sigma')(\sigma\rho')}} & \text{for } \mathcal{F}_7(n - n') < \mathcal{F}_7(n' - n) \\ \begin{bmatrix} x_2 y_1 \\ x_1 y_2 \end{bmatrix}_{\mathbf{2}_{(\mathbf{n}'-\mathbf{n})(\sigma\rho')(\rho\sigma')}} & \text{for } \mathcal{F}_7(n - n') > \mathcal{F}_7(n' - n) \end{cases} ,$$

where

$$\mathcal{F}_7(k) = \begin{cases} k & \text{for } k > 0 \\ 7 + k & \text{for } k < 0 \end{cases} . \quad (137)$$

A.3 $\Delta(96)$

We note that, as indicated above in Section 4.1, we follow the notation of [65] where the generators are identified by smallcase latin letters, but to help avoid confusion with our notation for coefficients (also smallcase latin letters), we refer to the generators of $\Delta(96)$ in bold. Here we present some of the product rules for irreducible representations of $\Delta(96)$ in the basis we used in Section 4.1, which matches the basis in [65] where the element $\mathbf{a}_{\mathbf{3}_1}^2 \mathbf{c}_{\mathbf{3}_1} \mathbf{d}_{\mathbf{3}_1}$ is diagonal. This basis has

$$\mathbf{a}_{\mathbf{3}_1} = \frac{1}{3} \begin{pmatrix} \omega & 1 + \sqrt{3} & (1 - \sqrt{3}) \omega^2 \\ 1 - \sqrt{3} & \omega^2 & (1 + \sqrt{3}) \omega \\ (1 + \sqrt{3}) \omega^2 & (1 - \sqrt{3}) \omega & 1 \end{pmatrix} ,$$

$$\begin{aligned}
\mathbf{b}_{\mathbf{3}_1} &= \frac{1}{3} \begin{pmatrix} -1 - \sqrt{3} & -\omega & (\sqrt{3} - 1)\omega^2 \\ -\omega^2 & \sqrt{3} - 1 & -(1 + \sqrt{3})\omega \\ (\sqrt{3} - 1)\omega - (1 + \sqrt{3})\omega^2 & -1 & \end{pmatrix}, \\
\mathbf{c}_{\mathbf{3}_1} &= \frac{1}{3} \begin{pmatrix} 1 & (1 - \sqrt{3})\omega^2 & (1 + \sqrt{3})\omega \\ (1 + \sqrt{3})\omega & 1 & (1 - \sqrt{3})\omega^2 \\ (1 - \sqrt{3})\omega^2 & (1 + \sqrt{3})\omega & 1 \end{pmatrix}, \tag{138}
\end{aligned}$$

and $\mathbf{d}_{\mathbf{3}_1} = \mathbf{a}_{\mathbf{3}_1}^{-1} \mathbf{c}_{\mathbf{3}_1} \mathbf{a}_{\mathbf{3}_1}$. In this basis, the group rules we have used in our models are all the rules for $\mathbf{3}_1 \times \bar{\mathbf{3}}_1 \sim \mathbf{1} + \mathbf{2} + \mathbf{6}$, $\mathbf{3}_1 \times \bar{\mathbf{3}}'_1 \sim \mathbf{1}' + \mathbf{2} + \mathbf{6}$, and only the rules for the trivial singlet built from $(\mathbf{1}' \times \mathbf{1}') \sim \mathbf{1}$, $(\mathbf{2} \times \mathbf{2})_1$ and $(\mathbf{6} \times \mathbf{6})_1$:

$$\begin{aligned}
(\mathbf{3}_1 \times \bar{\mathbf{3}}_1)_1 &\sim x_1 y_1 + x_2 y_2 + x_3 y_3, \\
(\mathbf{3}_1 \times \bar{\mathbf{3}}_1)_2 &\sim \begin{pmatrix} x_1 y_3 + x_2 y_1 + x_3 y_2 \\ \omega(x_1 y_2 + x_2 y_3 + x_3 y_1) \end{pmatrix}, \\
(\mathbf{3}_1 \times \bar{\mathbf{3}}_1)_6 &\sim \begin{pmatrix} x_1 y_3 + \omega x_2 y_1 + \omega^2 x_3 y_2 \\ \omega x_1 y_2 + \omega^2 x_2 y_3 + x_3 y_1 \\ \omega^2 x_1 y_1 + x_2 y_2 + \omega x_3 y_3 \\ x_1 y_3 + \omega^2 x_2 y_1 + \omega x_3 y_2 \\ \omega^2 x_1 y_2 + \omega x_2 y_3 + x_3 y_1 \\ \omega x_1 y_1 + x_2 y_2 + \omega^2 x_3 y_3 \end{pmatrix}, \\
(\mathbf{3}_1 \times \bar{\mathbf{3}}'_1)_1 &\sim x_1 y_1 + x_2 y_2 + x_3 y_3, \\
(\mathbf{3}_1 \times \bar{\mathbf{3}}'_1)_2 &\sim \begin{pmatrix} x_1 y_3 + x_2 y_1 + x_3 y_2 \\ -\omega(x_1 y_2 + x_2 y_3 + x_3 y_1) \end{pmatrix}, \\
(\mathbf{3}_1 \times \bar{\mathbf{3}}'_1)_6 &\sim \begin{pmatrix} x_1 y_3 + \omega x_2 y_1 + \omega^2 x_3 y_2 \\ \omega x_1 y_2 + \omega^2 x_2 y_3 + x_3 y_1 \\ \omega^2 x_1 y_1 + x_2 y_2 + \omega x_3 y_3 \\ -x_1 y_3 - \omega^2 x_2 y_1 - \omega x_3 y_2 \\ -\omega^2 x_1 y_2 - \omega x_2 y_3 - x_3 y_1 \\ -\omega x_1 y_1 - x_2 y_2 - \omega^2 x_3 y_3 \end{pmatrix}, \\
(\mathbf{2} \times \mathbf{2})_1 &\sim x_1 y_2 + x_2 y_1, \\
(\mathbf{6} \times \mathbf{6})_1 &\sim x_1 y_5 + x_2 y_4 + x_3 y_6 + x_4 y_2 + x_5 y_1 + x_6 y_3. \tag{139}
\end{aligned}$$

A.4 D_{15}

Following [63], D_N has $2 + (N - 1)/2$ representations when N is odd: 2 singlets $\mathbf{1}_+$ and $\mathbf{1}_-$ ($\mathbf{1}_+$ is the trivial singlet) and $(N - 1)/2$ doublets labeled $\mathbf{2}_k$. In our case we therefore have 7

doublet representations, within which our group generators take the form

$$\mathbf{b} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} \cos \frac{2k\pi}{15} & -\sin \frac{2k\pi}{15} \\ \sin \frac{2k\pi}{15} & \cos \frac{2k\pi}{15} \end{pmatrix}.$$

On the other hand, for the singlet representations we have that

$$\begin{aligned} \mathbf{1}_+ : \quad \mathbf{a} = 1 \quad \text{and} \quad \mathbf{b} = 1, \\ \mathbf{1}_- : \quad \mathbf{a} = 1 \quad \text{and} \quad \mathbf{b} = -1. \end{aligned} \tag{140}$$

One can work out the kronecker products for these different representations. In our case, the relevant ones for contracting two doublets will be given by

$$\begin{aligned} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_k} \times \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}_k} \right)_{\mathbf{1}} &\sim x_1 y_1 + x_2 y_2, \\ \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_k} \times \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}_k} \right)_{\mathbf{1}'} &\sim x_2 y_1 - x_1 y_2, \end{aligned} \tag{141}$$

while two non-trivial singlets contract to a trivial singlet

$$\mathbf{1}' \times \mathbf{1}' \sim \mathbf{1}. \tag{142}$$

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