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The mass ratios parametrization

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Abstract

The quark and lepton mixing matrices are here both parametrised in terms of the corresponding four independent mass ratios of each fermion sector. This realization is studied under two different approaches: a general discussion on the conditions to achieve it and through the examination of a particular example. In the former approach, the sole dependence on the mass ratios is exploited to investigate the properties that a mixing matrix will possess under such a parametrization. Already at this first stage, the observed fermion mixing can be roughly understood. Thereafter, in the second approach, a particular implementation which was recently proposed is considered. The procedure is revisited and both its weak and strong points are discussed. The purpose of this work is not to present a model but to explore the possibility of reviving an old idea, that of parametrizing the mixing matrices with the fermion masses.

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1 Introduction

The problem is simple: from the theoretical point of view we do not understand the observed values in fermion mixing. We wish to do it though but the nature of the answer seems to evade all our trials. In contrast, the experimental situation of the theory describing the fundamental interactions is quite remarkable. The Standard Model (SM) of the weak, color, and electromagnetic interactions keeps most of its original form; with the masses of neutrinos being the most needed change plus the inclusion of lepton mixing, but no deep connections have been really accomplished. This is equivalent to say that the initial number of arbitrary parameters has only increased from nineteen to twenty six (in the least worst scenario). On the other hand, after the phenomenological observation of a strong hierarchy simultaneously occurring in the masses and the mixings of quarks, one is tempted to relate both sets of parameters. Unfortunately, this picture is lost as soon as lepton mixing with its anarchical structure is incorporated as this seems to point to a completely independent origin. Explaining both sectors within a unifying picture is thus a difficult challenge. In this work, we investigate and realize the idea of relating mixing phenomena to the masses of fermions, thus obtaining a unified picture.

All begins from the complete arbitrariness in which the Yukawa couplings are introduced,

$$-\mathcal{L}_Y \supset \mathbf{Y}_f^{ij} \bar{F}_{L,i} \Phi f_{R,j} \ , \tag{1}$$

with $\mathbf{Y}_f^{ij}$ a complex number and entry of a three by three matrix. Assignment of a non-zero vacuum expectation value to the neutral component of the scalar field,

$$\Phi(x) = \begin{pmatrix} G^+(x) \\ \frac{v+h(x)+iG^0(x)}{\sqrt{2}} \end{pmatrix} \ , \tag{2}$$

spontaneously breaks the electroweak symmetry and brings about the massive nature of fermions,

$$\mathbf{M}_f = \frac{v}{\sqrt{2}} \mathbf{Y}_f \ , \tag{3}$$

where $v \simeq 246$ GeV. Diagonalization of the mass matrices,

$$\mathbf{\Sigma}_f = \mathbf{L}_f \mathbf{M}_f \mathbf{R}_f^\dagger \ , \tag{4}$$

occurs via a biunitary transformation each acting independently in the left or right-handed corresponding field,

$$F_L \to \mathbf{L}_f F_L \qquad \text{and} \qquad f_R \to \mathbf{R}_f f_R \ . \tag{5}$$

In this new basis, the mass basis, the quark and leptonic charged currents have changed to,

$$\mathcal{J}^{\mu,-}_{\text{cc-q}} = -\frac{g_w}{\sqrt{2}} \bar{u}_L \gamma^\mu \mathbf{V} d_L \ , \qquad \mathcal{J}^{\mu,+}_{\text{cc-}\ell} = -\frac{g_w}{\sqrt{2}} \bar{e}_L \gamma^\mu \mathbf{U} \nu_L \ , \tag{6}$$

where $\mathbf{V} = \mathbf{L}_u \mathbf{L}_d^\dagger$ and $\mathbf{U} = \mathbf{L}_e \mathbf{L}_\nu^\dagger$. These matrices parametrize how likely are the transitions between any two given unequal flavors via the interactions with the $W^\pm$ bosons. As the masses of the different up and down type quarks and neutrinos and the charged leptons are different we do not expect the unitary transformations, acting independently in the left handed fields, to be the same and therefore $\mathbf{V}$ and $\mathbf{U}$ should, in general, be distant from the unit matrix. From this picture, it becomes clear that the key to understanding fermion mixing lies in further understanding either the mass matrices or for all general purposes, in understanding the Yukawa matrices, see for example Ref. [1].

As this work is about a mixing parametrization, let us briefly discuss their history. The first parametrization one can encounter goes back to the days where Cabibbo proposed a way to preserve universality in the weak interactions [2]. Although not a parametrization proposal on its own the consideration of adding the charm quark in order to avoid tree level flavor changing neutral currents [3] came into represent an important step in the need for a better description of the charged current interactions. Later on, through the work of Kobayashi and Maskawa, who noticed that in order to introduce Charge-Parity (CP) violation three fermion families were needed, the first three by three mixing parametrization was proposed [4].

Among the different proposals that came after [4–13] there is one particular parametrization which has served, in the quark sector, to provide a better connection to the flavor parameters [7, 9, 11],

$$\mathbf{V} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \,. \tag{7}$$

The Wolfenstein parametrization exploits the hierarchical structure of the mixing angles, $\theta^q_{ij} \ll 1$, and takes one of them, the Cabibbo angle, $\lambda \equiv \theta^q_{12} \approx 0.22$, as a mixing and expansion parameter along with other three real parameters A, ρ, and η of order $\mathcal{O}(1)$. This parametrization was improved in Ref. [9] in order to guarantee unitarity of the quark mixing matrix to all orders in λ.

The standard parametrization, for both quarks and leptons, as suggested by the Particle Data Group (PDG), follows Chau and Keung's proposal [8],

$$\mathbf{W} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{8}$$

where we have denoted $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. In the case of Majorana neutrinos a second matrix is introduced $\mathbf{U} = \mathbf{WK}$ where $\mathbf{K} = \mathrm{diag}(e^{i\alpha}, e^{i\beta}, 1)$. On the other hand, it has been shown that in the lepton sector a symmetrical parametrization first introduced by Schechter and Valle [6] and later revisited [12] gives a similar description but with an additional feature when considering Majorana neutrinos, which is, that the effective mass parameter characterizing the amplitude for neutrinoless double beta decay only depends, as it should, in the two Majorana phases, α and β, whereas the PDG parametrization also includes the Dirac phase.

We know the following theoretical facts about the mixing matrix: it is unitary, there is no unique parametrization, and it requires of four independent parameters. In this regard, among the different parametrizations, there is one property which must be shared among all of them, that is, the unitary description should be made by four real parameters and still provide one non-removable complex phase. To this end, an invariant measure of CP violation (CPV), independent of the parametrization, is the Jarlskog invariant [14],

$$J_X = \frac{\mathrm{Im}\left(\det\left[\mathbf{M}_a\mathbf{M}_a^\dagger, \mathbf{M}_b\mathbf{M}_b^\dagger\right]\right)}{-2\Pi_{i>j}(m_{a,i}^2 - m_{a,j}^2)\Pi_{k>l}(m_{b,k}^2 - m_{b,l}^2)} \ , \tag{9}$$

where $X = q, \ell$, $a = u, \nu$, and $b = d, e$.

Mixing parametrizations including mass ratios as mixing parameters have been explicitly proposed [10, 13, 15–20]. Nevertheless, only one of them has really gathered all the four independent fermion mass ratios as mixing parameters [13] in full agreement with the observed mixing phenomena. Here we consider the work of Ref. [13] and study both its implications and relations to the most used parametrizations. Moreover, we start by first discussing the reality of such instance by only assuming the masses as mixing parameters and examine the properties of the mixing matrices under them. As the reader will find out, our main result strongly suggests a connection between the mixing parameters and the corresponding fermion masses.

This work is organized as follows. In the next section, Section 2, we discuss the main features on the procedure to relate the masses of fermions to their mixing. Then, in Section 3, we assume the corresponding four independent mass ratios of each sector as the mixing parameters without any explicit realization and investigate all the implied consequences and emergent properties. Afterwards, in Section 4, we take the particular example of Ref. [13] and discuss further aspects in the implied theoretical mixing. In Section 5, we discuss the main problematic of the parametrization in Ref. [13] along with a manner to solve it. Finally, in Section 6, we conclude.

2 Preamble: The nature of the solution

How could fermion mixing be understood through the corresponding fermion masses? In the following, we provide some general theoretical observations on the nature of the answer to this question. Some of them are trivial but we shall not omit them for the sake of completeness. A discussion on similar and complementary criteria may be found in Ref. [21].

2.1 The matrix invariants are the key

The usual procedure to reparametrize comes from the matrix invariants which are the coefficients of the characteristic polynomial, $\det[\mathbf{M}\mathbf{M}^\dagger - \lambda\mathbf{I}] = 0$. In the n family case, the set of n invariants provides n equations which can be used, although not always easily, to write the matrix parameters in terms of the singular values (masses). For simplicity, the $n = 3$ case would be given by,

$$\lambda^3 - \mathrm{tr}[\mathbf{H}_f]\lambda^2 + \frac{1}{2}\left(\mathrm{tr}[\mathbf{H}_f]^2 - \mathrm{tr}[\mathbf{H}_f\mathbf{H}_f]\right)\lambda - \det[\mathbf{H}_f] = 0 \ , \tag{10}$$

where $\mathbf{H}_f = \mathbf{M}_f\mathbf{M}_f^\dagger$ is the hermitian product and the roots of the equation are the eigenvalues (squared masses) of $\mathbf{H}_f$.

The matrix invariants in terms of the masses are written as,

$$\mathrm{tr}[\mathbf{H}] = m_1^2 + m_2^2 + m_3^2 \ , \tag{11}$$

$$\det[\mathbf{H}] = m_1^2 m_2^2 m_3^2 \ , \tag{12}$$

$$\frac{1}{2}\left(\mathrm{tr}[\mathbf{H}]^2 - \mathrm{tr}[\mathbf{H}\mathbf{H}]\right) = m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2 \ , \tag{13}$$

where in these last expressions we have suppressed the subscript f as this applies to all fermions.

To illustrate this, let us consider the Weinberg ansatz [22],

$$\mathbf{m} = \begin{pmatrix} 0 & |a| \\ |a| & |b| \end{pmatrix} \quad \rightarrow \quad \begin{matrix} \mathrm{tr}[\mathbf{m}] = |b| = m_1 + m_2 \\ \det[\mathbf{m}] = -|a|^2 = m_1 m_2 \end{matrix} \quad \rightarrow \quad \begin{pmatrix} 0 & \sqrt{m_1 m_2} \\ \sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \ , \tag{14}$$

where we have considered without any loss of generality $m_1 \rightarrow -m_1$.[1] In return, we immediately obtain a relation between the angle of rotation and a mass ratio, $\tan\theta = \sqrt{m_1/m_2}$. In fact, this ansatz was made to reproduce the well known Gatto–Sartori–Tonin (GST) relation for the Cabibbo angle, $\theta_C \approx \sqrt{m_d/m_s}$ [23]. We must add another remark to this example, even though we have two different masses, we can always consider the largest mass as setting the scale of the matrix whereas the ratio with the lighter one a relevant parameter,

$$\mathbf{m} = m_2 \begin{pmatrix} 0 & \gamma \\ \gamma & 1-\gamma^2 \end{pmatrix} = m_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + m_2 \begin{pmatrix} 0 & \gamma \\ \gamma & -\gamma^2 \end{pmatrix} \ , \tag{15}$$

where we defined it as $\gamma \equiv \sqrt{\frac{m_1}{m_2}}$; it provides all the internal structure of the mass matrix. To see this consider a hierarchy in the masses $m_2 \gg m_1$. The smallness could point to a model where the lighter masses arise from radiative corrections, for example [24, 25].

2.2 The SM is not enough

Consider the n family case. We are asking ourselves how possible it is to reparametrize the initial mass matrix solely in terms of its singular values (masses). A complex $n \times n$ matrix has n^2 phases and n^2 magnitudes. By virtue of the n invariants, it is clear, that the proposed task is impossible. The system is underdetermined.

However, a further reduction of the arbitrariness is still possible if we recall that the kinetic terms per fermion sector posses a $[U(n)]^3$ due to universality of the gauge couplings. This accidental symmetry group describes the nature of the transformations leaving invariant the weak interaction basis.[2] We have at our disposal: $\frac{3n(n-1)}{2}$ and $\frac{3n(n+1)-2}{2}$ arbitrary magnitudes and complex phases, respectively, to choose whatever basis we require. As these transformations are involved in the two kinds of fermions of a given sector, we continue our counting by summing up all the parameters of the corresponding two mass matrices: $2n^2$ magnitudes and $2n^2$ complex phases. A careful choice of basis, with both mass matrices still not fully diagonal, would have $\frac{n(n+3)}{2}$ and $\frac{(n-1)(n-2)}{2}$ arbitrary parameters in magnitudes and complex phases, respectively. Reparametrization with the $2n$ invariants would still leave $(n-1)^2$ arbitrary parameters. In

[1] This change of sign can be easily achieved by a global chiral transformation.

[2] By weak interaction basis, we mean those bases where the weak interactions are diagonal in flavor space.

particular, for $n = 3$, this means that there are *special* bases where the mass matrices can be reexpressed in terms of its singular values plus four unknown *physical* parameters. In fact, notice how these parameters are equivalent to the mixing parameters appearing in the mass basis.

An example of such a basis would be the following couple of matrices,

$$\mathbf{M}_a = \begin{pmatrix} m_1^a & 0 & 0 \\ 0 & m_2^a & 0 \\ 0 & 0 & m_3^a \end{pmatrix} , \qquad \mathbf{M}_b = \begin{pmatrix} m_{11}^b & 0 & 0 \\ m_{21}^b & m_{22}^b e^{-i\delta} & 0 \\ m_{31}^b & m_{32}^b & m_{33}^b \end{pmatrix} , \tag{16}$$

where we have employed Ref. [26] to find such a basis. In this case, one finds that,

$$\begin{aligned} m_{33}^b &= m_3^b \, , \\ m_{11}^b m_{22}^b &= m_1^b m_2^b \, , \\ (m_{11}^b)^2 + (m_{22}^b)^2 + (m_{21}^b)^2 + (m_{31}^b)^2 + (m_{32}^b)^2 &= (m_1^b)^2 + (m_2^b)^2 \, . \end{aligned} \tag{17}$$

We reach the known conclusion that the SM framework is not enough to fully reparametrize the mass matrices in terms of only its singular values and that if such possibility really occurs in Nature then the theory will require extending it in a smart way. Of course extending the SM is already so very well motivated by many other facts like, for example, neutrino masses. So the importance of this conclusion is only valid within the present context.

2.3 Mixing parameters: physical but not necessarily independent

A long standing fact is that the mixing parameters besides being physical are also completely arbitrary. The former aspect is true, however, the second one should be left as soon as we have extended the theory and the number of parameters, in the special basis, is less or equal than the general arbitrariness $(n-1)^2$ contained in each fermion sector of the SM (considering n families). Take for example the left-right symmetric models in which the matrices are naturally hermitian and thus a significant reduction of the Yukawa parameters appears, see for example Refs. [1, 27, 28].

2.4 Full reparametrization is only possible for two or three families

Curiously enough the possibility of fully reparametrizing a mass matrix in terms of its singular values only occurs for two and three fermion generations [13]. Assume again n fermion generations. The number of independent mass ratios, $2(n-1)$, grows much slower than the number of mixing parameters, $(n-1)^2$. As a consequence, being able to completely reparametrize depends on the inequality $2(n-1) \geq (n-1)^2$ and thus $1 < n \leq 3$. Lucky we, that we live in a universe with three fermion families.

2.5 Pursue the minimal description

It is difficult to reconcile into a single description both mixing sectors when they share no similarities and their mixing angles largely depart the one from the other, see Appendix A for the present status of fermion mixing. Much easier is then to naively consider that their origins should be unconnected. However, following the generality principle, where if allowed it should be included, we see that both possibilities, i.e. equal and different origins, should have an equal footing from the theoretical viewpoint. In the end, the real criteria to distinguish which description is better is that which less assumptions and free parameters requires. In the following, we then pursue for such a minimal explanation.

2.6 Naturalness links mixing with the mass ratios

A small number is natural only if an exact symmetry emerges when it is set to zero. This is 't Hooft's criteria for naturalness [29]. Regarding it, in the weak interaction basis, the SM lagrangian acquires an exact symmetry when all Yukawa couplings are set to zero,

$$\mathcal{G}_F = U_L^Q(3) \times U_R^u(3) \times U_R^d(3) \times U_L^E(3) \times U_R^e(3) \;, \tag{18}$$

Thus, the smallness of the Yukawa couplings is natural. The top Yukawa coupling is of course not small compared to the rest of fermion masses but it should be small compared to a flavor scale, $\Lambda_F \gtrsim 1$ TeV, of a more fundamental theory. Of course, it is also possible to consider subsets of Yukawa couplings zero such that we have the intermediate steps,

$$\mathcal{G}_{\mathcal{F}} \xrightarrow{m_3} U(2)^5 \xrightarrow{m_2} U(1)^5 \xrightarrow{m_1} U(1)_B \times U(1)^3_{L_\alpha} \;, \tag{19}$$

as suggested from the hierarchical fermion masses, $m_3 \gg m_2 \gg m_1$, where $\alpha = e, \mu, \tau$.

The aforementioned argument requires a modification when instead of the Yukawa couplings we consider the mixing matrix elements together with the masses. It is possible to relate the previous naturalness of the small Yukawa couplings to the mixing matrix elements if mixing obeys equivalent limits such that when $m_1 \to 0$ or $m_1, m_2 \to 0$ either no mixing with the first family occurs or the charged current interactions become diagonal even if the third family is massive, respectively. Regarding this, small mixing angles could become explicitly natural if they are expressible in terms of the masses. A similar argumentation can be found in Ref. [30] however their remark appears more in the connection to a discrete flavor symmetry.

Furthermore, to properly understand, for example, the observed quark mixing, the correct connection is not with masses themselves but with their ratios. In this way, the hierarchical structure of the quark masses manifests in the hierarchical pattern of their mixing. One can already guess how the anarchical pattern in lepton mixing should be related to either a very mild hierarchy or anarchical structure of neutrino masses [31–33].

3 First approach: An arbitrary relation

Hereafter, we assume that a mixing parametrization depending *solely* on the masses exists and explore its properties and implications.

3.1 Four independent mass ratios

The first important realization we should have in mind is the fact that only four independent mass ratios exist and perfectly match the number of required mixing parameters. Our first result can then be written as,

$$\mathbf{V} = \mathbf{V}\left(\frac{m_u}{m_c}, \frac{m_c}{m_t}, \frac{m_d}{m_s}, \frac{m_s}{m_b}\right) \tag{20}$$

and[3]

$$\mathbf{U} = \mathbf{U}\left(\frac{m_e}{m_\mu}, \frac{m_\mu}{m_\tau}, \frac{m_{\nu 1}}{m_{\nu 2}}, \frac{m_{\nu 2}}{m_{\nu 3}}\right) \ . \tag{21}$$

In order to see the latter, we can start by thinking that three mass ratios smaller than one can be formed within a given fermion species, $m_i/m_j < 1$. However, one of the three can always be made with the other two either through a product or a ratio, for example,

$$\frac{m_1}{m_3} = \frac{m_1}{m_2}\left(\frac{m_2}{m_3}\right) \ . \tag{22}$$

3.2 The need for a special weak basis

The new given dependence on the mass ratios has an immediate consequence: both fermions within a sector must equally contribute to fermion mixing. This in return means the departing weak interaction basis should have all matrices as non-diagonal; and hence, the unitary transformations acting in the left-handed fields and diagonalizing the mass matrices,

$$\mathbf{L}_u \mathbf{M}_u \mathbf{M}_u^\dagger \mathbf{L}_u^\dagger = \mathbf{\Sigma}_u^2 \ , \qquad \mathbf{L}_d \mathbf{M}_d \mathbf{M}_d^\dagger \mathbf{L}_d^\dagger = \mathbf{\Sigma}_d^2 \ , \tag{23}$$

$$\mathbf{L}_e \mathbf{M}_e \mathbf{M}_e^\dagger \mathbf{L}_e^\dagger = \mathbf{\Sigma}_e^2 \ , \qquad \mathbf{L}_\nu \mathbf{M}_\nu \mathbf{M}_\nu^\dagger \mathbf{L}_\nu^\dagger = \mathbf{\Sigma}_\nu^2 \ , \tag{24}$$

could give the desired dependence on the four mass ratios,

$$\mathbf{V}\left(\frac{m_u}{m_c}, \frac{m_c}{m_t}, \frac{m_d}{m_s}, \frac{m_s}{m_b}\right) = \mathbf{L}_u\left(\frac{m_u}{m_c}, \frac{m_c}{m_t}\right)\mathbf{L}_d^\dagger\left(\frac{m_d}{m_s}, \frac{m_s}{m_b}\right) \ , \tag{25}$$

$$\mathbf{U}\left(\frac{m_e}{m_\mu}, \frac{m_\mu}{m_\tau}, \frac{m_{\nu 1}}{m_{\nu 2}}, \frac{m_{\nu 2}}{m_{\nu 3}}\right) = \mathbf{L}_e\left(\frac{m_e}{m_\mu}, \frac{m_\mu}{m_\tau}\right)\mathbf{L}_\nu^\dagger\left(\frac{m_{\nu 1}}{m_{\nu 2}}, \frac{m_{\nu 2}}{m_{\nu 3}}\right) \ . \tag{26}$$

But it *seems* we have reached some kind of inconsistency as we all know that there are an infinite amount of weak interaction bases and none of them is more preferable than other. The choice of basis should not matter, nevertheless, here we have something that seems to imply that. For such a principle or symmetry dictating the initial basis, see for example Ref. [34].

To cure this problematic we consider the minimal breaking of the maximal flavor symmetry

[3] For the sake of illustration let us assume for the moment a normal ordering in neutrino masses.

group,[4]

$$U(3)^3 \to U(2)^3 \to U(1)^3 \to U(1)_{B(L)} \tag{27}$$

in which the sequential breaking simultaneously occurring for the two components in a weak doublet and the right-handed (singlet) counterparts, sets the initial basis, see Refs. [35, 36]. From the bottom-up perspective this would be pointing out to a class of models which had this breaking in their framework and would give concrete realizations of the Principle of Minimal Flavor Violation [37–41], which basically states that all sources of New Physics producing flavor transitions should obey the SM flavor structure. In the following, we will assume this to set our basis.

A last feature to be aware of is that,

$$\mathbf{L}\left(\frac{m_1}{m_2}, \frac{m_2}{m_3}\right) = \mathbf{L}\left(\frac{m_1}{m_2}, \frac{m_2}{m_3}, \frac{m_1}{m_3}\right) , \tag{28}$$

that is, despite having an explicit dependence on only two mass ratios we can still think of all the unitary matrices as having three different rotation angles, for example,

$$\begin{aligned}\mathbf{L} = \begin{pmatrix} \cos\Theta_{12} & \sin\Theta_{12} e^{-i\delta_{12}} & 0 \\ -\sin\Theta_{12} e^{i\delta_{12}} & \cos\Theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\Theta_{13} & 0 & \sin\Theta_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -\sin\Theta_{13} e^{-i\delta_{13}} & 0 & \cos\Theta_{13} \end{pmatrix} \times \\ \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta_{23} & \sin\Theta_{23} e^{-i\delta_{23}} \\ 0 & -\sin\Theta_{23} e^{i\delta_{23}} & \cos\Theta_{23} \end{pmatrix} .\end{aligned} \tag{29}$$

3.3 Three different limits

The Singular Value Decomposition (SVD) of our hermitian matrix is given as,

$$\mathbf{H}_f = \sum_i m_i^2 \vec{v}_{f,i} \vec{v}_{f,i}^{\dagger} , \tag{30}$$

where $\vec{v}_{f,i}$ is an eigenvector by which $\mathbf{L}_f$ is built.

Three different limits can teach us important structural aspects of the parametrization and a way to extend the naturalness of the Yukawa sector to the fermion mixing matrices:

- $\mathbf{m_1}, \mathbf{m_2} \to \mathbf{0}$: From the SVD, when taking the mass of the lightest families equal to zero we get,

$$\mathbf{H}_f = m_3^2 \vec{v}_{f,3} \vec{v}_{f,3}^{\dagger} . \tag{31}$$

  In general, the normalized singular vector can be written in spherical coordinates as,

$$\vec{v}_{f,3} = \begin{pmatrix} \sin\Omega_f \sin\omega_f \\ \sin\Omega_f \cos\omega_f \\ \cos\Omega_f \end{pmatrix} , \tag{32}$$

[4]In order to have a symmetrical treatment we are assuming Dirac neutrinos. However, a $U(3)$ factor should be left out when considering Majorana neutrinos.

where the complex phases are been omitted without any loss of generality as we only care on the magnitudes themselves.

Therefore, generally speaking, having a single generation with mass can still imply fermion mixing, see Refs. [24, 25] where this idea is exploited within a two Higgs doublet model. Notice, however, that on one hand both angles should be related to the mass ratios while on the other as m_1 and m_2 are already zero there is no way to have an adimensional number made out of only m_3. Then, either both angles are zero or these vectors point in the same direction irrespective of the fermion type such that after a weak basis transformation we reach the important conclusion that a single massive fermion family cannot imply mixing,

$$\mathbf{V}(0,0,0,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{33}$$

and

$$\mathbf{U}(0,0,0,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{34}$$

- $\mathbf{m_1 \to 0}$: In this limit, the two heaviest families are left with a non-zero mass. Therefore, all ratios of the kind m_1/m_j with $j = 2, 3$ will be zero. This case is actually the sequel of the previous one. So we can ask what modifications should we do in our mixing parametrization such that m_2 is now taken into account? That is,

$$\mathbf{V}\left(0,0,\frac{m_c}{m_t},\frac{m_s}{m_b}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathbf{A}(\frac{m_c}{m_t},\frac{m_s}{m_b}) \ , \tag{35}$$

$$\mathbf{U}\left(0,0,\frac{m_\mu}{m_\tau},\frac{m_{\nu 2}}{m_{\nu 3}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathbf{A}'(\frac{m_\mu}{m_\tau},\frac{m_{\nu 2}}{m_{\nu 3}}) \ , \tag{36}$$

where $\mathbf{A}$ and $\mathbf{A}'$ are unknown matrices with the expected dependence on the new contributions. But how should these additions look like? To answer this we need to turn our attention to the mass matrices. From the previous case, we know that, generically, our departing point should then be,

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + m_2^2 \vec{v}_2 \vec{v}_2^\dagger \ , \tag{37}$$

where $\vec{v}_2$ is the corresponding singular vector to m_2. As before, a similar expression holds for $\vec{v}_2$,

$$\vec{v}_2 = \begin{pmatrix} \sin\Omega' \sin\omega' \\ \cos\Omega' \\ \sin\Omega' \cos\omega' \end{pmatrix} \ , \tag{38}$$

however, in this case, the good new is that we have the non-zero mass ratio m_2/m_3 to connect to. Therefore, we can relate both angles to a mass ratio. In general, for a very small angle,

$$\mathbf{H} \approx m_3^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\Omega \\ 0 & -\Omega & 1 \end{pmatrix} + m_2^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \Omega' \\ 0 & \Omega' & 0 \end{pmatrix} , \tag{39}$$

mixing only appears between the two heaviest generations. Here, we have also considered $\vec{v}_3$ to depend on $\frac{m_2}{m_3}$.

The way in which masses appear directly suggest the order of diagonalization required,

$$\mathbf{L}_f = \mathbf{L}_{12}\mathbf{L}_{13}\mathbf{L}_{23} , \tag{40}$$

so the first two transformations are the unit matrix when $m_1 \to 0$. This property can be achieved if $\mathbf{L}_{12}$ and $\mathbf{L}_{13}$ are homogeneous functions in m_1 up to an unknown degree.[5]

- $\mathbf{m_3} \to \infty$: Taking $m_3 \to \infty$ is not equivalent to the first case $m_1, m_2 \to 0$. From the functional dependence we see that we have been left with,

$$\mathbf{V}\left(\frac{m_u}{m_c}, 0, \frac{m_d}{m_s}, 0\right) \qquad \text{and} \qquad \mathbf{U}\left(\frac{m_e}{m_\mu}, 0, \frac{m_{\nu 1}}{m_{\nu 2}}, 0\right) . \tag{41}$$

The third family has decoupled from the first two and therefore this case corresponds to a unique mixing angle in the $1-2$ sector,

$$\mathbf{V}\left(\frac{m_u}{m_c}, 0, \frac{m_d}{m_s}, 0\right) = \begin{pmatrix} \cos\theta_{12}^q & \sin\theta_{12}^q & 0 \\ -\sin\theta_{12}^q & \cos\theta_{12}^q & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{42}$$

$$\mathbf{U}\left(\frac{m_e}{m_\mu}, 0, \frac{m_{\nu 1}}{m_{\nu 2}}, 0\right) = \begin{pmatrix} \cos\theta_{12}^\ell & \sin\theta_{12}^\ell & 0 \\ -\sin\theta_{12}^\ell & \cos\theta_{12}^\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{43}$$

From this limit, we deduce the condition that the mixing angles in the $1-3$ and $2-3$ sectors should be only proportional to the ratios m_1/m_3 and m_2/m_3 such that they can vanish whenever $m_3 \to \infty$.

Another feature which can be introduced within this case is to also ask what happens when comparing the two left ratios in Eqs. (42) and (43) one of them may be neglected? We can infer then that mixing in this sector should majorly be due to only one fermion type. For example, in the quark sector, we have $\frac{m_d}{m_s} \gg \frac{m_u}{m_c}$, so we should expect the dominant contribution to be mainly given by the down quark sector,

$$\mathbf{V}\left(0, 0, \frac{m_d}{m_s}, 0\right) \simeq \begin{pmatrix} \cos\Theta_{12}^d & \sin\Theta_{12}^d & 0 \\ -\sin\Theta_{12}^d & \cos\Theta_{12}^d & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{44}$$

[5] A function, $f(x)$, is said to be homogeneous with degree q if its argument is multiplied by a number t and this is equivalent to have multiplied the original function by t^q, $f(tx) = t^q f(x)$.

where Θ_{12}^d is the angle coming from $\mathbf{L}_d$.

3.4 The Cabibbo–Kobayashi–Maskawa matrix

Let us apply the previous limits and see what we can infer from this picture in which we have related mixing angles to mass ratios. Our only input is the phenomenological observation that all the quark masses fulfill the hierarchy,

$$m_3^2 \gg m_2^2 \gg m_1^2 \,, \tag{45}$$

together with the fact that the ratios coming from the up-quark sector are all negligible compared to the ones in the down sector, see Appendix B. Then, we are left with only two rotations $(\Theta_{23}^d, \Theta_{12}^d \ll 1)$,[6]

$$\begin{aligned}\mathbf{V}_{\mathrm{CKM}} &\simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{(\Theta_{23}^d)^2}{2} & -\Theta_{23}^d \\ 0 & \Theta_{23}^d & 1-\frac{(\Theta_{23}^d)^2}{2} \end{pmatrix} \begin{pmatrix} 1-\frac{(\Theta_{12}^d)^2}{2} & -\Theta_{12}^d & 0 \\ \Theta_{12}^d & 1-\frac{(\Theta_{12}^d)^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\simeq \begin{pmatrix} 1-\frac{(\Theta_{12}^d)^2}{2} & -\Theta_{12}^d & 0 \\ \Theta_{12}^d & 1-\frac{(\Theta_{12}^d)^2}{2} & -\Theta_{23}^d \\ \Theta_{23}^d\Theta_{12}^d & \Theta_{23}^d & 1-\frac{(\Theta_{23}^d)^2}{2} \end{pmatrix} \,,\end{aligned} \tag{46}$$

where we identify the mixing sum rules,

$$\Theta_{12}^d \simeq \theta_{12}^{\mathrm{CKM}} \qquad \Theta_{23}^d \simeq \theta_{23}^{\mathrm{CKM}} \,, \tag{47}$$

in agreement to the more general form given in [44]. We may now roughly understand how the hierarchy in the Cabibbo–Kobayashi–Maskawa (CKM) matrix is a direct consequence of the strong hierarchy in the quark masses. In this sense, the smallness of m_u/m_t and m_d/m_b compared to the other ratios, and the fact that $|V_{\mathrm{ub}}|$ is also observed to be the smallest element in the mixing matrix supports this conclusion.

3.5 The Pontecorvo–Maki–Nakagawa–Sakata matrix

For the sake of illustration, in the following we consider Dirac neutrinos with normal ordered masses. We will only assume the charged lepton masses as known parameters and look for any possible hint into the spectra of neutrino masses. Again, as in the quark sector, the charged lepton masses satisfy the same hierarchical pattern, $m_e^2 \ll m_\mu^2 \ll m_\tau^2$, see Appendix B. From the three ratios, the largest one is $m_\mu/m_\tau \sim 10^{-2}$. So we safely neglect the other two ratios and take the limit $m_e \to 0$. The Pontecorvo–Maki–Nakawa–Sakata (PMNS) matrix is then

[6] Realize that if one takes $\Theta_{23}^d \sim (\Theta_{12}^d)^2 \sim \lambda^2$ one can directly reproduce the well established Wolfenstein parametrization. Note that the $|V_{\mathrm{ub}}|$ element rather has a hierarchy of $\mathcal{O}(\lambda^4)$ instead of the conventional $\mathcal{O}(\lambda^3)$ so we can safely neglect it at this order [42, 43].

estimated as,

$$\mathbf{U}_{\rm PMNS} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \Theta_{23}^{\ell} \\ 0 & -\Theta_{23}^{\ell} & 1 \end{pmatrix} \begin{pmatrix} c_{12}^{\nu}c_{13}^{\nu} & -s_{12}^{\nu}c_{13}^{\nu} & -s_{13}^{\nu}e^{i\delta_{\rm CP}^{\nu}} \\ s_{12}^{\nu}c_{23}^{\nu} - c_{12}^{\nu}s_{23}^{\nu}s_{13}^{\nu}e^{-i\delta_{\rm CP}^{\nu}} & c_{12}^{\nu}c_{23}^{\nu} + s_{12}^{\nu}s_{23}^{\nu}s_{13}^{\nu}e^{-i\delta_{\rm CP}^{\nu}} & -s_{23}^{\nu}c_{13}^{\nu} \\ s_{12}^{\nu}s_{23}^{\nu} + c_{12}^{\nu}c_{23}^{\nu}s_{13}^{\nu}e^{-i\delta_{\rm CP}^{\nu}} & c_{12}^{\nu}s_{23}^{\nu} - s_{12}^{\nu}c_{23}^{\nu}s_{13}^{\nu}e^{-i\delta_{\rm CP}^{\nu}} & c_{23}^{\nu}c_{13}^{\nu} \end{pmatrix} . \tag{48}$$

The latter matrix product will only slightly modify the second and third rows of the unitary neutrino matrix. From which we can find the new mixing sum rules,

$$\tan\Theta_{12}^{\nu} \simeq \tan\theta_{12}^{\rm PMNS} , \quad \sin\Theta_{13}^{\nu} \simeq \sin\theta_{13}^{\rm PMNS} , \quad \frac{-\Theta_{23}^{\ell} + \tan\Theta_{23}^{\nu}}{1 + \Theta_{23}^{\ell}\tan\Theta_{23}^{\nu}} \simeq \tan\theta_{23}^{\rm PMNS} . \tag{49}$$

For more examples on mixing sum rules we refer the interested reader to Refs. [44–47].

Hence, from the observed values of the leptonic mixing matrix, $|\mathbf{U}_{\alpha k}| > |\mathbf{V}_{\rm us}|$ ($\alpha = e, \mu, \tau$, $k = 1, 2, 3$), it is evident that neutrino masses should follow a rather different pattern from the charged fermion ones.

3.5.1 $\mu - \tau$ reflection symmetry

Let us introduce a $\mu - \tau$ reflection symmetry in the neutrino sector [48], $\tan\theta_{23}^{\nu} = 1$. Through the known values, $|\mathbf{U}_{\mu 3}| = 0.656$ and $|\mathbf{U}_{\tau 3}| = 0.739$, we can estimate the contribution coming from the charged lepton matrix,

$$\Theta_{23}^{\ell} \simeq 0.059 . \tag{50}$$

Curiously enough, the same value may be reached through the ratio $\frac{m_\mu}{m_\tau} = 0.059$. This meaning that the reflection symmetry can be easily cured by adding a rotation equal to the previous ratio, $\Theta_{23}^{\ell} = \frac{m_\mu}{m_\tau}$. This could be seen as a first possibility on how individual mixing angles could be related to mass ratios.

3.6 CP violation

At this point, we need to understand how CP violation should be introduced. As we have already chosen the mass ratios as the four mixing parameters there is no room left to have a complex phase as an independent parameter. We then introduce by hand the assumption that it should suffice to limit all complex phases to take only one of the four possibilities, $\beta_{ij} \in \{0, \pi/2, \pi, 3\pi/2\}$,

$$\begin{aligned} \mathbf{L} = &\begin{pmatrix} \cos\Theta_{12} & \sin\Theta_{12}e^{-i\beta_{12}} & 0 \\ -\sin\Theta_{12}e^{i\beta_{12}} & \cos\Theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\Theta_{13} & 0 & \sin\Theta_{13}e^{-i\beta_{13}} \\ 0 & 1 & 0 \\ -\sin\Theta_{13}e^{i\beta_{13}} & 0 & \cos\Theta_{13} \end{pmatrix} \times \\ &\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta_{23} & \sin\Theta_{23}e^{-i\beta_{23}} \\ 0 & -\sin\Theta_{23}e^{i\beta_{23}} & \cos\Theta_{23} \end{pmatrix} , \end{aligned} \tag{51}$$

when bringing the mass matrices to diagonal form [13]. In fact, the nature of our ansatz is strictly related to works where their mass matrix elements are either real or purely imaginary [44, 49, 50].

We now show the statement that it is sufficient to keep one subsector of the mass matrices as purely imaginary and take the rest as real. It is already well known that only one phase is needed to produce CP violation, that is, from the nine complex phases each mass matrix has only one from the eighteen is really independent.[7] It can be shown by considering the following hermitian matrix for a given fermion kind,

$$\mathbf{H} = \mathbf{M}\mathbf{M}^\dagger = \begin{pmatrix} R_1 & ae^{i\beta_1} & be^{i\beta_2} \\ ae^{-i\beta_1} & R_2 & ce^{i\beta_3} \\ be^{-i\beta_2} & ce^{-i\beta_3} & R_3 \end{pmatrix} , \tag{52}$$

where R_1, R_2 and R_3 are real elements. We then compute the weak basis transformation $\mathbf{KHK}^\dagger$ where

$$\mathbf{K} = \begin{pmatrix} e^{-i\alpha_1} & 0 & 0 \\ 0 & e^{-i\alpha_2} & 0 \\ 0 & 0 & e^{-i\alpha_3} \end{pmatrix} . \tag{53}$$

After that we can see that the phases $\beta_{1,2,3}$ can be easily reabsorbed through a convenient choice of $\alpha_{1,2,3}$. That is, an orthogonal transformation is sufficient to diagonalize this mass matrix, i.e. all complex phases from a sector could be taken as zero. On the other hand, the other fermion type within the same sector has also three arbitrary phases, $\beta'_{1,2,3}$. Two of the three can be reabsorbed during the diagonalization process through phase redefinitions of the fields. Hence, one unique phase remains that we intentionally localize it in the 1-2 sector,

$$\mathbf{V} = \mathbf{L}^u_{12}\mathbf{L}^u_{13}\mathbf{L}^u_{23}(\mathbf{L}^d_{23})^\dagger(\mathbf{L}^d_{13})^\dagger(\mathbf{L}^d_{12}(\frac{\pi}{2} \text{ or } \frac{3\pi}{2}))^\dagger \tag{54}$$

$$\mathbf{U} = \mathbf{L}^e_{12}\mathbf{L}^e_{13}\mathbf{L}^e_{23}(\mathbf{L}^\nu_{23})^\dagger(\mathbf{L}^\nu_{13})^\dagger(\mathbf{L}^\nu_{12}(\frac{\pi}{2} \text{ or } \frac{3\pi}{2}))^\dagger . \tag{55}$$

Our mixing matrices will then be built as the product of five orthogonal matrices with a special unitary one. For last, a freedom is left for the sense of rotation in the orthogonal matrices, i.e. clockwise or counterclockwise.

Our choice for the phase in the 1-2 sector could also have been anticipated from the broken symmetry chain $U(3)^3 \xrightarrow{m_3} U(2)^3 \xrightarrow{m_2} U(1)^3 \xrightarrow{m_1} U(1)_{B(L)}$:

$$\begin{aligned} &\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{m_3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \blacksquare \end{pmatrix} \xrightarrow{m_2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare \end{pmatrix} \xrightarrow[\mathbf{L}_{23}]{} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \end{pmatrix} \xrightarrow{m_1} \\ &\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \xrightarrow[\mathbf{L}'_{23}]{} \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & 0 \\ \blacksquare & 0 & \blacksquare \end{pmatrix} \xrightarrow[\mathbf{L}_{13}]{} \begin{pmatrix} \blacksquare & \blacksquare & 0 \\ \blacksquare & \blacksquare & 0 \\ 0 & 0 & \blacksquare \end{pmatrix} \xrightarrow[\mathbf{L}_{12}]{} \begin{pmatrix} \blacksquare & 0 & 0 \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \end{pmatrix} \end{aligned} \tag{56}$$

[7] In fact, this complex phase is a linear combination of the nine complex phases, but without any loss of generality, one can put eight equal to zero.

in which the last part of the diagonalization lies in the 1-2 sector. Then, one can only keep a single independent phase located in the 1-2 sector of one fermion species within a given sector.

3.7 Relating angles to mass ratios

The rank of a matrix is strictly related to the number of non-zero singular values. It is through this property that we will see what does hierarchical masses imply in the properties of a matrix and extract the relation between the mass ratios and the mixing angles.

As we have noted, the full diagonalization occurs via rotations in three different two family subspaces. Thus, it is enough to study the relation between eigenvectors and eigenvalues in a two dimensional scenario. We denote the complex 2×2 matrix,

$$\mathbf{m} = \begin{pmatrix} m_{ss} & m_{sl} \\ m_{ls} & m_{ll} \end{pmatrix} , \tag{57}$$

where its four matrix elements are complex numbers and its SVD is given as,

$$\mathbf{m} = \mathbf{L}^\dagger \mathbf{\Sigma} \mathbf{R} , \tag{58}$$

with $\mathbf{\Sigma} = \mathrm{diag}(m_s, m_l)$, $m_i \geq 0$, $m_l > m_s$, and

$$\mathbf{L}\mathbf{m}\mathbf{m}^\dagger\mathbf{L}^\dagger = \mathbf{\Sigma}^2 , \qquad \mathbf{R}^\dagger\mathbf{m}^\dagger\mathbf{m}\mathbf{R} = \mathbf{\Sigma}^2 . \tag{59}$$

Its two invariants,

$$\mathrm{tr}\left(\mathbf{m}\mathbf{m}^\dagger\right) = m_s^2 + m_l^2 , \qquad \det\left(\mathbf{m}\mathbf{m}^\dagger\right) = m_s^2 m_l^2 . \tag{60}$$

In general, the left hermitian product is computed as,

$$\mathbf{m}\mathbf{m}^\dagger = \mathbf{L}^\dagger\mathbf{\Sigma}^2\mathbf{L} = \begin{pmatrix} m_s^2 c^2 + m_l^2 s^2 & e^{-i\delta} cs(m_s^2 - m_l^2) \\ e^{i\delta} cs(m_s^2 - m_l^2) & m_s^2 s^2 + m_l^2 c^2 \end{pmatrix} , \tag{61}$$

where we have defined $c = \cos\Theta$ and $s = \sin\Theta$. We will now consider that due to $m_l \gg m_s$ we can take the matrix as a rank one, i.e. $m_s = 0$. This implies,

$$\mathbf{m}\mathbf{m}^\dagger \approx \begin{pmatrix} m_l^2 s^2 & -m_l^2 cse^{-i\delta} \\ -m_l^2 cse^{i\delta} & m_l^2 c^2 \end{pmatrix} = m_l^2 \begin{pmatrix} s^2 & -cse^{-i\delta} \\ -cse^{i\delta} & c^2 \end{pmatrix} , \tag{62}$$

whose invariants are now $\mathrm{tr}(\mathbf{m}\mathbf{m}^\dagger) = m_l^2$ and $\det(\mathbf{m}\mathbf{m}^\dagger) = 0$.

As already discussed, whenever $m_s = 0$ we should also have $\Theta = 0$. In other words, if we are trying to build a unitary transformation whose angle of rotation is given by a ratio of the masses then we should expect $\mathbf{L}\left(0\right) = \mathbf{I}_{2\times2}$ or for small ratios, $m_s \ll m_l$, to first order,

$$\mathbf{L}\left(\frac{m_s}{m_l}, \delta\right) \approx \frac{1}{\sqrt{1+\Theta^2}} \begin{pmatrix} 1 - \frac{\Theta^2}{2} & \Theta e^{-i\delta} \\ -\Theta e^{i\delta} & 1 - \frac{\Theta^2}{2} \end{pmatrix} , \tag{63}$$

where the relation between Θ and the mass ratio is to be found next.

After introducing m_s we get,

$$\frac{\mathbf{mm}^\dagger}{m_l^2} \approx \frac{1}{1+\Theta^2}\begin{pmatrix} \frac{m_s^2}{m_l^2}+\Theta^2 & e^{-i\delta}\Theta(\frac{m_s^2}{m_l^2}-1) \\ e^{i\delta}\Theta(\frac{m_s^2}{m_l^2}-1) & 1+\frac{m_s^2}{m_l^2}\Theta^2 \end{pmatrix}, \tag{64}$$

and where we have $\Theta = \Theta(\frac{m_s}{m_l})$. Notice how all the matrix elements won contributions related to m_s. This is important as we need to keep track of this effect.

We now need to establish a functional relation between the angle and the mass ratio. For that, we want Θ to behave as $\Theta \to 0$ whenever either $m_s \to 0$ or $m_l \to \infty$. Thus, let us consider the following simple kind of relation which behaves just as needed,

$$\Theta \sim \left(\frac{m_s}{m_l}\right)^n. \tag{65}$$

After substitution we obtain,

$$\frac{\mathbf{mm}^\dagger}{m_l^2} \approx \frac{1}{1+\Theta^2}\begin{pmatrix} \Theta^{2/n}+\Theta^2 & e^{-i\delta}\Theta(\Theta^{2/n}-1) \\ e^{i\delta}\Theta(\Theta^{2/n}-1) & 1+\Theta^{2(n+1)/n} \end{pmatrix}. \tag{66}$$

For $n > 1$ ($n \in \mathcal{Z}$), then,

$$\frac{\mathbf{mm}^\dagger}{m_l^2} \approx \begin{pmatrix} \Theta^{2/n} & -e^{-i\delta}\Theta \\ -e^{i\delta}\Theta & 1+\Theta^{2(n+1)/n} \end{pmatrix}. \tag{67}$$

For example, consider the particular case of $n = 2$,

$$\frac{\mathbf{mm}^\dagger}{m_l^2} \approx \begin{pmatrix} \Theta & -e^{-i\delta}\Theta \\ -e^{i\delta}\Theta & 1 \end{pmatrix}. \tag{68}$$

On the other hand, for $n = 1$, we have,

$$\frac{\mathbf{mm}^\dagger}{m_l^2} \sim \frac{1}{1+\Theta^2}\begin{pmatrix} \Theta^2 & -e^{-i\delta}\Theta \\ -e^{i\delta}\Theta & 1 \end{pmatrix}. \tag{69}$$

Last, for the case $\Theta \sim (\frac{m_s}{m_l})^{1/n}$ $(n > 1)$, we find,

$$\frac{\mathbf{mm}^\dagger}{m_l^2} \approx \frac{1}{1+\Theta^2}\begin{pmatrix} \Theta^{2n}+\Theta^2 & e^{-i\delta}\Theta(\Theta^{2n}-1) \\ e^{i\delta}\Theta(\Theta^{2n}-1) & 1+\Theta^{2(n+1)} \end{pmatrix}. \tag{70}$$

which, in general, can be approximated as,

$$\frac{\mathbf{mm}^\dagger}{m_l^2} \approx \frac{1}{1+\Theta^2}\begin{pmatrix} \Theta^2 & -e^{-i\delta}\Theta \\ -e^{i\delta}\Theta & 1 \end{pmatrix}. \tag{71}$$

Then, we arrive at the following conclusion. *Similar* hermitian mass matrix structures can have *different* relations between the angle and the mass ratio. For example, for the linear relation, $\Theta \sim m_s/m_l$, and the one given by $\Theta \sim (m_s/m_l)^{1/n}$, we have the same left hermitian matrix.

To clarify this ambiguity we consider, in the quark sector, the limit $m_{b,t} \to \infty$ and only study a two family mixing scenario,

$$|V_{us}| = \sqrt{\frac{(\frac{m_u}{m_c})^{2n} + (\frac{m_d}{m_s})^{2n} - 2(\frac{m_u}{m_c})^n(\frac{m_d}{m_s})^n \cos(\delta_u - \delta_d)}{\left(1 + (\frac{m_u}{m_c})^{2n}\right)\left(1 + (\frac{m_d}{m_s})^{2n}\right)}} \,. \tag{72}$$

Figure 1 shows the allowed range for Cabibbo(-like) mixing, due to the given mass ratios m_u/m_c ($m_{\nu 1}/m_{\nu 2}$) and m_d/m_s (m_e/m_μ), see Table 2 (Table 3), in the quark (lepton) sector. Two main observations are now given: i) Not any pair $(\delta_u - \delta_d, n)$ is able to explain mixing. In fact, for quark mixing close to the value $n \simeq 0.55$ and $n \simeq 0.6$ for the lepton sector, there is no chance to reproduce the amount of observed mixing and ii) For very small values of n and a certain choice of the phase difference one could reproduce any observed value for fermion mixing, however, the exploited hierarchy in all the previous discussion would be lost and so its applicability. The best scenario, of course, is that in which the same relation holds in both the quark and lepton sectors.

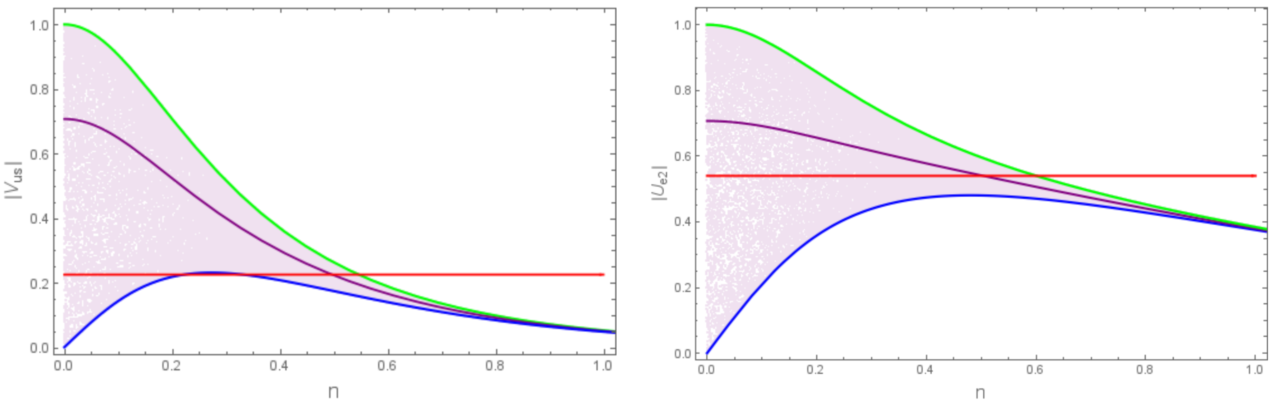

Figure 1: The left plot shows the allowed range for Cabibbo mixing, due to the given mass ratios m_u/m_c and m_d/m_s. We have allowed the complex phases to take any possible value from $[0, 2\pi)$ which is the cause for the filling in between the extremum lines. The bottom (blue) line corresponds to a phase difference equal to zero, the upper low opacity (green) line to a phase difference equal to π, and the medium (purple) line to a phase difference equal to $\pi/2$. The horizontal (red) line corresponds to the observed value. The right plot shows a similar situation but for a given hypothetical value for the mass ratio of the two lightest neutrino masses (considering normal ordering).

4 Second approach: A particular implementation

The most known relation among an angle and a mass ratio is the GST relation [23], $\theta_C \sim (m_d/m_s)^{1/2}$, and it is in a way related to the Cheng–Sher ansatz in multi-Higgs models, $\mathbf{M}_{ij} \sim \sqrt{m_i m_j}$, which is normally introduced to help suppress flavor changing neutral transitions thanks to the hierarchy in the fermion masses [51]. For a recent discussion of this relation and its possible symmetrical origin please refer to [34].

In the following, we revisit the parametrization proposed in Ref. [13]. Its basic idea is to exploit the phenomenological observation of hierarchical fermion masses, $m_3^2 \gg m_2^2 \gg m_1^2$, together with the matrix properties related to approximations. In this sense, the low-rank approximation theorem helps to build a systematic approach with a strong control on the errors made by neglecting at different stages all contributions proportional to either m_1 or m_1 and m_2. Moreover, the implementation of this idea can only be made through the assumption of Minimal Flavor Violation, as it is required to set the initial weak basis.

4.1 The low-rank approximation theorem

For a better understanding of the second approach is worth studying first the low rank approximation theorem [52–55]. The goal of the low-rank approximation theorem is to approximate as close as possible a given matrix $\mathbf{S}$ with rank r with another matrix $\widehat{\mathbf{S}}$ of lower rank $(r-p) \leq r$. Its results states that the closest matrix is the initial matrix, $\mathbf{S}$, with its p smallest singular values replaced by zeros,

$$||\mathbf{S} - \hat{\mathbf{S}}||_X \geq ||\mathbf{S} - \mathbf{S}(\{\sigma_1, ..., \sigma_p\} = 0)||_X \ , \tag{73}$$

where we have denoted by σ_i the singular values of $\mathbf{S}$ and ordered them as $\sigma_r \geq \sigma_{r-1} \geq \cdots \gg \sigma_p \geq \cdots \geq \sigma_1 > 0$, for any given norm $||\mathbf{A}||_X$.

The proof can be *sketched* as follows. Let us define the singular value decomposition of our initial matrix as,

$$\mathbf{S} = \mathbf{L}^\dagger \mathbf{D} \mathbf{R} \tag{74}$$

where $\mathbf{D}$ is a diagonal matrix with their elements ordered as previously indicated and we assume non-degeneracy. We only treat here the square complex case, $\mathbf{S}_{r\times r}$. $\mathbf{L}$ and $\mathbf{R}$ are the left and right singular unitary matrices, respectively. There is a one to one correspondence between the non-degenerate singular values and the corresponding singular vectors, up to a different phase which must be shared between $\vec{\ell}_i$ and $\vec{r}_i$. Recall that the rank of a matrix is defined as the number of non-zero singular values. For our purpose we introduce the Frobenius norm,

$$||\mathbf{A}||_F = \sqrt{\mathrm{tr}[\mathbf{A}\mathbf{A}^\dagger]} = \sqrt{\sum_{i=1}^{\mathrm{rank}(\mathbf{A})} a_i^2} \ . \tag{75}$$

Now, take $\hat{\mathbf{S}}$ as a matrix of lower rank $(r-p) \leq r$ with its singular value decomposition given as,

$$\hat{\mathbf{S}} = \mathbf{L}'^\dagger \mathbf{D}' \mathbf{R}' \ , \tag{76}$$

where we denote by λ_i its singular values and assume them ordered, $\lambda_{r-p} > \cdots > \lambda_1 > 0$.

Then, we have,

$$
\begin{aligned}
||\mathbf{S}-\hat{\mathbf{S}}||_F &= \sqrt{\operatorname{tr}\left[\left(\mathbf{S}-\hat{\mathbf{S}}\right)\left(\mathbf{S}^\dagger-\hat{\mathbf{S}}^\dagger\right)\right]}\ , \\
&= \sqrt{\operatorname{tr}\left[\mathbf{S}\mathbf{S}^\dagger\right]-\operatorname{tr}\left[\mathbf{S}\hat{\mathbf{S}}^\dagger\right]-\operatorname{tr}\left[\mathbf{S}^\dagger\hat{\mathbf{S}}\right]+\operatorname{tr}\left[\hat{\mathbf{S}}\hat{\mathbf{S}}^\dagger\right]}\ , \\
&= \sqrt{\sum_{i=1}^{p}\sigma_i^2+\sum_{i=1}^{r-p}\left(\sigma_{p+i}^2+\lambda_i^2\right)-\operatorname{tr}\left[\mathbf{S}\hat{\mathbf{S}}^\dagger\right]-\operatorname{tr}\left[\mathbf{S}^\dagger\hat{\mathbf{S}}\right]}\ .
\end{aligned}
\tag{77}
$$

In general, the upper value of the traces, $\operatorname{tr}\left[\mathbf{S}^\dagger\hat{\mathbf{S}}\right]$ and $\operatorname{tr}\left[\mathbf{S}\hat{\mathbf{S}}^\dagger\right]$, can be associated with that in which both matrices can be simultaneously diagonalized by the same biunitary transformation. This can be understood via the Cauchy–Schwarz inequality, $|\langle\vec{u},\vec{v}\rangle| \leq ||\vec{u}||||\vec{v}|| = 1$, among the pair of products of the different left and right singular vectors. Therefore,

$$
||\mathbf{S}-\hat{\mathbf{S}}||_F = \sqrt{\sum_{i=1}^{p}\sigma_i^2+\sum_{i=1}^{r-p}\left(\sigma_{p+i}-\lambda_i\right)^2}\ . \tag{78}
$$

where we have used Von Neumann's trace inequality [56, 57],

$$
|\operatorname{tr}[\mathbf{A}\mathbf{B}]| \leq \sum_{i=1}^{n} a_i b_i\ . \tag{79}
$$

Hence, we conclude that if $\lambda_i = \sigma_{(p+i)}$ $(i = 1, ..., r-p)$ we reach the bottom limit and thus we get,

$$
||\mathbf{S}-\hat{\mathbf{S}}||_F \geq \sqrt{\sum_{i=1}^{p}\sigma_i^2}\ , \tag{80}
$$

where we now know the equality is only satisfied when the p smallest singular values of the original matrix are set to zero, $\hat{\mathbf{S}} = \mathbf{S}(\{\sigma_1, ..., \sigma_p\} = 0)$.

The importance of using this theorem lies in the fact that we wish to connect the properties of the mixing matrices to the ones in the mass matrices under the limits $m_1 \to 0$ and $m_{1,2} \to 0$. This theorem guarantees that there will be no loss of generality within the approach and thus we can trust our deductions.

4.2 Building the parametrization

Through the approach discussed in Section 3 we can now easily work our way to reproduce a first example of such a parametrization as found in Ref. [13]. Although we will be a little bit redundant we will repeat many of the previous arguments.

Our first step, as we have many times emphasized, is the idea of having a mixing parametrization in terms of the mass ratios,

$$
\mathbf{V} = \mathbf{V}\left(\frac{m_u}{m_c}, \frac{m_c}{m_t}, \frac{m_d}{m_s}, \frac{m_s}{m_b}\right) = \mathbf{L}_u\mathbf{L}_d^\dagger\ , \tag{81}
$$

and

$$\mathbf{U} = \mathbf{U}(\frac{m_e}{m_\mu}, \frac{m_\mu}{m_\tau}, \frac{m_{\nu 1}}{m_{\nu 2}}, \frac{m_{\nu 2}}{m_{\nu 3}}) = \mathbf{L}_e \mathbf{L}_\nu^\dagger \ . \tag{82}$$

The first implication of such functional dependence is the following ansatz: complex phases δ can only take one of the four possible discrete values $\delta = 0, \pi/2, \pi,$ and $3\pi/2$. This ansatz has shown to be sufficient to obtain an excellent agreement to the most recent global fits [13].

The next step is to break the arbitrariness in the left hermitian products of the mass matrices,

$$\mathbf{L}\mathbf{M}\mathbf{M}^\dagger\mathbf{L}^\dagger = \mathrm{diag}(m_1^2, m_2^2, m_3^2) \ , \tag{83}$$

and find a way to express $\mathbf{L}$ in terms of the masses.

Fortunately, in spite of this situation, one can work an approximate solution via the low-rank approximation theorem. The mass spectra of all charged fermion species satisfy the double mass hierarchy pattern, $m_1^2 \ll m_2^2 \ll m_3^2$, which perfectly fits our problem and provide a way to study the mass matrices as either rank one, two, or three. Neutrino masses do not stay behind as their squared mass differences also satisfy an inequality $\Delta m_{21}^2 \ll \Delta m_{31}^2$. Moreover, the cosmological limit on the sum of their masses, $\sum_i m_{\nu i} < 0.23$ eV, favors the hierarchical case over quasidegeneration [58]. Regarding different examples on how to produce hierarchical masses we refer the interested reader to the Refs. [24, 25, 59–62].

The parametrization is constructed by a series of successive three by three unitary rotations in the three different two family planes. The transition from rank one to rank two implies rotating first the $2-3$ sector. Thereafter, two more rotations in this sector are introduced, as an ansatz, in order to consider possible contributions proportional to the mass of the first family which was neglected at this point. Then, the $1-3$ sector is rotated and followed by two more rotations proportional to the mass of the second family, contributions neglected in the previous rotation. Finally, the $1-2$ sector is the only one left and needs no further rotations as no mass is neglected at this step. This sequence of steps is written as,

$$\mathbf{L} = \mathbf{L}_{12}\mathbf{L}_{13}\mathbf{L}_{23} \ , \tag{84}$$

with,

$$\mathbf{L}_{23} = \mathbf{L}_{23}^{(2)}\left(\frac{m_1 m_2}{m_3^2}, \delta_{23}^{(2)}\right)\mathbf{L}_{23}^{(1)}\left(\frac{m_1}{m_3}, \delta_{23}^{(1)}\right)\mathbf{L}_{23}^{(0)}\left(\frac{m_2}{m_3}, \delta_{23}^{(0)}\right), \tag{85}$$

$$\mathbf{L}_{13} = \mathbf{L}_{13}^{(2)}\left(\frac{m_1 m_2}{m_3^2}, \delta_{13}^{(2)}\right)\mathbf{L}_{13}^{(1)}\left(\frac{m_2^2}{m_3^2}, \delta_{13}^{(1)}\right)\mathbf{L}_{13}^{(0)}\left(\frac{m_1}{m_3}, \delta_{13}^{(0)}\right), \tag{86}$$

$$\mathbf{L}_{12} = \mathbf{L}_{12}^{(0)}\left(\frac{m_1}{m_2}, \delta_{12}^{(0)}\right). \tag{87}$$

Each of the latter rotations has a two by two submatrix of the kind,

$$\mathbf{L}\left(\frac{m_i}{m_j}, \delta_{ij}^{(k)}\right) = \begin{pmatrix} \cos\Theta_{ij} & e^{-i\delta_{ij}^{(k)}}\sin\Theta_{ij} \\ -e^{i\delta_{ij}^{(k)}}\sin\Theta_{ij} & \cos\Theta_{ij} \end{pmatrix}, \tag{88}$$

where the angle of rotation is chosen to satisfy the GST relation, $\tan^2\Theta_{ij} = m_i/m_j$. For example,

$$\mathbf{L}_{12}^{(0)}\left(\frac{m_1}{m_2}, \delta_{12}^{(0)}\right) = \begin{pmatrix} \cos\Theta_{12} & e^{-i\delta_{12}^{(0)}}\sin\Theta_{12} & 0 \\ -e^{i\delta_{12}^{(0)}}\sin\Theta_{12} & \cos\Theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{89}$$

with $\tan^2\Theta_{12} = m_1/m_2$.

4.2.1 Complex phases, mixing, and CP violation

In total, either in the quark or lepton sector, fourteen complex phases has appeared at this point, seven for each fermion type. The study of their effect can be briefly summarized into this:

1. Complex phases appear always as a difference between two of them, $\delta^a_{ij} - \delta^b_{ij}$, one for each fermion species,
2. the latter implies that we only need to use seven of them per fermion sector with the rest equal to zero, $\delta^a_{ij} = 0$.
3. The study of two fermion families mixing helps us to introduce two kinds of mixing: minimal ($\delta^b_{ij} = 0$) or maximal ($\delta^b_{ij} = \pi$), as it can be seen from,

$$\tan^2\theta_{ij} = \frac{\hat{m}^a_{ij} + \hat{m}^b_{ij} - 2\sqrt{\hat{m}^a_{ij}\hat{m}^b_{ij}}\cos(\delta^a_{ij} - \delta^b_{ij})}{1 + \hat{m}^a_{ij}\hat{m}^b_{ij} + 2\sqrt{\hat{m}^a_{ij}\hat{m}^b_{ij}}\cos(\delta^a_{ij} - \delta^b_{ij})} = \begin{cases} \delta^b_{ij} = 0, & \left[\frac{\sqrt{\hat{m}^a_{ij}} - \sqrt{\hat{m}^b_{ij}}}{1 + \sqrt{\hat{m}^a_{ij}\hat{m}^b_{ij}}}\right]^2 \\ \delta^b_{ij} = \pi, & \left[\frac{\sqrt{\hat{m}^a_{ij}} + \sqrt{\hat{m}^b_{ij}}}{1 - \sqrt{\hat{m}^a_{ij}\hat{m}^b_{ij}}}\right]^2 \end{cases} \tag{90}$$

   where $\hat{m}^f_{ij} = m_i/m_j$ with $m_j > m_i$ ($j > i$) and where in the last step we took $\delta^a_{ij} = 0$.
4. For the two other cases, $\delta^b_{ij} = \pi/2$ or $3\pi/2$, one finds that the mixing angle has the same output,

$$\tan^2\theta_{ij} = \frac{\hat{m}^a_{ij} + \hat{m}^b_{ij}}{1 + \hat{m}^a_{ij}\hat{m}^b_{ij}}, \tag{91}$$

   whereas the Jarlskog invariant changes its sign from positive to negative, respectively.

Moreover, one finds out when defining the concepts of minimal or maximal mixing that three complex phases are enough to determine the kind of mixing, the other four will automatically

	δ_{12}	$\delta_{13}^{(0)}$	$\delta_{13}^{(1)}$	$\delta_{13}^{(2)}$	$\delta_{23}^{(0)}$	$\delta_{23}^{(1)}$	$\delta_{23}^{(2)}$
Quarks	$\pi/2$	0	π	π	0	π	π
Leptons	$3\pi/2$	0	π	π	π	π	0

Table 1: The choice of phases in Eq. (85) leading to the mixing matrices shown in Eqs. (94) and (118). Their value can be understood through the concepts of minimal, maximal, and CP violating mixing as shown in Eqs. (92) and (93).

follow them; so our unitary rotations can be written as,

$$\begin{aligned}
\mathbf{L}_{23}^a &= \mathbf{L}_{23}^{(2)}\left(\frac{m_{a,1}m_{a,2}}{m_{a,3}^2},0\right)\mathbf{L}_{23}^{(1)}\left(\frac{m_{a,1}}{m_{a,3}},0\right)\mathbf{L}_{23}^{(0)}\left(\frac{m_{a,2}}{m_{a,3}},0\right),\\
\mathbf{L}_{13}^a &= \mathbf{L}_{13}^{(2)}\left(\frac{m_{a,1}m_{a,2}}{m_{a,3}^2},0\right)\mathbf{L}_{13}^{(1)}\left(\frac{m_{a,2}^2}{m_{a,3}^2},0\right)\mathbf{L}_{13}^{(0)}\left(\frac{m_{a,1}}{m_{a,3}},0\right),\\
\mathbf{L}_{12}^a &= \mathbf{L}_{12}^{(0)}\left(\frac{m_{a,1}}{m_{a,2}},0\right),
\end{aligned} \tag{92}$$

and

$$\begin{aligned}
\mathbf{L}_{23}^b &= \mathbf{L}_{23}^{(2)}\left(\frac{m_{b,1}m_{b,2}}{m_{b,3}^2},\delta_{23}^b-\pi\right)\mathbf{L}_{23}^{(1)}\left(\frac{m_{b,1}}{m_{b,3}},\delta_{23}^b-\pi\right)\mathbf{L}_{23}^{(0)}\left(\frac{m_{b,2}}{m_{b,3}},\delta_{23}^b\right),\\
\mathbf{L}_{13}^b &= \mathbf{L}_{13}^{(2)}\left(\frac{m_{b,1}m_{b,2}}{m_{b,3}^2},\delta_{13}^b-\pi\right)\mathbf{L}_{13}^{(1)}\left(\frac{m_{b,2}^2}{m_{b,3}^2},\delta_{13}^b-\pi\right)\mathbf{L}_{13}^{(0)}\left(\frac{m_{b,1}}{m_{b,3}},\delta_{13}^b\right),\\
\mathbf{L}_{12}^b &= \mathbf{L}_{12}^{(0)}\left(\frac{m_{b,1}}{m_{b,2}},\delta_{12}^b\right),
\end{aligned} \tag{93}$$

where $\mathbf{V}_F = \mathbf{L}_a\mathbf{L}_b^\dagger$ and $\mathbf{L}_f = \mathbf{L}_{12}^f\mathbf{L}_{13}^f\mathbf{L}_{23}^f$ with $F = q,\ell$, $a = u,e$, $b = d,\nu$, and $f = a,b$. Four possible values for each complex phase give $4^3 = 64$ possible combinations. Nevertheless, due to the fact that we have, in general, $m_3 \gg m_{1,2}$ all mass matrices can be approximated as rank one matrices implying an approximate $U(2)^3$ accidental global symmetry in the kinetic terms. Due to this property, we expect that only in the $1-2$ family sector our complex phases will take one of the two values $\delta_{ij} = \pi/2$ or $3\pi/2$ whereas in the $2-3$ and $1-3$ sectors $\delta_{ij} = 0$ or π. This, then, restrains our combinations to $2^3 = 8$ possible cases.

Only one of the eight possibilities gives an agreement to the most recent global fits and is shown in Table 1. One finds that in the quark sector mixing phenomena is produced as two initial minimal mixings with $\delta_{23}^d = \delta_{13}^d = 0$ and in order to have a positive Jarlskog invariant we should choose $\delta_{12}^d = \pi/2$. On the other hand, as we will shortly see in Section 4.5, leptonic mixing can be produced with maximal mixing in the $2-3$ sector, $\delta_{23}^\nu = \pi$, minimal mixing in the $1-3$ sector, $\delta_{13}^\nu = 0$, and in order to have a negative Jarlskog invariant the value of the complex phase should be $\delta_{12}^\nu = 3\pi/2$. We need to stress here that the leptonic mixing has an unresolved feature which is that in the $2-3$ sector the phase appearing in $\mathbf{L}_{23}^{(1)}(\frac{m_{\nu 1}}{m_{\nu 3}})$ is π instead of zero. Nevertheless, until the precision in the neutrino mass squared differences and the leptonic mixing angles do not improve it is an impossible task to try to solve it.

Usually, against the fact of having fourteen complex phases, people is lead to consider that these relations between mixing angles and mass ratios could just generate any desirable mixing value under the right choice of complex phases. However, it is now our purpose to show that given some range of mass ratios the mixing angles are restricted to be inside a region as large as the order of the largest square root of a mass ratio. That is, the quark sector will then be much more limited than the lepton one. Figure 2 shows how the possibilities of different mixing values get reduced by restricting the value complex phases may take. If complex phases were allowed to take on any value $[0, 2\pi)$ one obtains the (blue) background. Next, if we impose on the complex phases the ansatz to take only one of the four possible values, $0, \pi/2, \pi,$ and $3\pi/2$, the plots transform into the (yellow) small extended regions. For last, when introducing the concepts of CP conserving with minimal or maximal mixing and CP violating mixing then the plots become the (red) solitary dots.

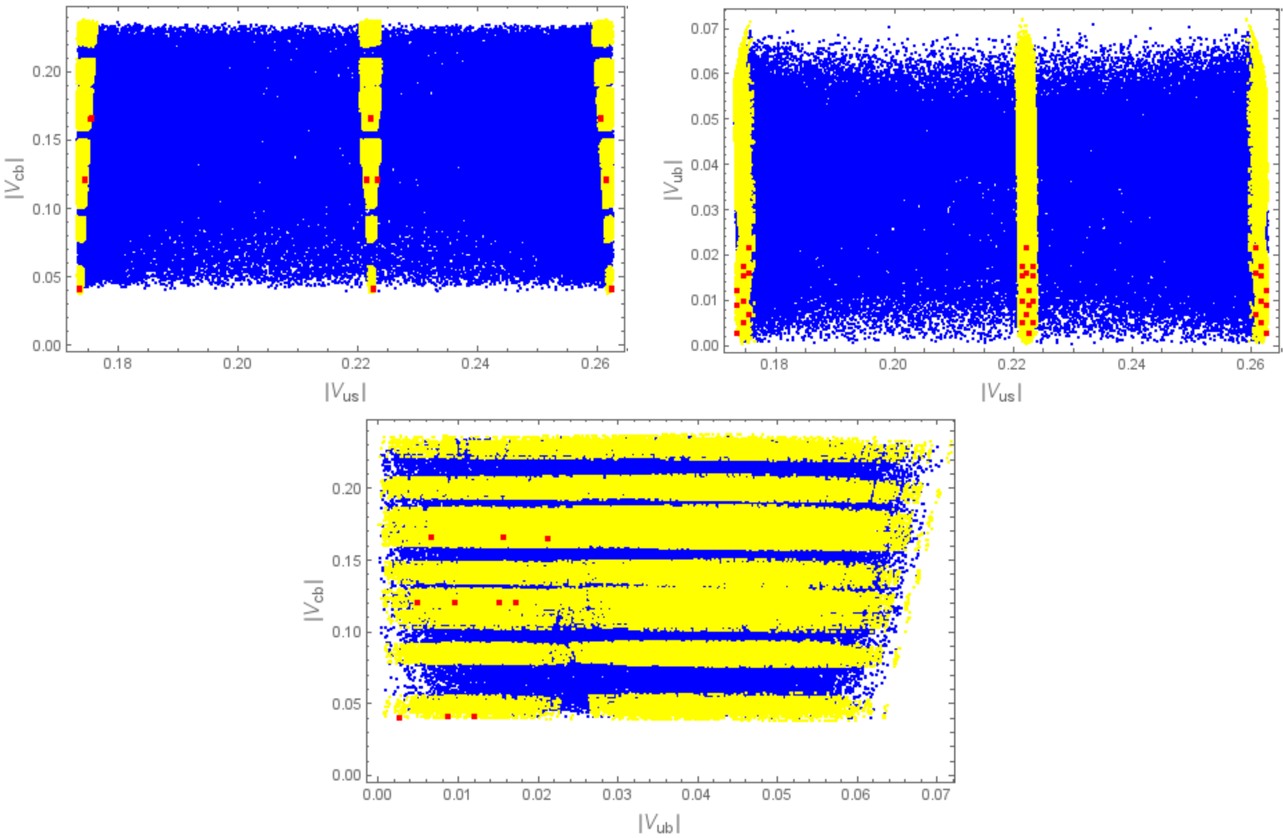

Figure 2: Allowed regions for the quark mixing matrix elements due to the four quark mass ratios. The main assumptions behind this particular parametrization are: Minimal Flavor Violation, all rotations have a GST-like relation [23], and that complex phases have only four possible values. To show the effect of the complex phases we consider them under three different scenarios. The background (blue) points refer to considering the fourteen complex phases as taking any given value, $\delta \in [0, 2\pi)$. On the other hand, the small extended regions (yellow) show the possible mixing values when considering the fourteen complex phases assuming only one of the four cases $0, \pi/2, \pi$ and $3\pi/2$. Already here one finds out that it becomes quite limited, specially for the two upper plots. For last, the solitary (red) square points, refer to the introduced ansatz where one has considered seven null phases while the rest in relation to the concepts of minimal, maximal, and CP violating mixing.

4.3 Quark mixing angles

To calculate the theoretical mixing matrix we need to run all quark masses to the M_Z scale, see Appendix B. We do this via the `RunDec 3.0` package [63], which considers five-loop corrections to the QCD beta functions and four-loop decoupling effects. Thereafter through the substitution of the mass values the following theoretical values of the CKM matrix are obtained as,

$$|\mathbf{V}^{\mathrm{th}}_{\mathrm{CKM}}| = \begin{pmatrix} 0.975^{+0.003}_{-0.002} & 0.22 \pm 0.01 & 0.003 \pm 0.001 \\ 0.22 \pm 0.01 & 0.974^{+0.003}_{-0.002} & 0.039 \pm 0.003 \\ 0.0086^{+0.0006}_{-0.0005} & 0.038 \pm 0.003 & 0.999 \pm 0.0001 \end{pmatrix} , \tag{94}$$

which is in good agreement within the present precision to the experimental values as depicted in the Appendix A. Also we find the following amount of CP violation as measured by the Jarlskog invariant,

$$J_q^{\mathrm{th}} \equiv \Im(V_{\mathrm{us}} V_{\mathrm{cb}} V^*_{\mathrm{ub}} V^*_{\mathrm{cs}}) = \left(2.1^{+1.2}_{-0.9}\right) \times 10^{-5}, \tag{95}$$

to be in good agreement. In the standard or PDG parametrization,

$$\sin\theta_{12}^{\mathrm{q,th}} = 0.22 \pm 0.01 , \qquad \sin\theta_{13}^{\mathrm{q,th}} = 0.003 \pm 0.001 , \tag{96}$$

$$\sin\theta_{23}^{\mathrm{q,th}} = 0.039 \pm 0.003 , \qquad \delta_{\mathrm{CP}}^{\mathrm{q,th}} = (62^{+28}_{-30})^\circ . \tag{97}$$

A curious and interesting feature of these formulae is that they are very stable against the running of the masses to the scale of grand unification (10^{19} GeV) [64].

4.4 Connection to other parametrizations

In the following, we show useful expressions of other parametrizations in terms of the mass ratios.

4.4.1 The strong hierarchical approximation

Let us show the short expressions one gets by considering the dominant contributions coming from the strong hierarchies in the quark masses,

$$|\mathbf{V}^{\mathrm{th}}_{\mathrm{us}}| \approx \sqrt{\frac{m_d}{m_s}}, \qquad |\mathbf{V}^{\mathrm{th}}_{\mathrm{cb}}| \approx -\left(\sqrt{\frac{m_u}{m_t}} + \sqrt{\frac{m_d}{m_b}} + \sqrt{\frac{m_c}{m_t}} - \sqrt{\frac{m_s}{m_b}}\right), \tag{98}$$

$$|\mathbf{V}^{\mathrm{th}}_{\mathrm{ub}}| \approx |\mathbf{V}^{\mathrm{th}}_{\mathrm{cb}}|\sqrt{\frac{m_u}{m_c}} - \left(\sqrt{\frac{m_u}{m_t}} + \frac{m_c}{m_t} - \sqrt{\frac{m_d}{m_b}} + \sqrt{\frac{m_d m_s}{m_b^2}} + \frac{m_s}{m_b}\right). \tag{99}$$

4.4.2 Connection to the standard parameters

The standard convention for the mixing parameters can be expressed in terms of the quark mass ratios as,

$$\delta_{\mathrm{CP}}^{\mathrm{q,th}} \approx \arctan\left[\sqrt{\frac{\frac{m_d}{m_s}(1+\frac{m_d}{m_s})}{\frac{m_u}{m_c}(1+\frac{m_u}{m_c})}}\right], \tag{100}$$

$$\sin\theta_{12}^{\mathrm{q,th}} \approx \sqrt{\frac{m_d/m_s + m_u/m_c}{(1+m_d/m_s)(1+m_u/m_c)}}\,, \tag{101}$$

$$\sin\theta_{23}^{\mathrm{q,th}} \approx -\left(\sqrt{\frac{m_u}{m_t}} + \sqrt{\frac{m_d}{m_b}} + \sqrt{\frac{m_c}{m_t}} - \sqrt{\frac{m_s}{m_b}}\right), \tag{102}$$

$$\sin\theta_{13}^{\mathrm{q,th}} \approx \sin\theta_{23}^{\mathrm{q,th}}\sqrt{\frac{m_u}{m_c}} - \left(\sqrt{\frac{m_u}{m_t}} + \frac{m_c}{m_t} - \sqrt{\frac{m_d}{m_b}} + \sqrt{\frac{m_d m_s}{m_b^2}} + \frac{m_s}{m_b}\right). \tag{103}$$

We may immediately observed the following three good features and one not adequate from the naturalness viewpoint in this particular parametrization:

1. when $m_{u,d,s,c} \to 0$ there is no mixing, $\mathbf{V}(0,0,0,0) = \mathbf{1}$,
2. when $m_{t,b} \to \infty$ we only get contributions from the $1-2$ sector, $\mathbf{V}(\frac{m_u}{m_c}, 0, \frac{m_d}{m_s}, 0) = \mathbf{L}_{12}$,
3. CP violation is directly related to the masses of the first two families and its large value is a major consequence of $\frac{m_u}{m_c} \to 0$, and for last,
4. when $m_{u,d} \to$ we do not only have contributions from the $2-3$ sector as needed but also from the $1-3$ sector. This is not adequate from the naturalness point of view and we consider it as the main aspect which suggests a careful revision of the procedure and the ansätze taken.

4.4.3 Connection to the Wolfenstein parameters

Two of the four Wolfenstein parameters can be directly expressed in terms of the quark mass ratios,

$$\lambda \approx \sqrt{\frac{m_d/m_s + m_u/m_c}{(1+m_d/m_s)(1+m_u/m_c)}}\,, \tag{104}$$

$$A \approx -\frac{(1+m_d/m_s)(1+m_u/m_c)}{m_d/m_s + m_u/m_c}\left(\sqrt{\frac{m_u}{m_t}} + \sqrt{\frac{m_d}{m_b}} + \sqrt{\frac{m_c}{m_t}} - \sqrt{\frac{m_s}{m_b}}\right). \tag{105}$$

However, in order to find the other two we first need to rephase both the up and down type quark fields,

$$\mathbf{V}' = \chi_u \mathbf{V} \chi_d^\dagger\,, \tag{106}$$

in such a way that we are able to produce the following structure,

$$\mathbf{V}' \sim \begin{pmatrix} \Re & \Re & \mathcal{C} \\ \mathcal{C} & \mathcal{C} & \Re \\ \mathcal{C} & \mathcal{C} & \Re \end{pmatrix} , \tag{107}$$

where $\chi_f = \mathrm{diag}(e^{i\phi_f}, 1, 1)$ and $\Re$ and $\mathcal{C}$ mean real and complex entries. It is actually possible to calculate the approximate expressions for the two phases in terms of the masses,

$$\phi_u \approx -\arctan\left[\sqrt{\frac{\frac{m_d}{m_s}(1+\frac{m_d}{m_s})}{\frac{m_u}{m_c}(1+\frac{m_u}{m_c})}}\right] , \tag{108}$$

$$\phi_d \approx \arctan\left[\sqrt{\frac{\frac{m_d}{m_s}\frac{m_u}{m_c}}{(1+\frac{m_d}{m_s})(1+\frac{m_u}{m_c})}}\right] - \arctan\left[\sqrt{\frac{\frac{m_d}{m_s}(1+\frac{m_d}{m_s})}{\frac{m_u}{m_c}(1+\frac{m_u}{m_c})}}\right] . \tag{109}$$

Then, the other two Wolfenstein parameters are,

$$\rho \approx u\cos\left[\sqrt{\frac{\frac{m_d}{m_s}(1+\frac{m_d}{m_s})}{\frac{m_u}{m_c}(1+\frac{m_u}{m_c})}}\right] , \qquad \eta \approx -u\sin\left[\sqrt{\frac{\frac{m_d}{m_s}(1+\frac{m_d}{m_s})}{\frac{m_u}{m_c}(1+\frac{m_u}{m_c})}}\right] , \tag{110}$$

where we have defined,

$$u \equiv \frac{\left(\sqrt{\frac{m_u}{m_t}}+\sqrt{\frac{m_d}{m_b}}+\sqrt{\frac{m_c}{m_t}}-\sqrt{\frac{m_s}{m_b}}\right)\sqrt{\frac{m_u}{m_c}}+\left(\sqrt{\frac{m_u}{m_t}}+\frac{m_c}{m_t}-\sqrt{\frac{m_d}{m_b}}+\sqrt{\frac{m_d m_s}{m_b^2}}+\frac{m_s}{m_b}\right)}{\sqrt{\frac{m_d/m_s+m_u/m_c}{(1+m_d/m_s)(1+m_u/m_c)}}\left(\sqrt{\frac{m_u}{m_t}}+\sqrt{\frac{m_d}{m_b}}+\sqrt{\frac{m_c}{m_t}}-\sqrt{\frac{m_s}{m_b}}\right)} , \tag{111}$$

and the relation between the bar parameters and the ones expressed before is,

$$\bar{\rho} \approx \rho\left[1-\frac{\lambda^2}{2}+\mathcal{O}(\lambda^4)\right] , \qquad \bar{\eta} \approx \eta\left[1-\frac{\lambda^2}{2}+\mathcal{O}(\lambda^4)\right] . \tag{112}$$

4.5 Leptonic mixing angles and neutrino masses

The elusive nature of massive neutrinos is as elusive as the problem at hand [65]. From the theory side, neutrinos are electrically neutral and except for the possibility of having total lepton number as a conserved charge, all indications point to the fact that they should satisfy the Majorana condition [66],

$$\nu^c = \nu , \tag{113}$$

and be their own antiparticle, thus violating total lepton number. As this work does not need to specify their nature and only focuses on their effective mass matrix, we put aside this question and consider both possibilities and refer the interested reader to some reviews on this subject [21, 67].

In the following, we study the two allowed cases coming from the oscillation picture, see

Appendix B: Normal Ordering (NO) and Inverted Ordering (IO),

$$\text{NO:} \qquad m_{\nu_3} > m_{\nu_2} > m_{\nu_1} \ , \tag{114}$$

$$\text{IO:} \qquad m_{\nu_2} > m_{\nu_1} > m_{\nu_3} \ . \tag{115}$$

4.5.1 Normal ordering

Although it might seem impossible to study lepton mixing if the absolute scale of neutrino masses is still unknown, through the measured squared mass differences Δm^2_{ij} and the $\sin^2\theta_{12}^{\text{PMNS}} = 0.307^{+0.013}_{-0.012}$ mixing angle is possible to determine them as first shown in Ref. [13],

$$\frac{m_{\nu_1}}{m_{\nu_2}} = \frac{\sin^2\theta_{12}^{\text{PMNS}}\left(1+\frac{m_e}{m_\mu}\right) - \frac{m_e}{m_\mu}}{1-\sin^2\theta_{12}^{\text{PMNS}}\left(1+\frac{m_e}{m_\mu}\right)} \ . \tag{116}$$

There are three main advantages for computing it in the $1-2$ sector: i) it is the cleanest part as there is no need to work in a lower rank approximation, ii) we expect the phase to be either $\pi/2$ or $3\pi/2$, and iii) we know that the neutrino mass ratio $\frac{m_{\nu 1}}{m_{\nu 2}}$ is the dominant contribution to this mixing angle (see Section 3.5),

$$m_{\nu_1} = (4.2 \pm 0.5)\ \text{meV},\ m_{\nu_2} = (9.6 \pm 0.2)\ \text{meV},\ m_{\nu_3} = (50.1 \pm 0.3)\ \text{meV} \ , \tag{117}$$

with their sum being way below the cosmological limit, $\sum m_\nu = 0.064 \pm 0.001$ eV < 0.23 eV [58].

Then, the mixing matrix for the lepton sector is,

$$|\mathbf{U}_{\text{PMNS}}^{\text{th}}| = \begin{pmatrix} 0.83 \pm 0.01 & 0.53 \pm 0.01 & 0.14 \pm 0.01 \\ 0.38^{+0.26}_{-0.15} & 0.59^{+0.25}_{-0.49} & 0.71 \pm 0.28 \\ 0.40^{+0.14}_{-0.27} & 0.61^{+0.29}_{-0.23} & 0.68^{+0.29}_{-0.62} \end{pmatrix} , \tag{118}$$

with a Jarlskog invariant given as,

$$J_\ell^{\text{th}} = -(0.03^{+0.01}_{-0.02}) \ , \tag{119}$$

and where the mixing angles in the PDG parametrization are:

$$\sin^2\theta_{12}^{\ell,\text{th}} = 0.54 \pm 0.01 \ , \quad \sin^2\theta_{13}^{\ell,\text{th}} = 0.14 \pm 0.01 \ , \quad \sin^2\theta_{23}^{\ell,\text{th}} = 0.72 \pm 0.28 \ . \tag{120}$$

We have considered an additional source of error in the $2-3$ sector due to the size of the assumption taken when changing a phase from 0 to π, see Table 1. In Figure 3, we show the allowed regions for the magnitude of the lepton mixing matrix elements as implied by the four leptonic mass ratios. There, we have only taken into account the propagation of error coming from the computed neutrino masses.

4.5.2 Inverted ordering

Dirac neutrinos follow a symmetrical treatment compared to the quark sector, that is, we may only expect them to have a normal ordering. Therefore, the inverted scenario to be studied at this stage uniquely applies to the Majorana nature which furthermore has the well known

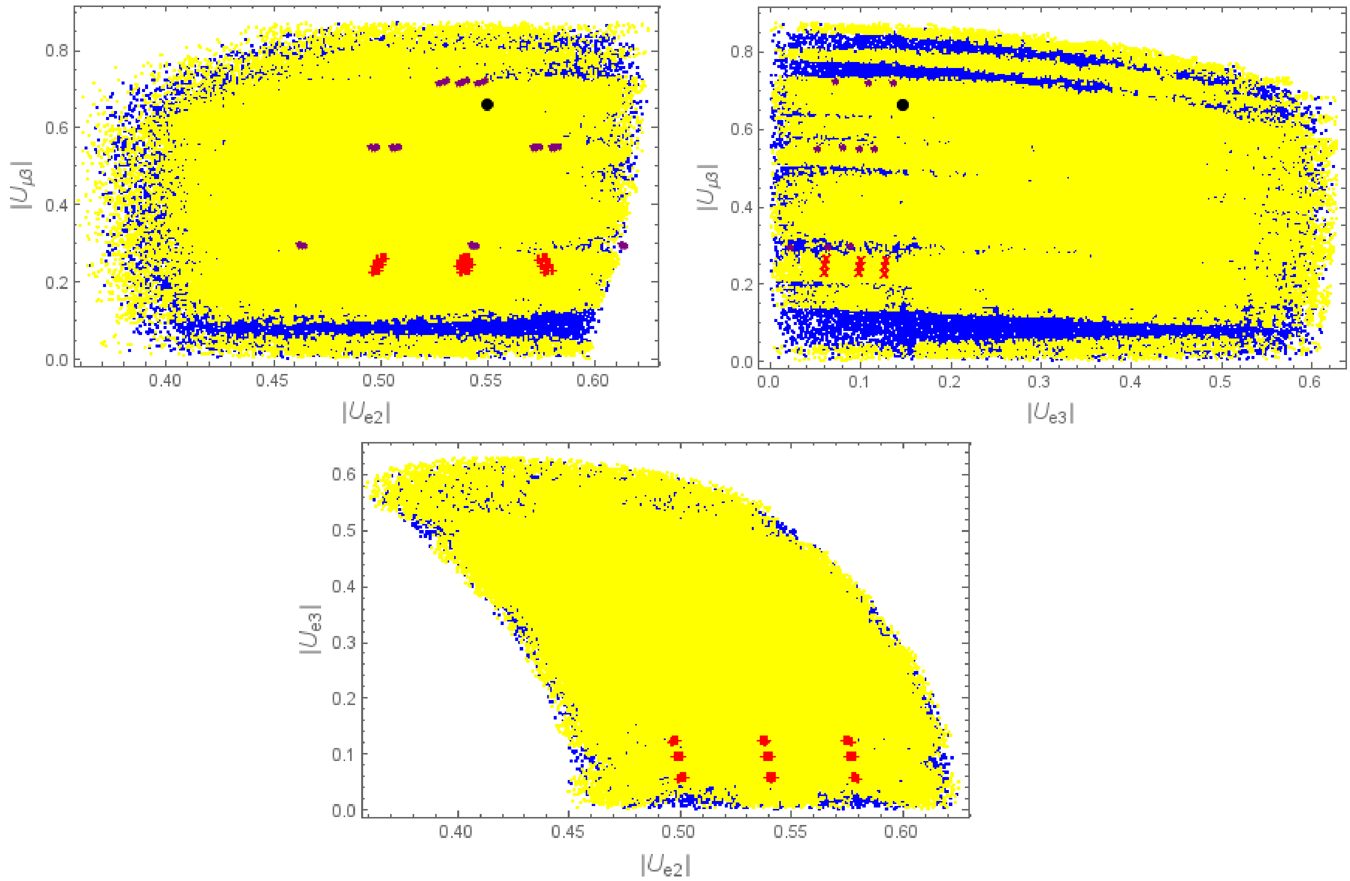

Figure 3: Allowed regions, in the normal ordering case, for the magnitude of the lepton mixing matrix elements as implied by the parametrisation with four leptonic mass ratios. The main assumptions behind this particular parametrization are: Minimal Flavor Violation, all rotations have a GST-like relation [23], and that complex phases have only four possible values. To show the effect of the complex phases we consider them under three different scenarios. The background (blue) points refer to considering the fourteen complex phases as taking any possible value, $\delta \in [0, 2\pi)$. On the other hand, the small extended regions (yellow) show the possible mixing values when considering the fourteen complex phases to take only one of the four possible phases: $0, \pi/2, \pi$ and $3\pi/2$. The large value of the neutrino mass ratios makes no clear distinction between these two cases (the blue and yellow points). On the other hand, the solitary (red) square points, refer to the introduced ansatz where one has considered seven null phases while the rest in relation to the concepts of minimal, maximal, and CP violating mixing ($3 \times 3 \times 3 = 27$ different combinations). The black (circular) spot is the observed value for mixing. Notice how the two upper plots show a discrepancy in the $2-3$ mixing angle. To correct this, a small modification is included, changing one phase from 0 to π, in the $2-3$ rotations. These are the purple points (closer to the black spot). By this the agreement is recovered.

advantage of producing small neutrino masses via the seesaw mechanism [6, 68, 69]. As the masses should satisfy,

$$m_{\nu_2} > m_{\nu_1} > m_{\nu_3} \, , \tag{121}$$

then we can infer the sequence of rotations that should be used in the neutrino sector,

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\xrightarrow{m_{\nu 2}}\begin{pmatrix}0&0&0\\0&\blacksquare&0\\0&0&0\end{pmatrix}\xrightarrow{m_{\nu 1}}\begin{pmatrix}\blacksquare&\blacksquare&0\\\blacksquare&\blacksquare&0\\0&0&0\end{pmatrix}\xrightarrow[\mathbf{L}_{12}]{}\begin{pmatrix}\blacksquare&0&0\\0&\blacksquare&0\\0&0&0\end{pmatrix}\xrightarrow{m_{\nu 3}}$$
$$\begin{pmatrix}\blacksquare&\blacksquare&\blacksquare\\\blacksquare&\blacksquare&\blacksquare\\\blacksquare&\blacksquare&\blacksquare\end{pmatrix}\xrightarrow[\mathbf{L}'_{12}]{}\begin{pmatrix}\blacksquare&0&\blacksquare\\0&\blacksquare&\blacksquare\\\blacksquare&\blacksquare&\blacksquare\end{pmatrix}\xrightarrow[\mathbf{L}_{23}]{}\begin{pmatrix}\blacksquare&0&\blacksquare\\0&\blacksquare&0\\\blacksquare&0&\blacksquare\end{pmatrix}\xrightarrow[\mathbf{L}_{13}]{}\begin{pmatrix}\blacksquare&0&0\\0&\blacksquare&0\\0&0&\blacksquare\end{pmatrix}, \tag{122}$$

where of course we have assumed a hierarchical pattern to provide a similar picture as before and make clearer how we should rotate, in general, however, we could have for example, quasi-degeneration in the heaviest masses and a very small one, in consistency with the cosmological limit.

Open to the possibility of having a different phase than $\pi/2$ we also estimate the neutrino masses for 0 and π, obtaining,

$$m_{\nu_2} = (49.7 \pm 0.3)\ \text{meV}\ , \quad m_{\nu_1} = (48.9 \pm 0.3)\ \text{meV}\ , \quad m_{\nu_3} = \begin{cases}(1.4 \pm 0.3)\ \text{meV} & \delta_\nu = 0\\(8.8 \pm 0.3)\ \text{meV} & \delta_\nu = \pi\\(1.1 \pm 0.3)\ \text{meV} & \delta_\nu = \frac{(3)\pi}{2}\end{cases} \tag{123}$$

which clearly shows that the heaviest masses are quasi-degenerate with the lightest one different in each case. In Figure 4, we show the allowed regions for the magnitude of the lepton mixing matrix elements as implied by the four leptonic mass ratios. In general, through the concepts of minimal, maximal, and CP violating mixing we cannot reproduce the observed values. Nevertheless, it is still possible to have an agreement if we limit ourselves to the values $\delta_\nu = 0, \pi/2, \pi, 3\pi/2$. Our intention is not to find the perfect combination but to comment on when it is or not possible.

4.6 Effective Majorana mass

A clear signal of neutrinos as their own antiparticles may be reached through the study of processes where total lepton number is violated. In this sense, the rare decay called neutrinoless double beta decay ($0\nu\beta\beta$), where total lepton number is violated by two units, offers a way to not only unveil the true massive nature of neutrinos but also to test predictions of left-right symmetric models and other models including right-handed currents and heavy neutral leptons [70, 71]. This rare process consists in an atom decaying into another one with the emission of two electrons in the following way,

$$(A, Z) \to (A, Z + 2) + 2e^-\ , \tag{124}$$

where (A, Z) are the mass and charge number, for the present status in these experiments see Ref. [72].

The survey of this decay is made by the effective mass parameter $\langle m_{ee}\rangle$ which, in the PDG

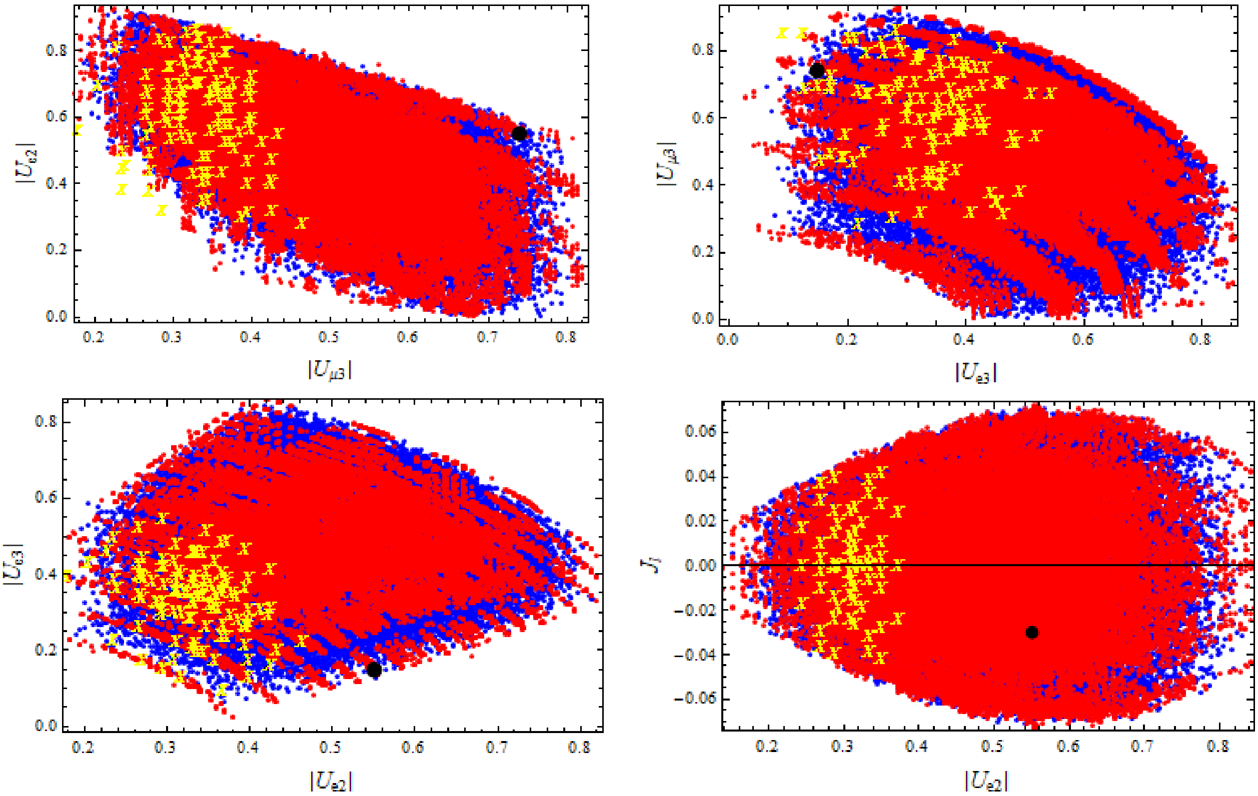

Figure 4: Allowed regions, in the inverted ordering case, for the magnitude of the lepton mixing matrix elements as implied by the parametrisation with four leptonic mass ratios. The main assumptions behind this particular parametrization are: Minimal Flavor Violation, all rotations have a GST-like relation [23], and that complex phases have only four possible values. To show the effect of the complex phases we consider them under three different scenarios. The background (blue) points refer to considering the fourteen complex phases as taking any possible value, $\delta \in [0, 2\pi)$. On the other hand, the not so extended regions (red) show the possible mixing values when considering the fourteen complex phases to take only one of the four possible phases: $0, \pi/2, \pi$ and $3\pi/2$. The large value of the neutrino mass ratios makes no clear distinction between these two cases (the blue and red points). On the other hand, the solitary (yellow) X points, refer to the introduced ansatz where one has considered seven null phases while the rest in relation to the concepts of minimal, maximal, and CP violating mixing ($3 \times 3 \times 3 = 27$ different combinations). The right bottom panel plots the Jarlskog invariant versus the $|U_{e2}|$ matrix element. The black (circular) spot is the observed value for each case. Notice how three of the four plots show a discrepancy.

parametrization, is expressed as,

$$\langle m_{ee} \rangle = |\sum_j U_{ej}^2 m_j| = |m_1 e^{2i\alpha} \cos^2\theta_{12} \cos^2\theta_{13} + m_2 e^{2i\beta} \sin^2\theta_{12} \cos^2\theta_{13} + m_3 e^{2i\delta} \sin^2\theta_{13}| \; , \quad (125)$$

where α and β are the Majorana phases and δ is the Dirac phase.

We now make a short account on the sensitivities of the leading experiments in $0\nu\beta\beta$. The Cryogenic Underground Observatory for Rare Events (CUORE) which makes use of TeO_2 crystals [73]. They recently found no evidence for this decay in a limit on the effective Majorana neutrino mass $\langle m_{ee} \rangle < (0.11 - 0.52)$ eV [74]. The GERmanium Detector Array (GERDA) uses high purity germanium detectors enriched with ^{76}Ge [75]. In their most present result they excluded the range $\langle m_{ee}\rangle < (0.12 - 0.26)$ eV [76]. The Enriched Xenon Observatory (EXO) experiment uses as a source and detector a pressurized time projection chamber filled with liquid Xenon [77]. In a first stage EXO-200 has established the limit $\langle m_{ee}\rangle < (0.15 - 0.40)$ eV where still no evidence of the rare decay has been seen [78]. The KAMioka Liquid Acintillator Anti-Neutrino Detector (KamLAND) is a multi-purpose detector that recently started the KamLAND-Zen experiment which in its second phase has reached the best measured limit so far, $\langle m_{ee}\rangle < (0.06 - 0.16)$ eV [79].

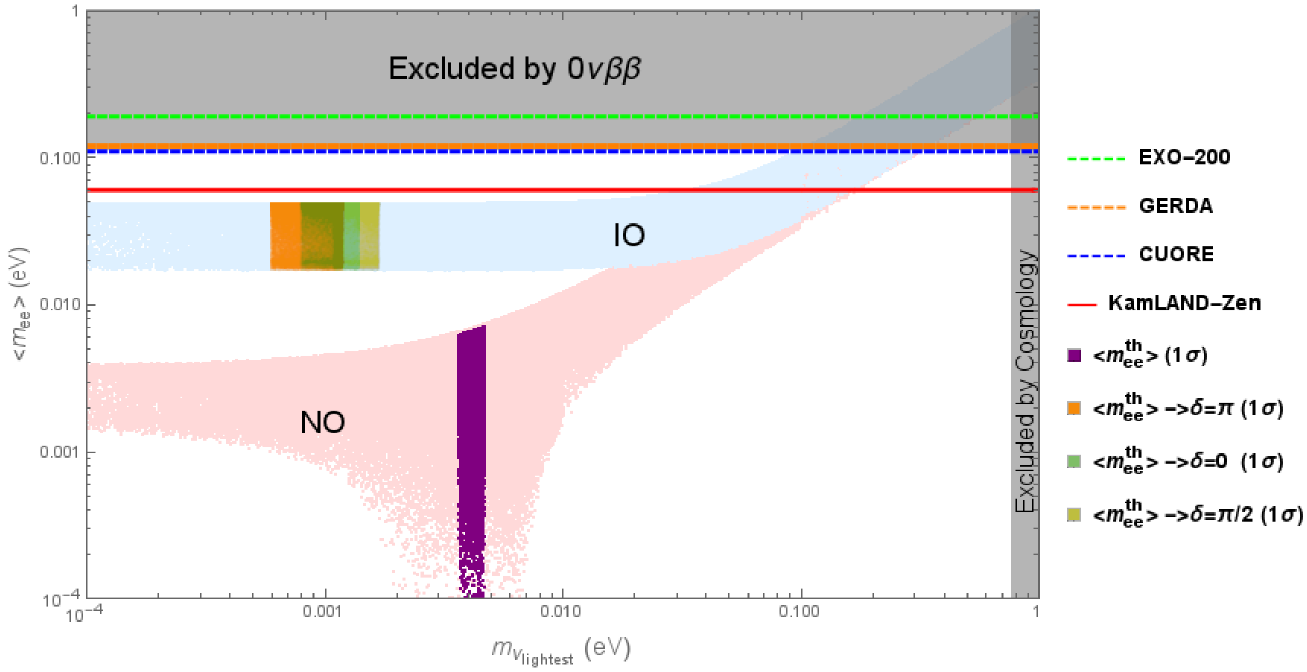

Figure 5: Allowed regions for the effective Majorana mass parameter, $\langle m_{ee}\rangle$, as a function of the lightest neutrino mass. The vertical (purple) band depicts the 1σ values of the masses and mixing angles, in the Normal Ordering (NO) case, as deduced from the mass ratios parametrization. The left (orange), center (green), and right (olive) correspond to the Inverted Ordering (IO) case, and show the 1σ regions for the three phase possibilities 0, $\pi/2$, and π. A significant overlap occurs among them and one could then take the three regions as a single one, as allowed by the present uncertainties. The light-blue (IO) and -red (NO) bands are computed from the present experimental data on neutrino oscillations at 1σ. The four horizontal lines correspond to the different experimental bounds. The gray vertical band corresponds to the upper boundary on the total of the three neutrino masses, $\sum_\nu m_\nu < 0.23$ eV as implied by cosmology [58].

5 Discussion

The purpose of this work is not to promote a particular parametrization but to promote an idea: the idea of understanding fermion mixing through the corresponding fermion mass ratios of each sector. In this sense, we consider the naturalness criteria of 't Hooft as the main support of such instance. Already its application to the parametrization of Ref. [13] shows how the limit $m_1 \to 0$ is not properly fulfill in the $1-3$ sector, $\theta_{13} \neq 0$ (recall we expect a single mixing angle acting in the $2-3$ sector). After a careful look at this problematic one finds that mixing in the $1-3$ sector is not simultaneously homogeneous in m_u, m_d or m_e, m_{ν_1}. A way to solve it, for example, if assuming the neutrino masses of Eq. (117), could be,

$$|\mathbf{V}_{\mathrm{ub}}| \simeq \sqrt{\frac{m_d m_s}{m_b^2}} - \sqrt{\frac{m_u m_c}{m_t^2}} = (4.2^{+0.2}_{-0.3}) \times 10^{-3} \tag{126}$$

$$|\mathbf{U}_{e3}| \simeq \sqrt{\frac{m_{\nu_1} m_{\nu_2}}{m_{\nu_3}^2}} - \sqrt{\frac{m_e m_\mu}{m_\tau^2}} = 0.12 \pm 0.01 \; . \tag{127}$$

It is beyond the scope of this work to find the correct form of the relations between the mixing angles and the mass ratios. Our emphasis has been put in the need to found some true guidelines to be followed and in which the theory should thus be extended until the real theory of flavor is found.

Another observation is the role this parametrization may play in the study of the strong CP problem [80, 81], in particular, within the Nelson–Barr type of models [82–84] which main challenge consists in understanding why the two parameters related to CP violation in the quark sector, δ_{KM} and $\bar{\Theta}$, are so different if they seem to share the same source [85].

6 Conclusions

The flavor puzzle stands as the most intriguing set of still not understood aspects of the SM. All of them originating from the fact that Nature has three fermion families. Here we have proposed and investigated, under two different approaches, the idea of connecting the mixing angles to the fermion mass ratios. That is, of building the concept of a new mixing parametrization whose four parameters are chosen to be the four independent mass ratios in each fermion sector. By virtue of it, the observed values in the Cabibbo–Kobayashi–Maskawa (CKM) together with the ones appearing in the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrices, can be clearly understood. The strong hierarchical nature in the masses of the quark sector, translated into four very small ratios, gives as a consequence very small mixing angles. Thus, the closeness of the CKM matrix to the identity. On the other hand, even though the absolute scale of neutrino masses is still unclear, through the same analysis we inferred that neutrino masses should have either a very mild hierarchy in the two ratios or at least in one of them, in order to produce such an anarchical structure in the PMNS matrix elements.

The first approach has consisted in exploring the consequences of solely demanding that the mixing matrices inherit the properties of the mass matrices under the limits of one, two, and/or three massless fermion families, and the third family with an infinite mass. To this end, the naturalness criteria of 't Hooft [29] is taken into consideration as the main argument to support this connection. The properties to be fulfilled by the mixing matrices then are: i) $\mathbf{V} = \mathbf{V}\left(\frac{m_u}{m_c}, \frac{m_c}{m_t}, \frac{m_d}{m_s}, \frac{m_s}{m_b}\right)$, ii) $\mathbf{V} = \mathbf{V}\left(0, \frac{m_c}{m_t}, 0, \frac{m_s}{m_b}\right) = \mathbf{L}_{23}$, iii) $\mathbf{V} = \mathbf{V}(0,0,0,0) = \mathbf{1}$, iv) if $m_{t,b} \to \infty$ then $\mathbf{V} = \mathbf{V}\left(\frac{m_u}{m_c}, 0, \frac{m_d}{m_s}, 0\right) = \mathbf{L}_{12}$, and a similar situation for the leptonic mixing matrix.

In the second approach, we have studied a recently proposed realization [13] to much greater detail. We have revisited it and discussed the importance that plays the low-rank approximation theorem in its construction, as it guarantees the robustness in which each approximation is made. Although some of the above "natural" properties are satisfied there is one which is not. In this sense, we see the importance which the first approach has in analyzing this or any particular new implementation. We have scrutinized the proposed parametrization and studied its connection to other well known parametrizations, thus reexpressing, for example, the Wolfenstein parameters in terms of the quark masses. Furthermore, in the lepton sector, we have studied the implied mixing in both possible scenarios: normal and inverted ordering. We have found that the former case is favored as it easily agrees with the observed mixing. For last, we have also included the allowed range for the effective Majorana mass parameter obtaining, $\langle m_{ee}^{\rm th}\rangle \approx 0.05$ eV and $\langle m_{ee}^{\rm th}\rangle \approx 0.009$ eV, for the inverted and normal ordering cases, respectively.

We have explictly not tried to give here a model but rather to leave the question open to model builders. As it has frequently happened in the past, the empirical relations have come first than the actual theory giving rise to them.

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A Present status in fermion mixing

The most recent global fit from the PDG for the updated values of the Cabibbo–Kobayashi–Maskawa (CKM) mixing matrix shows [86],

$$|\mathbf{V}_{\rm CKM}| = \begin{pmatrix} 0.97434^{+0.00011}_{-0.00012} & 0.22506 \pm 0.00050 & 0.00357 \pm 0.00015 \\ 0.22492 \pm 0.00050 & 0.97351 \pm 0.00013 & 0.0411 \pm 0.0013 \\ 0.00875^{+0.00032}_{-0.00033} & 0.0403 \pm 0.0013 & 0.99915 \pm 0.00005 \end{pmatrix} \tag{128}$$

with the Jarlskog invariant equal to $J_{\rm CKM} = (3.04^{+0.21}_{-0.20}) \times 10^{-5}$. This set of numbers can be summarized by virtue of the standard parametrization,

$$\sin\theta_{12}^{\rm CKM} = 0.22506 \pm 0.00050\ , \qquad \sin\theta_{13}^{\rm CKM} = 0.00357 \pm 0.00015\ , \tag{129}$$

$$\sin\theta_{23}^{\rm CKM} = 0.0411 \pm 0.0013\ , \qquad \text{and} \qquad \delta_{\rm CP}^{\rm CKM} = (71.6^{+1.3}_{-1.0})^\circ . \tag{130}$$

Whereas the fit for the improved Wolfenstein parameters gives [86],

$$\lambda = 0.22506 \pm 0.00050\ , \qquad A = 0.811 \pm 0.026\ , \tag{131}$$

$$\bar\rho = 0.124^{+0.019}_{-0.018}\ , \qquad \bar\eta = 0.356 \pm 0.011\ . \tag{132}$$

On the other hand, the current global fit values for the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) mixing matrix at 3σ are [87]:

$$|\mathbf{U}_{\rm PMNS}| = \begin{pmatrix} 0.799 \to 0.844 & 0.516 \to 0.582 & 0.141 \to 0.156 \\ 0.242 \to 0.494 & 0.467 \to 0.678 & 0.639 \to 0.774 \\ 0.284 \to 0.521 & 0.490 \to 0.695 & 0.615 \to 0.754 \end{pmatrix} , \tag{133}$$

with the Jarlskog invariant at 1σ as $J^{\rm max}_{\rm PMNS} = -(0.0329 \pm 0.0007)$. When expressed in the standard parametrization [87],

$$\sin^2\theta_{12}^{\rm PMNS} = 0.307^{+0.013}_{-0.012} , \qquad \sin^2\theta_{13}^{\rm PMNS} = 0.02206 \pm 0.00075 , \tag{134}$$

$$\sin^2\theta_{23}^{\rm PMNS} = 0.538^{+0.033}_{-0.069} , \qquad \text{and} \qquad \delta_{\rm CP}^{\rm PMNS} = (234^{+43}_{-31})^\circ . \tag{135}$$

B Present status in fermion masses

Using the mass values as shown in Tables 2 and 3, we can numerically define the four mass ratios that parametrize the mixing matrices for both quarks and leptons as we used them along the paper (we only show here the ratios for the Normal Ordering scenario),

$$\frac{m_u}{m_c} = 0.0021^{+0.0008}_{-0.0005} , \quad \frac{m_c}{m_t} = 0.0036^{+0.0002}_{-0.0001} , \quad \frac{m_s}{m_b} = 0.019^{+0.003}_{-0.001} , \quad \frac{m_d}{m_s} = 0.049^{+0.009}_{-0.006} , \tag{136}$$

$$\frac{m_e}{m_\mu} = 0.00473 , \quad \frac{m_\mu}{m_\tau} = 0.0588 , \quad \frac{m_{\nu_1}}{m_{\nu_2}} = 0.44^{+0.03}_{-0.02} , \quad \frac{m_{\nu_2}}{m_{\nu_3}} = 0.192 \pm 0.004 . \tag{137}$$

QUARK MASSES	
Experimental masses Input (GeV)	Masses at M_z scale Output (GeV)
$m_u(2\text{ GeV}) = 0.0022^{+0.0006}_{-0.0004}$	$m_u(M_z) = 0.0013^{+0.0003}_{-0.0002}$
$m_d(2\text{ GeV}) = 0.0047^{+0.0005}_{-0.0004}$	$m_d(M_z) = 0.0027^{+0.0003}_{-0.0002}$
$m_s(2\text{ GeV}) = 0.096^{+0.008}_{-0.004}$	$m_s(M_z) = 0.055^{+0.004}_{-0.002}$
$m_c(m_c) = 1.27 \pm 0.03$	$m_c(M_z) = 0.626 \pm 0.02$
$m_b(m_b) = 4.18^{+0.04}_{-0.03}$	$m_b(M_z) = 2.86^{+0.02}_{-0.02}$
$m_t(\text{OS}) = 173.21 \pm 0.87$	$m_t(M_z) = 172.29 \pm 0.06$

Table 2: Here we present in the left column the most recent measured masses as taken from [86]. By virtue of the `RunDec` package they are run to the Z boson mass scale [63]. `RunDec` takes into account the five-loop corrections of the QCD beta function and four-loop effects when decoupling the heavy quarks below their energy scale.

LEPTON MASSES	
Charged lepton (MeV)	Neutrino mass differences (eV^2)
$m_e(M_z) = 0.4861410527$	$\frac{\Delta m^2_{21}}{10^{-5}} = 7.40^{+0.21}_{-0.20}$
$m_\mu(M_z) = 102.627051$	IO: $\frac{\Delta m^2_{32}}{10^{-3}} = -2.465^{+0.032}_{-0.031}$
$m_\tau(M_z) = 1744.614156$	NO: $\frac{\Delta m^2_{31}}{10^{-3}} = +2.494^{+0.033}_{-0.031}$

Table 3: This table presents the charged lepton masses as taken from Ref. [88] and the updated neutrino mass differences [87]. We have denoted by NO and IO the Normal and Inverted Ordering scenarios, respectively. We have omitted the experimental error from the charged leptons due to their size which makes no difference in the error propagation.

References

[1] P. H. Frampton and C. Jarlskog, "Systematics of Quark Mass Matrices in the Standard Electroweak Model", *Phys. Lett.* **154B** (1985) 421–424.

[2] N. Cabibbo, "Unitary Symmetry and Leptonic Decays", *Phys. Rev. Lett.* **10** (1963) 531–533.

[3] S. L. Glashow, J. Iliopoulos, and L. Maiani, "Weak Interactions with Lepton-Hadron Symmetry", *Phys. Rev.* **D2** (1970) 1285–1292.

[4] M. Kobayashi and T. Maskawa, "CP Violation in the Renormalizable Theory of Weak Interaction", *Prog. Theor. Phys.* **49** (1973) 652–657.

[5] L. Maiani, "New Currents", in *Proceedings, 8th International Symposium on Lepton and Photon Interactions at High Energies: Hamburg, Germany, August 25-31, 1977*, pp. 867–894. 1977.

[6] J. Schechter and J. W. F. Valle, "Neutrino Masses in SU(2) x U(1) Theories", *Phys. Rev.* **D22** (1980) 2227.

[7] L. Wolfenstein, "Parametrization of the Kobayashi-Maskawa Matrix", *Phys. Rev. Lett.* **51** (1983) 1945.

[8] L.-L. Chau and W.-Y. Keung, "Comments on the Parametrization of the Kobayashi-Maskawa Matrix", *Phys. Rev. Lett.* **53** (1984) 1802.

[9] A. J. Buras, M. E. Lautenbacher, and G. Ostermaier, "Waiting for the top quark mass, K+ —> pi+ neutrino anti-neutrino, B(s)0 - anti-B(s)0 mixing and CP asymmetries in B decays", *Phys. Rev.* **D50** (1994) 3433–3446, `arXiv:hep-ph/9403384 [hep-ph]`.

[10] H. Fritzsch and Z.-z. Xing, "On the parametrization of flavor mixing in the standard model", *Phys. Rev.* **D57** (1998) 594–597, `arXiv:hep-ph/9708366 [hep-ph]`.

[11] **CKMfitter Group** , J. Charles, A. Hocker, H. Lacker, S. Laplace, F. R. Le Diberder, J. Malcles, J. Ocariz, M. Pivk, and L. Roos, "CP violation and the CKM matrix: Assessing the impact of the asymmetric B factories", *Eur. Phys. J.* **C41** no. 1, (2005) 1–131, arXiv:hep-ph/0406184 [hep-ph].

[12] W. Rodejohann and J. W. F. Valle, "Symmetrical Parametrizations of the Lepton Mixing Matrix", *Phys. Rev.* **D84** (2011) 073011, arXiv:1108.3484 [hep-ph].

[13] W. G. Hollik and U. J. Saldana-Salazar, "The double mass hierarchy pattern: simultaneously understanding quark and lepton mixing", *Nucl. Phys.* **B892** (2015) 364–389, arXiv:1411.3549 [hep-ph].

[14] C. Jarlskog, "Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Violation", *Phys. Rev. Lett.* **55** (1985) 1039.

[15] L. J. Hall and A. Rasin, "On the generality of certain predictions for quark mixing", *Phys. Lett.* **B315** (1993) 164–169, arXiv:hep-ph/9303303 [hep-ph].

[16] Z.-z. Xing, "Implications of the quark mass hierarchy on flavor mixings", *J. Phys.* **G23** (1997) 1563–1578, arXiv:hep-ph/9609204 [hep-ph].

[17] A. Rasin, "Diagonalization of quark mass matrices and the Cabibbo-Kobayashi-Maskawa matrix", arXiv:hep-ph/9708216 [hep-ph].

[18] J. L. Chkareuli and C. D. Froggatt, "Where does flavor mixing come from?", *Phys. Lett.* **B450** (1999) 158–164, arXiv:hep-ph/9812499 [hep-ph].

[19] A. Rasin, "Hierarchical quark mass matrices", *Phys. Rev.* **D58** (1998) 096012, arXiv:hep-ph/9802356 [hep-ph].

[20] J. L. Chkareuli, C. D. Froggatt, and H. B. Nielsen, "Minimal mixing of quarks and leptons in the SU(3) theory of flavor", *Nucl. Phys.* **B626** (2002) 307–343, arXiv:hep-ph/0109156 [hep-ph].

[21] S. F. King, "Unified Models of Neutrinos, Flavour and CP Violation", *Prog. Part. Nucl. Phys.* **94** (2017) 217–256, arXiv:1701.04413 [hep-ph].

[22] S. Weinberg, "The Problem of Mass", *Trans. New York Acad. Sci.* **38** (1977) 185–201.

[23] R. Gatto, G. Sartori, and M. Tonin, "Weak Selfmasses, Cabibbo Angle, and Broken SU(2) x SU(2)", *Phys. Lett.* **28B** (1968) 128–130.

[24] A. Ibarra and A. Solaguren-Beascoa, "Lepton parameters in the see-saw model extended by one extra Higgs doublet", *JHEP* **11** (2014) 089, arXiv:1409.5011 [hep-ph].

[25] A. Ibarra and A. Solaguren-Beascoa, "Radiative Generation of Quark Masses and Mixing Angles in the Two Higgs Doublet Model", *Phys. Lett.* **B736** (2014) 16–19, arXiv:1403.2382 [hep-ph].

[26] W. G. Hollik and U. J. Saldana-Salazar, "Texture zeros and hierarchical masses from flavour (mis)alignment", *Nucl. Phys.* **B928** (2018) 535–554, arXiv:1712.05387 [hep-ph].

[27] R. N. Mohapatra and G. Senjanovic, "Cabibbo Angle, CP Violation and Quark Masses", *Phys. Lett.* **73B** (1978) 176–180.

[28] F. Wilczek and A. Zee, "Discrete Flavor Symmetries and a Formula for the Cabibbo Angle", *Phys. Lett.* **70B** (1977) 418. [Erratum: Phys. Lett.72B,504(1978)].

[29] G. 't Hooft, "Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking", *NATO Sci. Ser. B* **59** (1980) 135–157.

[30] M. Leurer, Y. Nir, and N. Seiberg, "Mass matrix models: The Sequel", *Nucl. Phys.* **B420** (1994) 468–504, `arXiv:hep-ph/9310320 [hep-ph]`.

[31] A. de Gouvea and H. Murayama, "Statistical test of anarchy", *Phys. Lett.* **B573** (2003) 94–100, `arXiv:hep-ph/0301050 [hep-ph]`.

[32] A. de Gouvea and H. Murayama, "Neutrino Mixing Anarchy: Alive and Kicking", *Phys. Lett.* **B747** (2015) 479–483, `arXiv:1204.1249 [hep-ph]`.

[33] Y. Reyimuaji and A. Romanino, "Can an unbroken flavour symmetry provide an approximate description of lepton masses and mixing?", *JHEP* **03** (2018) 067, `arXiv:1801.10530 [hep-ph]`.

[34] U. J. Saldana-Salazar, "The flavor-blind principle: A symmetrical approach to the Gatto-Sartori-Tonin relation", *Phys. Rev.* **D93** no. 1, (2016) 013002, `arXiv:1509.08877 [hep-ph]`.

[35] E. Nardi, "Naturally large Yukawa hierarchies", *Phys. Rev.* **D84** (2011) 036008, `arXiv:1105.1770 [hep-ph]`.

[36] J. R. Espinosa, C. S. Fong, and E. Nardi, "Yukawa hierarchies from spontaneous breaking of the $SU(3)_L \times SU(3)_R$ flavour symmetry?", *JHEP* **02** (2013) 137, `arXiv:1211.6428 [hep-ph]`.

[37] G. D'Ambrosio, G. F. Giudice, G. Isidori, and A. Strumia, "Minimal flavor violation: An Effective field theory approach", *Nucl. Phys.* **B645** (2002) 155–187, `arXiv:hep-ph/0207036 [hep-ph]`.

[38] A. J. Buras, "Minimal flavor violation", *Acta Phys. Polon.* **B34** (2003) 5615–5668, `arXiv:hep-ph/0310208 [hep-ph]`.

[39] V. Cirigliano, B. Grinstein, G. Isidori, and M. B. Wise, "Minimal flavor violation in the lepton sector", *Nucl. Phys.* **B728** (2005) 121–134, `arXiv:hep-ph/0507001 [hep-ph]`.

[40] A. L. Kagan, G. Perez, T. Volansky, and J. Zupan, "General Minimal Flavor Violation", *Phys. Rev.* **D80** (2009) 076002, `arXiv:0903.1794 [hep-ph]`.

[41] R. Alonso, G. Isidori, L. Merlo, L. A. Munoz, and E. Nardi, "Minimal flavour violation extensions of the seesaw", *JHEP* **06** (2011) 037, `arXiv:1103.5461 [hep-ph]`.

[42] W.-S. Hou and G.-G. Wong, "Perspective on quark mass and mixing relations", *Phys. Rev.* **D52** (1995) 5269–5272, `arXiv:hep-ph/9412390 [hep-ph]`.

[43] G. W.-S. Hou, "Perspectives on quark mass and mixing relations", in *Masses and mixings of quarks and leptons. Proceedings, International Workshop, MMQL'97, Shizuoka, Japan, March 19-21, 1997*, pp. 92–107. 1997. `arXiv:hep-ph/9707528 [hep-ph]`.

[44] S. Antusch, S. F. King, M. Malinsky, and M. Spinrath, "Quark mixing sum rules and the right unitarity triangle", *Phys. Rev.* **D81** (2010) 033008, arXiv:0910.5127 [hep-ph].

[45] L. Dorame, S. Morisi, E. Peinado, J. W. F. Valle, and A. D. Rojas, "A new neutrino mass sum rule from inverse seesaw", *Phys. Rev.* **D86** (2012) 056001, arXiv:1203.0155 [hep-ph].

[46] F. Buccella, M. Chianese, G. Mangano, G. Miele, S. Morisi, and P. Santorelli, "A neutrino mass-mixing sum rule from SO(10) and neutrinoless double beta decay", *JHEP* **04** (2017) 004, arXiv:1701.00491 [hep-ph].

[47] J. Gehrlein and M. Spinrath, "Neutrino Mass Sum Rules and Symmetries of the Mass Matrix", *Eur. Phys. J.* **C77** no. 5, (2017) 281, arXiv:1704.02371 [hep-ph].

[48] P. F. Harrison and W. G. Scott, "mu - tau reflection symmetry in lepton mixing and neutrino oscillations", *Phys. Lett.* **B547** (2002) 219–228, arXiv:hep-ph/0210197 [hep-ph].

[49] I. Masina and C. A. Savoy, "Real and imaginary elements of fermion mass matrices", *Nucl. Phys.* **B755** (2006) 1–20, arXiv:hep-ph/0603101 [hep-ph].

[50] S. Antusch, S. F. King, C. Luhn, and M. Spinrath, "Right Unitarity Triangles and Tri-Bimaximal Mixing from Discrete Symmetries and Unification", *Nucl. Phys.* **B850** (2011) 477–504, arXiv:1103.5930 [hep-ph].

[51] T. P. Cheng and M. Sher, "Mass Matrix Ansatz and Flavor Nonconservation in Models with Multiple Higgs Doublets", *Phys. Rev.* **D35** (1987) 3484.

[52] E. Schmidt, "Zur Theorie der linearen und nichtlinearen Integralgleichungen - I. Teil: Entwicklung willkrlicher Funktionen nach Systemen vorgeschriebener", *Mathematische Annalen* **63** no. 4, (1907) 433–476.

[53] L. Mirsky, "Symmetric gauge functions and unitarily invariant norms", *Quart. J. Math.* **11** (1960) 50–59.

[54] G. Eckart and Y. G., "The approximation of one matrix by another of lower rank", *Psychometrika* **1** (1936) 211–218.

[55] G. Golub, A. Hoffman, and G. Stewart, "A generalization of the Eckart-Young-Mirsky matrix approximation theorem", *Linear Algebra and Its Applications* **88-89** no. C, (1987) 317–327.

[56] J. Von Neumann, *Some matrix-inequalities and metrization of matric space.* 1937.

[57] L. Mirsky, "A trace inequality of John von Neumann", *Monatshefte für Mathematik* **79** no. 4, (Dec, 1975) 303–306. https://doi.org/10.1007/BF01647331.

[58] **Planck** , P. A. R. Ade *et al.*, "Planck 2015 results. XIII. Cosmological parameters", *Astron. Astrophys.* **594** (2016) A13, arXiv:1502.01589 [astro-ph.CO].

[59] C. D. Froggatt and H. B. Nielsen, "Hierarchy of Quark Masses, Cabibbo Angles and CP Violation", *Nucl. Phys.* **B147** (1979) 277–298.

[60] N. Arkani-Hamed and M. Schmaltz, "Hierarchies without symmetries from extra dimensions", *Phys. Rev.* **D61** (2000) 033005, arXiv:hep-ph/9903417 [hep-ph].

[61] W. Altmannshofer, C. Frugiuele, and R. Harnik, "Fermion Hierarchy from Sfermion Anarchy", *JHEP* **12** (2014) 180, arXiv:1409.2522 [hep-ph].

[62] S. Knapen and D. J. Robinson, "Disentangling Mass and Mixing Hierarchies", *Phys. Rev. Lett.* **115** no. 16, (2015) 161803, arXiv:1507.00009 [hep-ph].

[63] F. Herren and M. Steinhauser, "Version 3 of RunDec and CRunDec", *Comput. Phys. Commun.* **224** (2018) 333–345, arXiv:1703.03751 [hep-ph].

[64] We thank Jakob Schwichtenberg for pointing this out.

[65] S. T. Petcov, "The Nature of Massive Neutrinos", *Adv. High Energy Phys.* **2013** (2013) 852987, arXiv:1303.5819 [hep-ph].

[66] E. Majorana, "Teoria simmetrica dell'elettrone e del positrone", *Nuovo Cim.* **14** (1937) 171–184.

[67] S. Morisi and J. W. F. Valle, "Neutrino masses and mixing: a flavour symmetry roadmap", *Fortsch. Phys.* **61** (2013) 466–492, arXiv:1206.6678 [hep-ph].

[68] P. Minkowski, "$\mu \to e\gamma$ at a Rate of One Out of 10^9 Muon Decays?", *Phys. Lett.* **67B** (1977) 421–428.

[69] R. N. Mohapatra and G. Senjanovic, "Neutrino Mass and Spontaneous Parity Violation", *Phys. Rev. Lett.* **44** (1980) 912. [,231(1979)].

[70] S.-F. Ge, M. Lindner, and S. Patra, "New physics effects on neutrinoless double beta decay from right-handed current", *JHEP* **10** (2015) 077, arXiv:1508.07286 [hep-ph].

[71] J. C. Helo and M. Hirsch, "LHC dijet constraints on double beta decay", *Phys. Rev.* **D92** no. 7, (2015) 073017, arXiv:1509.00423 [hep-ph].

[72] W. Maneschg, "Present status of neutrinoless double beta decay searches", in *Proceedings, Prospects in Neutrino Physics (NuPhys2016): London, UK, December 12-14, 2016*. 2017. arXiv:1704.08537 [physics.ins-det]. http://inspirehep.net/record/1597176/files/arXiv:1704.08537.pdf.

[73] **CUORE** , D. R. Artusa *et al.*, "Searching for neutrinoless double-beta decay of ^{130}Te with CUORE", *Adv. High Energy Phys.* **2015** (2015) 879871, arXiv:1402.6072 [physics.ins-det].

[74] **CUORE** , C. Alduino *et al.*, "First Results from CUORE: A Search for Lepton Number Violation via $0\nu\beta\beta$ Decay of ^{130}Te", *Phys. Rev. Lett.* **120** no. 13, (2018) 132501, arXiv:1710.07988 [nucl-ex].

[75] **GERDA** , M. Agostini *et al.*, "Results on Neutrinoless Double-β Decay of ^{76}Ge from Phase I of the GERDA Experiment", *Phys. Rev. Lett.* **111** no. 12, (2013) 122503, arXiv:1307.4720 [nucl-ex].

[76] **GERDA** , M. Agostini *et al.*, "Improved Limit on Neutrinoless Double-β Decay of ^{76}Ge from GERDA Phase II", *Phys. Rev. Lett.* **120** no. 13, (2018) 132503, arXiv:1803.11100 [nucl-ex].

[77] **EXO-200** , J. B. Albert *et al.*, "Search for Majorana neutrinos with the first two years of EXO-200 data", *Nature* **510** (2014) 229–234, arXiv:1402.6956 [nucl-ex].

[78] **EXO** , J. B. Albert *et al.*, "Search for Neutrinoless Double-Beta Decay with the Upgraded EXO-200 Detector", *Phys. Rev. Lett.* **120** no. 7, (2018) 072701, arXiv:1707.08707 [hep-ex].

[79] **KamLAND-Zen** , A. Gando *et al.*, "Search for Majorana Neutrinos near the Inverted Mass Hierarchy Region with KamLAND-Zen", *Phys. Rev. Lett.* **117** no. 8, (2016) 082503, arXiv:1605.02889 [hep-ex]. [Addendum: Phys. Rev. Lett.117,no.10,109903(2016)].

[80] J.-M. Gerard, "Mass Issues in Fundamental Interactions", in *2008 European School of High-Energy Physics, Herbeumont-sur-Semois, Belgium, 8-21 June 2008*, pp. 281–314. 2008. arXiv:0811.0540 [hep-ph]. http://inspirehep.net/record/801581/files/arXiv:0811.0540.pdf.

[81] J. L. Diaz-Cruz, W. G. Hollik, and U. J. Saldana-Salazar, "A bottom-up approach to the strong CP problem", arXiv:1605.03860 [hep-ph].

[82] A. E. Nelson, "Naturally Weak CP Violation", *Phys. Lett.* **136B** (1984) 387–391.

[83] S. M. Barr, "Solving the Strong CP Problem Without the Peccei-Quinn Symmetry", *Phys. Rev. Lett.* **53** (1984) 329.

[84] S. M. Barr, "A Natural Class of Nonpeccei-quinn Models", *Phys. Rev.* **D30** (1984) 1805.

[85] M. Dine and P. Draper, "Challenges for the Nelson-Barr Mechanism", *JHEP* **08** (2015) 132, arXiv:1506.05433 [hep-ph].

[86] **Particle Data Group** , C. Patrignani *et al.*, "Review of Particle Physics", *Chin. Phys.* **C40** no. 10, (2016) 100001.

[87] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, and T. Schwetz, "Updated fit to three neutrino mixing: exploring the accelerator-reactor complementarity", *JHEP* **01** (2017) 087, arXiv:1611.01514 [hep-ph].

[88] S. Antusch and V. Maurer, "Running quark and lepton parameters at various scales", *JHEP* **11** (2013) 115, arXiv:1306.6879 [hep-ph].