

$(g - 2)_\mu$  at four loops in QED

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**Abstract.** We review the four-loop QED corrections to the anomalous magnetic moment of the muon. The fermionic contributions with closed electron and tau contributions are discussed. Furthermore, we report on a new independent calculation of the universal four-loop contribution and compare with existing results.

## 1 Introduction

The anomalous magnetic moment of the muon, which is usually written as  $a_\mu = (g - 2)_\mu/2$ , measures the deviation from Dirac's prediction  $g = 2$ . Experimentally it is known with high precision from measurements at BNL [1, 2]

$$a_\mu^{\text{exp}} = 116\,592\,089(63) \times 10^{-11}. \quad (1)$$

It is expected that the uncertainty will be reduced in the coming years. Actually, there are two experiments which are currently under construction, one at Fermilab and one at J-PARC [3–5]

Also on the theory side an impressive precision has been reached. However, since many years there is a persistent discrepancy of the order of about 3 sigma. The uncertainty of the theory prediction is dominated by the hadronic contributions, both from the vacuum polarization [6–8] (see Refs. [9–11] for a recent compilations) and the so-called light-by-light part [12].

The numerically largest contribution to  $a_\mu$  is given by the QED part which is known up to five loops. One-, two- and three-loop corrections are known analytically from Refs. [13–16] and four- and five-loop contributions have been computed in Refs. [17–20] using numerical methods. The fermionic contributions involving closed tau and electron loops have been cross checked in Refs. [21–23]. Very recently, semi-analytic results for the universal contribution, i.e., the purely photonic and muon-loop contribution, have been obtained in a remarkable calculation by Laporta [24]. It is based on an evaluation of Feynman integrals with high-precision (several thousand digits) which was enough to reconstruct rational coefficients of known transcendental constants with the help of the PSLQ algorithm [25]. In addition, there were several contributions which were not recognized as known constants. The final result for the four-loop contribution to  $a_\mu$  from [24] is known to 1100 digits.

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In this work we present results of an independent calculation of the universal contribution.

In order to fix the notation we provide numerical results for  $a_\mu$  up to five-loop order which are given by (numbers are taken from Refs. [19, 20])

$$\begin{aligned}
 a_\mu &= \frac{(g-2)_\mu}{2} \\
 &= \frac{\alpha}{2\pi} \\
 &\quad + (-0.328\,478\dots + 1.094\,336\dots|_{e,\tau}) \left(\frac{\alpha}{\pi}\right)^2 + (1.181\,241\dots + 22.869\,268\dots|_{e,\tau}) \left(\frac{\alpha}{\pi}\right)^3 \\
 &\quad + (-1.912\,98(84) + 132.790\,3(60)|_{e,\tau}) \left(\frac{\alpha}{\pi}\right)^4 + (9.168(571) + 744.123(870)|_{e,\tau}) \left(\frac{\alpha}{\pi}\right)^5,
 \end{aligned} \tag{2}$$

where the ellipses indicate that the numbers are truncated and actually more digits are known. The universal part (first number in the brackets) has been separated from electron and tau contributions (second number), which appears for the first time at two loops. Note that the latter is numerically dominant due to unsuppressed large logarithms of the ratio of the electron and muon mass,  $\log(m_\mu/m_e) \approx 5.332$ . At  $\ell$ -loop order such logarithms occur up to the  $(\ell - 1)$ th order. On the other hand, heavy virtual particles are decoupled and thus the tau contributions are suppressed by  $m_\mu^2/m_\tau^2$ . They are numerically small.

Let us note that up to terms suppressed by  $m_e^2/m_\mu^2$  the first numbers in the coefficients of Eq. (2) coincide with the anomalous magnetic moment of the electron,  $a_e$ .

It is interesting to note that after inserting the fine structure constant the four-loop coefficient evaluates to

$$a_\mu^{(8)} = (-1.912\,98 + 132.790\,3|_{e,\tau}) \left(\frac{\alpha}{\pi}\right)^4 \approx 381 \times 10^{-11}, \tag{3}$$

which is of the same order of magnitude as the current difference between the Standard Model prediction of  $a_\mu$  and the experimental value given in Eq. (1). Furthermore, it is larger than the uncertainties of the hadronic vacuum polarization and light-by-light contributions which are both of the order of  $40 \times 10^{-11}$ . Thus, an independent cross check of the four-loop QCD contributions is indispensable.

## 2 Technical remarks

The techniques used to obtain the results in Refs. [21–23] and for the universal part, which we report below, have largely been developed in the context of the  $\overline{\text{MS}}$ -on-shell quark mass relation in QCD. To obtain the mass relation one has to evaluate on-shell integrals up to four loops which are also the basis for the anomalous magnetic moment. In fact, we use the same integral families as defined in Refs. [26, 27] and express the four-loop expression for  $a_\mu$  as a linear combination of scalar integrals. The latter are reduced to master integrals with the help of FIRE [28] and Crusher [29]. Let us mention that the reduction of the integrals contributing to  $a_\mu$  is more expensive since vertex integrals (instead of two-point functions) are considered which are expanded around vanishing momentum transfer of the photon. Thus, in the corresponding integrals the total power of the propagators is increased by at least two as compared to the integrals needed for the  $\overline{\text{MS}}$ -on-shell relation. For the  $\overline{\text{MS}}$ -on-shell relation we have to evaluate 386 master integrals; a subset of 357 master integrals contribute to  $a_\mu$ . For details concerning their evaluation we refer to Ref. [27]. To obtain the precision mentioned below some of the master integrals had to be evaluated with higher precision following the methods of [27].

Additional work is needed for the fermionic contributions with closed electron or tau loops. In both cases it is appealing to perform an asymptotic expansion either for  $m_e \ll m_\mu$  or  $m_\mu \ll m_\tau$ . The

latter is a Euclidean-like asymptotic expansion which can be performed with the help of the program `exp` [30, 31]. The most complicated integrals which have to be evaluated are four-loop vacuum integrals which are well studied in the literature (see, for example, Ref. [27] and references therein). All other contributions are of lower loop order and also known analytically. Thus, the four-loop contribution to  $a_\mu$  containing tau leptons is known analytically as a series in  $m_\mu/m_\tau$  which is rapidly converging [21].

To obtain an expansion of the electron-loop contribution in  $m_e/m_\mu$  an asymptotic expansion around the on-shell limit has to be performed. The complicated integrals one has to compute are either of on-shell type (as for the universal contribution) or integrals which contain linear propagators of the form  $1/p \cdot q$  where  $q^2 = m_\mu^2$  is the external momentum and  $p$  is a loop momentum. Some integrals of this type can be computed analytically, others are computed numerically using FIESTA [32]. In Refs. [22, 23] expansion terms up to order  $m_e^3/m_\mu^3$  have been computed which show a good convergence behaviour. Let us remark that the numerically dominant contribution arises from the light-by-light-type contributions which have been computed in [22].

In Refs. [17–19] a completely different technique has been used to compute the four-loop corrections to  $a_\mu$ . In a first step a finite expression is constructed by generating the proper counterterms together with four-loop diagrams which is afterwards integrated numerically.

In Ref. [24], similar to our approach, all occurring integrals are reduced to a small set of master integrals. However, different software is used and most probably also a different basis of master integrals is chosen. Furthermore, Ref. [24] manages to obtain high-precision numerical expressions for all master integrals whereas we have chosen a more automated approach and stopped manipulating the integrals once the desired precision has been reached.

### 3 Results

Let us start with discussing the universal part to  $a_\mu$  which consists of the pure photon contribution and the contribution with closed muon loops. It can be subdivided into six gauge invariant subsets; a representative diagram for each one is shown in the first column of Tab. 1. The second column in Tab. 1 contains the corresponding results from Ref. [33], this work, and Ref. [24], respectively (from top to bottom). The results from Ref. [33] are taken from Table I of that reference and the uncertainties are added in quadrature in case several contributions had to be combined. The uncertainty of the results obtained in this work are the quadratically combined results from the individual  $\epsilon$  coefficients of the master integrals. We refrain from introducing a “security factor” (as, e.g., in Ref. [27]) for the universal contribution since the four-loop result for  $a_\mu$  has also been computed by two other groups. There is no uncertainty in the result provided in Ref. [24].

Within the given uncertainties the results from [33] and this work agree with the semi-analytic expressions of [24]. In most cases our uncertainty is at the per cent level or below, except for the contribution in the second row where a 40% uncertainty is observed. Note, that the absolute size of the uncertainty is of the same order as the one in the first and third row. However, due to cancellations from individual contributions, the central value is significantly smaller.

In the following we summarize the four-loop QED contributions and compare the results from the different groups. Denoting the coefficient of  $(\alpha/\pi)^4$  by  $a_\mu^{(8)}$  we have

universal	$e^-$	$\tau$	$e^- + \tau$	
$a_\mu^{(8)} = -1.87(12)$	$+ 132.86(48)$	$+ 0.0424941(53)$	$+ 0.062722(10)$	this work and [21–23]
$a_\mu^{(8)} = -1.912\,98(84)$	$+ 132.6852(60)$	$+ 0.04234(12)$	$+ 0.06272(4)$	[19]
$a_\mu^{(8)} = -1.9122457649264\dots$				[24]

Representative Feynman diagram	Contribution of $a_\mu$
	$-2.1755 \pm 0.0020$ $-2.161 \pm 0.065$ $-2.176866027739540077443259355895893938670$
	$0.05596 \pm 0.0001$ $0.077 \pm 0.031$ $0.056110899897828364831469274418908842233$
	$-0.3162 \pm 0.0002$ $-0.3048 \pm 0.021$ $-0.316538390648940158843260382381513284828$
	$-0.074665 \pm 0.000006$ $-0.07461 \pm 0.00008$ $-0.074671184326105513860159965722793126809$
	$0.598838 \pm 0.000019$ $0.597204 \pm 0.0012$ $0.598842072031421820464649513201747727836$
	$0.000876865858889990697913748939713726165$ $0.000876865858889990697913748939713726165$ $0.000876865858889990697913748939713726165$

**Table 1.** The three numbers given in each row (from top to bottom) are taken from [33], this work, and [24], respectively.

Note that the uncertainties in the first line in the parts involving a tau lepton are due to the lepton masses only. After multiplication with  $(\alpha/\pi)^4$  we obtain for the three equations

$$(-5.44(35) + 386.77(1.40) + 0.12371(15) + 0.182592(29)) \times 10^{-11} \quad \text{this work and [21–23]}$$

$$(-5.56894(245) + 386.264(17) + 0.12326(35) + 0.18259(12)) \times 10^{-11} \quad [19]$$

$$(-5.56679893738506 \dots + \dots) \times 10^{-11} \quad [24]$$

The uncertainty of our result is about two orders of magnitudes larger. It is nevertheless much smaller than the current and foreseen uncertainties from both experiment and the hadronic contributions. This can be seen by considering the difference between the experimental result and the Standard Model prediction which is given by (see, e.g., Ref. [19])

$$a_\mu(\text{exp}) - a_\mu(\text{SM}) \approx 250(90) \times 10^{-11}.$$

The uncertainty is about two orders of magnitude larger than our numerical uncertainty cited above. This remains even true after applying the improvements by a factor 4. Thus, it can be claimed that the four-loop contribution for  $a_\mu$  is cross-checked: There are three independent calculations for the universal part and the electron and tau contributions have been computed by two independent groups.

Let us finally remark on  $a_e$ . The Standard Model prediction given in Ref. [24] reads

$$a_e(\text{SM}) = 115\,965\,218.1664(23)(16)(763) \times 10^{-11}, \quad (4)$$

where the three uncertainties have their origin in the numerical accuracy of the five-loop calculation, the hadronic and electroweak corrections and the fine structure constant. Due to the result of Ref. [24] an additional uncertainty of “(60)”, which is still present in [33], has been removed. Note that our result for the universal part of  $a_\mu$  can also be applied to  $a_e$ . However, since it has an uncertainty which is two orders of magnitude larger than the one cited in [33] it is not competitive to [33] and [24].

## 4 Conclusions

We summarize all four-loop QED contributions to the anomalous magnetic moment of the muon. They have been computed for the first time in Refs. [17–19]. An independent cross check of the tau-loop contributions can be found in Ref. [21] where analytic results are provided for the expansion in  $m_\mu/m_\tau$ . The electron-loop contributions have been cross checked in Refs. [22, 23] where an asymptotic expansion in  $m_e/m_\mu$  has been used. An independent semi-analytic calculation of the universal (purely photonic and muon-loop) contribution has been obtained in Ref. [24]. In this work we provide yet another independent cross check. In summary, all four-loop QED contributions to  $a_\mu$  have been computed by at least two groups independently using completely different methods.

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