

# High energy behaviour of form factors

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**ABSTRACT:** We solve renormalization group equations that govern infrared divergences of massless and massive form factors. By comparing to recent results for planar massive three-loop and massless four-loop form factors in QCD, we give predictions for the high-energy limit of massive form factors at the four- and for the massless form factor at five-loop order. Furthermore, we discuss the relation which connects infrared divergences regularized dimensionally and via a small quark mass and extend results present in the literature to higher order.

**KEYWORDS:** QCD, form factor, massive quarks, infrared divergences

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## 1 Introduction

In perturbative QCD the knowledge of the infrared divergences of scattering amplitudes are of utmost importance. In the recent past this issue has obtained significant attention both from formal considerations and explicit calculations up to four-loop order.

We consider form factors of quarks of mass  $m$  at total energy squared  $q^2$ . The simplest example is the correlator of the electromagnetic current with two massive quarks that is parametrized by two form factors  $F_1$  and  $F_2$  which enter the photon quark vertex as follows

$$\Gamma^\mu(q_1, q_2) = Q_q \left[ F_1(q^2) \gamma^\mu - \frac{i}{2m} F_2(q^2) \sigma^{\mu\nu} q_\nu \right]. \quad (1.1)$$

$F_1$  is a building block for a variety of observables. Among them are the cross section of hadron production in electron-positron annihilation and derived quantities like the forward-backward asymmetry. The form factor  $F_2$  is of particular interest in the limit  $q^2 = 0$  where it describes the quark magnetic anomalous moment. In the massless case only one form factor proportional to  $\gamma^\mu$  is sufficient to parametrize the photon quark vertex. In the remainder of the paper we call this form factor  $\overline{F}$  to avoid confusion with the massive case.

Exchanges of soft particles between the massive quarks can lead to infrared divergences. The latter are conveniently regulated by dimensional regularization, with  $d = 4 - 2\epsilon$ , where  $d$  is the space-time dimension. The divergences can be effectively described by cusped Wilson lines and their associated cusp anomalous dimensions [1]. In the high-energy, or massless limit, additional collinear divergences appear, that give rise to large logarithms involving the mass and the momentum transfer. Alternatively, if  $m = 0$  is chosen from the start, the latter are replaced by higher poles in the dimensional regularization parameter  $\epsilon$ . While at leading order in the coupling this correspondence between poles in  $\epsilon$  and logarithms of the mass is straightforward, making it quantitative at higher orders requires the use of renormalization group equations, see Refs. [2–4]. One obtains conversion factors

between infrared divergences regularized with a small quark mass and those regularized using dimensional regularization. Due to the universal nature of infrared divergences, once obtained from one quantity, the conversion factors can be used in other calculations as well.

Renormalization group equations allow the use of information from lower-loop corrections in order to predict poles in  $\epsilon$  and logarithms in the mass at higher loop orders. In this way, high-energy terms of massive form factors at three loops were predicted in Refs. [2, 4] based on two-loop computations. Similarly, the pole structure of three-loop massless form factors in dimensional regularization is available in the literature (see, e.g., Ref. [5]). We note that in conformal theories, the renormalization group equation can be solved exactly [6, 7].

Recently, new perturbative results at higher loop orders have become available, such as the planar massless four-loop form factors [8, 9], and the planar massive three-loop form factors [10]. Motivated by this, we determine the solution of the aforementioned renormalization group equations to higher orders of perturbation theory. We then use the wealth of new information to determine the integration constants appearing in the latter to higher orders. In this way, we are able to make new predictions about the high-energy behavior of massive form factors at four loops, as well as about the infrared terms appearing in five-loop massless form factors.

The paper is organized as follows. In section 2, we review renormalization group equations satisfied by massive form factors, and their solution. In section 3, we perform the analysis for the massless case. Then, in section 4, we use the new planar results to perform a matching, and give the new predictions at higher loop orders. In section 5, we explicitly compute the universal conversion factors between massive and massless regularizations. We conclude in section 6.

## 2 Renormalization group equation: massive case

The form factors satisfy the KG integro-differential equation [2, 4] which is merely a consequence of the factorization property and of gauge and renormalization group (RG) invariances. It reads

$$-\frac{d}{d \ln \mu^2} \ln \tilde{F} \left( \hat{a}_s, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \epsilon \right) = \frac{1}{2} \left[ \tilde{K} \left( \hat{a}_s, \frac{m^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon \right) + \tilde{G} \left( \hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon \right) \right], \quad (2.1)$$

where the quantity  $\tilde{F}$  is related to form factor<sup>1</sup>  $F$  through a matching coefficient  $C$  (see also below) via the relation

$$F = C(a_s(m^2), \epsilon) e^{\ln \tilde{F}}. \quad (2.2)$$

In Eq. (2.1), we have  $Q^2 = -q^2 = -(p_1 + p_2)^2$  where  $p_i$  are the momenta of the external massive partons satisfying  $p_i^2 = m^2$  with  $m$  being the on-shell quark mass. The momentum

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<sup>1</sup>In this section we generically write  $F$  which stands for the form factor  $F_1$  in Eq. (1.1). Note that  $F_2$  is suppressed by  $m^2/q^2$  in the high-energy limit.

of the colorless particle, i.e. the virtual photon, is represented by  $q$ . The quantities

$$\hat{a}_s \equiv \frac{\hat{\alpha}_s}{4\pi} \quad \text{and} \quad a_s \equiv \frac{\alpha_s}{4\pi} \quad (2.3)$$

are the bare and renormalized strong coupling constants, respectively. To keep  $\hat{a}_s$  dimensionless in  $d = 4 - 2\epsilon$ , the mass scale  $\mu$  is introduced.  $\mu_R$  is the renormalization scale. In Eq. (2.1), all the  $m^2$  dependence of the  $\ln \tilde{F}$  is captured through the function  $\tilde{K}$ , whereas  $\tilde{G}$  contains the  $Q^2$  dependent part. The RG invariance of the form factor with respect to  $\mu_R$  implies

$$\lim_{m \rightarrow 0} \frac{d}{d \ln \mu_R^2} \tilde{K} \left( \hat{a}_s, \frac{m^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon \right) = - \frac{d}{d \ln \mu_R^2} \tilde{G} \left( \hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon \right) = -A(a_s(\mu_R^2)), \quad (2.4)$$

where  $A$  is the light-like cusp anomalous dimension. The renormalized and bare strong coupling constants are related through

$$\hat{a}_s S_\epsilon = a_s(\mu_R^2) Z_{a_s}(\mu_R^2) \left( \frac{\mu^2}{\mu_R^2} \right)^{-\epsilon}, \quad (2.5)$$

with  $S_\epsilon = \exp[-(\gamma_E - \ln 4\pi)\epsilon]$ . The renormalization constant  $Z_{a_s}(\mu_R^2)$  [11] is given by

$$\begin{aligned} Z_{a_s}(\mu_R^2) = & 1 + a_s(\mu_R^2) \left\{ -\frac{1}{\epsilon} \beta_0 \right\} + a_s^2(\mu_R^2) \left\{ \frac{1}{\epsilon^2} \beta_0^2 - \frac{1}{2\epsilon} \beta_1 \right\} + a_s^3(\mu_R^2) \left\{ -\frac{1}{\epsilon^3} \beta_0^3 + \frac{7}{6\epsilon^2} \beta_0 \beta_1 \right. \\ & \left. - \frac{1}{3\epsilon} \beta_2 \right\} + a_s^4(\mu_R^2) \left\{ \frac{1}{\epsilon^4} \beta_0^4 - \frac{23}{12\epsilon^3} \beta_0^2 \beta_1 + \frac{1}{\epsilon^2} \left( \frac{3}{8} \beta_1^2 + \frac{5}{6} \beta_0 \beta_2 \right) - \frac{1}{4\epsilon} \beta_3 \right\} \end{aligned} \quad (2.6)$$

up to  $\mathcal{O}(a_s^4)$  with the first coefficient of QCD  $\beta$  function given by

$$\beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_f. \quad (2.7)$$

$C_A = N$  and  $C_F = (N^2 - 1)/2N$  are the eigenvalues of the quadratic Casimir operators of SU( $N$ ) group of the underlying gauge theory.  $n_f$  is the number of active quark flavors. For our calculation of the massive form factors up to four-loop order, we need  $Z_{a_s}$  to  $\mathcal{O}(a_s^3)$ . However, for the massless case at five loop, which is discussed in Section 3, the term to  $\mathcal{O}(a_s^4)$  is required if the bare coupling constant is replaced by the renormalized one. The  $\beta$  functions to three and four loops can be found in Refs. [11] and [12, 13], respectively.

We solve the RG equation (2.4) and consequently (2.1) following the methodology used for the massless case which has been discussed in [14, 15] (see also [16] for details). The solutions of  $\tilde{K}$  and  $\tilde{G}$  are obtained as<sup>2</sup>

$$\tilde{K} \left( \hat{a}_s, \frac{m^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon \right) = K(a_s(m^2), \epsilon) - \int_{m^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A(a_s(\mu_R^2)),$$

<sup>2</sup>In the following, we will tacitly assume that  $m^2$  is small with respect to  $Q^2$ .

$$\tilde{G}\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = G(a_s(Q^2), \epsilon) + \int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A(a_s(\mu_R^2)), \quad (2.8)$$

where the functions  $K$  and  $G$  are determined at the boundaries  $\mu_R^2 = m^2$  and  $\mu_R^2 = Q^2$ , respectively. Our initial goal is to solve for  $\ln \tilde{F}$  in Eq. (2.1) in powers of the bare coupling  $\hat{a}_s$ . In order to achieve that we need to obtain the solutions of  $\tilde{K}$  and  $\tilde{G}$  in powers of  $\hat{a}_s$ . We begin by expanding the relevant quantities in powers of the renormalized strong coupling constant as

$$\mathcal{B}(a_s(\lambda^2)) \equiv \sum_{k=1}^{\infty} a_s^k(\lambda^2) \mathcal{B}_k \quad (2.9)$$

with  $\mathcal{B} \in \{K, G, A\}$  and the argument  $\lambda$  of  $a_s$  refers to the corresponding parameter, i.e.,  $\lambda \in \{m, Q, \mu_R\}$ . The dependence of  $G$  and  $K$  on  $\epsilon$  is implicit in  $\mathcal{B}_k$ . In order to obtain the expansion of  $\mathcal{B}$  in powers of  $\hat{a}_s$ , we require the  $Z_{a_s}^{-1}(\lambda^2)$  which is obtained as

$$Z_{a_s}^{-1}(\lambda^2) = 1 + \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{\lambda^2}{\mu^2}\right)^{-k\epsilon} \hat{Z}_{a_s}^{-1,(k)} \quad (2.10)$$

with

$$\begin{aligned} \hat{Z}_{a_s}^{-1,(1)} &= \frac{1}{\epsilon} \beta_0, \\ \hat{Z}_{a_s}^{-1,(2)} &= \frac{1}{\epsilon^2} \beta_0^2 + \frac{1}{2\epsilon} \beta_1, \\ \hat{Z}_{a_s}^{-1,(3)} &= \frac{1}{\epsilon^3} \beta_0^3 + \frac{4}{3\epsilon^2} \beta_0 \beta_1 + \frac{1}{3\epsilon} \beta_2, \\ \hat{Z}_{a_s}^{-1,(4)} &= \frac{1}{\epsilon^4} \beta_0^4 + \frac{29}{12\epsilon^3} \beta_0^2 \beta_1 + \frac{1}{\epsilon^2} \left( \frac{3}{8} \beta_1^2 + \frac{7}{6} \beta_0 \beta_2 \right) + \frac{1}{4\epsilon} \beta_3, \end{aligned} \quad (2.11)$$

up to  $\mathcal{O}(\hat{a}_s^4)$ . Employing Eq. (2.5) and  $Z_{a_s}^{-1}$ , we can express  $\mathcal{B}$  in powers of  $\hat{a}_s$  as

$$\mathcal{B}(a_s(\lambda^2)) = \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{\lambda^2}{\mu^2}\right)^{-k\epsilon} \hat{\mathcal{B}}_k \quad (2.12)$$

where

$$\begin{aligned} \hat{\mathcal{B}}_1 &= \mathcal{B}_1, \\ \hat{\mathcal{B}}_2 &= \mathcal{B}_2 + \mathcal{B}_1 \hat{Z}_{a_s}^{-1,(1)}, \\ \hat{\mathcal{B}}_3 &= \mathcal{B}_3 + 2\mathcal{B}_2 \hat{Z}_{a_s}^{-1,(1)} + \mathcal{B}_1 \hat{Z}_{a_s}^{-1,(2)}, \\ \hat{\mathcal{B}}_4 &= \mathcal{B}_4 + 3\mathcal{B}_3 \hat{Z}_{a_s}^{-1,(1)} + \mathcal{B}_2 \left\{ \left( \hat{Z}_{a_s}^{-1,(1)} \right)^2 + 2\hat{Z}_{a_s}^{-1,(2)} \right\} + \mathcal{B}_1 \hat{Z}_{a_s}^{-1,(3)}, \\ \hat{\mathcal{B}}_5 &= \mathcal{B}_5 + 4\mathcal{B}_4 \hat{Z}_{a_s}^{-1,(1)} + 3\mathcal{B}_3 \left\{ \left( \hat{Z}_{a_s}^{-1,(1)} \right)^2 + \hat{Z}_{a_s}^{-1,(2)} \right\} + 2\mathcal{B}_2 \left\{ \hat{Z}_{a_s}^{-1,(1)} \hat{Z}_{a_s}^{-1,(2)} + \hat{Z}_{a_s}^{-1,(3)} \right\} \\ &\quad + \mathcal{B}_1 \hat{Z}_{a_s}^{-1,(4)}. \end{aligned} \quad (2.13)$$

With the help of Eq. (2.12) we evaluate the integral appearing on the right hand side of Eq. (2.8) and we obtain

$$\int_{\lambda^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A(a_s(\mu_R^2)) = \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \frac{1}{k\epsilon} \left[ \left( \frac{\lambda^2}{\mu^2} \right)^{-k\epsilon} - \left( \frac{\mu_R^2}{\mu^2} \right)^{-k\epsilon} \right] \hat{A}_k, \quad (2.14)$$

where we either have  $\lambda^2 = m^2$  or  $\lambda^2 = Q^2$ . At this point it is straightforward to solve for  $\tilde{K}$  and  $\tilde{G}$  using Eqs. (2.12) and (2.14) which consequently leads us to the solution for the KG equation (2.1) in powers of  $\hat{a}_s$  as

$$\ln \tilde{F} \left( \hat{a}_s, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \epsilon \right) = \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left[ \left( \frac{Q^2}{\mu^2} \right)^{-k\epsilon} \hat{\mathcal{L}}_k^Q(\epsilon) + \left( \frac{m^2}{\mu^2} \right)^{-k\epsilon} \hat{\mathcal{L}}_k^m(\epsilon) \right], \quad (2.15)$$

with

$$\begin{aligned} \hat{\mathcal{L}}_k^Q(\epsilon) &= -\frac{1}{2k\epsilon} \left[ \hat{G}_k + \frac{1}{k\epsilon} \hat{A}_k \right], \\ \hat{\mathcal{L}}_k^m(\epsilon) &= -\frac{1}{2k\epsilon} \left[ \hat{K}_k - \frac{1}{k\epsilon} \hat{A}_k \right]. \end{aligned} \quad (2.16)$$

Equivalently, we can express the solution of  $\ln \tilde{F}$  in powers of renormalized coupling constant  $a_s(\mu_R^2)$  using Eq. (2.5). Without loss of generality, we present the results for  $\mu_R^2 = m^2$  and write

$$\ln \tilde{F} = \sum_{k=1}^{\infty} a_s^k(m^2) \tilde{\mathcal{L}}_k. \quad (2.17)$$

This is achieved with the help of the  $d$ -dimensional evolution of  $a_s(\mu_R^2)$  satisfying the RG equation

$$\frac{d}{d \ln \mu_R^2} a_s(\mu_R^2) = -\epsilon a_s(\mu_R^2) - \sum_{k=0}^{\infty} \beta_k a_s^{k+2}(\mu_R^2) \quad (2.18)$$

which is solved iteratively. The solution up to  $\mathcal{O}(a_s^4)$  reads

$$\begin{aligned} a_s(\mu_R^2) &= a_s(m^2) \left\{ 1 - \epsilon L_R + \frac{\epsilon^2 L_R^2}{2} - \frac{\epsilon^3 L_R^3}{6} + \frac{\epsilon^4 L_R^4}{24} - \frac{\epsilon^5 L_R^5}{120} + \frac{\epsilon^6 L_R^6}{720} \right\} \\ &+ a_s^2(m^2) \left\{ -\beta_0 L_R + \frac{3}{2} \beta_0 \epsilon L_R^2 - \frac{7}{6} \beta_0 \epsilon^2 L_R^3 + \frac{5}{8} \beta_0 \epsilon^3 L_R^4 - \frac{31}{120} \beta_0 \epsilon^4 L_R^5 \right. \\ &+ \left. \frac{7}{80} \beta_0 \epsilon^5 L_R^6 \right\} + a_s^3(m^2) \left\{ \beta_0^2 L_R^2 - \beta_1 L_R + \epsilon \left( 2\beta_1 L_R^2 - 2\beta_0^2 L_R^3 \right) \right. \\ &+ \epsilon^2 \left( \frac{25\beta_0^2 L_R^4}{12} - \frac{13\beta_1 L_R^3}{6} \right) + \epsilon^3 \left( \frac{5\beta_1 L_R^4}{3} - \frac{3\beta_0^2 L_R^5}{2} \right) + \epsilon^4 \left( \frac{301\beta_0^2 L_R^6}{360} \right. \\ &+ \left. \left. - \frac{121\beta_1 L_R^5}{120} \right) \right\} + a_s^4(m^2) \left\{ -\beta_0^3 L_R^3 + \frac{5}{2} \beta_0 \beta_1 L_R^2 - \beta_2 L_R + \epsilon \left( \frac{5\beta_0^3 L_R^4}{2} \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{19}{3}\beta_0\beta_1L_R^3 + \frac{5\beta_2L_R^2}{2} \Big) + \epsilon^2 \left( -\frac{13}{4}\beta_0^3L_R^5 + \frac{205}{24}\beta_0\beta_1L_R^4 - \frac{7\beta_2L_R^3}{2} \right) \\
& + \epsilon^3 \left( \frac{35\beta_0^3L_R^6}{12} - \frac{97}{12}\beta_0\beta_1L_R^5 + \frac{85\beta_2L_R^4}{24} \right) \Big\}, \tag{2.19}
\end{aligned}$$

with  $L_R = \ln(\mu_R^2/m^2)$ . We have presented Eq. (2.19) only up to the order in  $\epsilon$  relevant for our calculation. The terms up to  $\mathcal{O}(a_s^3)$  can be found in [5, 17]. Upon employing Eq. (2.19), we get

$$\begin{aligned}
\tilde{\mathcal{L}}_1 &= \frac{1}{\epsilon} \left\{ -\frac{1}{2} \left( G_1 + K_1 - A_1L \right) \right\} + \frac{L}{2} \left( G_1 - \frac{A_1L}{2} \right) - \epsilon \left\{ \frac{L^2}{4} \left( G_1 - \frac{A_1L}{3} \right) \right\} \\
&+ \epsilon^2 \left\{ \frac{L^3}{12} \left( G_1 - \frac{A_1L}{4} \right) \right\} - \epsilon^3 \left\{ \frac{L^4}{48} \left( G_1 - \frac{A_1L}{5} \right) \right\} + \epsilon^4 \left\{ \frac{L^5}{240} \left( G_1 - \frac{A_1L}{6} \right) \right\} + \mathcal{O}(\epsilon^5), \\
\tilde{\mathcal{L}}_2 &= \frac{1}{\epsilon^2} \left\{ \frac{\beta_0}{4} \left( G_1 + K_1 - A_1L \right) \right\} - \frac{1}{\epsilon} \left\{ \frac{1}{4} \left( G_2 + K_2 - A_2L \right) \right\} + \frac{L}{2} \left( G_2 - \frac{A_2L}{2} \right) \\
&- \frac{\beta_0L^2}{4} \left( G_1 - \frac{A_1L}{3} \right) - \epsilon \left\{ \frac{L^2}{2} \left( G_2 - \frac{A_2L}{3} \right) - \frac{\beta_0L^3}{4} \left( G_1 - \frac{A_1L}{4} \right) \right\} \\
&+ \epsilon^2 \left\{ \frac{L^3}{3} \left( G_2 - \frac{A_2L}{4} \right) - \frac{7\beta_0L^4}{48} \left( G_1 - \frac{A_1L}{5} \right) \right\} - \epsilon^3 \left\{ \frac{L^4}{6} \left( G_2 - \frac{A_2L}{5} \right) \right. \\
&\left. - \frac{\beta_0L^5}{16} \left( G_1 - \frac{A_1L}{6} \right) \right\} + \mathcal{O}(\epsilon^4), \\
\tilde{\mathcal{L}}_3 &= \frac{1}{\epsilon^3} \left\{ -\frac{\beta_0^2}{6} \left( G_1 + K_1 - A_1L \right) \right\} + \frac{1}{\epsilon^2} \left\{ \frac{\beta_1}{6} \left( G_1 + K_1 - A_1L \right) + \frac{\beta_0}{6} \left( G_2 + K_2 \right. \right. \\
&\left. \left. - A_2L \right) \right\} - \frac{1}{\epsilon} \left\{ \frac{1}{6} \left( G_3 + K_3 - A_3L \right) \right\} + \frac{L}{2} \left( G_3 - \frac{A_3L}{2} \right) - \frac{\beta_0L^2}{2} \left( G_2 - \frac{A_2L}{3} \right) \\
&+ \frac{\beta_0^2L^3}{6} \left( G_1 - \frac{A_1L}{4} \right) - \frac{\beta_1L^2}{4} \left( G_1 - \frac{A_1L}{3} \right) - \epsilon \left\{ \frac{3L^2}{4} \left( G_3 - \frac{A_3L}{3} \right) \right. \\
&\left. - \frac{5\beta_0L^3}{6} \left( G_2 - \frac{A_2L}{4} \right) + \frac{\beta_0^2L^4}{4} \left( G_1 - \frac{A_1L}{5} \right) - \frac{\beta_1L^3}{3} \left( G_1 - \frac{A_1L}{4} \right) \right\} \\
&+ \epsilon^2 \left\{ \frac{3L^3}{4} \left( G_3 - \frac{A_3L}{4} \right) - \frac{19\beta_0L^4}{24} \left( G_2 - \frac{A_2L}{5} \right) + \frac{5\beta_0^2L^5}{24} \left( G_1 - \frac{A_1L}{6} \right) \right. \\
&\left. - \frac{13\beta_1L^4}{48} \left( G_1 - \frac{A_1L}{5} \right) \right\} + \mathcal{O}(\epsilon^3), \\
\tilde{\mathcal{L}}_4 &= \frac{1}{\epsilon^4} \left\{ \frac{\beta_0^3}{8} \left( G_1 + K_1 - A_1L \right) \right\} - \frac{1}{\epsilon^3} \left\{ \frac{\beta_0^2}{8} \left( G_2 + K_2 - A_2L \right) + \frac{\beta_0\beta_1}{4} \left( G_1 + K_1 \right. \right. \\
&\left. \left. - A_1L \right) \right\} + \frac{1}{\epsilon^2} \left\{ \frac{\beta_0}{8} \left( G_3 + K_3 - A_3L \right) + \frac{\beta_1}{8} \left( G_2 + K_2 - A_2L \right) + \frac{\beta_2}{8} \left( G_1 + K_1 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - A_1 L \Big) \Big\} - \frac{1}{\epsilon} \left\{ \frac{1}{8} \left( G_4 + K_4 - A_4 L \right) \right\} + \frac{L}{2} \left( G_4 - \frac{A_4 L}{2} \right) - \frac{3\beta_0 L^2}{4} \left( G_3 - \frac{A_3 L}{3} \right) \\
& + \frac{\beta_0^2 L^3}{2} \left( G_2 - \frac{A_2 L}{4} \right) - \frac{\beta_0^3 L^4}{8} \left( G_1 - \frac{A_1 L}{5} \right) + \frac{5\beta_0 \beta_1 L^3}{12} \left( G_1 - \frac{A_1 L}{4} \right) \\
& - \frac{\beta_1 L^2}{2} \left( G_2 - \frac{A_2 L}{3} \right) - \frac{\beta_2 L^2}{4} \left( G_1 - \frac{A_1 L}{3} \right) - \epsilon \left\{ L^2 \left( G_4 - \frac{A_4 L}{3} \right) \right. \\
& - \frac{7\beta_0 L^3}{4} \left( G_3 - \frac{A_3 L}{4} \right) + \frac{9\beta_0^2 L^4}{8} \left( G_2 - \frac{A_2 L}{5} \right) - \frac{\beta_0^3 L^5}{4} \left( G_1 - \frac{A_1 L}{6} \right) \\
& \left. + \frac{19\beta_0 \beta_1 L^4}{24} \left( G_1 - \frac{A_1 L}{5} \right) - \beta_1 L^3 \left( G_2 - \frac{A_2 L}{4} \right) - \frac{5\beta_2 L^3}{12} \left( G_1 - \frac{A_1 L}{4} \right) \right\} + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{2.20}$$

with  $L = \log(Q^2/m^2)$ . Up to three-loop order we find agreement with the results provided in Refs. [2, 4]. The four-loop expression  $\tilde{\mathcal{L}}_4$  is new.

Before proceeding further, let us make some remarks on the solution of KG integro-differential equation. The solution provided in Eq. (2.15) relies on the fact that we have a through-going heavy quark line from the external quark to the photon-quark coupling and then to the external anti-quark. In particular, we do not consider contributions originating from closed heavy-quark loops or so-called singlet contributions where the photon does not couple to the external quark line. Note that, these contributions also contain Sudakov logarithms which obey an exponentiation similar to the case under consideration [2]. However, in the large- $N$  limit they are sub-leading.

To arrive at the solution of the KG equation, Eq. (2.20), we have used the standard  $\overline{\text{MS}}$  coupling running with  $n_l$  light flavours. On the other hand, the explicit fixed order results of the form factors depend on  $\alpha_s$  with  $n_f = n_l + 1$  active flavours. Hence, to compare these two results (in particular, to perform the matching) it is necessary to use the  $d$ -dimensional decoupling relation [18–21] (see also [22]) which establishes the connection between  $\alpha_s$  in the full and effective theory. Note, however, that the decoupling relation generates contributions which are sub-dominant in the large- $N$  limit. Hence, in this article we can ignore the difference between  $\alpha_s$  defined with  $n_f$  or  $n_l$  active quark flavours.

Results for the form factor are obtained with the help of Eq. (2.2) where the matching coefficient  $C$  is expanded in powers of  $a_s(m^2)$  according to

$$C(a_s(m^2), \epsilon) = 1 + \sum_{k=1}^{\infty} a_s^k(m^2) C_k(\epsilon). \tag{2.21}$$

The coefficients  $C_k, G_k$  and  $K_k$  are obtained from comparing Eq. (2.2) with explicit calculations for  $F$ . We determine  $F$  up to the  $1/\epsilon^2$  pole at four loops which requires the following input:  $G_1$  to  $\mathcal{O}(\epsilon^2)$ ,  $G_2$  to  $\mathcal{O}(\epsilon)$ , and  $G_3$  to  $\mathcal{O}(\epsilon^0)$ . Furthermore we need  $K$  and  $A$  to three-loop order and  $C_1$  to  $\mathcal{O}(\epsilon)$  and  $C_2$  up to the constant term in  $\epsilon$ . The explicit results for  $G_k, K_k, A_k$  and  $C_k$  to the relevant order in  $\epsilon$  are presented in the Section 4.



It is interesting to note that in the conformal case, i.e.  $\beta_i = 0$ , the above considerations simplify, and one obtains the following all-order solution

$$\tilde{\mathcal{L}}_k = \sum_{l=0}^{\infty} (-\epsilon k)^{l-1} \frac{L^l}{2^l l!} \left( G_k + \delta_{0l} K_k - \frac{A_k L}{l+1} \right) \quad (2.22)$$

where  $\delta_{ij}$  is the Kronecker delta function.

### 3 Renormalization group equation: massless case

The massless form factors also satisfy KG integro-differential equation [23–26], similar to the massive one (see Eq. (2.1)). It is also dictated by the factorization property and by gauge and RG invariance

$$\frac{d}{d \ln Q^2} \ln \bar{F} \left( \hat{a}_s, \frac{Q^2}{\mu^2}, \epsilon \right) = \frac{1}{2} \left[ \bar{K} \left( \hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon \right) + \bar{G} \left( \hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon \right) \right], \quad (3.1)$$

with  $Q^2 = -q^2 = -(p_1 + p_2)^2$  and  $p_i$  are the momenta of the external massless partons satisfying  $p_i^2 = 0$ . The quantities  $\bar{K}$  and  $\bar{G}$  play similar role to those of  $\tilde{K}$  and  $\tilde{G}$  in Eq. (2.1). Because of the dependence of the form factor on the quantities  $Q^2$  and  $\mu^2$  through the ratio  $Q^2/\mu^2$ , the KG equation can equally be written as

$$-\frac{d}{d \ln \mu^2} \ln \bar{F} \left( \hat{a}_s, \frac{Q^2}{\mu^2}, \epsilon \right) = \frac{1}{2} \left[ \bar{K} \left( \hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon \right) + \bar{G} \left( \hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon \right) \right]. \quad (3.2)$$

This equation is the analogue to Eq. (2.1) with the difference that there is no mass dependence. Hence, it can be solved in a similar way as discussed in the previous section for the massive case. The general solution is obtained as

$$\ln \bar{F} \left( \hat{a}_s, \frac{Q^2}{\mu^2}, \epsilon \right) = \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left( \frac{Q^2}{\mu^2} \right)^{-k\epsilon} \hat{\mathcal{L}}_k(\epsilon) \quad (3.3)$$

which corresponds to Eq. (2.15) with  $\hat{\mathcal{L}}_k^Q = \hat{\mathcal{L}}_k$  and vanishing  $\hat{\mathcal{L}}_k^m$ . This is consistent with the existing solutions up to four loops [5, 14]. The solution at the five loop level reads

$$\begin{aligned} \hat{\mathcal{L}}_5 = & \frac{1}{\epsilon^6} \left\{ -\frac{A_1 \beta_0^4}{50} \right\} + \frac{1}{\epsilon^5} \left\{ -\frac{29A_1 \beta_0^2 \beta_1}{600} - \frac{2A_2 \beta_0^3}{25} - \frac{\beta_0^4 G_1}{10} \right\} + \frac{1}{\epsilon^4} \left\{ -\frac{7A_1 \beta_0 \beta_2}{300} \right. \\ & \left. - \frac{3A_1 \beta_1^2}{400} - \frac{11A_2 \beta_0 \beta_1}{150} - \frac{3A_3 \beta_0^2}{25} - \frac{2\beta_0^3 G_2}{5} - \frac{29\beta_0^2 \beta_1 G_1}{120} \right\} + \frac{1}{\epsilon^3} \left\{ -\frac{A_1 \beta_3}{200} - \frac{A_2 \beta_2}{75} \right. \\ & \left. - \frac{3A_3 \beta_1}{100} - \frac{2A_4 \beta_0}{25} - \frac{3\beta_0^2 G_3}{5} - \frac{11\beta_0 \beta_1 G_2}{30} - \frac{7\beta_0 \beta_2 G_1}{60} - \frac{3\beta_1^2 G_1}{80} \right\} + \frac{1}{\epsilon^2} \left\{ -\frac{A_5}{50} \right. \\ & \left. - \frac{2\beta_0 G_4}{5} - \frac{3\beta_1 G_3}{20} - \frac{\beta_2 G_2}{15} - \frac{\beta_3 G_1}{40} \right\} - \frac{G_5}{10\epsilon}. \end{aligned} \quad (3.4)$$

The form factor can be obtained by exponentiating the  $\ln \bar{F}$ :

$$\bar{F} = e^{\ln \bar{F}} = 1 + \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left( \frac{Q^2}{\mu^2} \right)^{-k\epsilon} \bar{F}^{(k)}. \quad (3.5)$$

Note that in the massless case the matching coefficient is identical to 1. In the next section the results of the  $\bar{F}^{(5)}$  is presented in the planar limit including terms up to  $1/\epsilon^3$ , where we restrict ourselves, as in the massive case, to the non-singlet contributions, since the singlet terms only contribute to sub-leading colour structures.

In the conformal case, for which  $\beta_i = 0$ , the above considerations simplify, and one obtains an all order result [7, 27]

$$\hat{\mathcal{L}}_k = \frac{1}{\epsilon^2} \left\{ -\frac{1}{2k^2} A_k \right\} + \frac{1}{\epsilon} \left\{ -\frac{1}{2k} G_k \right\}. \quad (3.6)$$

#### 4 Matching of perturbative results: four- and five-loop predictions

In this section, we use the results of the recent three-loop computation of the planar massive form factors [10] in order to determine the undetermined coefficients in section 2, in the planar limit. This will allow us to make concise four-loop predictions. For the massless form factor the results from Ref. [8, 9] are used to predict the leading five-loop terms.

A comment is due regarding the definition of the planar limit. The easiest way of thinking about it is to consider  $SU(N)$  gauge group, and take the 't Hooft limit,  $N \rightarrow \infty$ , keeping  $a_s N$  fixed. In the presence of light fermions, we also want to keep the planar diagrams involving fermion loops, which means that<sup>3</sup>  $n_l$  should count the same as  $N$ . This can be reformulated in the simple rule that we keep all terms  $n_l^{a_1} N^{a_2} a_s^{a_3}$  with  $a_1 + a_2 = a_3$ .

The cusp anomalous dimension is known to three loops from [28–34], and all the  $n_l$  terms in the planar limit at four loop are known from [8, 35, 36]. The  $n_l$  independent terms at four loops recently became available [9]. In the planar limit we have

$$\begin{aligned} A_1 &= 2N, \\ A_2 &= \left( \frac{134}{9} - \frac{2\pi^2}{3} \right) N^2 - \frac{20}{9} N n_l, \\ A_3 &= N^3 \left( \frac{44\zeta_3}{3} + \frac{245}{3} - \frac{268\pi^2}{27} + \frac{22\pi^4}{45} \right) + N^2 n_l \left( -\frac{32\zeta_3}{3} - \frac{1331}{54} + \frac{40\pi^2}{27} \right) - \frac{8}{27} N n_l^2, \\ A_4 &= N^4 \left( -16\zeta_3^2 - \frac{88\pi^2\zeta_3}{9} + \frac{10496\zeta_3}{27} - 176\zeta_5 - \frac{146\pi^6}{315} + \frac{451\pi^4}{45} - \frac{22208\pi^2}{243} + \frac{42139}{81} \right) \\ &\quad + N^3 n_l \left( -\frac{8126\zeta_3}{27} + \frac{64\pi^2\zeta_3}{9} + 112\zeta_5 - \frac{39883}{162} + \frac{6673\pi^2}{243} - \frac{22\pi^4}{27} \right) + N^2 n_l^2 \left( \frac{640\zeta_3}{27} \right) \end{aligned}$$

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<sup>3</sup>In this paper we denote the number of massless quarks for the massless form factor by  $n_l$  to be consistent with the notation for the massive form factor.

$$+ \frac{2119}{162} - \frac{152\pi^2}{243} - \frac{16\pi^4}{135} \Big) + Nn_l^3 \left( \frac{32\zeta_3}{27} - \frac{16}{81} \right). \quad (4.1)$$

The coefficients  $C_k$ ,  $G_k$  and  $K_k$  are determined by comparing the general solutions obtained from solving the KG equations with the results of the explicit computations. The coefficients  $C_1$  up to  $\mathcal{O}(\epsilon^2)$  and  $C_2$  to  $\mathcal{O}(\epsilon)$  have been obtained in Ref. [4], in agreement with our findings. In this article we extend  $C_1$  to  $\mathcal{O}(\epsilon^4)$  and  $C_2$  to  $\mathcal{O}(\epsilon^2)$ , respectively, which is needed for the conversion factors discussed in Section 5. Moreover, using the results of Ref. [10] we obtain a new expression for  $C_3$  up to the constant term in  $\epsilon$ . For convenience we present explicit results in the planar limit. We have

$$\begin{aligned} C_1 = N & \left[ 2 + \frac{\pi^2}{12} + \epsilon \left\{ 4 + \frac{\pi^2}{24} - \frac{\zeta_3}{3} \right\} + \epsilon^2 \left\{ -\frac{\zeta(3)}{6} + 8 + \frac{\pi^2}{6} + \frac{\pi^4}{160} \right\} + \epsilon^3 \left\{ -\frac{\pi^2\zeta_3}{36} - \frac{2\zeta_3}{3} \right. \right. \\ & \left. \left. - \frac{\zeta_5}{5} + \frac{\pi^4}{320} + \frac{\pi^2}{3} + 16 \right\} + \epsilon^4 \left\{ \frac{\zeta_3^2}{18} - \frac{\pi^2\zeta_3}{72} - \frac{4\zeta_3}{3} - \frac{\zeta_5}{10} + \frac{61\pi^6}{120960} + \frac{\pi^4}{80} + \frac{2\pi^2}{3} + 32 \right\} \right. \\ & \left. + \mathcal{O}(\epsilon^5) \right], \\ C_2 = N^2 & \left( \frac{31\zeta_3}{9} + \frac{45275}{2592} + \frac{557\pi^2}{216} - \frac{39\pi^4}{160} \right) + Nn_l \left( -\frac{13\zeta_3}{9} - \frac{1541}{648} - \frac{37\pi^2}{108} \right) \\ & + \epsilon \left\{ N^2 \left( \frac{125\zeta_3}{54} + \frac{5\pi^2\zeta_3}{18} - \frac{211\zeta_5}{2} + \frac{1124291}{15552} + \frac{26383\pi^2}{2592} + \frac{727\pi^4}{1440} \right) + Nn_l \left( -\frac{91\zeta(3)}{27} \right. \right. \\ & \left. \left. - \frac{46205}{3888} - \frac{673\pi^2}{648} - \frac{49\pi^4}{360} \right) \right\} + \epsilon^2 \left\{ N^2 \left( -\frac{341\zeta_3^2}{18} + \frac{323\pi^2\zeta_3}{54} + \frac{37969\zeta_3}{648} + \frac{403\zeta_5}{15} \right. \right. \\ & \left. \left. - \frac{150767\pi^6}{362880} + \frac{6205\pi^4}{3456} + \frac{524455\pi^2}{15552} + \frac{23123195}{93312} \right) + Nn_l \left( -\frac{28\pi^2\zeta_3}{27} - \frac{1711\zeta_3}{162} - \frac{361\zeta_5}{15} \right. \right. \\ & \left. \left. - \frac{1607\pi^4}{4320} - \frac{15481\pi^2}{3888} - \frac{1063589}{23328} \right) \right\} + \mathcal{O}(\epsilon^3), \\ C_3 = N^3 & \left( -\frac{15743\zeta_3}{486} + \frac{533\pi^2\zeta(3)}{108} + \frac{298\zeta_3^2}{9} - 357\zeta(5) + \frac{10907077}{52488} + \frac{15763067\pi^2}{279936} \right. \\ & \left. - \frac{116957\pi^4}{38880} + \frac{145051\pi^6}{653184} \right) + N^2 n_l \left( -\frac{296\zeta_3}{9} - \frac{77\pi^2\zeta_3}{108} + \frac{602\zeta_5}{9} - \frac{6389497}{104976} \right. \\ & \left. - \frac{1160555\pi^2}{69984} - \frac{1337\pi^4}{6480} \right) + Nn_l^2 \left( \frac{1072\zeta(3)}{243} + \frac{58883}{26244} + \frac{145\pi^2}{243} + \frac{221\pi^4}{2430} \right) + \mathcal{O}(\epsilon). \end{aligned} \quad (4.2)$$

The coefficients  $G_1$  to  $\mathcal{O}(\epsilon^3)$  and  $G_2$  to  $\mathcal{O}(\epsilon^0)$  are presented in Ref. [2]. In this article, we extend the results to higher orders in  $\epsilon$  and obtain the  $G_3$  for the first time from the recent results of the massive form factors [10] in planar limit. It has been observed in Ref. [2] that the  $G_k$  for the massive quark form factor coincide with those of the massless ones [5, 31].

This is also true for the newly computed coefficients. Note that within our method this feature is not surprising since the massless results are obtained from the massive one by putting the mass-dependent part ( $\tilde{\mathcal{L}}_k^m$ ) of the solution (2.15) to zero and by setting  $C = 1$ . It is interesting to note that the quantities  $K_i$ , which capture the mass dependence of  $\ln \tilde{F}$ , only enter into the pole terms of  $\tilde{\mathcal{L}}_k$  in Eq. (2.17). As a consequence, the constant and  $\epsilon^k$  term can be determined from the massless calculation and are thus universal. This could lead to deeper understanding of the connection between the massive and massless form factors.

In order to predict the four loop massive form factors to  $1/\epsilon^2$ , we require  $G_1$  to  $\mathcal{O}(\epsilon^2)$ ,  $G_2$  to  $\mathcal{O}(\epsilon)$  and  $G_3$  to  $\mathcal{O}(\epsilon^0)$ . Moreover, to obtain the predictions of the massless quark form factors at five loop order up to  $1/\epsilon^3$ , we need  $G_4$  to  $\mathcal{O}(\epsilon^0)$  and in addition  $G_1$ ,  $G_2$  and  $G_3$  to  $\mathcal{O}(\epsilon^6)$ ,  $\mathcal{O}(\epsilon^4)$  and  $\mathcal{O}(\epsilon^2)$ , respectively. With the help of the results of the massless quark form factors to three loops [33, 34, 37], we calculate  $G_k$  ( $k = 1, 2, 3$ ) to the required orders in  $\epsilon$ .  $G_4$  is obtained from the recent results of four loop massless quark form factors in the planar limit [8, 9]. These quantities to the relevant orders in  $\epsilon$  are given by

$$\begin{aligned}
G_1 &= 3N + \epsilon \left\{ N \left( 8 - \frac{\pi^2}{6} \right) \right\} + \epsilon^2 \left\{ N \left( -\frac{14\zeta(3)}{3} + 16 - \frac{\pi^2}{4} \right) \right\} + \epsilon^3 \left\{ N \left( -7\zeta_3 - \frac{47\pi^4}{720} \right. \right. \\
&\quad \left. \left. - \frac{2\pi^2}{3} + 32 \right) \right\} + \epsilon^4 \left\{ N \left( \frac{7\pi^2\zeta_3}{18} - \frac{56\zeta_3}{3} - \frac{62\zeta_5}{5} - \frac{47\pi^4}{480} - \frac{4\pi^2}{3} + 64 \right) \right\} \\
&\quad + \epsilon^5 \left\{ N \left( \frac{49\zeta_3^2}{9} + \frac{7\pi^2\zeta_3}{12} - \frac{112\zeta_3}{3} - \frac{93\zeta_5}{5} - \frac{949\pi^6}{60480} - \frac{47\pi^4}{180} - \frac{8\pi^2}{3} + 128 \right) \right\} \\
&\quad + \epsilon^6 \left\{ N \left( \frac{49\zeta_3^2}{6} + \frac{329\pi^4\zeta_3}{2160} + \frac{14\pi^2\zeta_3}{9} - \frac{224\zeta_3}{3} + \frac{31\pi^2\zeta_5}{30} - \frac{248\zeta_5}{5} - \frac{254\zeta_7}{7} - \frac{949\pi^6}{40320} \right. \right. \\
&\quad \left. \left. - \frac{47\pi^4}{90} - \frac{16\pi^2}{3} + 256 \right) \right\} + \mathcal{O}(\epsilon^7), \\
G_2 &= N^2 \left( -14\zeta_3 + \frac{5171}{108} + \frac{2\pi^2}{9} \right) + Nn_l \left( -\frac{209}{27} - \frac{2\pi^2}{9} \right) + \epsilon \left\{ N^2 \left( -\frac{170\zeta_3}{3} + \frac{140411}{648} \right. \right. \\
&\quad \left. \left. + \frac{53\pi^2}{108} - \frac{11\pi^4}{45} \right) + Nn_l \left( \frac{8\zeta_3}{3} - \frac{5813}{162} - \frac{37\pi^2}{54} \right) \right\} + \epsilon^2 \left\{ N^2 \left( \frac{47\pi^2\zeta_3}{9} - \frac{7511\zeta_3}{27} - 90\zeta_5 \right. \right. \\
&\quad \left. \left. - \frac{1969\pi^4}{2160} + \frac{2795\pi^2}{648} + \frac{3069107}{3888} \right) + Nn_l \left( \frac{602\zeta_3}{27} + \frac{7\pi^4}{216} - \frac{425\pi^2}{162} - \frac{129389}{972} \right) \right\} \\
&\quad + \epsilon^3 \left\{ N^2 \left( \frac{334\zeta_3^2}{3} + \frac{161\pi^2\zeta_3}{18} - \frac{162443\zeta_3}{162} - 402\zeta_5 - \frac{907\pi^6}{7560} - \frac{55187\pi^4}{12960} + \frac{69131\pi^2}{3888} \right. \right. \\
&\quad \left. \left. + \frac{61411595}{23328} \right) + Nn_l \left( \frac{11\pi^2\zeta_3}{9} + \frac{8170\zeta_3}{81} + 24\zeta_5 + \frac{1873\pi^4}{6480} - \frac{8405\pi^2}{972} - \frac{2628821}{5832} \right) \right\} \\
&\quad + \epsilon^4 \left\{ N^2 \left( \frac{12197\zeta_3^2}{27} + \frac{1909\pi^4\zeta_3}{540} + \frac{8279\pi^2\zeta_3}{324} - \frac{3218759\zeta_3}{972} + \frac{467\pi^2\zeta_5}{15} - \frac{78683\zeta_5}{45} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -606\zeta_7 - \frac{27737\pi^6}{36288} - \frac{579427\pi^4}{38880} + \frac{1506131\pi^2}{23328} + \frac{1175293667}{139968} \Big) + Nn_l \left( -\frac{578\zeta_3^2}{27} \right. \\
& \left. + \frac{791\pi^2\zeta_3}{162} + \frac{91985\zeta_3}{243} + \frac{6218\zeta_5}{45} + \frac{953\pi^6}{12960} + \frac{12781\pi^4}{9720} - \frac{160493\pi^2}{5832} - \frac{50947325}{34992} \right) \Big\} \\
& + \mathcal{O}(\epsilon^5), \\
G_3 = & N^3 \left( -\frac{10237\zeta_3}{27} + \frac{44\pi^2\zeta_3}{9} + 136\zeta_5 + \frac{4317713}{5832} + \frac{7165\pi^2}{972} - \frac{22\pi^4}{45} \right) + N^2 n_l \left( \frac{416\zeta_3}{9} \right. \\
& - \frac{361489}{1458} - \frac{1933\pi^2}{243} + \frac{8\pi^4}{45} \Big) + Nn_l^2 \left( \frac{16\zeta_3}{27} + \frac{9838}{729} + \frac{76\pi^2}{81} \right) + \epsilon \left\{ N^3 \left( \frac{656\zeta_3^2}{3} \right. \right. \\
& \left. \left. + \frac{748\pi^2\zeta_3}{27} - \frac{8816\zeta_3}{3} - 224\zeta_5 + \frac{674\pi^6}{1215} - \frac{3461\pi^4}{405} + \frac{718195\pi^2}{5832} + \frac{197609041}{34992} \right) \right. \\
& \left. + N^2 n_l \left( -\frac{82\pi^2\zeta_3}{27} + \frac{39110\zeta_3}{81} + \frac{400\zeta_5}{3} + \frac{55\pi^4}{54} - \frac{416425\pi^2}{5832} - \frac{16790213}{8748} \right) \right. \\
& \left. + Nn_l^2 \left( -\frac{536\zeta_3}{81} + \frac{10\pi^4}{81} + \frac{1733\pi^2}{243} + \frac{258445}{2187} \right) \right\} + \epsilon^2 \left\{ N^3 \left( \frac{6340\zeta_3^2}{3} + \frac{952\pi^4\zeta_3}{135} \right. \right. \\
& \left. \left. + \frac{48055\pi^2\zeta_3}{324} - \frac{17023405\zeta_3}{972} - \frac{302\pi^2\zeta_5}{9} - \frac{16355\zeta_5}{3} + \frac{7436\zeta_7}{3} - \frac{5867\pi^6}{7560} - \frac{949937\pi^4}{19440} \right. \right. \\
& \left. \left. + \frac{64740335\pi^2}{69984} + \frac{2200841219}{69984} \right) + N^2 n_l \left( -152\zeta_3^2 + \frac{34\pi^2\zeta_3}{81} + \frac{184573\zeta_3}{54} + 1376\zeta_5 \right. \right. \\
& \left. \left. + \frac{1963\pi^6}{3780} + \frac{78947\pi^4}{19440} - \frac{3943345\pi^2}{8748} - \frac{188719859}{17496} \right) + Nn_l^2 \left( -\frac{188\pi^2\zeta_3}{81} - \frac{22220\zeta_3}{243} \right. \right. \\
& \left. \left. - \frac{112\zeta_5}{9} + \frac{269\pi^4}{324} + \frac{29452\pi^2}{729} + \frac{1024873}{1458} \right) \right\} + \mathcal{O}(\epsilon^3), \\
G_4 = & N^4 \left( \frac{1622\zeta_3^2}{3} - \frac{164\pi^4\zeta_3}{45} + \frac{4240\pi^2\zeta_3}{27} - \frac{169547\zeta_3}{18} - 48\pi^2\zeta_5 + \frac{10100\zeta_5}{3} - 1410\zeta_7 \right. \\
& \left. + \frac{6175\pi^6}{6804} - \frac{577\pi^4}{45} + \frac{571537\pi^2}{1458} + \frac{1913092765}{139968} \right) + N^3 n_l \left( \frac{16\zeta_3^2}{3} - \frac{608\pi^2\zeta_3}{27} \right. \\
& \left. + \frac{406621\zeta_3}{162} + \frac{670\zeta_5}{9} - \frac{3679\pi^6}{17010} + \frac{7547\pi^4}{1080} - \frac{1903985\pi^2}{5832} - \frac{164021633}{23328} \right) \\
& + N^2 n_l^2 \left( \frac{28\pi^2\zeta_3}{27} - \frac{9298\zeta_3}{81} - \frac{772\zeta_5}{9} - \frac{103\pi^4}{405} + \frac{98581\pi^2}{1458} + \frac{11076787}{11664} \right) \\
& + Nn_l^3 \left( -\frac{40\zeta_3}{81} - \frac{26\pi^4}{405} - \frac{812\pi^2}{243} - \frac{69874}{2187} \right) + \mathcal{O}(\epsilon). \tag{4.3}
\end{aligned}$$

The remaining quantities for the predictions of the massive form factors,  $K_k$  in the large- $N$

limit, are obtained as

$$\begin{aligned}
K_1 &= -N, \\
K_2 &= N^2 \left( 18\zeta_3 - \frac{827}{108} - \frac{3\pi^2}{2} \right) + Nn_l \left( \frac{5}{27} + \frac{\pi^2}{3} \right), \\
K_3 &= N^3 \left( \frac{949\zeta_3}{3} - \frac{32\pi^2\zeta_3}{9} - 172\zeta_5 - \frac{93823}{2916} - \frac{14929\pi^2}{972} + \frac{11\pi^4}{135} \right) + N^2 n_l \left( -\frac{1108\zeta_3}{27} \right. \\
&\quad \left. - \frac{859}{2916} + \frac{2963\pi^2}{486} - \frac{2\pi^4}{135} \right) + Nn_l^2 \left( -\frac{8\zeta_3}{27} + \frac{2201}{729} - \frac{10\pi^2}{27} \right), \tag{4.4}
\end{aligned}$$

consistent with the existing results up to two loop from Ref. [4]. The corresponding quantities for the massless case, see  $\bar{K}$  in Eq. (3.1), can be expressed in terms of the cusp anomalous dimensions and  $\beta$ -function [14]. They do not appear in the final expressions of the massless form factors (as can be seen in Eq. (3.4)) since they get canceled against the similar terms arising from  $\bar{G}$ . Hence, we do not present the results for  $\bar{K}_k$ .

Expanding the massive quark form factor, Eq. (2.2), in powers of  $a_s$  as

$$F = 1 + \sum_{k=1}^{\infty} a_s(m^2) F^{(k)} \tag{4.5}$$

and using the results of the above quantities, we predict  $F$  to four-loop order in the planar limit. The result reads

$$\begin{aligned}
F^{(4)} &= \frac{1}{\epsilon^4} \left\{ N^4 \left( \frac{L^4}{24} - \frac{13L^3}{12} + \frac{1979L^2}{216} - \frac{2977L}{108} + \frac{175}{9} \right) + N^3 n_l \left( \frac{L^3}{6} - \frac{74L^2}{27} \right. \right. \\
&\quad \left. \left. + \frac{316L}{27} - \frac{493}{54} \right) + N^2 n_l^2 \left( \frac{11L^2}{54} - \frac{44L}{27} + \frac{77}{54} \right) + Nn_l^3 \left( \frac{2L}{27} - \frac{2}{27} \right) \right\} \\
&\quad + \frac{1}{\epsilon^3} \left\{ N^4 \left( -\frac{L^5}{12} + \frac{17L^4}{12} - \frac{\pi^2 L^3}{18} - \frac{170L^3}{27} - \frac{L^2 \zeta_3}{2} + \frac{31\pi^2 L^2}{54} - \frac{1265L^2}{324} \right. \right. \\
&\quad \left. \left. + \frac{95L\zeta_3}{18} - \frac{439\pi^2 L}{324} + \frac{9475L}{162} - \frac{23\zeta_3}{2} + \frac{271\pi^2}{324} - \frac{4633}{108} \right) + N^3 n_l \left( -\frac{L^4}{6} \right. \right. \\
&\quad \left. \left. + \frac{37L^3}{27} - \frac{2\pi^2 L^2}{27} + \frac{199L^2}{81} - \frac{7L\zeta_3}{9} + \frac{23\pi^2 L}{81} - \frac{2471L}{81} + \frac{29\zeta_3}{9} - \frac{17\pi^2}{81} \right. \right. \\
&\quad \left. \left. + \frac{3953}{162} \right) + N^2 n_l^2 \left( -\frac{2L^3}{27} - \frac{11L^2}{81} - \frac{\pi^2 L}{81} + \frac{322L}{81} - \frac{2\zeta_3}{9} + \frac{\pi^2}{81} - \frac{287}{81} \right) \right. \\
&\quad \left. + Nn_l^3 \left( \frac{10}{81} - \frac{10L}{81} \right) \right\} + \frac{1}{\epsilon^2} \left\{ N^4 \left( \frac{13L^6}{144} - \frac{15L^5}{16} + \frac{\pi^2 L^4}{9} + \frac{659L^4}{1296} - \frac{8L^3 \zeta_3}{3} \right. \right. \\
&\quad \left. \left. - \frac{49\pi^2 L^3}{108} + \frac{13051L^3}{648} + \frac{379L^2 \zeta_3}{18} - \frac{\pi^4 L^2}{54} - \frac{431\pi^2 L^2}{432} - \frac{23401L^2}{432} + 6L\zeta_5 \right. \right. \\
&\quad \left. \left. - \frac{2\pi^2 L \zeta_3}{9} - \frac{3037L\zeta_3}{54} + \frac{637\pi^4 L}{3240} + \frac{9593\pi^2 L}{3888} + \frac{4735L}{648} - \frac{45\zeta_5}{2} + \frac{\zeta_3^2}{2} + \frac{67\pi^2 \zeta_3}{54} \right) \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1918\zeta_3}{27} - \frac{577\pi^4}{3240} - \frac{4171\pi^2}{1944} + \frac{35}{8} \Big) + N^3 n_l \left( \frac{L^5}{24} + \frac{259L^4}{648} - \frac{\pi^2 L^3}{54} - \frac{4421L^3}{648} \right. \\
& - \frac{35L^2\zeta_3}{9} + \frac{223\pi^2 L^2}{324} + \frac{2819L^2}{162} + 17L\zeta_3 - \frac{47\pi^4 L}{1620} - \frac{2609\pi^2 L}{1944} + \frac{15697L}{1296} \\
& + 3\zeta_5 - \frac{5\pi^2\zeta_3}{27} - \frac{1325\zeta_3}{54} + \frac{47\pi^4}{1620} + \frac{1667\pi^2}{1944} - \frac{5921}{432} \Big) + N^2 n_l^2 \left( -\frac{L^4}{81} + \frac{61L^3}{162} \right. \\
& - \frac{4\pi^2 L^2}{81} - \frac{215L^2}{162} - \frac{28L\zeta_3}{27} + \frac{55\pi^2 L}{486} - \frac{1019L}{648} + \frac{16\zeta_3}{9} - \frac{31\pi^2}{486} + \frac{1163}{648} \Big) \\
& \left. + N n_l^3 \left( \frac{2}{81} - \frac{2L}{81} \right) \right\} + \mathcal{O}\left(\frac{1}{\epsilon}\right), \tag{4.6}
\end{aligned}$$

where  $L$  is defined after Eq. (2.20). Similarly, we obtain predictions for the massless quark form factor at five-loop order,  $\overline{F}^{(5)}$  in Eq. (3.5), in the planar limit including pole terms up to  $1/\epsilon^3$ . It is given by

$$\begin{aligned}
\overline{F}^{(5)} = & \frac{1}{\epsilon^{10}} \left\{ -\frac{N^5}{120} \right\} + \frac{1}{\epsilon^9} \left\{ \frac{13}{144} N^5 - \frac{1}{36} N^4 n_l \right\} + \frac{1}{\epsilon^8} \left\{ \left( -\frac{169}{2592} - \frac{\pi^2}{96} \right) N^5 + \frac{55}{324} N^4 n_l \right. \\
& \left. - \frac{25}{648} N^3 n_l^2 \right\} + \frac{1}{\epsilon^7} \left\{ N^5 \left( -\frac{35\zeta_3}{72} + \frac{629\pi^2}{5184} - \frac{7417}{5184} \right) + N^4 n_l \left( \frac{1547}{1944} - \frac{47\pi^2}{1296} \right) \right. \\
& + N^3 n_l^2 \frac{91}{3888} - N^2 n_l^3 \frac{13}{486} \Big\} + \frac{1}{\epsilon^6} \left\{ N^5 \left( \frac{2297\zeta_3}{432} - \frac{2383\pi^4}{103680} - \frac{9323\pi^2}{31104} - \frac{4013269}{777600} \right) \right. \\
& + N^4 n_l \left( -\frac{38\zeta_3}{27} + \frac{11\pi^2}{36} - \frac{11641}{16200} \right) + N^3 n_l^2 \left( \frac{1301}{900} - \frac{451\pi^2}{7776} \right) - \frac{3901}{24300} N^2 n_l^3 \\
& \left. - \frac{16}{2025} N n_l^4 \right\} + \frac{1}{\epsilon^5} \left\{ N^5 \left( -\frac{241\pi^2\zeta_3}{864} + \frac{70163\zeta_3}{7776} - \frac{593\zeta_5}{40} + \frac{185783\pi^4}{622080} - \frac{7337729\pi^2}{4665600} \right) \right. \\
& + \frac{9986195}{559872} \Big\} + N^4 n_l \left( \frac{9569\zeta_3}{1944} - \frac{12743\pi^4}{155520} + \frac{226423\pi^2}{583200} - \frac{23487251}{699840} \right) \\
& + N^3 n_l^2 \left( -\frac{2821\zeta_3}{1944} + \frac{280753\pi^2}{1166400} + \frac{3532007}{349920} \right) + N^2 n_l^3 \left( -\frac{235}{2916} - \frac{7679\pi^2}{145800} \right) \\
& \left. - \frac{136}{1215} N n_l^4 \right\} + \frac{1}{\epsilon^4} \left\{ N^5 \left( -\frac{7211\zeta_3^2}{432} + \frac{2357\pi^2\zeta_3}{15552} - \frac{62527457\zeta_3}{388800} + \frac{10855\zeta_5}{48} \right) \right. \\
& - \frac{1749887\pi^6}{26127360} - \frac{59053001\pi^4}{93312000} + \frac{123222683\pi^2}{27993600} + \frac{7667063497}{27993600} \Big\} + N^4 n_l \left( -\frac{145\pi^2\zeta_3}{1944} \right. \\
& + \frac{29885263\zeta_3}{291600} - \frac{1031\zeta_5}{18} + \frac{2774111\pi^4}{3888000} - \frac{14290127\pi^2}{1749600} - \frac{4362993917}{20995200} \Big) \\
& \left. + N^3 n_l^2 \left( -\frac{3395129\zeta_3}{291600} - \frac{3038281\pi^4}{23328000} + \frac{31751\pi^2}{10800} + \frac{3134216}{164025} \right) + N^2 n_l^3 \left( -\frac{12019\zeta_3}{36450} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{8827\pi^2}{58320} + \frac{5620601}{656100} \Big) + Nn_l^4 \left( -\frac{19312}{18225} - \frac{28\pi^2}{1215} \right) \Big\} \\
& + \frac{1}{\epsilon^3} \left\{ N^5 \left( \frac{712567\zeta_3^2}{7200} - \frac{302281\pi^4\zeta_3}{311040} + \frac{81003121\pi^2\zeta_3}{2332800} - \frac{934296631\zeta_3}{777600} - \frac{2327\pi^2\zeta_5}{864} \right. \right. \\
& - \frac{23099237\zeta_5}{64800} - \frac{205279\zeta_7}{504} + \frac{4029584131\pi^6}{3919104000} - \frac{3176537179\pi^4}{559872000} + \frac{14248468873\pi^2}{167961600} \\
& + \frac{24207802321}{18662400} \Big) + N^4 n_l \left( -\frac{168383\zeta_3^2}{5400} - \frac{200671\pi^2\zeta_3}{12960} + \frac{208671203\zeta_3}{583200} + \frac{8158807\zeta_5}{16200} \right. \\
& - \frac{257266573\pi^6}{979776000} + \frac{134354611\pi^4}{69984000} - \frac{838588711\pi^2}{13996800} - \frac{11993770663}{41990400} \Big) \\
& + N^3 n_l^2 \left( \frac{993089\pi^2\zeta_3}{583200} + \frac{97579\zeta_3}{2025} - \frac{1483877\zeta_5}{16200} + \frac{58802923\pi^4}{139968000} + \frac{123959209\pi^2}{20995200} \right. \\
& - \frac{3380652283}{10497600} \Big) + N^2 n_l^3 \left( -\frac{1029499\zeta_3}{72900} - \frac{1961437\pi^4}{17496000} + \frac{2125111\pi^2}{874800} + \frac{72339173}{656100} \right) \\
& + Nn_l^4 \left( \frac{1904\zeta_3}{6075} - \frac{238\pi^2}{729} - \frac{447136}{54675} \right) \Big\} + \mathcal{O}\left(\frac{1}{\epsilon^2}\right). \tag{4.7}
\end{aligned}$$

All results presented in this paper can be found in the ancillary file submitted to the `arXiv` and can be downloaded in computer-readable form from <https://www.ttp.kit.edu/preprints/2017/ttp17-023/>.

## 5 Regularization scheme independent ratio functions

We use the results derived in the previous sections and extend the conversion formula which relates dimensionally regularized amplitudes to those where the infrared divergence has been regularized with a small quark mass. In Ref. [2] the following formula has been derived which relates amplitudes computed in the two regularization schemes

$$\mathcal{M}^{(m)} = \prod_{i \in \{\text{all legs}\}} \left[ Z_{[i]}^{(m|0)} \left( \frac{m^2}{\mu^2} \right) \right]^{1/2} \mathcal{M}^{(0)}, \tag{5.1}$$

where for simplicity most of the arguments are suppressed. Note that the amplitudes  $\mathcal{M}^{(m)}$  and  $\mathcal{M}^{(0)}$  depend on all kinematical variables and the regularization scale  $\mu$ . The universal factor  $Z_{[i]}^{(m|0)}$ , however, only depends on the ratio of the (small) mass  $m$  and  $\mu$ . Of course, all three quantities in Eq. (5.1) are expansions in  $\alpha_s$  and  $\epsilon$ . It is an important observation of Ref. [2] that the  $Z^{(m|0)}$  are process independent and can thus be computed with the help of the simplest possible amplitudes, the form factors. In particular, for the photon quark form factor we have

$$Z_{[q]}^{(m|0)} = \frac{F(Q^2, m^2, \mu^2)}{\bar{F}(Q^2, \mu^2)}. \tag{5.2}$$



Note that the two quantities on the right-hand side of this equation depend on  $Q$  which has to cancel in the ratio. The cancellations of  $Q^2$  is obvious from the general solutions of the massive, Eq. (2.15), and massless form factors, Eq. (3.3), which show that the  $Q^2$  dependent parts of  $\ln \tilde{F}$  and  $\ln \bar{F}$  are identical. Thus they drop out from  $\ln Z_{[q]}^{(m|0)}$  given by

$$\ln Z_{[q]}^{(m|0)} = \ln C + \ln \tilde{F} - \ln \bar{F}. \quad (5.3)$$

Note that  $C$  is independent of  $Q^2$ .

In Refs. [2, 4] the quantity  $Z_{[q]}^{(m|0)}$  has been computed including  $\mathcal{O}(\epsilon)$  terms at two loops and up to the pole part at order  $\alpha_s^3$ . We are in the position to add the constant term in the large- $N$  limit and furthermore extend the considerations to four loops up to order  $1/\epsilon^2$ . For convenience we present the results for  $\mu = m$  and write

$$Z_{[q]}^{(m|0)} = 1 + \sum_{k=1}^{\infty} a_s(m^2) Z_{[q]}^{(k)}, \quad (5.4)$$

with

$$\begin{aligned} Z_{[q]}^{(1)} &= \frac{N}{\epsilon^2} + \frac{1}{\epsilon} \left\{ \frac{N}{2} \right\} + N \left( 2 + \frac{\pi^2}{12} \right) + \epsilon \left\{ N \left( -\frac{\zeta_3}{3} + \frac{\pi^2}{24} + 4 \right) \right\} + \epsilon^2 \left\{ N \left( -\frac{\zeta_3}{6} + \frac{\pi^4}{160} \right. \right. \\ &\quad \left. \left. + \frac{\pi^2}{6} + 8 \right) \right\} + \epsilon^3 \left\{ N \left( -\frac{\pi^2 \zeta_3}{36} - \frac{2\zeta_3}{3} - \frac{\zeta_5}{5} + \frac{\pi^4}{320} + \frac{\pi^2}{3} + 16 \right) \right\} + \epsilon^4 \left\{ N \left( \frac{\zeta_3^2}{18} - \frac{\pi^2 \zeta_3}{72} \right. \right. \\ &\quad \left. \left. - \frac{4\zeta_3}{3} - \frac{\zeta_5}{10} + \frac{61\pi^6}{120960} + \frac{\pi^4}{80} + \frac{2\pi^2}{3} + 32 \right) \right\} + \mathcal{O}(\epsilon^5), \\ Z_{[q]}^{(2)} &= \frac{N^2}{2\epsilon^4} + \frac{1}{\epsilon^3} \left\{ \frac{N n_l}{2} - \frac{9N^2}{4} \right\} + \frac{1}{\epsilon^2} \left\{ \frac{221N^2}{72} - \frac{N n_l}{9} \right\} + \frac{1}{\epsilon} \left\{ N^2 \left( -\frac{29\zeta_3}{6} + \frac{11\pi^2}{24} \right. \right. \\ &\quad \left. \left. + \frac{2987}{432} \right) + N n_l \left( -\frac{5}{108} - \frac{\pi^2}{12} \right) \right\} + N^2 \left( \frac{28\zeta_3}{9} - \frac{19\pi^4}{80} + \frac{1195\pi^2}{432} + \frac{71195}{2592} \right) \\ &\quad + N n_l \left( -\frac{13\zeta_3}{9} - \frac{37\pi^2}{108} - \frac{1541}{648} \right) + \epsilon \left\{ N^2 \left( \frac{\pi^2 \zeta_3}{4} + \frac{169\zeta_3}{108} - \frac{1057\zeta_5}{10} + \frac{23\pi^4}{45} \right. \right. \\ &\quad \left. \left. + \frac{27463\pi^2}{2592} + \frac{1435331}{15552} \right) + N n_l \left( -\frac{91\zeta_3}{27} - \frac{49\pi^4}{360} - \frac{673\pi^2}{648} - \frac{46205}{3888} \right) \right\} \\ &\quad + \epsilon^2 \left\{ N^2 \left( -\frac{170\zeta_3^2}{9} + \frac{643\pi^2 \zeta_3}{108} + \frac{36889\zeta_3}{648} + \frac{80\zeta_5}{3} - \frac{2689\pi^6}{6480} + \frac{7817\pi^4}{4320} + \frac{537415\pi^2}{15552} \right. \right. \\ &\quad \left. \left. + \frac{26855675}{93312} \right) + N n_l \left( -\frac{28\pi^2 \zeta_3}{27} - \frac{1711\zeta_3}{162} - \frac{361\zeta_5}{15} - \frac{1607\pi^4}{4320} - \frac{15481\pi^2}{3888} \right. \right. \\ &\quad \left. \left. - \frac{1063589}{23328} \right) \right\} + \mathcal{O}(\epsilon^3), \\ Z_{[q]}^{(3)} &= \frac{N^3}{6\epsilon^6} + \frac{1}{\epsilon^5} \left\{ \frac{N^2 n_l}{2} - \frac{5N^3}{2} \right\} + \frac{1}{\epsilon^4} \left\{ N^3 \left( \frac{2887}{324} - \frac{\pi^2}{24} \right) - N^2 n_l \left( \frac{923}{324} \right) + N n_l^2 \left( \frac{22}{81} \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\epsilon^3} \left\{ N^3 \left( -\frac{14\zeta_3}{3} + \frac{41\pi^2}{81} - \frac{5089}{486} \right) + N^2 n_l \left( \frac{4393}{972} - \frac{67\pi^2}{648} \right) - N n_l^2 \left( \frac{32}{243} \right) \right\} \\
& + \frac{1}{\epsilon^2} \left\{ N^3 \left( \frac{1477\zeta_3}{108} - \frac{5713\pi^4}{25920} + \frac{4705\pi^2}{3888} + \frac{13153}{972} \right) + N^2 n_l \left( -\frac{227\zeta_3}{54} + \frac{311\pi^2}{3888} \right. \right. \\
& \left. \left. - \frac{149}{486} \right) + \left( -\frac{1}{27} - \frac{\pi^2}{27} \right) N n_l^2 \right\} + \frac{1}{\epsilon} \left\{ N^3 \left( \frac{55\pi^2\zeta_3}{108} - 58\zeta_3 - \frac{1154\zeta_5}{15} + \frac{9971\pi^4}{25920} \right. \right. \\
& \left. \left. + \frac{168431\pi^2}{11664} + \frac{2956649}{34992} \right) + N^2 n_l \left( \frac{875\zeta_3}{324} - \frac{3563\pi^4}{25920} - \frac{27035\pi^2}{11664} - \frac{167279}{17496} \right) \right. \\
& \left. + N n_l^2 \left( \frac{4\zeta_3}{81} + \frac{5\pi^2}{81} - \frac{2201}{4374} \right) \right\} + N^3 \left( \frac{565\zeta_3^2}{36} + \frac{173\pi^2\zeta_3}{16} + \frac{9113\zeta_3}{972} - \frac{22949\zeta_5}{60} \right. \\
& \left. - \frac{1265051\pi^6}{6531840} - \frac{148157\pi^4}{155520} + \frac{6704201\pi^2}{69984} + \frac{102449365}{209952} \right) + N^2 n_l \left( -\frac{125\pi^2\zeta_3}{72} \right. \\
& \left. - \frac{4907\zeta_3}{108} + \frac{769\zeta_5}{18} - \frac{33643\pi^4}{51840} - \frac{744325\pi^2}{34992} - \frac{11072359}{104976} \right) + N n_l^2 \left( \frac{1072\zeta_3}{243} + \frac{221\pi^4}{2430} \right. \\
& \left. + \frac{145\pi^2}{243} + \frac{58883}{26244} \right) + \mathcal{O}(\epsilon),
\end{aligned}$$

$$\begin{aligned}
Z_{[q]}^{(4)} &= \frac{N^4}{24\epsilon^8} + \frac{1}{\epsilon^7} \left\{ \frac{N^3 n_l}{4} - \frac{31N^4}{24} \right\} + \frac{1}{\epsilon^6} \left\{ N^4 \left( \frac{29783}{2592} - \frac{\pi^2}{36} \right) - N^3 n_l \left( \frac{2701}{648} \right) \right. \\
& \left. + N^2 n_l^2 \left( \frac{257}{648} \right) \right\} + \frac{1}{\epsilon^5} \left\{ N^4 \left( -\frac{83\zeta_3}{36} + \frac{665\pi^2}{1296} - \frac{146849}{3888} \right) + N^3 n_l \left( \frac{69571}{3888} - \frac{67\pi^2}{648} \right) \right. \\
& \left. - N^2 n_l^2 \left( \frac{2525}{972} \right) + N n_l^3 \left( \frac{25}{162} \right) \right\} + \frac{1}{\epsilon^4} \left\{ N^4 \left( \frac{5285\zeta_3}{216} - \frac{1259\pi^4}{12960} - \frac{6763\pi^2}{3888} \right. \right. \\
& \left. \left. + \frac{1881821}{31104} \right) + N^3 n_l \left( -\frac{619\zeta_3}{108} + \frac{557\pi^2}{648} - \frac{248017}{7776} \right) + N^2 n_l^2 \left( \frac{8917}{1944} - \frac{187\pi^2}{1944} \right) \right. \\
& \left. - N n_l^3 \left( \frac{47}{486} \right) \right\} + \frac{1}{\epsilon^3} \left\{ N^4 \left( \frac{163\pi^2\zeta_3}{216} - \frac{403273\zeta_3}{3888} - \frac{1447\zeta_5}{60} + \frac{52597\pi^4}{77760} + \frac{117481\pi^2}{15552} \right. \right. \\
& \left. \left. - \frac{8208995}{279936} \right) + N^3 n_l \left( \frac{6017\zeta_3}{243} - \frac{6443\pi^4}{38880} - \frac{1483\pi^2}{729} + \frac{450337}{17496} \right) + N^2 n_l^2 \left( -\frac{1109\zeta_3}{486} \right. \right. \\
& \left. \left. + \frac{1403\pi^2}{5832} - \frac{160379}{69984} \right) + \left( -\frac{2}{81} - \frac{\pi^2}{54} \right) N n_l^3 \right\} + \frac{1}{\epsilon^2} \left\{ N^4 \left( \frac{7511\zeta_3^2}{216} + \frac{8075\pi^2\zeta_3}{1944} \right. \right. \\
& \left. \left. + \frac{3428065\zeta_3}{23328} - \frac{4217\zeta_5}{24} + \frac{17951\pi^6}{816480} - \frac{770389\pi^4}{233280} + \frac{5764783\pi^2}{139968} + \frac{574001923}{3359232} \right) \right. \\
& \left. + N^3 n_l \left( -\frac{253\pi^2\zeta_3}{972} - \frac{1146223\zeta_3}{11664} + \frac{3571\zeta_5}{180} + \frac{38467\pi^4}{233280} - \frac{417503\pi^2}{69984} - \frac{18669917}{839808} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + N^2 n_l^2 \left( \frac{10111\zeta_3}{1458} + \frac{175\pi^4}{7776} - \frac{2231\pi^2}{2916} - \frac{275065}{419904} \right) + N n_l^3 \left( \frac{5\zeta_3}{81} + \frac{5\pi^2}{162} - \frac{2255}{8748} \right) \Big\} \\
& + \mathcal{O}\left(\frac{1}{\epsilon}\right). \tag{5.5}
\end{aligned}$$

The analytic expressions of these equations (both explicit and generic) can be found in the ancillary file to this paper.

## 6 Conclusions and outlook

It is among the primary goals of modern quantum field theory to investigate the structure of perturbation theory. QCD corrections to the photon-quark form factors, both with massless and massive quarks, constitute important quantities in this context. In this paper, we discuss in detail the equations which govern the renormalization group dependence both of the massless and massive form factors and present an elegant derivation of explicit analytic solutions valid for a general gauge group  $SU(N)$ . The key idea of the derivation is the use of the bare coupling for the solution of the integrals in Eq. (2.8). The solutions are expressed in terms of a function  $G$  governing the  $Q^2$  dependence of the RG equation and the cusp anomalous dimension  $A$ . Both of them are universal in the sense that they are equal for the massive and massless form factors. The solution contains furthermore the function  $K$  which is different for the massless and massive case. In the massive case one has in addition a non-trivial matching condition, parametrized with the function  $C$ , which is determined from the comparison with the explicit calculation.

The comparison of the generic formula with explicit calculation to three (massive) and four (massless) loops, and the knowledge of the cusp anomalous dimension, enables us to extend  $K$ ,  $G$  and  $C$  to higher orders in  $\alpha_s$  and  $\epsilon$ , which in turn leads to new four and five loop predictions for the massive and massless form factors, respectively. Since the highest loop order of the form factors are only known for large  $N$  our predictions are restricted to this limit. The new results for the form factors are used to extract new information about the universal conversion factors between amplitudes where infrared singularities are regularized dimensionally or with the help of a small quarks mass.

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