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Heavy quark form factors in the large β_0 limit

A. G. Grozin[1]
Budker Institute of Nuclear Physics, Novosibirsk, Russia[2];
Institut für Theoretische Teilchenphysik,
Karlsruher Institut für Technologie, Germany;
THEP, Institut für Physik, Universität Mainz, Germany

Abstract

Heavy quark form factors are calculated at $\beta_0\alpha_s \sim 1$ to all orders in α_s at the first order in $1/\beta_0$. The $n_f^2\alpha_s^3$ terms in the recent results [1] for the vector form factors are confirmed, and $n_f^{L-1}\alpha_s^L$ terms for higher L are predicted.

1 Introduction

Quark form factors are building blocks for various production cross sections and decay widths in QCD. Recently massive-quark vector form factors have been calculated to to 3 loops [1].

We'll consider heavy-quark form factors in the large β_0 limit, where $\beta_0\alpha_s \sim 1$, and $1/\beta_0$ is an expansion parameter (see the reviews [2, 3]). A bare form factor can be written as

$$F = 1 + \sum_{L=1}^{\infty} \sum_{n=0}^{L-1} a_{Ln} \beta_0^n \left(\frac{g_0^2}{(4\pi)^{d/2}} \right)^L . \tag{1}$$

Keeping terms with the highest degree of β_0 in each order of perturbation theory, we get

$$F = 1 + \frac{1}{\beta_0} f\left(\frac{\beta_0 g_0^2}{(4\pi)^{d/2}} \right) + \mathcal{O}\left(\frac{1}{\beta_0^2} \right) . \tag{2}$$

The leading coefficients $a_{L,L-1}$ can be easily obtained from n_f^{L-1} terms (Fig. 1). We shall consider only the first $1/\beta_0$ order[3].

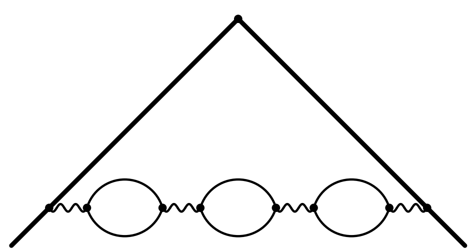

Figure 1: Diagrams producing the highest degree of n_f in each order of perturbation theory.

[1] A.G.Grozin@inp.nsk.su
[2] Also Novosibirsk State University, Novosibirsk, Russia
[3] In some cases it is possible to obtain results for $1/\beta_0^2$ corrections, see, e. g., [4, 5].

2 Heavy-quark bilinear currents

We consider the QCD currents

$$J_0 = \bar{Q}_0 \Gamma Q_0 = Z(\alpha_s^{(n_f)}(\mu)) J(\mu) \,, \quad \Gamma = \gamma^{[\mu_1} \cdots \gamma^{\mu_n]} \,, \tag{3}$$

where Q_0 is a bare heavy-quark field. The antisymmetrized product of n γ matrices has the property

$$\gamma^\mu \Gamma \gamma_\mu = \eta (d - 2n) \Gamma \,, \quad \eta = (-1)^n \,. \tag{4}$$

In situations when the initial heavy-quark momentum p_1 and the final one p_2 can be written as $p_{1,2} = m v_{1,2} + k_{1,2}$ (m is the on-shell mass) with $k_{1,2} \ll m$, these currents can be expanded in HQET ones [6, 7]:

$$J(\mu) = \sum_{i=0}^{2} H_i(\mu,\mu') \tilde{J}_i(\mu') + \frac{1}{2m} \sum_i G_i(\mu,\mu') \tilde{O}_i(\mu') + \mathcal{O}\left(\frac{1}{m^2}\right) \,, \tag{5}$$

where the leading HQET currents are

$$\tilde{J}_{i0} = \bar{h}_{v_2 0} \Gamma_i h_{v_1 0} = \tilde{Z}(\alpha_s^{(n_l)}(\mu)) \tilde{J}_i(\mu) \,, \quad \Gamma_i = \Gamma \,,\ \not{v}_1 \Gamma + \Gamma \not{v}_2 \,,\ \not{v}_1 \Gamma \not{v}_2 \,, \tag{6}$$

and $\tilde{O}_i$ are local and bilocal dimension-4 HQET operators with appropriate quantum numbers. The HQET current renormalization constant $\tilde{Z}$ does not depend on the Dirac structure and is a function of the Minkowski angle ϑ: $v_1 \cdot v_2 = \cosh\vartheta = w$.

The coefficients in (5) can be obtained by matching the on-shell matrix elements ($k_{1,2} = 0$) in QCD and HQET:

$$\begin{aligned} &<Q(p_2 = m v_2)|J_0|Q(p_1 = m v_1)> = \sum_{i=0}^{2} F_i \, \bar{u}_2 \Gamma_i u_1 \,, \\ &<Q(k_2 = 0)|\tilde{J}_{i0}|Q(k_1 = 0)> = \tilde{F}_i \, \bar{u}_2 \Gamma_i u_1 \,, \quad \tilde{F}_i = 1 \end{aligned} \tag{7}$$

(all loop corrections to $\tilde{F}_i$ vanish because they contain no scale). Therefore the bare matching coefficients (in the relation similar to (5) but for the bare currents) are $H_i^0 = F_i/\tilde{F}_i = F_i$. The renormalized matching coefficients are

$$H_i(\mu,\mu') = H_i^0 \frac{\tilde{Z}(\alpha_s^{(n_l)}(\mu'))}{Z(\alpha_s^{(n_f)}(\mu))} = \frac{F_i \tilde{Z}}{\tilde{F}_i Z} \,. \tag{8}$$

UV divergences cancel in the ratio F_i/Z as well as in the ratio $\tilde{F}_i/\tilde{Z}$. Both F_i and $\tilde{F}_i$ contain IR divergences which cancel in the ratio $F_i/\tilde{F}_i$ because HQET is constructed to reproduce the IR behaviour of QCD ($\tilde{F}_i$ have no loop corrections because their UV and IR divergences cancel each other).

The dependence of $H_i(\mu,\mu')$ on μ and μ' is determined by the RG equations. Their solution can be written as

$$\begin{aligned} H_i(\mu,\mu') = \hat{H}_i &\left(\frac{\alpha_s^{(n_f)}(\mu)}{\alpha_s^{(n_f)}(\mu_0)} \right)^{\gamma_{n0}/(2\beta_0^{(n_f)})} K_{\gamma_n}^{(n_f)}(\alpha_s^{(n_f)}(\mu)) \\ \times &\left(\frac{\alpha_s^{(n_l)}(\mu')}{\alpha_s^{(n_l)}(\mu_0)} \right)^{-\tilde{\gamma}_0/(2\beta_0^{(n_l)})} K_{-\tilde{\gamma}}^{(n_l)}(\alpha_s^{(n_l)}(\mu')) \,, \end{aligned} \tag{9}$$

where for any anomalous dimension $\gamma(\alpha_s) = \gamma_0 \alpha_s/(4\pi) + \gamma_1 (\alpha_s/(4\pi))^2 + \cdots$ we define

$$K_\gamma(\alpha_s) = \exp \int_0^{\alpha_s} \frac{d\alpha_s}{\alpha_s} \left(\frac{\gamma(\alpha_s)}{2\beta(\alpha_s)} - \frac{\gamma_0}{2\beta_0} \right) = 1 + \frac{\gamma_0}{2\beta_0} \left(\frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0} \right) \frac{\alpha_s}{4\pi} + \cdots \tag{10}$$

Matrix elements of the currents with $n = 0, 1$ can be written via smaller numbers of form factors:

$$<Q(mv_2)|J|Q(mv_1)> = F^S \bar{u}_2 u_1 \,, \quad F^S = F_0 + 2F_1 + (2w-1)F_2 \tag{11}$$

(where F_i with $n = 0$, $\eta = 1$ are used), and

$$\begin{aligned} &<Q(mv_2)|J^\mu|Q(mv_1)> = (F_1^V + F_2^V)\bar{u}_2\gamma^\mu u_1 - F_2^V \bar{u}_2 u_1 \frac{(v_1+v_2)^\mu}{2} \,, \\ &F_1^V = F_0 + 2F_1 - (2w-3)F_2 \,, \quad F_2^V = -4(F_1 + F_2) \end{aligned} \tag{12}$$

(where F_i with $n = 1$, $\eta = -1$ are used).

3 Inversion relations

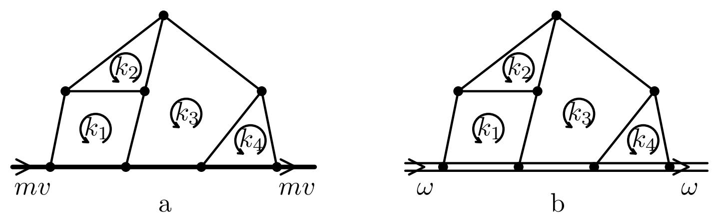

Figure 2: On-shell massive self-energy integrals and off-shell HQET ones.

On-shell massive self-energy integrals with one massive line and any number of massless ones in some cases can be expressed via similar off-shell HQET integrals. Suppose all massless lines can be drawn at one side of the massive one and the resulting graph is planar (e. g., the diagram in Fig. 2a). Lines of such a diagram subdivide the plane into a number of polygonal cells (plus the exterior); with each cell we can associate a loop momentum (flowing counterclockwise). Then outer massless edges of the diagram correspond to the denominators $-k_i^2 - i0$; inner massless edges – to $-(k_i - k_j)^2 - i0$; and massive edges – to $m^2 - (k_i + mv)^2 - i0$ (Table 1). The corresponding HQET diagram (Fig. 2b) has HQET denominators $-2k_i \cdot v - 2\omega - i0$ instead of massive ones. First we perform Wick rotation of all loop momenta $k_{i0} \to ik_{i0}$ (in the v rest frame). Then, in Euclidean momentum space, we invert each loop momentum [8]:

$$k_i \to \frac{k_i}{k_i^2} \,. \tag{13}$$

Inversion transforms massive denominators to HQET ones (and vice versa) if we identify

$$-2\omega = m^{-1} \,, \tag{14}$$

see Table 1. As a result, a massive on-shell diagram (Fig. 2a) becomes $m^{-\sum n_i}$ (the sum runs over all massive line segments, n_i are their indices, i. e. the powers of the denominators) times the off-shell HQET diagram (Fig. 2b) with $\omega = -(2m)^{-1}$ (14). The indices of all inner massless edges, as well as of all massive edges (which become HQET ones), remain intact (see Table 1). From the same table it is clear that the index of an outer massless edge becomes $d - \sum n_i$, where the sum runs over all edges of the cell to which this outer edge belongs (they can be all massless, or one of them can be massive). If there is a cell k_i bounded only by inner massless edges, and maybe one massive one, then the denominator $(k_i^2)^{d-\sum n_j}$ will appear (Fig. 3). This denominator does not correspond to any line, and hence the resulting integral is not a Feynman integral at all; in this case, the discussed relation becomes rather useless (though formally correct). The inversion relations [8] were used, e. g., in [9, 10]).

Table 1: Inversion relations.

	Minkowski	Euclidean	Inversion
outer massless	$-k_i^2 - i0$	k_i^2	$\frac{1}{k_i^2}$
inner massless	$-(k_i - k_j)^2 - i0$	$(k_i - k_j)^2$	$\frac{(k_i - k_j)^2}{k_i^2 k_j^2}$
massive	$-k_i^2 - 2mv \cdot k_i - i0$	$k_i^2 - 2imk_{i0}$	$m\frac{-2\omega - 2ik_{i0}}{k_i^2}$
HQET	$-2\omega - 2k_i \cdot v - i0$	$-2\omega - 2ik_{i0}$	$m^{-1}\frac{k_i^2 - 2ik_{i0}}{k_i^2}$
measure	$d^d k_i$	$id^d k$	$i\frac{d^d k_i}{(k_i^2)^d}$

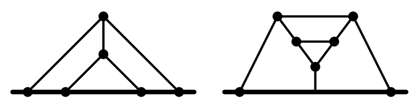

Figure 3: Examples of on-shell massive diagrams which cannot be transformed to off-shell HQET ones by inversion relations.

The inversion relations can be generalized to similar vertex integrals; the masses of the initial particle and the final one may differ. At one loop (Fig. 4), the integrals

$$\begin{aligned}
&M(n_1, n_2, n; \vartheta; m_1, m_2) = \int \frac{d^d k}{i\pi^{d/2}} \\
&\quad \times \frac{1}{[-k^2 - 2m_1 v_1 \cdot k - i0]^{n_1} [-k^2 - 2m_2 v_2 \cdot k - i0]^{n_2} (-k^2 - i0)^n}\,,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&I(n_1, n_2, n; \vartheta; \omega_1, \omega_2) = \int \frac{d^d k}{i\pi^{d/2}} \\
&\quad \times \frac{1}{[-2k \cdot v_1 - 2\omega_1 - i0]^{n_1} [-2k \cdot v_2 - 2\omega_2 - i0]^{n_2} (-k^2 - i0)^n}
\end{aligned} \tag{16}$$

are related by

$$M(n_1, n_2, n; \vartheta; m_1, m_2) = m_1^{-n_1} m_2^{-n_2} I(n_1, n_2, d - n_1 - n_2 - n; \vartheta; -(2m_1)^{-1}, -(2m_2)^{-1})\,. \tag{17}$$

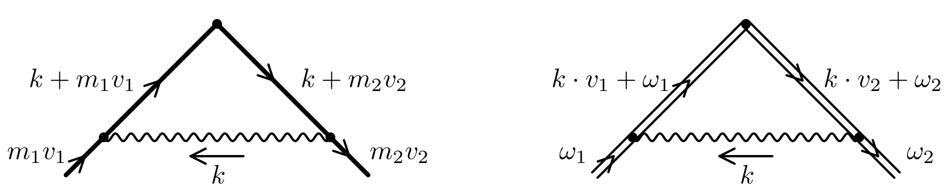

Figure 4: One-loop vertex integrals.

The integrals I (16) have been investigated in [11]. Here we need only the integrals M (15) with $m_1 = m_2$; they reduce to the integrals I (16) with $\omega_1 = \omega_2$ which are especially simple [11]:

$$I(n_1, n_2, n; \vartheta; \omega, \omega) = (-2\omega)^{d - n_1 - n_2 - 2n} I(n_1 + n_2, n)\, {}_3F_2 \left(\left. \begin{array}{c} n_1, n_2, \frac{d}{2} - n \\ \frac{n_1 + n_2}{2}, \frac{n_1 + n_2 + 1}{2} \end{array} \right| \frac{1 - \cosh\vartheta}{2} \right)\,, \tag{18}$$

where

$$I(n_1,n)=\frac{\Gamma(-d+n_1+2n)\Gamma(d/2-n)}{\Gamma(n_1)\Gamma(n)} \tag{19}$$

is the one-loop HQET self-energy integral. We only need integer $n_{1,2}$; in this case all I reduce by IBP to 2 master integrals [11]: $I(1,0,n)$ (trivial) and $I(1,1,n)$ (given by (18)).

Inversion relations can be generalized to diagrams with more external legs. For example, the one-loop massive box diagram with 2 on-shell legs and the corresponding off-shell HQET one (Fig. 5)

$$\begin{aligned}
&M(n_1,n_2,n_3,n_4;\vartheta;m_1,m_2;q^2,q\cdot v_1,q\cdot v_2)=\int\frac{d^dk}{i\pi^{d/2}}\times\\
&\frac{1}{(-k^2-2m_1v_1\cdot k)^{n_1}(-k^2-2m_2v_2\cdot k)^{n_2}(-(k+q)^2)^{n_3}(-k^2)^{n_4}}\,,
\end{aligned} \tag{20}$$

$$\begin{aligned}
&I(n_1,n_2,n_3,n_4;\vartheta;\omega_1,\omega_2;q^2,q\cdot v_1,q\cdot v_2)=\int\frac{d^dk}{i\pi^{d/2}}\times\\
&\frac{1}{(-2k\cdot v_1-2\omega_1)^{n_1}(-2k\cdot v_2)^{n_2}(-(k+q)^2)^{n_3}(-k^2)^{n_4}}
\end{aligned} \tag{21}$$

are related by

$$\begin{aligned}
&M(n_1,n_2,n_3,n_4;\vartheta;m_1,m_2;q^2,q\cdot v_1,q\cdot v_2)=m_1^{-n_1}m_2^{-n_2}(-q^2)^{n_3}\\
&I(n_1,n_2,n_3,d-n_1-n_2-n_3-n_4;\vartheta;-(2m_1)^{-1},-(2m_2)^{-1};1/q^2,q\cdot v_1/q^2,q\cdot v_2/q^2)\,.
\end{aligned} \tag{22}$$

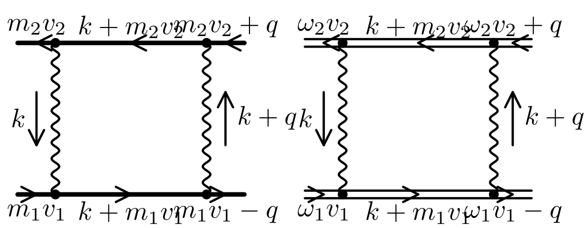

Figure 5: Box diagrams.

4 Large-β_0 limit

We need only terms with the highest degree of n_f; therefore, there is no need to distinguish between n_f and $n_l=n_f-1$, or any $n_f+\text{const}$. The gluon propagator can be written as

$$D_{\mu\nu}(k)=\frac{1}{k^2(1-\Pi(k^2))}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\,, \tag{23}$$

where the gluon self energy is

$$\begin{aligned}
\Pi(k^2)&=\beta_0\frac{g_0^2}{(4\pi)^{d/2}}e^{-\gamma\varepsilon}\frac{D(\varepsilon)}{\varepsilon}(-k^2)^{-\varepsilon}\,,\\
D(\varepsilon)&=e^{\gamma\varepsilon}\frac{(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{(1-2\varepsilon)(1-\frac{2}{3}\varepsilon)\Gamma(1-2\varepsilon)}=1+\frac{5}{3}\varepsilon+\cdots
\end{aligned} \tag{24}$$

At this leading large β_0 order, the coupling constant renormalization is simple:

$$\beta_0\frac{g_0^2}{(4\pi)^{d/2}}e^{-\gamma\varepsilon}=bZ_\alpha(b)\mu^{2\varepsilon}\,,\quad b=\beta_0\frac{\alpha_s(\mu)}{4\pi}\,,\quad Z_\alpha=\frac{1}{1+b/\varepsilon}\,. \tag{25}$$

The bare QCD matrix elements can be written in the form [12, 4]

$$F_i = \delta_{i0} + \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{f_i(\varepsilon, L\varepsilon)}{L} \Pi(-m^2)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) . \tag{26}$$

It is convenient to write the functions $f_i(\varepsilon, u)$ in the form usual for on-shell massive QCD problems (see [3])

$$f_i(\varepsilon, u) = C_F \frac{e^{\gamma\varepsilon}}{D(\varepsilon)} \frac{\Gamma(1-2u)\Gamma(1+u)}{\Gamma(3-u-\varepsilon)} N_i(\varepsilon, u) . \tag{27}$$

We calculate the vertex function (Fig. 1) and multiply it by Z_Q^{os} with the $1/\beta_0$ accuracy (see [3]). Reducing on-shell massive QCD integrals to off-shell HQET ones by the inversion relation (17) and then to the master integrals by IBP [11], we obtain

$$\begin{aligned}
N_0(\varepsilon, u) = {} & \biggl[-\eta u \frac{n-2+\varepsilon}{w-1} - 2(w+1)u(n-2)^2 - u\bigl(\eta u + 4(w+1)\varepsilon\bigr)(n-2) \\
& \quad + 2(2-u)\bigl(w + (w+1)u\bigr) - (6w + 2u + \eta u^2)\varepsilon + 2\bigl(w - (w+1)u\bigr)\varepsilon^2 \biggr] F \\
& + \eta u \frac{n-2+\varepsilon}{w-1} + 2(n-2)^2 + 4\varepsilon(n-2) - 6(1-u^2) + 2(1-u)(5+2u)\varepsilon - 2(1-2u)\varepsilon^2 , \\
N_1(\varepsilon, u) = {} & u \biggl[\eta w \frac{n-2+\varepsilon}{w-1} - \eta u(n-2) - 2 + u + \varepsilon - \eta u\varepsilon \biggr] F - \eta u \frac{n-2+\varepsilon}{w-1} , \\
N_2(\varepsilon, u) = {} & \eta u \frac{n-2+\varepsilon}{w-1} \Bigl[1 - \bigl(1 + (w-1)u\bigr) F \Bigr] ,
\end{aligned} \tag{28}$$

where

$$F = {}_2F_1 \left(\begin{array}{c} 1, 1+u \\ 3/2 \end{array} \middle| \frac{1-w}{2} \right) . \tag{29}$$

At $\vartheta = 0$ this result agrees with the result of [13] at $m_1 = m_2$, see also [3][4].

Re-expressing the bare form factors (26) via the renormalized coupling we obtain

$$F_i = \delta_{i0} + \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{f_i(\varepsilon, L\varepsilon)}{L} \left[D(\varepsilon) \left(\frac{\mu^2}{m^2}\right)^{\varepsilon} \frac{b}{\varepsilon + b} \right]^L . \tag{30}$$

We should have (see (8))

$$\log F_0 = \log(Z(\alpha_s(\mu))/\tilde{Z}(\alpha_s(\mu))) + \log H(\mu, \mu) \; : \tag{31}$$

negative degrees of ε go to $\log(Z/\tilde{Z})$, non-negative ones – to $\log H$. The function

$$f_0(\varepsilon, u) D(\varepsilon)^{u/\varepsilon} \left(\frac{\mu^2}{m^2}\right)^u = \sum_{n,m=0}^{\infty} f_{nm} \varepsilon^n u^m \tag{32}$$

is regular at the origin; expanding $(b/(\varepsilon + b))^L$ in b, we obtain a quadruple sum. In the coefficient of ε^{-1} all f_{nm} except f_{n0} cancel; differentiating this coefficient in $\log b$ we obtain the anomalous dimension corresponding to $Z/\tilde{Z}$ [12, 4]:

$$\gamma_n - \tilde{\gamma} = -2 \frac{b}{\beta_0} f_0(-b, 0) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) . \tag{33}$$

[4] Note a typo: the unnumbered formula below (8.93) should read

$$R_0 = \cosh(Lu) , \quad R_1 = \frac{\sinh[(1-2u)L/2]}{\sinh(L/2)} .$$

These anomalous dimensions at the $1/\beta_0$ order are [14, 15]

$$\gamma_n = 4C_F \frac{b}{\beta_0} \frac{(1+\frac{2}{3}b)\Gamma(2+2b)}{(1+b)^2(2+b)\Gamma^3(1+b)\Gamma(1-b)}(n-1)(3-n+2b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,, \tag{34}$$

$$\tilde{\gamma} = 4C_F \frac{b}{\beta_0} \frac{(1+\frac{2}{3}b)\Gamma(2+2b)}{(1+b)\Gamma^3(1+b)\Gamma(1-b)}(\vartheta\coth\vartheta - 1) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,. \tag{35}$$

Our results satisfy this requirement ($f_{1,2}(-b,0) = 0$ because the QCD current J does not mix with currents with other Dirac structures).

In the coefficient of ε^0 all f_{nm} except f_{n0} and f_{0m} cancel. The coefficients f_{n0} form $K_{\gamma_n - \tilde{\gamma}}(\alpha_s(\mu))$, see (9); we have [4]

$$\hat{H}_i = \delta_{i0} + \frac{1}{\beta_0}\int_0^\infty du\, e^{-u/b} S_i(u) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,, \tag{36}$$

where the Borel images of the perturbative series for $\hat{H}_i$ are

$$S_i(u) = \frac{1}{u}\left[\left(e^{5/3}\frac{\mu_0^2}{m^2}\right)^u f_i(0,u) - f_i(0,0)\right]\,. \tag{37}$$

The integral (36) is not well defined because of poles at the integration contour. The leading renormalon ambiguities are given by the residues at $u = 1/2$ [16] (see also [3]). It is easy to calculate these residues because F (29) at $u = 1/2$ is just $2/(w+1)$:

$$\Delta H_0 = \left(\frac{4}{w+1} - 3\right)\frac{\Delta\bar{\Lambda}}{2m}\,, \quad \Delta H_1 = \frac{1}{w+1}\frac{\Delta\bar{\Lambda}}{2m}\,, \quad \Delta H_2 = 0\,, \tag{38}$$

where

$$\Delta\bar{\Lambda} = -2\frac{C_F}{\beta_0}e^{5/6}\Lambda_{\overline{\mathrm{MS}}}\,.$$

As demonstrated in [16], these IR renormalon ambiguities are compensated by the UV renormalon ambiguities in the matrix elements of the HQET operators $\tilde{O}_i$ in (5).

Using the `Mathematica` package `HypExp` [17] (which uses uses `HPL` [18]) we expand F (29) in u:

$$\begin{aligned}
F = \frac{1}{\sinh\vartheta}\Bigg[&\vartheta - H_{-+}(\tau)u - (H_{-+-}(\tau) - 2H_{-+}(\tau)l)\,\frac{u^2}{2}\\
&- \left(H_{-+--}(\tau) - 2H_{-+-}(\tau)l + 2H_{-+}(\tau)l^2\right)\frac{u^3}{3}\\
&- \left(H_{-+---}(\tau) - 2H_{-+--}(\tau)l + 2H_{-+-}(\tau)l^2 - \frac{4}{3}H_{-+}(\tau)l^3\right)\frac{u^4}{4}\\
&- \left(H_{-+----}(\tau) - 2H_{-+---}(\tau)l + 2H_{-+--}(\tau)l^2 - \frac{4}{3}H_{-+-}(\tau)l^3 + \frac{2}{3}H_{-+}(\tau)l^4\right)\frac{u^5}{5}\\
&- \Bigg(H_{-+-----}(\tau) - 2H_{-+----}(\tau)l + 2H_{-+---}(\tau)l^2 - \frac{4}{3}H_{-+--}(\tau)l^3 + \frac{2}{3}H_{-+-}(\tau)l^4\\
&\quad - \frac{4}{15}H_{-+}(\tau)l^5\Bigg)\frac{u^6}{6} - \cdots\Bigg]\,,
\end{aligned} \tag{39}$$

where

$$\tau = \tanh\frac{\vartheta}{2}\,, \quad l = \frac{1}{2}H_-(\tau) = \log\cosh\frac{\vartheta}{2}\,, \quad H_+(\tau) = \vartheta\,, \tag{40}$$

and $H_{\dots}(\tau)$ are harmonic polylogarithms (see [19, 18]). The expansion (39) is sufficient up to 6 loops; it can be extended if desired. Only one new polylogarithm appears at each order.

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A Anticommuting γ_5 and 't Hooft–Veltman γ_5

For flavour-nonsinglet currents one may use the anticommuting γ_5 without encountering contradictions; they are related to the currents with the 't Hooft–Veltman γ_5 by a finite renormalization [20]:

$$\left(\bar{q}\gamma_5^{\mathrm{AC}}\Gamma_n\tau q\right)_\mu = Z_{2-n}(\alpha^{(n_f)}(\mu))\left(\bar{q}\gamma_5^{\mathrm{HV}}\Gamma_n\tau q\right)_\mu\,, \tag{41}$$

where τ is a flavour matrix with $\operatorname{Tr}\tau=0$. The currents with $\gamma_5^{\mathrm{AC}}\Gamma_n$ have anomalous dimensions γ_n, because they can be obtained from the case of massless quarks; $\gamma_5^{\mathrm{HV}}\Gamma_n$ is just Γ_{4-n} with reshuffled components. Equating the derivatives in $d\log\mu$ we obtain

$$Z_{2-n}(\alpha_s) = K^{(n_f)}_{\gamma_n-\gamma_{4-n}}(\alpha_s)\,, \tag{42}$$

where the anomalous dimensions γ_n and γ_{4-n} differ starting from 2 loops. In particular, $Z_0(\alpha_s)=1$. In HQET currents with γ_5^{AC} and with γ_5^{HV} have the same anomalous dimension $\tilde{\gamma}$, and the finite renormalization factor similar to (42) is 1. In the large β_0 limit (see (34))

$$Z_n(\alpha_s) = \exp\left[-\frac{8n}{\beta_0}\int_0^b db\frac{(1+\frac{2}{3}b)\Gamma(2+2b)}{(1+b)^2(2+b)\Gamma^3(1+b)\Gamma(1-b)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\right]. \tag{43}$$

At the leading $1/\beta_0$ order we may use these formulae for flavour singlet currents, too. The matrix $\gamma_5^{\mathrm{AC}}\Gamma_n$ has the same property (4) but with $\eta=-(-1)^n$. From our results (26)–(28) we see that, indeed,

$$\hat{H}_{\gamma_5^{\mathrm{AC}}\Gamma_n} = \hat{H}_{\Gamma_n}\Big|_{\eta\to-\eta} = \hat{H}_{\gamma_5^{\mathrm{HV}}\Gamma_n} = \hat{H}_{\Gamma_{4-n}}\,. \tag{44}$$

Matrix elements of the currents with γ_5^{AC} and $n=0$, 1 can be written via smaller numbers of form factors:

$$<Q(mv_2)|J|Q(mv_1)> = F^P\bar{u}_2\gamma_5^{\mathrm{AC}}u_1\,,\quad F^P = F_0 - 2F_1 - (2w+1)F_2 \tag{45}$$

(where F_i with $n=0$, $\eta=-1$ are used), and

$$\begin{gathered}<Q(mv_2)|J^\mu|Q(mv_1)> = F_1^A\bar{u}_2\gamma_5^{\mathrm{AC}}\gamma^\mu u_1 + F_2^A\bar{u}_2\gamma_5^{\mathrm{AC}}u_1\frac{(v_2-v_1)^\mu}{2}\,,\\ F_1^A = F_0 + 2F_1 + (2w-1)F_2\,,\quad F_2^A = 4(F_1-F_2)\end{gathered} \tag{46}$$

(where F_i with $n=1$, $\eta=1$ are used).

The divergence of the axial current is

$$i\partial_\mu\left(\bar{Q}_0\gamma_5^{\mathrm{AC}}\gamma^\mu Q_0\right) = 2m_0\bar{Q}_0\gamma_5^{\mathrm{AC}}Q_0\,, \tag{47}$$

where the bare mass $m_0=Z_m^{\mathrm{os}}m$. Taking the matrix element of this equation we obtain

$$F_1^A + \frac{w-1}{2}F_2^A = Z_m^{\mathrm{os}}F^P\,. \tag{48}$$

The on-shell mass renormalization constant Z_m^{os} at the first $1/\beta_0$ order is given by the formula similar to (26), (27) with $N_m(\varepsilon,u)=-2(3-2\varepsilon)(1-u)$, see, e.g., [3]. And indeed, from (28), (45–46) we obtain

$$N_1^A + \frac{w-1}{2}N_2^A = N^P + N_m\,. \tag{49}$$

B Vector form factors

The vector form factors $F^V_{1,2}$ (12) can be written in the form (26), (27); from (28), (12) we obtain

$$
\begin{aligned}
N^V_1(\varepsilon,u) &= 2\left[2w+u-3u^2-3w\varepsilon+2wu\varepsilon-(w-3)u^2\varepsilon+w\varepsilon^2-(w+1)u\varepsilon^2\right]F\\
&\quad -2\left[2+u-3u^2-3\varepsilon+2u\varepsilon+2u^2\varepsilon+\varepsilon^2-2u\varepsilon^2\right]\,, \qquad (50)\\
N^V_2(\varepsilon,u) &= 4u(1+u-2u\varepsilon)F\,. \qquad (51)
\end{aligned}
$$

All loop corrections to F^V_1 vanish at $\vartheta=0$, and hence $N^V_1=0$ at $w=1$.

The form factor $F^V_1 = H^V_1/\tilde{Z}$, where $\tilde{Z}$ at the $1/\beta_0$ order is determined by the anomalous dimension (35), and H^V_1 contains only non-negative powers of ε. We choose $\mu=\mu'=\mu_0=m$. H^V_1 at $\varepsilon=0$ is given by the coefficients f_{n0} (which produce $K_{-\tilde{\gamma}}$ (10)) and f_{0n} (which produce $\hat{H}^V_1$ (36)); ε^n terms ($n>0$) require all f_{nm}. Writing the expansion (39) as $F=f_0-f_1u-f_2u^2/2-f_3u^3/3-\cdots$ we obtain up to 4 loops

$$
\begin{aligned}
H^V_1 &= 1+C_F\frac{b}{\beta_0}\Bigg\{-2wf_1+(3w+1)f_0-4-\left(wf_2+(3w+1)f_1-\left(\frac{\pi^2}{6}+8\right)wf_0+\frac{\pi^2}{6}+8\right)\varepsilon\\
&\quad-\left(\frac{2}{3}wf_3+\frac{3w+1}{2}f_2+\left(\frac{\pi^2}{6}+8\right)wf_1+\left(\frac{2}{3}\zeta_3w-\frac{\pi^2}{4}w-\frac{\pi^2}{12}-16w\right)f_0-\frac{2}{3}\zeta_3+\frac{\pi^2}{3}+16\right)\varepsilon^2\\
&\quad-\Bigg(\frac{w}{2}f_4+\left(w+\frac{1}{3}\right)f_3+\left(\frac{\pi^2}{12}+4\right)wf_2-\left(\frac{2}{3}\zeta_3w-\frac{\pi^2}{4}w-\frac{\pi^2}{12}-16w\right)f_1\\
&\qquad-\left(\frac{\pi^4}{80}w-\zeta_3w-\frac{1}{3}\zeta_3+\frac{2}{3}\pi^2w+32w\right)f_0+\frac{\pi^4}{80}-\frac{4}{3}\zeta_3+\frac{2}{3}\pi^2+32\Bigg)\varepsilon^3+\cdots\\
&\quad-b\Bigg[wf_2+\left(\frac{19}{3}w+1\right)f_1-\frac{1}{3}\left(2\pi^2w+\frac{209}{6}w+\frac{1}{2}\right)f_0+\frac{2}{3}\left(\pi^2+\frac{53}{3}\right)\\
&\quad+\Bigg(2wf_3+\frac{1}{2}\left(\frac{47}{3}w+3\right)f_2+\left(\frac{3}{2}\pi^2w+\frac{281}{9}w+\frac{1}{3}\right)f_1\\
&\qquad-\left(8\zeta_3w+\frac{131}{36}\pi^2w+\frac{3}{4}\pi^2+\frac{5813}{108}w-\frac{203}{36}\right)f_0+8\zeta_3+\frac{79}{18}\pi^2+\frac{1301}{27}\Bigg)\varepsilon\\
&\quad+\Bigg(\frac{7}{2}wf_4+\frac{1}{3}\left(\frac{103}{3}w+7\right)f_3+\frac{1}{3}\left(\frac{19}{4}\pi^2w+\frac{317}{3}w+1\right)f_2\\
&\qquad+\frac{1}{3}\left(46\zeta_3w+\frac{271}{12}\pi^2w+\frac{19}{4}\pi^2+\frac{6677}{18}w-\frac{203}{6}\right)f_1\\
&\qquad-\frac{1}{3}\left(\frac{199}{80}\pi^4w+\frac{317}{3}\zeta_3w+23\zeta_3+\frac{1693}{36}\pi^2w+\frac{5}{12}\pi^2+\frac{129389}{216}w-\frac{6563}{72}\right)f_0\\
&\qquad+\frac{1}{3}\left(\frac{199}{80}\pi^4+\frac{386}{3}\zeta_3+\frac{427}{9}\pi^2+\frac{27425}{54}\right)\Bigg)\varepsilon^2+\cdots\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&- b^2 \Bigg[\frac{4}{3} w f_3 + \left(\frac{19}{3} w + 1 \right) f_2 + \frac{1}{3} \left(4\pi^2 w + \frac{203}{3} w + 1 \right) f_1 \\
&\quad - \frac{1}{3} \left(28\zeta_3 w + \frac{38}{3}\pi^2 w + 2\pi^2 + \frac{4919}{54} w - \frac{139}{6} \right) f_0 + \frac{1}{3} \left(28\zeta_3 + \frac{44}{3}\pi^2 + \frac{1834}{27} \right) \\
&\quad + \Bigg(6 w f_4 + 4 \left(\frac{52}{9} w + 1 \right) f_3 + \frac{1}{2} \left(7\pi^2 w + \frac{1171}{9} w + \frac{5}{3} \right) f_2 \\
&\quad + \left(44\zeta_3 w + \frac{359}{18}\pi^2 w + \frac{7}{2}\pi^2 + \frac{5366}{27} w - \frac{310}{9} \right) f_1 \\
&\quad - \left(\frac{92}{45}\pi^4 w + \frac{1114}{9}\zeta_3 w + 22\zeta_3 + \frac{4075}{108}\pi^2 w + \frac{17}{36}\pi^2 + \frac{258445}{972} w - \frac{9473}{108} \right) f_0 \\
&\quad + \frac{1}{9} \left(\frac{92}{5}\pi^4 + 1312\zeta_3 + \frac{2063}{6}\pi^2 + \frac{43297}{27} \right) \Bigg) \varepsilon + \cdots \Bigg] \\
&- b^3 \Bigg[3 w f_4 + 2 \left(\frac{19}{3} w + 1 \right) f_3 + \left(2\pi^2 w + \frac{203}{6} w + \frac{1}{2} \right) f_2 \\
&\quad + \left(24\zeta_3 w + \frac{38}{3}\pi^2 w + 2\pi^2 + \frac{4955}{54} w - \frac{139}{6} \right) f_1 \\
&\quad - \left(\frac{71}{60}\pi^4 w + \frac{233}{3}\zeta_3 w + 12\zeta_3 + \frac{203}{9}\pi^2 w + \frac{\pi^2}{3} + \frac{34937}{324} w - \frac{6007}{108} \right) f_0 \\
&\quad + \frac{1}{3} \left(\frac{71}{20}\pi^4 + 269\zeta_3 + \frac{206}{3}\pi^2 + \frac{4229}{27} \right) + \cdots \Bigg] + \cdots \Bigg\} . \qquad (52)
\end{aligned}
$$

The form factor $F_2^V = H_2^V$ is finite at $\varepsilon = 0$ (this requirement explains why N_2^V (51) vanishes at $u = 0$). We obtain

$$
\begin{aligned}
F_2^V &= C_F \frac{b}{\beta_0} \Bigg\{ 2 f_0 - 2(f_1 - 4 f_0)\varepsilon - \left(f_2 + 8 f_1 - \left(\frac{\pi^2}{6} + 16 \right) f_0 \right) \varepsilon^2 \\
&\quad - \frac{2}{3} \left(f_3 + 6 f_2 + \left(\frac{\pi^2}{4} + 24 \right) f_1 + \left(\zeta_3 - \pi^2 - 48 \right) f_0 \right) \varepsilon^3 + \cdots \\
&\quad - b \Bigg[2 f_1 - \frac{25}{3} f_0 + \left(3 f_2 + \frac{74}{3} f_1 - \frac{1}{2} \left(3\pi^2 + \frac{961}{9} \right) f_0 \right) \varepsilon \\
&\qquad + \frac{1}{3} \left(14 f_3 + 86 f_2 + \left(\frac{19}{2}\pi^2 + \frac{1105}{3} \right) f_1 - \left(46\zeta_3 + \frac{233}{6}\pi^2 + \frac{23545}{36} \right) f_0 \right) \varepsilon^2 + \cdots \Bigg] \\
&\quad - b^2 \Bigg[2 f_2 + \frac{50}{3} f_1 - \frac{1}{3} \left(4\pi^2 + \frac{317}{3} \right) f_0 \\
&\qquad + \left(8 f_3 + \frac{149}{3} f_2 + \left(7\pi^2 + \frac{1912}{9} \right) f_1 - \left(44\zeta_3 + \frac{521}{18}\pi^2 + \frac{18451}{54} \right) f_0 \right) \varepsilon + \cdots \Bigg] \\
&\quad - b^3 \Bigg[4 f_3 + 25 f_2 + \left(4\pi^2 + \frac{317}{3} \right) f_1 - \left(24\zeta_3 + \frac{50}{3}\pi^2 + \frac{8609}{54} \right) f_0 + \cdots \Bigg] + \cdots \Bigg\} . \qquad (53)
\end{aligned}
$$

We use `HPL` [18] to convert harmonic polylogarithms with $\pm$ indices to the usual ones (with indices $0, \pm 1$), then convert the argument to $x = e^{-\vartheta}$ from $\tau = (1-x)/(1+x)$, and finally transform products of harmonic polylogarithms to linear combinations; we have successfully reproduced all $n_l^{L-1}\alpha_s^L$ terms with $L = 1$, 2, 3 in $F_{1,2}^V$ from [1].

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