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Four-loop renormalization of QCD with a reducible fermion representation of the gauge group: anomalous dimensions and renormalization constants

K. G. Chetyrkin[a,b] **, M. F. Zoller**[c]

[a] *Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), Germany*

[b] *II Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany*

[c] *Institut für Physik, University of Zurich (UZH), Switzerland*

E-mail: konstantin.chetyrkin@kit.edu, zoller@physik.uzh.ch

Abstract: We present analytical results at four-loop level for the renormalization constants and anomalous dimensions of an extended QCD model with one coupling constant and an arbitrary number of fermion representations. One example of such a model is the QCD plus gluinos sector of a supersymmetric theory where the gluinos are Majorana fermions in the adjoint representation of the gauge group.

The renormalization constants of the gauge boson (gluon), ghost and fermion fields are analytically computed as well as those for the ghost-gluon vertex, the fermion-gluon vertex and the fermion mass. All other renormalization constants can be derived from these. Some of these results were already produced in Feynman gauge for the computation of the β-function of this model, which was recently published [1]. Here we present results for an arbitrary ξ-parameter.

Keywords: Renormalization Group, QCD

Contents

1 Introduction

The behaviour of Green's functions wrt a shift of the renormalization scale is described by the anomalous dimensions of the fields and parameters of the theory, which enter the Renormalization Group Equations (RGE). For QCD the full set of four-loop renormalization constants and anomalous dimensions was presented in [2]. The results for the four-loop QCD β-function [3, 4] and the four-loop quark mass and field anomalous dimensions had already been available [5–7]. [1]

In this paper we consider a model with a non-abelian gauge group, one coupling constant and a reducible fermion representation, i. e. any number of irreducible fermion representations. The β-function for the coupling this model was computed in an earlier work [1]. Here we provide the remaining Renormalization Group (RG) functions in full dependence on the gauge parameter ξ.

Apart from completing the set of renormalization constants and the RGE of the theory, which is important in itself, the gauge boson and ghost propagator anomalous dimensions serve another purpose. These quantities are essential ingredients in comparing the momentum dependence of the corresponding propagators derived in non-perturbative calculations on the lattice, with perturbative results (see e. g. [11–18]).

This paper is structured as follows: First, we will give the notation and definitions for the model and the computed RG functions We will also repeat how the special case of QCD

[1]Recently, the five-loop QCD β-function has been obtained for QCD colour factors [8] as well as for a generic gauge group [9] (see, also, [10]).

plus Majorana gluinos in the adjoint representation of the gauge group can be derived from our more general results. Then we will present analytical results for the four-loop anomalous dimensions of the gauge boson, ghost and fermion field as well as the ones for the ghost-gluon vertex, the fermion-gluon vertex and the fermion mass in Feynman gauge for compactness. The renormalization constants and anomalous dimensions for a generic gauge parameter ξ can be found in machine readable form in an accompanying file, which can be downloaded together with our source files on `www.arxiv.org`.

2 Notation and definitions

2.1 QCD with several fermion representations

The Lagrangian of a QCD-like model extended to include several fermion representations of the gauge group is given by

$$\begin{aligned}\mathcal{L}_{QCD} &= -\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} - \frac{1}{2\lambda}\left(\partial_\mu A^{a\,\mu}\right)^2 + \partial_\mu \bar{c}^a \partial^\mu c^a + g_s f^{abc}\,\partial_\mu \bar{c}^a A^{b\,\mu} c^c \\ &+ \sum_{r=1}^{N_{\rm rep}} \sum_{q=1}^{n_{f,r}} \left\{ \frac{i}{2}\bar{\psi}_{q,r} \overleftrightarrow{\not{\partial}}\, \psi_{q,r} - m_{q,r}\bar{\psi}_{q,r}\psi_{q,r} + g_s \bar{\psi}_{q,r} \not{A}^a T^{a,r}\psi_{q,r} \right\}, \end{aligned} \tag{2.1}$$

with the gluon field strength tensor

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu. \tag{2.2}$$

The index r specifies the fermion representation and the index q the fermion flavour, $\psi_{q,r}$ is the corresponding fermion field and $m_{q,r}$ the corresponding fermion mass. The number of fermion flavours in representation r is $n_{f,r}$ for any of the $N_{\rm rep}$ fermion representations.

The generators $T^{a,r}$ of each fermion representation r fulfill the defining anticomuting relation of the Lie Algebra corresponding to the gauge group:

$$\left[T^{a,r}, T^{b,r}\right] = i f^{abc} T^{c,r} \tag{2.3}$$

with the structure constants f^{abc}. We have one quadratic Casimir operator $C_{F,r}$ for each fermion representation, defined through

$$T^{a,r}_{ik} T^{a,r}_{kj} = \delta_{ij} C_{F,r}, \tag{2.4}$$

and C_A for the adjoint representation. The dimensions of the fermion representations are given by $d_{F,r}$ and the dimension of the adjoint representation by N_A. The traces of the different representations are defined as

$$T_{F,r}\delta^{ab} = \mathbf{Tr}\left(T^{a,r}T^{b,r}\right) = T^{a,r}_{ij} T^{b,r}_{ji}. \tag{2.5}$$

At four-loop level we also encounter higher order invariants in the gauge group factors which are expressed in terms of symmetric tensors

$$d_R^{a_1 a_2 \ldots a_n} = \frac{1}{n!} \sum_{\text{perm } \pi} \mathrm{Tr}\left\{ T^{a_{\pi(1)},R} T^{a_{\pi(2)},R} \ldots T^{a_{\pi(n)},R} \right\}, \tag{2.6}$$

where R can be any fermion representation r, noted as $R = \{F, r\}$, or the adjoint representation, $R = A$, where $T^{a,A}_{bc} = -i\, f^{abc}$.

An important special case of this model is the QCD plus gluinos sector of a supersymmetric theory where the gluinos are Majorana fermions in the adjoint representation of the gauge group. Here we have $N_{rep} = 2$ and

$$\begin{array}{rclrcl} n_{f,1} &=& n_f, & n_{f,2} &=& \frac{n_{\tilde{g}}}{2}, \\ T_{F,1} &=& T_F, & T_{F,2} &=& C_A, \\ C_{F,1} &=& C_F, & C_{F,2} &=& C_A, \end{array} \tag{2.7}$$

the factor $\frac{1}{2}$ in front of the number of gluinos $n_{\tilde{g}}$ being a result of the Majorana nature of the gluinos (see e. g. [19]).

By adding counterterms to the Lagrangian (2.1) in order to remove all possible UV divergences we arrive at the bare Lagrangian expressed through renormalized fields, masses and the coupling constant:

$$\begin{aligned} \mathcal{L}_{QCD,B} = &-\frac{1}{4} Z_3^{(2g)} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \frac{1}{2\lambda} \left(\partial_\mu A^{a\,\mu}\right)^2 \\ &- \frac{1}{2} Z_1^{(3g)} g_s f^{abc} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^b_\mu A^c_\nu \\ &- \frac{1}{4} Z_1^{(4g)} g_s^2 \left(f^{abc} A^b_\mu A^c_\nu\right)^2 + Z_3^{(2c)} \partial_\mu \bar{c}^a \partial^\mu c^a + Z_1^{(ccg)} g_s f^{abc}\, \partial_\mu \bar{c}^a A^{b\,\mu} c^c \\ &+ \sum_{r=1}^{N_{\rm rep}} \sum_{q=1}^{n_{f,r}} \left\{ Z_2^{(q,r)} \frac{i}{2} \bar{\psi}_{q,r} \overleftrightarrow{\partial\!\!\!/}\, \psi_{q,r} - m_{q,r} Z_m^{(q,r)} Z_2^{(q,r)} \bar{\psi}_{q,r} \psi_{q,r} \right. \\ &+ \left. g_s Z_1^{(q,r)} \bar{\psi}_{q,r} A\!\!\!/^a T^{a,r} \psi_{q,r} \right\}, \end{aligned} \tag{2.8}$$

were we have already used the fact that $Z_\lambda = Z_3^{(2g)}$.

Due to the Slavnov-Taylor identities all vertex renormalization constants are connected and can be expressed through the renormalization constant of the coupling constant and the renormalization constants of the fields appearing in the respective vertex:

$$Z_{g_s} = Z_1^{(3g)} \left(Z_3^{(2g)}\right)^{-\frac{3}{2}} \tag{2.9}$$

$$Z_{g_s} = \sqrt{Z_1^{(4g)}} \left(Z_3^{(2g)}\right)^{-1} \tag{2.10}$$

$$Z_{g_s} = Z_1^{(ccg)} \left(Z_3^{(2c)} \sqrt{Z_3^{(2g)}}\right)^{-1} \tag{2.11}$$

$$Z_{g_s} = Z_1^{(q,r)} \left(Z_2^{(q,r)} \sqrt{Z_3^{(2g)}}\right)^{-1} \tag{2.12}$$

In the $\overline{\text{MS}}$-scheme using regularization in $D = 4 - 2\varepsilon$ space time dimensions all renormalization constants have the form

$$Z(a, \lambda) = 1 + \sum_{n=1}^{\infty} \frac{z^{(n)}(a, \lambda)}{\varepsilon^n}, \tag{2.13}$$

where $a = \frac{g_s^2}{16\pi^2}$. From the fact that the bare parameter $a_B = Z_a a \mu^{2\varepsilon}$ (with $Z_a = Z_{g_s}^2$) does not depend on the renormalization scale μ one finds

$$\beta^{(D)}(a) = \mu^2 \frac{da}{d\mu^2} = -\varepsilon a + \beta(a), \tag{2.14}$$

$$\beta(a) = a^2 \frac{d}{da} z_a^{(1)}(a) \tag{2.15}$$

Given a renormalization constant Z the corresponding anomalous dimension is defined as

$$\gamma(a, \lambda) = -\mu^2 \frac{d \log Z(a, \lambda)}{d\mu^2} = a \frac{\partial z^{(1)}(a)}{\partial a} := -\sum_{n=1}^{\infty} \gamma^{(n)}(\lambda)\, a^n \tag{2.16}$$

From the definition of anomalous dimensions (2.16) it follows that

$$\gamma(a, \lambda) = (\varepsilon a - \beta(a)) \frac{d \log Z(a, \lambda)}{da} - \gamma_3^{(2g)}(a, \lambda) \lambda \frac{d \log Z(a, \lambda)}{d\lambda}, \tag{2.17}$$

where we use the fact that the evolution of any parameter (or field) – here λ – is described by its anomalous dimension, i. e.

$$\lambda_B = Z_\lambda \lambda \;\Rightarrow\; \mu^2 \frac{d}{d\mu^2} \lambda = \gamma_\lambda \lambda, \tag{2.18}$$

and the fact that $\gamma_\lambda = \gamma_3^{(2g)}$. Using (2.17) one can reconstruct renormalization constants from the corresponding anomalous dimension, a finite and usually more compact quantity, and the β-function of the model.

2.2 Technicalities

The 1-particle-irreducible Feynman diagrams needed for this project were generated with QGRAF [20]. We compute $Z_3^{(2c)}$, $Z_3^{(2g)}$ and $Z_2^{(q,r)}$ from the 1PI self-energies of the fields A_μ^a, c and $\psi_{q,r}$ as well as $Z_1^{(ccg)}$ and $Z_1^{(q,r)}$ from the respective vertex corrections and $Z_m^{(q,r)}$ from the 1PI corrections to a Green's function with an insertion of one operator $\bar{\psi}_{q,r}\psi_{q,r}$ and an external fermion line of type (q, r). We used two different methods to calculate these objects, first a direct four-loop calculation in Feynman gauge with massive tadpoles and then an indirect method where four-loop objects are constructed from propagator-like three-loop objects to derive the full dependence on the gauge parameter $\xi := 1 - \lambda$.

2.2.1 Direct four-loop calculation in the Feynman gauge with massive tadpoles

For $\xi = 0$ (Feynman gauge) the topologies of the diagrams were identified with the C++ programs Q2E and EXP [21, 22]. In this approach all diagrams were expanded in the external momenta in order to factor out the momentum dependence of the tree-level vertex or propagator, e. g. $q^\mu q^\nu - q^2 g^{\mu\nu}$ for the gluon self-energy. Then the tensor integrals were projected onto scalar integrals, using e. g. $\frac{q^\mu q^\nu}{q^4}$ as well as $\frac{g^{\mu\nu}}{q^2}$ as projectors for the gluon self-energy. After this we set all external momenta to zero since the UV divergent part of the integral does not depend on finite external momenta. We then use the method of introducing the same auxiliary mass parameter M^2 in every propagator denominator [23, 24]. Subdivergencies $\propto M^2$ are cancelled by an unphysical gluon mass counterterm $\frac{M^2}{2}\delta Z^{(2g)}_{M^2}\, A^a_\mu A^{a\,\mu}$ restoring the correct UV divergent part of the diagrams. This method was e. g. used in [3, 4, 25–29] and is explained in detail in [30].

For the expansions, application of projectors, evaluation of fermion traces and counterterm insertions in lower loop diagrams we used FORM [31, 32]. The scalar tadpole integrals were computed with the FORM-based package MATAD [33] up to three-loop order. At four loops we use the C++ version of FIRE 5 [34, 35] in order to reduce the scalar integrals to Master Integrals which can be found in [4]. Technical details of the reduction are described in the previous paper [28].

2.2.2 Indirect four-loop calculation using three-loop massless propagators

The case of a generic gauge parameter ξ is certainly possible to treat in the same *massive* way but calculations then require significantly more time and computer resources[2]. As a result we have chosen an alternative *massless* approach which reduces the evaluation of any L-loop Z-factor to the calculation of some properly constructed set of $(L-1)$-loop massless propagators [37–40]. As is well-known (starting already from $L = 2$ [41]) calculation of L-loop massive vacuum diagrams is significantly more complicated and time-consuming than the one of corresponding $(L-1)$-loop massless propagators.

The approach is easily applicable for any Z-factor except for Z_3 [2]. The latter problem is certainly doable within the massless approach but requires significantly more human efforts in resolving rather sophisticated combinatorics[3]. On the other hand, one could restore the full ξ-dependence of Z_3 from all other renormalization constants and from the fact that the charge renormalization constant Z_g is gauge invariant [2, 36]. As Z_g in QCD with fermions transforming under arbitrary reducible representation of the gauge group has been recently found in [1] we have proceeded in this way. For calculation of 3-loop massless propagator we have used the FORM version of MINCER [42].

2.2.3 computation of the gauge group factors

The calculation of the gauge group factors was done with an extended version of the FORM package COLOR [43] already used and presented in [1]. We take the following steps:

1. For the generation of the diagrams in QGRAF [20] we use one field A for the adjoint representation (gauge boson) and one field ψ for all the fermion representations. This has the advantage that we do not produce more Feynman diagrams than in QCD. Each fermion line in a diagram gets a line number and is treated as a different representation from the other fermion lines. Since we compute diagrams up to four-

[2]Nevertheless, it has been done recently along theses lines in [36] for the case of one irreducible fermion representation.

[3]Very recently the problem has been successfully solved in two radically different ways [8] and [9].

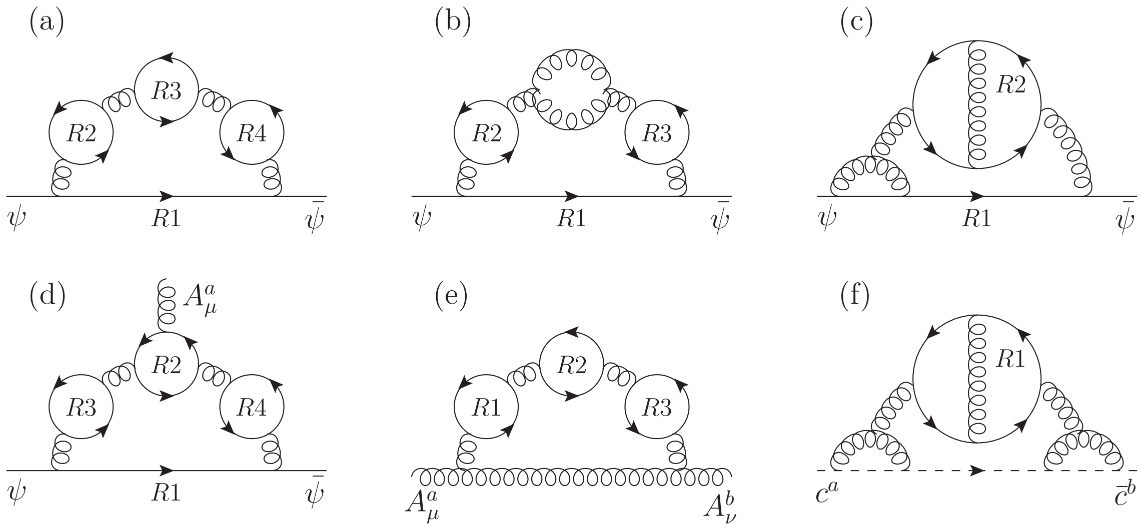

Figure 1: Four-loop diagrams contributing to the fermion self-energy (a,b,c), the fermion-gauge-boson-vertex (d), the gluon self-energy (e) and the ghost self-energy (f). Each fermion line is initially treated as a different representation $R1, \ldots, R4$.

loop order we need up to four different line representations $R1, \ldots, R4$ (see Fig. 1) with the generators $T^{a,R1}_{ij}$ = `T1(i,j,a)`, $T^{a,R2}_{ij}$ = `T2(i,j,a)`, $T^{a,R3}_{ij}$ = `T3(i,j,a)` and $T^{a,R4}_{ij}$ = `T4(i,j,a)`. Each fermion loop gets assigned a factor n_f.

2. The modified version of COLOR [1, 43] then writes the generators into traces

$$\mathbf{Tr}\left\{T^{a_1,R}\ldots T^{a_n,R}\right\} = \texttt{TR\{R\}(a1,...,an)}, \quad (R = R1, \ldots, R4) \tag{2.19}$$

which are then reduced as outlined in [43] yielding colour factors expressed through traces `TF{R}`, the Casimir operators `cF{R}` and `cA`, the dimensions of the representations `dF{R}` and `NA`.

3. Now we change from fermion line numbers $R1, \ldots, R4$ to four explicit physical fermion representations r by substituting each of the line numbers $R1, \ldots, R4$ by the sum over all representations $r = 1, \ldots, 4$. An example of the substitution of $\{R1, \ldots, R4\}$-colour factors with those of the physical representaions in a one-loop diagram is

$$\texttt{Nf*TF1} \to n_{f,1}T_{F,1} + n_{f,2}T_{F,2} + n_{f,3}T_{F,3} + n_{f,4}T_{F,4}. \tag{2.20}$$

At higher orders this subtitution becomes much more involved[4]. Diagram (a) from Fig. 1 now corresponds to a sum of $4^4 = 256$ diagrams with explicit fermion representations. This lengthy representation of our results is needed for the renormalization procedure, since e. g. a one loop counterterm to the gluon self-energy, computed from

[4] For this reason it is convenient to collect all combinations $\texttt{Nf}^{\texttt{x1}}\texttt{*TF1}^{\texttt{x2}}\texttt{*CF1}^{\texttt{x3}}\texttt{*TF2}^{\texttt{x4}}\texttt{*CF2}^{\texttt{x5}}\texttt{*TF3}^{\texttt{x6}}\texttt{*CF3}^{\texttt{x7}}\texttt{*TF4}^{\texttt{x8}}\texttt{*CF4}^{\texttt{x9}}$ in a function `C(x1,...,x9)`. The factors `C(x1,...,x7)` are then substituted by the proper combinations of $n_{f,1}$, $T_{F,1}$, $c_{F,1}$, etc.

a diagram with only $R1$, must be applied to all the fermion loops in Fig. 1 (a,b,d,e). This is most conveniently achieved if each fermion-loop is considered a sum over all physical fermion representations just as it is considered a some over all (massless) fermion flavours.[5] The factors involving $d_{F,r}^{a_1a_2a_3a_4}$, $d_{F,r}^{a_1a_2a_3}$, $d_A^{a_1a_2a_3a_4}$ and $d_A^{a_1a_2a_3}$ appear only at four-loop level and do hence not interfere with lower order diagrams with counterterm insertions. They can be treated directly in the next step.

4. After all subdivergencies are cancelled by adding the lower-loop diagrams with counterterm insertions we simplify and generalize the notation. The explicit colour factors are collected in sums of terms built from $n_{f,r}$, $C_{F,r}$ and $T_{F,r}$ over all physical representations r, e. g.[6]

$$n_{f,1}T_{F,1} \to \sum n_{f,i}T_{F,i} - n_{f,2}T_{F,2} - n_{f,3}T_{F,3} - n_{f,4}T_{F,4}. \tag{2.21}$$

Since we used the maximum number of different fermion representations which can appear in any diagram the result is valid for any number of fermion representations N_{rep}.

3 Results

In this section we give the results for the anomalous dimensions of the QCD-like model with an arbitrary number of fermion representations as described above to four-loop level. The number of active fermion flavours of representation i is denoted by $n_{f,i}$. Apart from the Casimir operators C_A and $C_{F,i}$ and the trace $T_{F,i}$ the following invariants appear in our results:

$$\begin{aligned} d_{AA}^{(4)} &= \frac{d_A^{abcd}d_A^{abcd}}{N_A}, \quad d_{FA,i}^{(4)} = \frac{d_{F,i}^{abcd}d_A^{abcd}}{N_A}, \quad d_{FF,ij}^{(4)} = \frac{d_{F,i}^{abcd}d_{F,j}^{abcd}}{N_A}, \\ \tilde{d}_{FA,r}^{(4)} &= \frac{d_{F,r}^{abcd}d_A^{abcd}}{d_{F,r}}, \quad \tilde{d}_{FF,ri}^{(4)} = \frac{d_{F,r}^{abcd}d_{F,i}^{abcd}}{d_{F,r}}, \end{aligned} \tag{3.1}$$

where r is fixed and i, j will be summed over all fermion representations. In this section we give the results for $\lambda = 1$ (Feynman gauge), the general case $\lambda = (1-\xi)$ can be found in the accompanying source files on `www.arxiv.org`.

From the gauge boson field strength renormalization constant $Z_3^{(2g)}$ we compute the anomalous dimension according to (2.16)

$$\left(\gamma_3^{(2g)}\right)^{(1)} = -\frac{5}{3}C_A + \sum_i \frac{4}{3}n_{f,i}T_{F,i}, \tag{3.2}$$

$$\left(\gamma_3^{(2g)}\right)^{(2)} = -\frac{23}{4}C_A^2 + \sum_i n_{f,i}T_{F,i}\left(4C_{F,i} + 5C_A\right), \tag{3.3}$$

[5] Since renormalization constants in the $\overline{\text{MS}}$-scheme do not depend on masses all fermion flavours can be treated as massless for their computation.

[6] For convenience we collect $n_{f,1}^{x1}n_{f,2}^{x2}n_{f,3}^{x3}n_{f,4}^{x4}T_{F,1}^{y1}T_{F,2}^{y2}T_{F,3}^{y3}T_{F,4}^{y4}C_{F,1}^{z1}C_{F,2}^{z2}C_{F,3}^{z3}C_{F,4}^{z4}$ in a function `CR(x1,...,x4,y1,...,y4,z1...,44)` which are then substituted by the proper sums of colour factors over all representations r.

$$\left(\gamma_3^{(2g)}\right)^{(3)} = -C_A^3\left(\frac{4051}{144}-\frac{3}{2}\zeta_3\right)+\sum_i n_{f,i}T_{F,i}\left[-2C_{F,i}^2+C_AC_{F,i}\left(\frac{5}{18}+24\zeta_3\right)\right.$$
$$\left.+C_A^2\left(\frac{875}{18}-18\zeta_3\right)\right]-\sum_{i,j}n_{f,i}n_{f,j}T_{F,i}T_{F,j}\left(\frac{44}{9}C_{F,j}+\frac{76}{9}C_A\right), \quad (3.4)$$
$$\left(\gamma_3^{(2g)}\right)^{(4)} = -C_A^4\left(\frac{252385}{1944}-\frac{1045}{12}\zeta_3+\frac{111}{16}\zeta_4+\frac{5125}{48}\zeta_5\right)+d_{AA}^{(4)}\left(\frac{131}{36}-\frac{307}{6}\zeta_3\right.$$
$$\left.-\frac{335}{2}\zeta_5\right)+\sum_i n_{f,i}\left\{T_{F,i}\left[-46C_{F,i}^3+C_AC_{F,i}^2\left(\frac{10847}{54}+\frac{980}{9}\zeta_3-240\zeta_5\right)\right.\right.$$
$$-C_A^2C_{F,i}\left(\frac{363565}{1944}-\frac{2492}{9}\zeta_3+126\zeta_4-120\zeta_5\right)+C_A^3\left(\frac{1404961}{3888}\right.$$
$$\left.\left.\left.-\frac{1285}{4}\zeta_3+\frac{387}{4}\zeta_4+110\zeta_5\right)\right]-d_{FA,i}^{(4)}\left(\frac{512}{9}-\frac{1376}{3}\zeta_3-120\zeta_5\right)\right\}$$
$$+\sum_{i,j}n_{f,i}n_{f,j}\left\{T_{F,i}T_{F,j}\left[C_{F,j}^2\left(\frac{304}{27}+\frac{128}{9}\zeta_3\right)-C_{F,i}C_{F,j}\left(\frac{184}{3}-64\zeta_3\right)\right.\right.$$
$$\left.-C_AC_{F,j}\left(\frac{15082}{243}+\frac{1168}{9}\zeta_3-48\zeta_4\right)-C_A^2\left(\frac{41273}{486}-\frac{340}{9}\zeta_3+36\zeta_4\right)\right]$$
$$\left.+d_{FF,ij}^{(4)}\left(\frac{704}{9}-\frac{512}{3}\zeta_3\right)\right\}$$
$$-\sum_{i,j,k}n_{f,i}n_{f,j}n_{f,k}T_{F,i}T_{F,j}T_{F,k}\left[\frac{1232}{243}C_{F,i}+C_A\left(\frac{1420}{243}-\frac{64}{9}\zeta_3\right)\right]. \quad (3.5)$$

From the ghost field strength renormalization constant $Z_3^{(2c)}$ we compute

$$\left(\gamma_3^{(2c)}\right)^{(1)} = -\frac{1}{2}C_A, \quad (3.6)$$
$$\left(\gamma_3^{(2c)}\right)^{(2)} = -\frac{49}{24}C_A^2+\frac{5}{6}C_A\sum_i n_{f,i}T_{F,i}, \quad (3.7)$$
$$\left(\gamma_3^{(2c)}\right)^{(3)} = -C_A^3\left(\frac{229}{27}+\frac{3}{4}\zeta_3\right)+C_A\sum_i n_{f,i}T_{F,i}\left[C_{F,i}\left(\frac{45}{4}-12\zeta_3\right)\right.$$
$$\left.+C_A\left(\frac{5}{216}+9\zeta_3\right)\right]+\frac{35}{27}C_A\sum_{i,j}n_{f,i}n_{f,j}T_{F,i}T_{F,j}, \quad (3.8)$$
$$\left(\gamma_3^{(2c)}\right)^{(4)} = -C_A^4\left(\frac{256337}{3888}+\frac{2485}{72}\zeta_3-\frac{123}{32}\zeta_4-\frac{4505}{96}\zeta_5\right)+d_{AA}^{(4)}\left(\frac{21}{8}-\frac{299}{4}\zeta_3\right.$$
$$\left.+\frac{265}{4}\zeta_5\right)+\sum_i n_{f,i}\left\{T_{F,i}C_A\left[-C_{F,i}^2\left(\frac{271}{12}+74\zeta_3-120\zeta_5\right)\right.\right.$$
$$+C_AC_{F,i}\left(\frac{22517}{432}-86\zeta_3+69\zeta_4-60\zeta_5\right)+C_A^2\left(\frac{449239}{7776}+\frac{2983}{24}\zeta_3\right.$$
$$\left.\left.\left.-\frac{423}{8}\zeta_4-55\zeta_5\right)\right]+d_{FA,i}^{(4)}\left(48\zeta_3-60\zeta_5\right)\right\}$$
$$-C_A\sum_{i,j}n_{f,i}n_{f,j}T_{F,i}T_{F,j}\left[C_{F,j}\left(\frac{115}{27}-40\zeta_3+24\zeta_4\right)\right.$$
$$\left.+C_A\left(\frac{8315}{972}+\frac{86}{3}\zeta_3-18\zeta_4\right)\right]$$

$$
+\sum_{i,j,k} n_{f,i}n_{f,j}n_{f,k}T_{F,i}T_{F,j}T_{F,k}C_A\left(\frac{166}{81}-\frac{32}{9}\zeta_3\right). \tag{3.9}
$$

From the fermion field strength renormalization constant $Z_2^{(q,r)}$ we find

$$
\begin{aligned}
\left(\gamma_2^{(q,r)}\right)^{(1)} &= C_{F,r}, &(3.10)\\
\left(\gamma_2^{(q,r)}\right)^{(2)} &= -\frac{3}{2}C_{F,r}^2+\frac{17}{2}C_AC_{F,r}-2\,C_{F,r}\sum_i n_{f,i}T_{F,i}, &(3.11)\\
\left(\gamma_2^{(q,r)}\right)^{(3)} &= \frac{3}{2}C_{F,r}^3+C_AC_{F,r}^2\left(-\frac{143}{4}+12\zeta_3\right)+C_A^2C_{F,r}\left(\frac{10559}{144}-\frac{15}{2}\zeta_3\right)\\
&\quad -C_{F,r}\sum_i n_{f,i}T_{F,i}\left(6C_{F,i}-9C_{F,r}+\frac{1301}{36}C_A\right)\\
&\quad +\frac{20}{9}C_{F,r}\sum_{i,j} n_{f,i}n_{f,j}T_{F,i}T_{F,j}, &(3.12)\\
\left(\gamma_2^{(q,r)}\right)^{(4)} &= -C_{F,r}^4\left(\frac{1027}{8}+400\zeta_3-640\zeta_5\right)+C_AC_{F,r}^3\left(\frac{5131}{12}+848\zeta_3-1440\zeta_5\right)\\
&\quad -C_A^2C_{F,r}^2\left(\frac{23777}{36}+214\zeta_3+66\zeta_4-790\zeta_5\right)+C_A^3C_{F,r}\left(\frac{10059589}{15552}\right.\\
&\quad \left.-\frac{1489}{24}\zeta_3+\frac{173}{4}\zeta_4-\frac{1865}{12}\zeta_5\right)-\tilde{d}_{FA,r}^{(4)}\left(66-190\zeta_3+170\zeta_5\right)\\
&\quad +\sum_i n_{f,i}\left\{T_{F,i}C_{F,r}\left[3C_{F,i}^2+C_{F,r}C_{F,i}\left(62-48\zeta_3\right)-C_{F,r}^2\left(\frac{119}{3}+16\zeta_3\right)\right.\right.\\
&\quad -C_AC_{F,i}\left(\frac{2945}{12}-156\zeta_3-12\zeta_4\right)+C_AC_{F,r}\left(\frac{1607}{9}-112\zeta_3+24\zeta_4\right.\\
&\quad \left.\left.+160\zeta_5\right)-C_A^2\left(\frac{1365691}{3888}+\frac{119}{3}\zeta_3+25\zeta_4+80\zeta_5\right)\right]+128\,\tilde{d}_{FF,ri}^{(4)}\Bigg\}\\
&\quad -\sum_{i,j} n_{f,i}n_{f,j}T_{F,i}T_{F,j}C_{F,r}\left[\frac{92}{9}C_{F,r}-C_{F,j}\left(44-32\zeta_3\right)\right.\\
&\quad \left.-C_A\left(\frac{6835}{243}+\frac{112}{3}\zeta_3\right)\right]+\frac{280}{81}C_{F,r}\sum_{i,j,k} n_{f,i}n_{f,j}n_{f,k}T_{F,i}T_{F,j}T_{F,k} &(3.13)
\end{aligned}
$$

for the anomalous dimension of a representation r fermion field.

The fermion field-gauge boson-vertex renormalization constant $Z_1^{(q,r)}$ yields

$$
\begin{aligned}
\left(\gamma_1^{(q,r)}\right)^{(1)} &= C_{F,r}+C_A, &(3.14)\\
\left(\gamma_1^{(q,r)}\right)^{(2)} &= -\frac{3}{2}C_{F,r}^2+\frac{17}{2}C_AC_{F,r}+\frac{67}{24}C_A^2-\sum_i n_{f,i}T_{F,i}\left(2C_{F,r}+\frac{5}{6}C_A\right), &(3.15)\\
\left(\gamma_1^{(q,r)}\right)^{(3)} &= \frac{3}{2}C_{F,r}^3-C_AC_{F,r}^2\left(\frac{143}{4}-12\zeta_3\right)+C_A^2C_{F,r}\left(\frac{10559}{144}-\frac{15}{2}\zeta_3\right)\\
&\quad +C_A^3\left(\frac{10703}{864}+\frac{3}{4}\zeta_3\right)+\sum_i n_{f,i}T_{F,i}\left[-6C_{F,r}C_{F,i}+9C_{F,r}^2\right.\\
&\quad \left.-C_AC_{F,i}\left(\frac{45}{4}-12\zeta_3\right)-\frac{1301}{36}C_AC_{F,r}-C_A^2\left(\frac{205}{108}+9\zeta_3\right)\right]
\end{aligned}
$$

$$+\sum_{i,j} n_{f,i} n_{f,j} T_{F,i} T_{F,j} \left(\frac{20}{9} C_{F,r} - \frac{35}{27} C_A \right), \tag{3.16}$$

$$\begin{aligned}\left(\gamma_1^{(q,r)}\right)^{(4)} = & -C_{F,r}^4 \left(\frac{1027}{8} + 400\zeta_3 - 640\zeta_5 \right) + C_A C_{F,r}^3 \left(\frac{5131}{12} + 848\zeta_3 - 1440\zeta_5 \right) \\ & -C_A^2 C_{F,r}^2 \left(\frac{23777}{36} + 214\zeta_3 + 66\zeta_4 - 790\zeta_5 \right) + C_A^3 C_{F,r} \left(\frac{10059589}{15552} \right. \\ & \left. -\frac{1489}{24}\zeta_3 + \frac{173}{4}\zeta_4 - \frac{1865}{12}\zeta_5 \right) + C_A^4 \left(\frac{350227}{3888} + \frac{2959}{72}\zeta_3 - \frac{111}{32}\zeta_4 - \frac{5125}{96}\zeta_5 \right) \\ & -d_{AA}^{(4)} \left(\frac{21}{8} - \frac{367}{4}\zeta_3 + \frac{335}{4}\zeta_5 \right) - \tilde{d}_{FA,r}^{(4)} \left(66 - 190\zeta_3 + 170\zeta_5 \right) \\ & +\sum_i n_{f,i} \left\{ T_{F,i} \left[3C_{F,r} C_{F,i}^2 + C_{F,r}^2 C_{F,i} \left(62 - 48\zeta_3 \right) - C_{F,r}^3 \left(\frac{119}{3} + 16\zeta_3 \right) \right. \right. \\ & +C_A C_{F,i}^2 \left(\frac{271}{12} + 74\zeta_3 - 120\zeta_5 \right) - C_A C_{F,r} C_{F,i} \left(\frac{2945}{12} - 156\zeta_3 - 12\zeta_4 \right) \\ & +C_A C_{F,r}^2 \left(\frac{1607}{9} - 112\zeta_3 + 24\zeta_4 + 160\zeta_5 \right) - C_A^2 C_{F,i} \left(\frac{34109}{432} - 102\zeta_3 + 63\zeta_4 \right. \\ & \left. -60\zeta_5 \right) - C_A^2 C_{F,r} \left(\frac{1365691}{3888} + \frac{119}{3}\zeta_3 + 25\zeta_4 + 80\zeta_5 \right) - C_A^3 \left(\frac{473903}{7776} + \frac{3311}{24}\zeta_3 \right. \\ & \left. \left. \left. -\frac{387}{8}\zeta_4 - 55\zeta_5 \right) \right] + 128\, \tilde{d}_{FF,ri}^{(4)} - d_{FA,i}^{(4)} \left(48\zeta_3 - 60\zeta_5 \right) \right\} \\ & +\sum_{i,j} n_{f,i} n_{f,j} T_{F,i} T_{F,j} \left[C_{F,r} C_{F,j} \left(44 - 32\zeta_3 \right) - \frac{92}{9} C_{F,r}^2 + C_A C_{F,j} \left(\frac{115}{27} - 40\zeta_3 \right. \right. \\ & \left. \left. +24\zeta_4 \right) + C_A C_{F,r} \left(\frac{6835}{243} + \frac{112}{3}\zeta_3 \right) + C_A^2 \left(\frac{6307}{972} + \frac{94}{3}\zeta_3 - 18\zeta_4 \right) \right] \\ & +\sum_{i,j,k} n_{f,i} n_{f,j} n_{f,k} T_{F,i} T_{F,j} T_{F,k} \left[\frac{280}{81} C_{F,r} - C_A \left(\frac{166}{81} - \frac{32}{9}\zeta_3 \right) \right] \end{aligned} \tag{3.17}$$

for each representation r and the ghost-gauge boson-vertex renormalization constant $Z_1^{(ccg)}$ yields

$$\left(\gamma_1^{(ccg)}\right)^{(1)} = \frac{1}{2} C_A, \tag{3.18}$$

$$\left(\gamma_1^{(ccg)}\right)^{(2)} = \frac{3}{4} C_A^2, \tag{3.19}$$

$$\left(\gamma_1^{(ccg)}\right)^{(3)} = \frac{125}{32} C_A^3 - \frac{15}{8} C_A^2 \sum_i n_{f,i} T_{F,i}, \tag{3.20}$$

$$\begin{aligned}\left(\gamma_1^{(ccg)}\right)^{(4)} = & \, C_A^4 \left(\frac{46945}{1944} + \frac{79}{12}\zeta_3 + \frac{3}{8}\zeta_4 - \frac{155}{24}\zeta_5 \right) + d_{AA}^{(4)} \left(17\zeta_3 - \frac{35}{2}\zeta_5 \right) \\ & -\sum_i n_{f,i} T_{F,i} C_A^2 \left[C_{F,i} \left(\frac{161}{6} - 16\zeta_3 - 6\zeta_4 \right) + C_A \left(\frac{3083}{972} \right. \right. \\ & \left. \left. +\frac{41}{3}\zeta_3 + \frac{9}{2}\zeta_4 \right) \right] - \sum_{i,j} n_{f,i} n_{f,j} T_{F,i} T_{F,j} C_A^2 \left(\frac{502}{243} - \frac{8}{3}\zeta_3 \right). \end{aligned} \tag{3.21}$$

Finally, the mass anomalous dimension computed from $Z_m^{(q,r)}$ is found to be

$$\left(\gamma_m^{(q,r)}\right)^{(1)} = 3\, C_{F,r}, \tag{3.22}$$

$$\left(\gamma_m^{(q,r)}\right)^{(2)} = \frac{3}{2}C_{F,r}^2 + \frac{97}{6}C_A C_{F,r} - \frac{10}{3}C_{F,r}\sum_i n_{f,i}T_{F,i}, \tag{3.23}$$

$$\begin{aligned}\left(\gamma_m^{(q,r)}\right)^{(3)} = &\frac{129}{2}C_{F,r}^3 - \frac{129}{4}C_A C_{F,r}^2 + \frac{11413}{108}C_A^2 C_{F,r} \\ &-C_{F,r}\sum_i n_{f,i}T_{F,i}\left[C_{F,r} + C_{F,i}\left(45 - 48\zeta_3\right) + C_A\left(\frac{556}{27} + 48\zeta_3\right)\right] \\ &-\frac{140}{27}C_{F,r}\sum_{i,j} n_{f,i}n_{f,j}T_{F,i}T_{F,j},\end{aligned} \tag{3.24}$$

$$\begin{aligned}\left(\gamma_m^{(q,r)}\right)^{(4)} = &-C_{F,r}^4\left(\frac{1261}{8} + 336\zeta_3\right) + C_A C_{F,r}^3\left(\frac{15349}{12} + 316\zeta_3\right) - C_A^2 C_{F,r}^2\left(\frac{34045}{36}\right. \\ &\left.+152\zeta_3 - 440\zeta_5\right) + C_A^3 C_{F,r}\left(\frac{70055}{72} + \frac{1418}{9}\zeta_3 - 440\zeta_5\right) - \tilde{d}_{FA,r}^{(4)}\left(32 - 240\zeta_3\right) \\ &+\sum_i n_{f,i}\left\{T_{F,i}C_{F,r}\left[C_{F,i}^2\left(\frac{271}{3} + 296\zeta_3 - 480\zeta_5\right) - C_{F,r}C_{F,i}\left(38 - 48\zeta_3\right)\right.\right. \\ &-C_{F,r}^2\left(\frac{437}{3} - 208\zeta_3\right) - C_A C_{F,i}\left(\frac{13106}{27} - 592\zeta_3 + 264\zeta_4 - 240\zeta_5\right) \\ &\left.+C_A C_{F,r}\left(\frac{1429}{9} - 224\zeta_3 - 160\zeta_5\right) - C_A^2\left(\frac{65459}{162} + \frac{2684}{3}\zeta_3 - 264\zeta_4 - 400\zeta_5\right)\right] \\ &\left.+\tilde{d}_{FF,ri}^{(4)}\left(64 - 480\zeta_3\right)\right\} + C_{F,r}\sum_{i,j} n_{f,i}n_{f,j}T_{F,i}T_{F,j}\left[C_{F,j}\left(\frac{460}{27} - 160\zeta_3 + 96\zeta_4\right)\right. \\ &\left.-\frac{52}{9}C_{F,r} + C_A\left(\frac{1342}{81} + 160\zeta_3 - 96\zeta_4\right)\right] \\ &-\sum_{i,j,k} n_{f,i}n_{f,j}n_{f,k}T_{F,i}T_{F,j}T_{F,k}C_{F,r}\left(\frac{664}{81} - \frac{128}{9}\zeta_3\right).\end{aligned} \tag{3.25}$$

We checked that the well known relations

$$\frac{\beta(a)}{a} = 2\gamma_1^{(ccg)}(a,\lambda) - 2\gamma_3^{(2c)}(a,\lambda) - \gamma_3^{(2g)}(a,\lambda), \tag{3.26}$$

$$\frac{\beta(a)}{a} = 2\gamma_1^{(q,r)}(a,\lambda) - 2\gamma_2^{(q,r)}(a,\lambda) - \gamma_3^{(2g)}(a,\lambda) \tag{3.27}$$

are fulfilled with the β-function from [1]. This is also true if we include the full dependence on the gauge parameter $\xi = 1 - \lambda$ in the anomalous dimensions. This dependence cancels in the β-function. We provide renormalization constants and anomalous dimensions with the full gauge dependence in the attached files, which can be downloaded with the source files of this paper from `www.arxiv.org`. We compared these fully ξ-dependent results with [36] for one fermion representation and find full agreement.

4 Conclusions

We have presented analytical results for the field anomalous dimensions $\gamma_3^{(2g)}$, $\gamma_3^{(2c)}$, $\gamma_2^{(q,r)}$, the vertex anomalous dimensions $\gamma_1^{(ccg)}$ and $\gamma_1^{(q,r)}$ and the mass anomalous dimension $\gamma_m^{(q,r)}$ in a QCD-like model with arbitrarily many fermion representations and with the full dependence on the gauge parameter ξ.

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