

COMBINATORICS OF R -, R^{-1} -, AND R^* -OPERATIONS AND ASYMPTOTIC EXPANSIONS OF FEYNMAN INTEGRALS IN THE LIMIT OF LARGE MOMENTA AND MASSES*

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Abstract

A generalization of the forest technique procedure — the R^{-1} -operation—is elaborated and then employed to treat a variety of problems. First, it is employed to reveal the underlying simple structure of the Bogoliubov-Parasiuk renormalization prescription based on momentum subtractions. Second, we use this structure to derive a generalized Zimmermann identity connecting two different renormalized versions of a given Feynman integral. Third, the recursive procedure to minimally subtract the ultraviolet and infrared divergences from euclidean, dimensionally regularized Feynman integrals—the R^* -operation—is simplified by reformulating it in terms of the R -operation alone. The new formulation is shown to lead immediately to a simple and regular algorithm for evaluating the overall ultraviolet divergences of arbitrary dimensionally regularized Feynman integrals, (including the ones appearing in two-dimensional field-theoretical models), the algorithm neatly reducing the problem to computing some massless propagator-type integrals. Finally, we construct a brief and concise proof of a general theorem which gives an explicitly finite large momenta and/or masses asymptotic expansion of an arbitrary (minimally subtracted) euclidean Feynman integral.

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Contents

1	INTRODUCTION	3
2	STRUCTURE OF FEYNMAN GRAPHS AND INTEGRALS	5
2.1	Basic graph-theoretical notations and definitions	5
2.2	Dimensionally regularized Feynman integrals	7
2.3	c -operation and its properties	8
3	R- AND R^{-1}- OPERATIONS	9
3.1	R -operation	9
3.2	R^{-1} -operation	11
3.3	R - and R^{-1} -operations within the MS-scheme	12
3.4	BPHZ renormalization and the R^{-1} -operation	13
4	R^*-OPERATION	17
4.1	R^* - primer	17
4.2	Generalized forest technique	19
4.3	Extra notation and definitions for Feynman integrals	20
4.4	R^* -operation in the MS-scheme	22
4.5	R^* -operation and evaluation of UV and IR counterterms	26
5	ASYMPTOTIC EXPANSIONS OF FEYNMAN INTEGRALS	28
5.1	Dimensionally regularized Feynman integrals	29
5.2	R^* -normalized Feynman integrals	31
6	RELATED WORKS: A COMPARATIVE DISCUSSION OF RESULTS	33
7	CONCLUSIONS	35
8	Comments	39

1 INTRODUCTION

One of cornerstones of the local quantum field theory is renormalization, i.e. proper identification and self-consistent subtraction of infinities that plague perturbation series. A rigorous all-order treatment of the renormalization problem was began with the classic papers of Bogoliubov and Parasiuk [1–3] who constructed a recursive subtraction scheme — the R -operation — to remove ultraviolet (UV) divergences from a given Feynman integral in a way compatible with adding local counterterms to the Lagrangian. Unfortunately their proof of the main theorem of the renormalization theory, — the fact that the R -operation does subtract *all* infinities — had included an intermediate statement which in fact was not true. This has been corrected by Hepp [4] and hence the theorem is known as BPH (a simpler version of the proof have been presented by Anikin, Zavalov and Polivanov [5]).

A major step in elaborating the discussed approach to renormalization has been made by Zimmermann [6–10]. In particular, he has introduced the concept of oversubtractions and developed a graph-theoretical forest technique to disentangle the complicated recurrence structure of the R -operation with oversubtractions. (Note that the forest formula for the R -operation without oversubtractions was first derived by Zavalov and Stepanov in a somewhat disguised form [11]). The forest technique and oversubtractions form the basis for the normal product method which has been of great use in treating theories with massless particles [12–15], in deriving rigorously Wilson expansion [7, 9, 16], and in proving various general relations between renormalized Green functions meet (Zimmermann identities [7, 8, 17, 18], renormalization group equations [19], the Quantum Action Principle [19–21], etc.). Further important development of the method has been made by Anikin, Polivanov and Zavalov [22–25].

A notorious problem of the renormalization theory is the choice of the ultraviolet cut-off and of the renormalization scheme. Though in perturbation theory one can manage to avoid any cut-off (by using the so-called regulator-free formalism which proves to especially useful in rigorous study of supersymmetric theories [26]) in general there is no a preferred choice of the renormalization scheme. On the other hand, a suitable choice of regularization and of the renormalization prescription can facilitate doing field theory considerably.

In practice, the dimensional regularization [27–31] (DR) has become a very useful tool in perturbative treatment of various field theories, including the non-abelian gauge ones. In particular, the DR is used to perform virtually all complicated perturbative calculations. There are at least two good reasons for doing so, in addition to its explicit gauge invariance. The first one is that within the DR the divergent FI's can be treated, in many aspects, as if they were convergent. In other words, such operations as cancellation of identical factors in the numerator and denominator of the integrand, (formal) integration by parts, and replacement of the integral of a sum by the sum of the corresponding integrals are well-defined (a more complete list is given in [32]). Another very useful property of the DR is its ability to regularize simultaneously both the UV and the infrared (IR) divergences by transforming these into poles in $\epsilon = (D_0 - D)/2$, where $D_0 > 0$ is the integer dimension of space-time while D is the running dimension.

As for the choice of the renormalization prescription, it is the minimal subtraction (MS) scheme [33] along with its straightforward modifications such as the \overline{MS} -scheme [34] and the G -scheme [35] that prove to be very convenient both for calculation and for phenomenological applications.

A remarkable feature of the MS-scheme is the fact that in its framework all UV counterterms (or, equivalently, UV renormalization constants) are polynomial *both* in momenta (which must be the case for every meaningful renormalization prescription) *and* in masses [36]. It is this property, along with the useful features listed above, that gives the DR its calculational power in such problems as evaluation of various renormalization group functions. Indeed, the overall divergence of a log-divergent Feynman integral (i.e. the divergence remaining after minimally subtraction of all its subdivergences) must be a polynomial in ϵ^{-1} , with purely numerical coefficients *without* any dependence on dimensional parameters. Thus, when computing the UV renormalization constants one is free to perform arbitrary rearrangements of masses and external momenta, e.g to nullify some of them *provided* this does not lead to IR divergences. This observation, first made in ref. [37] (see also [35,38]) has since then been used repeatedly in a number of important calculations. Among the latter are the analytical evaluation of the α_s^2 -correction to the total cross-section of the e^+e^- process [39] and the calculation of the β -functions for QCD and N=4 supersymmetric Yang-Mills theory [40,41] both made at the three-loop level.

However, the condition of that the IR divergences do not appear restricts considerably our calculational abilities, since for complicated FI's this requirement prevents one from reducing a given FI to a simpler one (see e.g. ref. [42] and Sect.4).

The R^* -operation — a generalization of the R -operation for subtracting both UV and IR divergences — was invented [42] just to overcome this difficulty and solved the problem by allowing for arbitrary rearrangements of masses and external momenta [42,43]. This, in turn, has greatly extended the class of problems amenable to analytical solution. It is sufficient to say that it was the use of the R^* -operation that has enabled one to evaluate analytically the β -function at the *five* loop level for two theories: the ϕ^4 model [44,45] and the supersymmetric two-dimensional sigma-model [46].

Another field of interest where the R^* -operation has been working successfully is the investigation of various short-distance expansions and heavy mass decoupling [47–55].

The so-called R^{-1} -operation — the inverted usual R -operation — was first introduced in [56,57] in the study of asymptotic expansions of minimally subtracted FI's. An explicit resolution of the involved recursion structure of the R^{-1} -operation was found in [58,59] and then applied to construct a rigorous algebraic derivation of the renormalization group equations in the MS-scheme [59].

The present work is aimed at making a regular and uniform treatment of the combinatorics of R -, R^{-1} -, and R^* - operations and of the generic large-mass and/or momentum asymptotic expansions of euclidean FI's.

The outline of this work is as follows. In the next section we introduce definitions and notations for Feynman graphs, integrals and the generic subtraction operation. Section 3 is devoted mainly to the derivation of a forest representation of the R^{-1} -operation and to studying its implications for the BPHZ renormalization formalism. We show that the R^{-1} -operation reveals the underlying simple structure of the BPHZ renormalization and is a natural tool for constructing general conversion formulas connecting two different renormalization versions of a given FI. As an example we give a simple derivation of the Zimmermann identities and the conversion formulas from the MS-scheme to the momentum subtraction and vice versa.

In Sect.4 we develop the theory of the R^* -operation in a general situation where the Feynman integrals to be renormalized may (i) have their external momenta put under an arbitrary number of linear constraints and (ii) may be formally expanded in some

external momenta and/or masses¹. It is shown that in the MS -scheme the R^* -operation may be naturally expressed through the usual R -operation and its inverse — the R^{-1} -operation. Then, as a result of this representation we get a simple and regular algorithm for evaluating the UV counterterms of arbitrary dimensionally regularized FI's, including the ones appearing in the two-dimensional field models.

Section 5 is devoted to combinatorics of the large momentum and/or heavy mass expansions of euclidean, minimally renormalized FI's in a very general setting. The integrals under consideration may be of arbitrary form, including the non-scalar and (completely or partially) massless ones, and with their external momenta may be subjected to any number of linear constraints. This also includes the case where these integrals are considered as being formally expanded in some of their external momenta and/or masses.

It should be stressed that the above integrals form a natural class for studying large momentum and/or heavy mass behavior. This is due to the fact that the resulting asymptotic series can be expressed exclusively in terms of the FI's from the same class.

Our starting point is Theorem 8 of subsect. 5.1 that describes the complete asymptotic expansion of FI's of the form specified above at $\rho \rightarrow 0$, in which some of their external momenta and/or masses are scaled by ρ^{-1} . The expansion is presented in a concise form convenient for practical calculations: in terms of the (again dimensionally regularized!) FI's for some subgraphs and reduced graphs. Once the expansion of regularized FI's at $\rho \rightarrow 0$ is at hand it becomes a purely combinatorial problem to derive the corresponding expansion for minimally renormalized FI's or Green functions. The complete solution of the problem given in subsect. 5.2 naturally leads to the appearance of the R^* -operation in the resulting, explicitly finite expansion. The section ends with a simple combinatorial proof of the main theorem of the R^* -operation theory — of the fact that this does subtract all divergences from the generic dimensionally regularized FI.

Section 6 contains a comparative discussion of the relevant results of previous studies. Finally, in Sect.7 we present our main conclusions together with a brief outline of some of the problems awaiting their solution in the developed framework.

2 STRUCTURE OF FEYNMAN GRAPHS AND INTEGRALS

In this section we briefly recall basic graph-theoretical notations relevant to (Feynman) graphs and integrals. The material is partially taken from refs. [38,60,61]. We also discuss the definition and main properties of the c -operation — a generalization of the standard ultraviolet counterterm operation Δ_U — which will be in constant use in the following.

2.1 Basic graph-theoretical notations and definitions

A graph Γ is a set of lines and vertices which can be associated with a Feynman integral (FI) — a term in the perturbation expansion. The collections of internal lines, vertices and external lines will be denoted as \mathcal{L}_Γ , \mathcal{V}_Γ and \mathcal{E}_Γ , while $L_\Gamma = |\mathcal{L}_\Gamma|$, $V_\Gamma = |\mathcal{V}_\Gamma|$ and $E_\Gamma = |\mathcal{E}_\Gamma|$ will stand for the numbers of elements in each set. Every internal line $l \in \mathcal{L}_\Gamma$ has two (possibly coinciding) vertices incident to, it, viz., the initial vertex $\pi_-(l) \in \mathcal{V}_\Gamma$ and the final vertex $\pi_+(l) \in \mathcal{V}_\Gamma$. Every external line $l \in \mathcal{E}_\Gamma$ has with one vertex, $\pi(l) \in \mathcal{V}_\Gamma$,

¹ That is one first expands the corresponding *integrand*s and then dimensionally regularizes resulting FI's.

incident to it, and a sign factor $f_\Gamma(l) = \pm 1$. If $f_\Gamma(l) = +1(-1)$, the corresponding external momentum is supposed to flow in(out) the vertex $v = \pi(l)$.

The empty graph Γ_\emptyset is the unique graph with $\mathcal{L}_\emptyset = \emptyset$, $\mathcal{V}_\emptyset = \emptyset$ and $\mathcal{E}_\emptyset = \emptyset$.

Let $l \in \mathcal{L}_\Gamma$. A subgraph $\Gamma' = \Gamma \setminus l$ of Γ is obtained via deleting l from \mathcal{L}_Γ and adding two new external lines l_- and l_+ produced by "cutting" the line l and incident to the vertices $\pi_-(l)$ and $\pi_+(l)$, respectively. The mappings π'_\pm and $\pi'(l)$ and the function $f_{\Gamma'}$ are defined in the natural way,² with $\pi'(l_\pm) = \pi_\pm(l)$, and $f_{\Gamma'}(l_\pm) = \pm 1$. A subgraph $\Gamma \setminus \mathcal{L}'$ is defined for any set $\mathcal{L}' \subset \mathcal{L}$ by repeated applications of this prescription. Every subgraph γ of Γ is (unambiguously) determined by a pair of subsets $\mathcal{L}_\gamma \subset \mathcal{L}_\Gamma$ and $\mathcal{V}_\gamma \subset \mathcal{V}_\Gamma$ (\mathcal{V}_γ is understood to comprise all the vertices incident to the lines from \mathcal{L}_γ). It is obtained from the subgraph $\Gamma' = \Gamma \setminus (\mathcal{L}_\Gamma \setminus \mathcal{L}_\gamma)$ by throwing away all the isolated vertices of Γ' not belonging to \mathcal{V}_γ . Thus, the external lines incident to a vertex v of γ are the original external lines together with some "fragments" of internal lines from \mathcal{L}_Γ . Sometimes we shall write \mathcal{L} , \mathcal{E} , \mathcal{V} and $\mathcal{L}(\gamma)$, $\mathcal{E}(\gamma)$, $\mathcal{V}(\gamma)$ instead of \mathcal{L}_Γ , \mathcal{V}_Γ , \mathcal{E}_Γ and \mathcal{L}_γ , \mathcal{E}_γ , \mathcal{V}_γ respectively. Every subgraph γ of Γ , except for the Γ itself is said to be a (proper) subgraph of Γ . If γ is a (proper) subgraph of Γ , we shall write $\gamma \subseteq \Gamma$ ($\gamma \subset \Gamma$). A graph γ is trivial if $\mathcal{L}_\gamma = \emptyset$. If $v \in \mathcal{V}$ then \dot{v} will stand for the (unique) trivial subgraph of Γ with $\mathcal{V}_{\dot{v}} = v$.

The number of *c-components* (that is of the maximal connected subgraphs of Γ) and independent loops (circles) of Γ will be denoted as $c(\Gamma)$ or, equivalently, c_Γ and $\mathcal{N}(\Gamma) = L_\Gamma - V_\Gamma + c_\Gamma$ respectively.

If γ and h are two subgraphs of Γ , their *union* $\delta = \gamma \cup h$ is the subgraph with $\mathcal{L}_\delta = \mathcal{L}_\gamma \cup \mathcal{L}_h$ and $\mathcal{V}_\delta = \mathcal{V}_\gamma \cup \mathcal{V}_h$; their *intersection* $\gamma \cap h$ is defined in a similar way. γ and h are called *disjoint* if $\gamma \cap h = \Gamma_\emptyset$.

We skip over the well-known definitions of a *connected* graph and of a *one-particle-irreducible* (1PI) graph. Given a connected graph $\gamma \subset \Gamma$, the *reduced* graph Γ/γ is obtained by reducing γ to a single vertex, v_γ . If γ is a disconnected subgraph than Γ/γ is produced by reducing each of its c-components γ_i into a single vertex, v_{γ_i} .

A (proper) *spinney* S of a graph Γ is a pairwise disjoint family of non-empty 1PI (proper) subgraphs of Γ . A (proper) *wood* $W\{\Gamma\}$ ($\bar{W}\{\Gamma\}$) is the collection of all (proper) spinneys of G . A (proper) *forest* F (\bar{F}) is a set of non-empty 1PI subgraphs of Γ such that if $\gamma, \gamma' \in F$ then either $\gamma \cap \gamma' = \Gamma_\emptyset$, or $\gamma \subset \gamma'$ or $\gamma' \subset \gamma$. The collection of all (proper) forests of Γ will be denoted as $F\{\Gamma\}$ ($\bar{F}\{\Gamma\}$). Given a forest F , we shall denote by $|F|$ and $(F)_{max}$, respectively, the number of elements in F and the (unique) maximal spinney S such that $S \subseteq F$. The empty spinney S_\emptyset , i.e. the unique spinney without elements, belongs to both $W\{\Gamma\}$ and $F\{\Gamma\}$.

In what follows we shall repeatedly deal with various relations between forests. Being a particular example of a forest, a spinney (or even a single graph) may also take part in these relations. Let F and F' be two forests. We shall define the relations between these forests in terms of their members $\gamma \in F$ and $\gamma' \in F'$. $F \sim F'$ if for all γ and γ' $\gamma \cap \gamma' = \Gamma_\emptyset$; $F \geq (>) F'$ if every γ' is a (proper) subgraph of an element from F ; $F \gtrsim F'$ if for every γ' either $F > \gamma'$ or $F \sim \gamma'$; $F \succ F'$ if $F \gtrsim \gamma'$ and there is no such element $\gamma \in F$ that $\gamma \sim F'$. For every nonempty forest F one has $S_\emptyset \sim F$, $F > S_\emptyset$, $F \gtrsim S_\emptyset$, $S_\emptyset \gtrsim F$ and $S_\emptyset \succ F$.

If S is a spinney of Γ , then Γ/S will stand for the graph obtained by reducing every non-trivial element of S . By $S \downarrow \gamma$ we shall mean a spinney formed by all the elements of S which are subgraphs of γ . If S_1 and S_2 are two spinneys of Γ such that $S_1 \gtrsim S_2$ or

² That is as restrictions of π_\pm , π , and f_Γ onto $\mathcal{L}' = \mathcal{L} \setminus l$, and $\mathcal{V}' = \mathcal{V}$ respectively.

$S_1 \geq S_2$, than the family S_1/S_2 is defined according to

$$S_1/S_2 = \{\gamma/(S_2 \downarrow \gamma) | \gamma \in S_1\}$$

and is to be, naturally, interpreted as a spinney from $W\{\Gamma/S_2\}$.

Finally, we shall denote the collection of graphs $\{\gamma | \gamma = \delta/S, \delta \subseteq \Gamma, S \in W\{\delta\}\}$ as $\mathcal{P}\{\Gamma\}$.

2.2 Dimensionally regularized Feynman integrals

Having recalled the graph-theoretical background, we turn to the second part of the notion of the Feynman diagram — the Feynman integral. Consider an unrenormalized Feynman amplitude $\bar{F}_\Gamma(\mathbf{q}, \mathbf{m})$ corresponding to a connected graph Γ with $\mathbf{q} = \{q_l | l \in \mathcal{E}\}$ and $\mathbf{m} = \{m_l | l \in \mathcal{L}\}$ standing for its external momenta and masses, respectively. Within the dimensional regularization it can be formally represented as³

$$\begin{aligned} \bar{F}_\Gamma(\mathbf{q}, \mathbf{m}, \mu, \epsilon) &= \delta_\mu \left(\sum_{l \in \mathcal{E}} f_\Gamma(l) q_l \right) \langle \Gamma \rangle (\mathbf{q}, \mathbf{m}, \mu, \epsilon), \\ \langle \Gamma \rangle &= \int Y_\Gamma(\mathbf{q}, \mathbf{k}) d_\mu \mathbf{k}, \end{aligned} \tag{2.1}$$

where

$$Y_\Gamma = \prod_{v \in \mathcal{V}} P_v(\mathbf{q}^{\dot{v}}) \prod_{l \in \mathcal{L}} D_l(p_l), \quad d_\mu \mathbf{k} = \prod_{i=1}^{\mathcal{N}(\Gamma)} d_\mu k_i, \quad d_\mu k_i = \frac{\mu^{2\epsilon}}{(2\pi)^D} dk_i,$$

and $D_l(p_l) = P_l(p_l)/(m_l^2 + p_l^2)$, with P_l being a polynomial of degree a_l . The momentum p_l is a linear combination of the loop momenta $\mathbf{k} = \{k_1, \dots, k_{\mathcal{N}_\Gamma}\}$ and of the external momenta flowing through the internal line l . $P_v(\mathbf{q}^{\dot{v}})$ is a polynomial of degree a_v in momenta $\mathbf{q}^{\dot{v}} = \{q_l^{\dot{v}} | l \in \mathcal{E}_{\dot{v}}\}$ — the set of external momenta of the (trivial) subgraph $\dot{v} \subset \Gamma$. By definition, $q_l^{\dot{v}} = q_l$ if $l \in \mathcal{E}_\Gamma \cap \mathcal{E}_{\dot{v}}$, and $q_l^{\dot{v}} = p_{l'}$ if $l \in \mathcal{E}_{\dot{v}}, l = l'_\pm$. $D = D_0 - 2\epsilon$ is the running space-time dimension, with the (positive) integer D_0 being the physical one.

We have introduced the mass μ to preserve the correct dimension of $\langle \Gamma \rangle$; after minimally subtracting the UV poles in ϵ , μ will serve as the MS-scheme renormalization parameter. Finally, every polynomial P_l (and P_v) should be considered as a member of the formal algebra of D -dimensional covariants (which includes the γ -matrices and the metric tensor in addition to the D -dimensional vectors in the case of theories with fermion fields). We shall assume that every polynomial that may serve as a particular example of P_l (or P_v) is written in the normal form [29] and, thus, does not contain any explicit dependence on ϵ . The index of UV divergence of the FI $\langle \Gamma \rangle$ reads

$$\omega(\Gamma) = D_0 \mathcal{N}_\Gamma - 2L_\Gamma + \sum_{l \in \mathcal{L}_\Gamma} a_l + \sum_{v \in \mathcal{V}_\Gamma} a_v.$$

Let us denote a formal algebra formed by polynomials in $q_l \in \mathbf{q}$ as $A(\mathbf{q}, \epsilon)$; the coefficients of these polynomials may depend meromorphically on ϵ . By $\{\mathbf{e}_i(\mathbf{q})\}$ we shall

³We use this formal presentation only as a convenient substitution for rigorous definitions of refs. [29,32,62]. The main role of the former is to help one to formulate concise definitions of various operations with dimensionally regularized FI's.

mean a monomial basis in the algebra, so that every element $e \in A(\mathbf{q}, \epsilon)$ can be uniquely presented in the form

$$e = \sum_i \xi_i(\epsilon) e_i(\mathbf{q}^\gamma),$$

with all its dependence on ϵ concentrated in the coefficients $\{\xi_i\}$.

For a given a connected subgraph $\gamma \subseteq \Gamma$, the corresponding Feynman subintegral $\langle \gamma \rangle$ is the FI defined by (2.1), with the polynomials P_v^γ and P_l^γ being equal to P_v and P_l , respectively. If S is a spinney, then we shall denote the FI corresponding to the (disconnected) graph $\bigcup_{\gamma \in S} \gamma$ as $\langle S \rangle = \prod_{\gamma \in S} \langle \gamma \rangle$.

Now let $S \in W\{\Gamma\}$ and let $\{e_{i_\gamma}(\mathbf{q}^\gamma) | \gamma \in S\}$ be a collection of basic monomials. Proceeding in the same manner we construct the FI $\langle \Gamma/S \rangle^{\mathbf{i}}$ with $\mathbf{i} = \{i_\gamma | \gamma \in S\}$ from propagators and vertex factors of $\langle \Gamma \rangle$, with the understanding that every element e_{i_γ} serves as the vertex polynomial $P_{v_\gamma}^{\Gamma/S}(\mathbf{q}^{i_\gamma})$ (missing in the initial FI $\langle F \rangle$). Finally, if $\{P_\gamma | \gamma \in S\}$ is a collection of polynomials such that

$$P_\gamma \in A(\mathbf{q}^\gamma, \epsilon), \quad P_\gamma = \sum_{i_\gamma} \xi_{i_\gamma}(\epsilon) e_{i_\gamma}(\mathbf{q}^\gamma),$$

then we define the FI $(\prod_\gamma P_\gamma)^* \langle \Gamma/S \rangle$ as the following multifold series of FI's

$$\sum_{\mathbf{i}} \left(\prod_{\gamma \in S} \xi_{i_\gamma} \right) \langle \Gamma/S \rangle^{\mathbf{i}}.$$

2.3 c -operation and its properties

Let us suppose that we are provided with a rule, Δ , which associates a polynomial from $A(\mathbf{q}^\gamma, \epsilon)$ to every FI $\langle \gamma \rangle$ with γ being a 1PI graph:

$$\Delta \langle \gamma \rangle = \sum_i \xi_i(\langle \gamma \rangle, \epsilon) \mathbf{e}_i(\mathbf{q}^\gamma), \quad (2.2)$$

where ξ_i is a meromorphic function of ϵ . Unless otherwise is explicitly stated we shall also demand that $\Delta \langle \gamma \rangle = 0$ if γ is a trivial graph.

If $\gamma \subseteq \Gamma$ then the action of the c (counterterm)-operation $\Delta(\gamma)$ corresponding to the rule Δ on a FI $\langle \Gamma \rangle$ is defined as follows

$$\Delta(\gamma) \langle \Gamma \rangle = \sum_i \xi_i(\langle \gamma \rangle, \epsilon) (\mathbf{e}_i^* \langle \Gamma/\gamma \rangle)$$

or, equivalently,

$$\Delta(\gamma) \langle \Gamma \rangle = \Delta \langle \gamma \rangle * \langle \Gamma/\gamma \rangle.$$

Next, let S be a spinney of Γ and $\langle \Gamma/S \rangle$ be an arbitrary FI corresponding to the graph $\langle \Gamma/S \rangle$. We define

$$\Delta(\gamma) \langle \Gamma/S \rangle = \begin{cases} \langle \gamma/(S \downarrow \gamma) \rangle * \langle \Gamma/(S \cup \gamma)_{max} \rangle, & \text{if } \Gamma \supseteq \gamma \tilde{>} S; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if $h \subset \gamma \subseteq \Gamma$ and Δ and Δ' are a pair of (possibly identical) c -operations then

$$\Delta(h) \Delta'(\gamma) = \Delta(\gamma) \Delta(h) = \Delta'(h) \Delta'(h) = 0,$$

and

$$\Delta'(\gamma)\Delta(h) \langle \Gamma \rangle = \sum_i \xi_i(\langle h \rangle, \epsilon) \Delta'(\gamma) \langle \Gamma/h \rangle^i .$$

Moreover, if $\gamma, h \subseteq \Gamma$ and $h \cap \gamma = \Gamma_\emptyset$ then

$$\Delta'(h)\Delta(\gamma) = \Delta(\gamma)\Delta'(h)$$

and

$$\Delta(\gamma)\Delta'(h) = \sum_{i,i'} \xi'_i(\langle \gamma \rangle, \epsilon) \xi'_{i'}(\langle h \rangle, \epsilon) \langle \Gamma/(\gamma \cup h) \rangle^{i,i'} .$$

If $S \in W\{\Gamma\}$, then we put

$$\Delta(S) = \prod_{\gamma \in S} \Delta(\gamma), \quad (2.3)$$

with

$$\Delta(S) \langle \Gamma \rangle = \Delta \langle S \rangle * \langle \Gamma/S \rangle \quad \text{and} \quad \Delta \langle S \rangle = \prod_{\gamma \in S} (\Delta \langle \gamma \rangle).$$

Similarly, if F is a forest, then

$$\Delta(F) = \prod_{\gamma \in F} \Delta(\gamma), \quad (2.4)$$

where it is understood that in the product in the r.h.s. of (2.4) the order is determined by the c -operation $\Delta(\gamma)$ for smaller graphs acting first (on the right). As a consequence of (2.3) we get that for the empty spinney S_\emptyset

$$\Delta(S_\emptyset) = 1$$

It is worth noting that as for their algebraic properties c -operations are, in fact, identical to the subtraction operators widely used in the renormalization theory (see, e.g. [25]).

3 R - AND R^{-1} - OPERATIONS

In this section we first give a precise definition of the R - operation corresponding to a given c -operation and elaborate on its inverse — the R^{-1} - operation —, which will prove to be a very convenient tool in treating both the R^* -operation and the asymptotic expansions of Feynman integrals. Then we consider the particular versions of these operations adopted to the MS-scheme and reveal the deep connection between the R^{-1} -operation and the BPHZ renormalization and finally derive general conversion formulas connecting the FI's renormalized according to different renormalization prescriptions, viz. by the minimal subtraction and by the BPHZ momentum subtraction.

3.1 R -operation

Let Δ be an arbitrary c -operation and Γ be a (not necessarily connected) graph. According to the notation developed above the corresponding R -operation reads as

$$R(\Gamma) = \sum_{S \in W_U\{\Gamma\}} \Delta(S), \quad (3.1)$$

where the wood⁴ $W_U\{\Gamma\}$ comprises only such spinneys from $W\{\Gamma\}$ that do not include trivial subgraphs of Γ as their members. In addition, it is also convenient to define

$$'R(\Gamma) = \sum_{\substack{S \neq \Gamma \\ S \in W_U\{\Gamma\}}} \Delta(S) \equiv R(\Gamma) - \Delta(\Gamma),$$

and write $R < \Gamma >$ and $'R < \Gamma >$ instead of $R(\Gamma) < \Gamma >$ and $'R(\Gamma) < \Gamma >$, respectively.

Theorem 1

(a) If Γ is a trivial graph, then

$$R(\Gamma) = \Delta(S_\emptyset) = 1. \quad (3.2)$$

(b) If Δ_1 and Δ_2 are two (not necessarily different) c -operations, then

$$R_1(\Gamma)R_2(\Gamma) = \sum_{S \in W_U\{\Gamma\}} \prod_{\gamma \in S} (\Delta_1(\gamma)'R_2(\gamma) + \Delta_2(\gamma)). \quad (3.3)$$

(c) Let γ be a graph of the form $\gamma = \delta/\Phi$ with $\delta \subset \Gamma$ and Φ being a spinney of Γ such that $\delta > \Phi$. Then there holds the following chain of equations:

$$R(\Gamma) < \delta/\Phi > = \sum_{S \in W_U\{\gamma\}} \Delta(S) < \delta/\Phi > = R < \gamma > . \quad (3.4)$$

Proof.

(a) Follows directly from the definition of the c -operation.

(b) Indeed, due to the a properties of the product of c -operations discussed above

$$R_1(\Gamma)R_2(\Gamma) = \sum_{S' \in W_U\{\Gamma\}} \sum_{\substack{S' \succ S \\ S \in W_U\{\Gamma\}}} \Delta_1(S')\Delta_2(S). \quad (3.5)$$

Further, if a spinney S obeys the relation $S' \succ S$ then it can be unambiguously decomposed into two "subspinneys" S_a and S_b such that $S = S_a \cup S_b$, $S_a < S'$ and $S_b \sim S'$. This allows one to transform the second sum over S in the r.h.s. of (3.5) as follows:

$$\sum_{S_a < S'} \sum_{S_b \sim S'} \Delta_1(S')\Delta_2(S_a)\Delta_2(S_b) = \left(\prod_{\gamma' \in S'} \Delta_1(\gamma)'R_2(\gamma') \right) \left(\sum_{S_b \sim S'} \Delta_2(S_b) \right).$$

Using this relation and going to the summation over $S = S' \cup S_b$ and S_b , one readily obtains (3.3).

(c) As a direct consequence of the properties of c -operations, one gets

$$R(\Gamma) < \gamma > = \sum_{\Phi \prec S \in W_U\{\delta\}} \Delta(S/\Phi) < \delta/(\Phi \cup S)_{max} > .$$

On the other hand, the mapping

$$\{S|\Phi \prec S \in W_U\{\delta\}\} \xrightarrow{r} W_U\{\delta/\Phi\}$$

with $r(S) = S/\Phi$ provides a one-to-one correspondence between these two sets, whence (3.4) follows.

⁴The subscript U is to recall that the genuine Ultraviolet subdivergences are connected with 1PI, non-trivial subgraphs.

3.2 R^{-1} -operation

Let us define

$$R^{-1}(\Gamma) = \sum_{S \in W_U\{\Gamma\}} \Delta^{-1}(S), \quad (3.6)$$

where Δ^{-1} is a c -operation⁵. It was shown in refs. [56, 57] that the c -operation Δ^{-1} is unambiguously determined by demanding that

$$R(\Gamma)R^{-1}(\Gamma) = 1 \quad (3.7)$$

and

$$R^{-1}(\Gamma)R(\Gamma) = 1. \quad (3.8)$$

We are going to find an explicit representation for the R^{-1} -operation in terms of the Δ -operation. We begin by proving that (3.7) is equivalent to the following equation

$$\Delta(\gamma)R^{-1}(\gamma) = -\Delta^{-1}(\gamma) \quad (3.9)$$

provided the latter holds for any 1PI graph γ . Indeed, given (3.9), (3.7) is derived by a straightforward application of (3.3)

$$R(\Gamma)R^{-1}(\Gamma) = 1 + \sum_{\emptyset \neq S \in W_U\{\Gamma\}} \prod_{\gamma \in S} (\Delta(\gamma)R^{-1}(\gamma) + \Delta^{-1}(\gamma)) \quad (3.10)$$

On the other hand, if (3.7) holds then (3.9) can be proved trivially by considering (3.10) with $\Gamma = \gamma$ and using induction in the loop number \mathcal{N}_γ .

The identity (3.9) is a convenient starting point for constructing Δ^{-1} . Let us rewrite it as follows

$$\Delta^{-1}(\Gamma) = -\Delta(\Gamma) - \sum_{\emptyset \neq S \in W_U\{\Gamma\}} \Delta(\Gamma)\Delta^{-1}(S). \quad (3.11)$$

Due to the identity $\Delta(\Gamma)\Delta^{-1}(\Gamma) = 0$, (3.11) expresses the Δ^{-1} -operation in terms of the Δ and Δ^{-1} -operations, the latter appearing only for graphs with their loop number less than \mathcal{N}_Γ . The next step is to use the same equation (with Γ substituted by γ with $\gamma \in S$) for every factor $\Delta^{-1}(\gamma)$ in the r.h.s of (3.11). This recursion process will stop when there remain only one-loop diagrams as arguments of the Δ^{-1} -operations. The resulting sum may be further simplified by collecting similar terms and the final result reads

$$\Delta^{-1}(\Gamma) = -\Delta(\Gamma) \sum_{F \in \bar{F}_U\{\Gamma\}} (-)^{|F|} \Delta(F), \quad (3.12)$$

where the the collections of forests $F_U\{\Gamma\}$ comprises only such forests from $F\{\Gamma\}$ that do not include trivial graphs as their elements.

Proof. By induction in the loop number \mathcal{N} . The statement is obviously true at the one-loop level. Assuming its validity for $\mathcal{N}(\Gamma) \leq n$, we transform the relation (3.11) with $\mathcal{N}(\Gamma) = n + 1$ as follows:

$$\Delta^{-1}(\Gamma) = -\Delta(\Gamma) \sum_{S \in W_U\{\Gamma\}} \prod_{\gamma \in S} \left\{ -\Delta(\gamma) \sum_{F \in \bar{F}\{\gamma\}} (-)^{|F|} \Delta(F) \right\},$$

⁵ Δ^{-1} is *not* an inverted Δ !

whence (3.12) follows immediately. Note that the representation (3.12) leads directly to *the forest formula* for the R^{-1} -operation, viz.

$$R^{-1}(\Gamma) = \sum_{F \in \mathcal{F}_U\{\Gamma\}} (-1)^{|F|} \Delta(F). \quad (3.13)$$

As we shall see later on, the remarkable similarity between (3.13) and the forest formula for the BP R -operation (see subsection 3.4) is by no means an accidental coincidence but, rather, reflects the deep relation of the R^{-1} -operation to the BPHZ formalism.

3.3 R - and R^{-1} -operations within the MS-scheme

The main application of the R -operation is to make a given FI finite at $\epsilon \rightarrow 0$ via subtracting from it the UV divergences in a way compatible with the general principles of quantum field theory (see, e.g. refs. [3, 25]). Any particular version of renormalization scheme can be obtained by making a proper choice of the corresponding c -operation. A very convenient choice, inherently connected with dimensional regularization, is the minimal subtraction scheme, with its main virtue of respecting formal symmetries⁶ of dimensionally regularized FI's and, hence, dimensionally regularized Green functions.

Let $A(\mathbf{q}^\gamma, \mathbf{m}^\gamma)$ denote the formal algebra $A(\mathbf{q}^\gamma)$ extended by allowing for every mass from \mathbf{m}^γ to act as an extra generator. In other words, every $\mathbf{e}_i \in A(\mathbf{q}^\gamma, \mathbf{m}^\gamma)$ is, in fact, a basis vector from $A(\mathbf{q}^\gamma)$ multiplied by a monomial in masses from \mathbf{m}^γ . The R -operation in the MS-scheme, R_U , which, when applied to a FI $\langle \Gamma \rangle$, converts the latter into the corresponding minimally subtracted FI, reads [35–37]

$$R_U(\Gamma) = \sum_{S \in \mathcal{W}_U\{\Gamma\}} \Delta_U(S). \quad (3.14)$$

Here the c -operation $\Delta_U(\gamma)$ is supposed to evaluate the ultraviolet MS-counterterm corresponding the overall UV divergence of the FI $\langle \gamma \rangle$. Considering the linearity of the R_U -operation [36], we may assume, without essential loss of generality, that polynomials P_v and P_l are homogeneous. This allows us to present the action of the c -operation Δ_U on a FI $\langle \Gamma \rangle$ in the following form

$$\Delta_U \langle \gamma \rangle = \sum_i Z_i(\langle \gamma \rangle, \epsilon) \mathbf{e}_i(\mathbf{q}^\gamma, \mathbf{m}^\gamma), \quad (3.15)$$

where $\{\mathbf{e}_i(\mathbf{q}^\gamma, \mathbf{m}^\gamma)\}$ are the basis vectors of mass dimension $\omega(\gamma)$ from $A(\mathbf{q}^\gamma, \mathbf{m}^\gamma)$. The dimensionless renormalization constants $\{Z_i\}$ are polynomials in $1/\epsilon$ such that

$$K Z_i = Z_i, \quad (3.16)$$

where $K f(\epsilon)$ stands for the singular part of the Laurent expansion of $f(\epsilon)$ in ϵ near $\epsilon = 0$.

The main theorem of the renormalization theory in the case of the MS-scheme can be formulated as follows [29]:

Theorem 2 *There exists a unique choice of the renormalization constants $\{Z_i\}$ fulfilling the minimality restriction (3.16) and such that the R_U -operation makes arbitrary (infrared convergent⁷) FI finite in the limit $\epsilon \rightarrow 0$.*

⁶With notorious exceptions being symmetries evoking chiral transformations.

⁷Any euclidean FI without massless lines is infrared convergent; the FI's that include massless lines are discussed in the next section.

The finiteness of $R_U \langle \Gamma \rangle$ at $\epsilon \rightarrow 0$ means that

$$\Delta_U \langle \gamma \rangle = -K' R_U \langle \gamma \rangle .$$

This equation may be used to define the UV counterterms and, thereby, the R_U -operation for any FI's including the IR divergent ones. This is achieved by (i) introducing an auxiliary non-zero mass μ_0 into every massless propagator to ensure suppression of all the IR divergences; (ii) computing $\Delta_U \langle \gamma \rangle (\mathbf{q}^\gamma, \mathbf{m}^\gamma, \mu_0, \mu, \epsilon)$ and (iii) setting $\mu_0 = 0$ in the final result. Being quite correct this *a brute force* prescription is surely not to be considered as a practical one; there exist much more subtle tricks that are more convenient for practical calculations (see refs. [35, 37, 42, 43] and below).

Now we turn to the R_U^{-1} — to the inverse of the R_U -operation. It reads

$$R_U^{-1}(\Gamma) = \sum_{S \in W_U\{\Gamma\}} \Delta_U^{-1}(S). \quad (3.17)$$

As a consequence of (3.16) we get that the Δ_U^{-1} -operation also possesses the minimality property, i.e.

$$\Delta_U^{-1} \langle \gamma \rangle = \sum_i Z_i^{(-1)}(\langle \gamma \rangle, \epsilon) \mathbf{e}_i(\mathbf{q}^\gamma, \mathbf{m}^\gamma).$$

with $K Z_i^{(-1)} = Z_i^{(-1)}$.

It was stressed by the authors of refs. [56, 57] that in the MS-scheme R_U^{-1} -operation plays a role which, in a sense, is similar to that of the Zimmermann identities in the momentum subtraction scheme⁸. It could be effectively employed to express an unrenormalized (or only partially renormalized) FI as a sum of appropriately constructed MS-renormalized FI's.

The main idea of these applications of the R_U^{-1} -operation is very simple and could be illustrated best by the following example. Let the product $R_U R_U^{-1}(\Gamma)$ act on an unrenormalized FI $\langle \Gamma \rangle$. Simple manipulations based on (3.4) give the identity

$$\langle \Gamma \rangle = \sum_{S \in W_U\{\Gamma\}} \left\{ \prod_{\gamma \in S} \Delta^{-1}(\gamma) \langle \gamma \rangle \right\} * R_U \langle \Gamma/S \rangle, \quad (3.18)$$

which expresses the initial FI $\langle \Gamma \rangle$ in the form of a linear combination of completely MS-renormalized FI's times some constants (divergent in the limit of ϵ going to zero).

3.4 BPHZ renormalization and the R^{-1} -operation

In this section we shall clarify the relation between the R^{-1} -operation and the good old Bogoliubov-Parasiuk renormalization prescription which is based on the momentum subtractions. It will be shown that the use of the combinatorial technique above developed uncovers a remarkable hidden structure of this renormalization scheme. We shall then employ this structure to obtain simple regular derivations of the Zimmermann identities and the conversion formulas establishing a connection between the minimal subtraction and momentum subtraction schemes.

⁸See in this connection the next subsection.

We begin, as usual, by fixing our notation. The BP R -operation, $\mathcal{R}(\Gamma)$, is defined in terms of the corresponding c -operation δ as follows

$$\mathcal{R}(\Gamma) = \sum_{S \in W_U\{\Gamma\}} \delta(S), \quad (3.19)$$

$$\delta(\gamma) = -M(\gamma)' \mathcal{R}(\gamma), \quad (3.20)$$

where $M(\gamma)$ is a subtraction operator (in fact another c -operation) which associates with a given FI $\langle \gamma \rangle$ its truncated Taylor expansion in the external momenta \mathbf{q}^γ . The formal definition of M is (note that this definition assumes that one first chooses an initial FI $\langle \Gamma \rangle$ so that Feynman subintegrals $\langle \gamma' \rangle$, $\gamma' \in \Gamma$ and, thereby, $\omega(\gamma')$ are well-defined)

$$\langle \gamma \rangle = \begin{cases} \mathcal{T}_{\mathbf{q}^\gamma}^{\omega(\gamma')} \langle \gamma \rangle, & \text{if } \gamma = \gamma'/S, \gamma' \subseteq \Gamma, \langle \gamma' \rangle, S \in W_U\{\Gamma\}, \\ 0, & \text{if } \gamma \text{ is an isolated vertex,} \end{cases} \quad (3.21)$$

$$\mathcal{T}_{\dots}^n = \sum_{i \geq 0}^{(n)} \mathcal{T}_{\dots}^{(i)}, \quad (3.22)$$

where ⁹ we have denoted by $\mathcal{T}_{\dots}^{(i)}$ the operator that picks out the terms of the i -th order of the Taylor expansion with respect to the corresponding variables, e.g.

$$\mathcal{T}_{\rho}^{(n)} f(\rho) = \frac{1}{n!} \left(\frac{d}{d\kappa} \right)^n f(\kappa\rho) \Big|_{\kappa=0}$$

and $\mathcal{T}_{\rho}^{(n)} f(\rho) \equiv 0$ if $n < 0$. The complicated recurrence structure of the \mathcal{R} -operation can be explicitly resolved via the forest formula [6, 11] which has the form

$$\mathcal{R}(\Gamma) = \sum_{F \in F\{\Gamma\}} (-)^{|F|} M(F). \quad (3.23)$$

The BPH theorem ensures that for every FI $\langle \Gamma \rangle$ without massless lines the combination $\mathcal{R} \langle \Gamma \rangle$ is finite at $\epsilon \rightarrow 0$.

In view of (3.12), it is clear that the inverse \mathcal{R}^{-1} -operation assumes a remarkably simple *explicit* form, viz.

$$\mathcal{R}^{-1}(\Gamma) = \sum_{S \in W_U\{\Gamma\}} M(S). \quad (3.24)$$

Below we shall see that it is the simplicity of the \mathcal{R}^{-1} -operation that distinguishes the BHZP renormalization prescription.

Indeed, as is well known, the concepts of oversubtractions and the so-called Zimmermann identities are of vital importance for most applications of the BPHZ approach. Now we employ the general formalism of the R^{-1} -operation to give a simple and regular way of deriving these identities and their generalizations.

Suppose we are given a rule which associates an integer $a(\gamma)$ with every 1PI subgraph $\gamma \subset \Gamma$. Let us put, in addition, $a(\gamma) = a(\gamma')$ for any graph $\gamma' = \gamma/S$ with $\gamma' \subseteq \Gamma$

⁹Throughout this subsection it is assumed that Feynman integrals encountered contain no massless lines and thus are free of any IR divergences. This qualification ensures that the operator $\mathcal{T}_{\mathbf{q}^\gamma}^{(i)}$ is well defined for any integer $i \geq 0$.

and $S \in W_U\{\Gamma\}$. The \mathcal{R} -operation with oversubtractions corresponding to the rule $a(\gamma)$ reads [6]

$$\mathcal{R}_a(\Gamma) = \sum_{F \in F_U\{\Gamma\}} (-)^{|F|} M_a(F), \quad (3.25)$$

where the c-operation M_a is defined as

$$M_a \langle \gamma \rangle = \begin{cases} \mathcal{T}_{\mathbf{q}^\gamma}^{a(\gamma)} \langle \gamma \rangle & \text{if } \gamma \text{ is a non-trivial 1PI graph from } \mathcal{P}\{\Gamma\}, \\ 0, & \text{otherwise.} \end{cases} \quad (3.26)$$

The following proposition is valid [6].

Theorem 3 *The Feynman integral $\mathcal{R}_a \langle \Gamma \rangle$ is finite at $\epsilon = 0$ if the subtraction degrees $a(\gamma)$ satisfy the inequalities*

$$a(\gamma) \geq \omega(\gamma) + \sum_{\delta \in S} (a(\delta) - \omega(\delta)) \quad (3.27)$$

for any 1PI subgraph $\gamma \in \Gamma$ and any spinney $S \in W_U\{\gamma\}$.

An example of choosing the function $f(\gamma)$ meeting the constraint (3.27) is, in fact, provided the \mathcal{R} -operation.

As a direct consequence of the theorem, we get that the \mathcal{R} -operation \mathcal{R}_a makes also finite every FI $\gamma = \left(\prod_{\gamma'' \in S} P_{\gamma''}(\mathbf{q}^{\gamma''}) \right) * \langle \gamma' / S \rangle$ with $\gamma' \subset \Gamma$ and $\gamma \in S \in W_U\{\gamma\}$ provided the degree of every polynomial $P_{\gamma''}$ is less than or equal to $a(\gamma'')$. In what follows we shall refer to such FI as *a-admissible* ones.

An equivalent form of (3.25) is (cf. eqs.(3.23) -(3.24))

$$\mathcal{R}_a(\Gamma) = (\mathcal{R}_a^{-1}(\Gamma))^{-1}, \quad (3.28)$$

with

$$\mathcal{R}_a^{-1}(\Gamma) = \sum_{S \in W_U\{\Gamma\}} M_a(S). \quad (3.29)$$

Let us now consider two different \mathcal{R} -operations with oversubtractions, viz. \mathcal{R}_a and \mathcal{R}_b such that both functions $a(\gamma)$ and $b(\gamma)$ are constrained by (3.27) and the inequality

$$a(\gamma) \geq b(\gamma) \quad (3.30)$$

holds for every $\gamma \subseteq \Gamma$. At the level of individual FI's the Zimmermann identity may be considered as a conversion formula to express an \mathcal{R}_a -normalized FI in terms of \mathcal{R}_b -normalized FI's . To derive the conversion relation it suffices to elaborate a trivial identity

$$\mathcal{R}_a(\Gamma) \equiv \mathcal{R}_b(\Gamma) \mathcal{R}_b^{(-1)}(\Gamma) \mathcal{R}_a(\Gamma) \quad (3.31)$$

with the help of (3.3) and (3.29). As a result we get

$$\mathcal{R}_a(\Gamma) = \mathcal{R}_b(\Gamma) \mathcal{R}_{b,a}(\Gamma), \quad (3.32a)$$

$$\mathcal{R}_{b,a}(\Gamma) = \mathcal{R}_b^{(-1)}(\Gamma) \mathcal{R}_a(\Gamma) = \sum_{S \in W_U\{\Gamma\}} \delta_{b,a}(S), \quad (3.32b)$$

$$\delta_{b,a}(\gamma) = (M_b - M_a)' \mathcal{R}_a(\gamma). \quad (3.32c)$$

Here $\mathcal{R}_{b,a}$ is an R -operation, with

$$\delta_{b,a} \langle \gamma \rangle = \sum_{i_\gamma} \xi_{i_\gamma}^{b,a} \mathbf{e}_{i_\gamma}(\mathbf{q}^\gamma) \quad (3.33)$$

and the coefficients $\xi_{i_\gamma}^{b,a}$ being finite at $\epsilon \rightarrow 0$ for every a -admissible FI $\langle \gamma \rangle$. Indeed, it follows from (3.32) that

$$\delta_{b,a} \langle \gamma \rangle = \sum_{i > a(\gamma)}^{b(\gamma)} \mathcal{T}_{\mathbf{q}^\gamma}^{(i)} \mathcal{R}_a \langle \gamma \rangle, \quad (3.34)$$

with $\mathcal{R}_a \langle \gamma \rangle$ being finite due to Theorem 3. We now combine (3.32) and (3.34) to come to the following conversion formula

$$\mathcal{R}_a \langle \Gamma \rangle = \sum_{S \in W_U \{\Gamma\}} \delta_{b,a} \langle S \rangle * \mathcal{R}_b \langle \Gamma/S \rangle \quad (3.35)$$

or, explicitly,

$$\mathcal{R}_a \langle \Gamma \rangle = \sum_{S \in W_U \{\Gamma\}} \sum_{\mathbf{i}} \prod_{\gamma \in S} \left(\xi_{i_\gamma}^{b,a} \right) \mathcal{R}_b \langle \Gamma/S \rangle^{\mathbf{i}}. \quad (3.36)$$

Let us turn to the derivation of the conversion formulas which allow one to express easily a minimally subtracted FI in the form of an appropriate linear combination of FI's renormalized by the \mathcal{R}_a -operation, and vice versa. Let us define

$$R_{a,U}(\Gamma) = \mathcal{R}_a^{-1}(\Gamma) R_U(\Gamma) \quad (3.37a)$$

$$R_{U,a}(\Gamma) = R_U^{-1}(\Gamma) \mathcal{R}_a(\Gamma). \quad (3.37b)$$

An almost literal repetition of the above argument shows that the corresponding c -operation is

$$\delta_{a,U} \langle \gamma \rangle = M_a' R_U \langle \gamma \rangle + \Delta_U \langle \gamma \rangle \quad (3.38a)$$

while

$$\delta_{U,a} \langle \gamma \rangle \equiv \delta_{a,U}^{-1} \langle \gamma \rangle = \delta_{a,U}(\gamma) \sum_{F \in F_U \{\gamma\}} (-1)^{|F|} \delta_{a,U}(F) \langle \gamma \rangle. \quad (3.38b)$$

Thus,

$$R_U \langle \Gamma \rangle = \sum_{S \in W_U \{\Gamma\}} \delta_{a,U} \langle S \rangle * \mathcal{R}_a \langle \Gamma/S \rangle, \quad (3.39a)$$

and

$$\mathcal{R}_a \langle \Gamma \rangle = \sum_{S \in W_U \{\Gamma\}} \delta_{U,a} \langle S \rangle * R_U \langle \Gamma/S \rangle. \quad (3.39b)$$

As a consequence of Theorem 2 the R -operations $R_{a,U}$ and $R_{U,a}$ are evidently finite¹⁰ on the class of a -admissible FI's if and only if the subtraction degrees $a(\gamma)$ meet condition (3.27). It is worth noting that Theorem 3 follows directly from this observation and the conversion relation (3.39b).

¹⁰That is both c -operations $\delta_{a,U}(\gamma)$ and $\delta_{U,a}(\gamma)$ transform any a -admissible FI $\langle \gamma \rangle$ into polynomials in \mathbf{q}^γ finite at $\epsilon \rightarrow 0$.

4 R^* -OPERATION

The purpose of this section is twofold. First, we formulate an extended version of the R^* -operation which minimally removes both the UV and IR divergences from a (dimensionally regularized) euclidean Feynman integral with arbitrary set of external momenta (including the exceptional ones) and formally expanded in some of its masses and external momenta. Second, we employ the forest technique developed above to simplify considerably the apparatus of the R^* -operation by explicitly expressing the corresponding IR counterterm operation in terms of its UV counterpart. This extended formulation of the R^* -operation is useful for the studying asymptotic expansions of generic Feynman integrals (see the next section) and computing the UV renormalization constants (see below).

Since the R^* -operation is comparatively a newcomer in the renormalization market we begin with a special introductory section which provides the reader with a short overview of the main ideas of the R^* -operation and a simple illustrative example of its use in calculations.

4.1 R^* - primer

A lot of information about the structure of the renormalized quantum field theory is contained in the renormalization group (RG) functions (the β -functions and anomalous dimensions). These functions, in their turn, are expressible through some combinations of UV renormalization constants (see any textbook on QFT). Historically, the R^* -operation was invented [42] in the attempt to get rid of some limitations of the so-called IR rearrangement (IRR) trick [37], where, in fact, it was first demonstrated that the problem of evaluating the UV renormalization constants can be drastically simplified by proper use of the rich possibilities offered by the MS-scheme. Coupled with the R^* -operation, the IRR trick became a regular powerful method of computing the UV renormalization constants [42, 43].

The starting ideas both of the IRR trick and of the R^* -operation can be illustrated best exposed by considering a couple of simple examples. Let us begin with a quadratically divergent FI,

$$\langle \Gamma_1 \rangle (q, m) = \int \frac{d^\mu k_1 d^\mu k_2}{((q - k_2)^2 + m^2)((k_1 - k_2)^2 + m^2)(k_1^2 + m^2)} \quad (4.1)$$

The corresponding UV counterterm reads

$$\Delta_U \langle \Gamma_1 \rangle = Z_m m^2 + Z_2 q^2 \quad (4.2)$$

where Z_m and Z_2 contribute to the mass and wave functions renormalizations, respectively, in the φ^4 -model¹¹. After differentiating $\langle \Gamma_1 \rangle$ with respect to m^2 , one comes to another, now only logarithmically divergent, FI:

$$\langle \Gamma_2 \rangle (q, m) = \int \frac{d^\mu k_1 d^\mu k_2}{((q - k_2)^2 + m^2)((k_1 - k_2)^2 + m^2)(k_1^2 + m^2)^2}.$$

The differentiated version of eq.(4.2) is

$$\Delta_U \langle \Gamma_2 \rangle = -Z_m/3. \quad (4.3)$$

¹¹Throughout this subsection we work in four-dimensional space-time and hence set $D_0 = 4$.

The idea of the IRR trick is quite simple. Since the UV renormalization constant of a logarithmically divergent FI does not depend on any dimensional parameters and since our aim is to calculate just this constant, nothing prevents us from introducing auxiliary external momenta and/or masses along with setting zero the initial mass and external momentum *provided* that this does not lead to any IR divergences. In many cases, the resulting FI becomes simpler for calculation. For the FI in question this can be easily achieved if we introduce a new non-zero external momentum q' to flow through *one* of the two lines in the series (both with one and the same propagator $1/(k_1^2 + m^2)$ in (4.3)) and set q and m zero. Indeed, the resulting FI

$$\langle \Gamma_3 \rangle (q') = \int \frac{d^\mu k_1 d^\mu k_2}{k_2^2 (k_1 - k_2)^2 (k_1 - q')^2 k_1^2} \quad (4.4)$$

is readily computed in terms of Γ -functions (see, e.g. ref. [35]).

Unfortunately, the condition of that the IR divergences do not appear restricts considerably the range of applicability of the IRR trick, since for complicated FI's this prevents one from transforming a given FI to the simplest form (see, e.g. ref. [42]).

A radical generalization of IRR, free of any limitations of this kind, is based on the R^* -operation. To understand the essence of this approach, let us try to compute Z_m starting directly from FI $\langle \Gamma_2 \rangle (q, m)$ and this time *avoiding* any manipulations with external momenta. By definition of Δ_U , we have

$$\Delta_U \langle \Gamma_2 \rangle = K_\epsilon \left[\langle \Gamma_2 \rangle (q, m) + Z_\gamma \int \frac{d_\mu k_1}{(k_1^2 + m^2)^2} \right], \quad (4.5)$$

where $Z_\gamma = -(16\pi^2\epsilon)^{-1}$ and the second term subtracts the UV subdivergence corresponding to the subintegration over the loop momentum k_2 . If now we tried to set $m = 0$ in the corresponding integrand, then the integration over the small k_1 region would lead to an IR divergence. Within the DR this IR divergence shows up as an extra pole in ϵ . As a result, a straightforward calculation of the rhs of (4.5) would give the wrong result, containing spurious IR poles along with the true UV ones. Now there arises a natural question: can one extract the IR pole and remove it by subtracting from $\langle \Gamma_2 \rangle (q, 0)$ an IR counterterm local in the x -space and not in the p -space? The affirmative answer was given in refs. [42,43] where the \tilde{R} -operation with the necessary properties was constructed. It may be easily checked that the combination

$$f(k_1) = k_1^{-4} + \tilde{Z}\delta(k_1), \quad \text{with} \quad \tilde{Z} = (16\pi^2\epsilon)^{-1} \quad (4.6)$$

does not lead to any IR poles after a formal D-dimensional integration with a function smooth at $k_1 = 0$. This means that the pole part of the integral

$$'R\tilde{R} \langle \Gamma_2(q, 0) \rangle = \int \left[\int (q - k_2)^{-2} (k_1 - k_2)^{-2} f(k_1) d_\mu k_2 + Z_\gamma f(k_1) \right] d_\mu k_1 \quad (4.7)$$

is determined by the behavior of the integrand in the region of large k_1, k_2 and coincides with the UV counterterm we are looking for. In (4.7) the symbol \tilde{R} stand for the IR \tilde{R} -operation that removes the IR divergencies by x-local counterterms and in our case comes to replacing $k_1^{(-4)}$ by combination (4.6).

Unfortunately, the representation of the IR counterterms through δ -functions of some loop momenta becomes rather cumbersome and unnecessarily complicated in general case. Another approach that is much more convenient and powerful was first outlined in ref. [58] and will be presented below.

4.2 Generalized forest technique

In this subsection we extend some of our previous considerations to the more general case of spinneys and forests formed by arbitrary connected graphs. Though such a generalization is quite straightforward it proves to be quite useful in the treatment of the R^* -operation and asymptotic expansions of Feynman integrals.

Let us denote as $C\{\Gamma\}$ the collection of all connected subgraphs of a graph Γ . From now on a spinney (forest) of Γ will refer to a collection of connected non-intersecting (non-overlapping) non-empty subgraphs of Γ . Thus, if S is a member of $W\{\Gamma\}$ then for any pair $\gamma, \gamma', \gamma \neq \gamma'$ from S one has

$$\gamma \in C, \gamma' \in C\{\Gamma\} \text{ and } \gamma \cap \gamma' = \Gamma_\emptyset. \quad (4.8)$$

It is an easy exercise to see that all the reasoning of sections 3.1 and 3.2 can be applied without any modifications to these extended definitions. Note also that even for a one-particle-reducible graph γ one has $\Delta(\gamma) \neq 0$, in spite of the fact that by definition $\Delta < \gamma > \equiv 0$. The reason is that for a subgraph $\delta \subseteq \gamma$ such that the reduced graph γ/δ is 1PI $\Delta(\gamma) < \gamma/\delta >$ is $\Delta < \gamma/\delta >$ and, hence, can be non-zero.

Let $C'\{\Gamma\} \subseteq C\{\Gamma\}$ be an arbitrary subset of the family of connected subgraphs of the graph Γ and $W'\{\Gamma\} \subseteq W\{\Gamma\}$ ($F'\{\Gamma\} \subseteq F\{\Gamma\}$) stand for the collection of all spinneys (forests) with their elements from $C'\{\Gamma\}$. Let us define

$$R(W'\{\Gamma\}, \Delta) = \sum_{S \in W'\{\Gamma\}} \Delta(S), \quad (4.9)$$

$$R(F'\{\Gamma\}, \Delta) = \sum_{F \in F'\{\Gamma\}} \Delta(F), \quad (4.10)$$

where Δ is a c -operation. A literal repetition of the argument given in section 3.3 immediately shows that

$$R^{-1}(W'\{\Gamma\}, \Delta) = R(F'(\{\Gamma\}, -\Delta), \quad (4.11)$$

$$R^{-1}(F'\{\Gamma\}, \Delta) = R(W'(\{\Gamma\}, -\Delta). \quad (4.12)$$

For any spinney $S \in W\{\Gamma\}$ we denote

$$C' \equiv C^\sim\{\Gamma, S\} = \{\gamma | S \sim \gamma \in C\{\Gamma\}\},$$

$$C'' \equiv C^\succ\{\Gamma, S\} = \{\gamma | S \succ \gamma \in C\{\Gamma\}\},$$

$$C''' \equiv C^\tilde{\succ}\{\Gamma, S\} = \{\gamma | S \tilde{\succ} \gamma \in C\{\Gamma\}\},$$

while

$$W^\sim\{\Gamma, S\}, W^\succ\{\Gamma, S\}, W^{\tilde{\succ}}\{\Gamma, S\}, F^\sim\{\Gamma, S\}, F^\succ\{\Gamma, S\} \text{ and } F^{\tilde{\succ}}\{\Gamma, S\}$$

will stand for the respective families

$$W'\{\Gamma, S\}, W''\{\Gamma, S\}, W'''\{\Gamma, S\}, \mathcal{F}'\{\Gamma, S\}, \mathcal{F}''\{\Gamma, S\} \text{ and } \mathcal{F}'''\{\Gamma, S\}.$$

Theorem 4. *For an arbitrary spinney $S \in W\{\Gamma\}$ the following relations are true*

$$R(W^{\tilde{\succ}}\{\Gamma, S\}, \Delta) < \Gamma/S > = R(W\{\Gamma/S\}, \Delta) < \Gamma/S >, \quad (4.13)$$

$$R(W^{\tilde{>}}\{\Gamma, S\}, \Delta) = R(W^{\sim}\{\Gamma, S\}, \Delta)R(W^{\succ}\{\Gamma, S\}, \Delta), \quad (4.14)$$

$$R^{-1}(W^{\tilde{>}}\{\Gamma, S\}, \Delta) = R(F^{\succ}\{\Gamma, S\}, -\Delta)R(F^{\sim}\{\Gamma, S\}, -\Delta). \quad (4.15)$$

Proof. Relation (4.13) is nothing but a shorthand of (3.4). Eq.(4.14) is a direct consequence of the fact that every spinney S from $W^{\tilde{>}}\{\Gamma, S\}$ can be unambiguously presented in a form

$$S = S^{\succ} \cup S^{\sim} \quad \text{with } S^{\succ} \in W^{\succ}\{\Gamma, S\} \quad \text{and } S^{\sim} \in W^{\sim}\{\Gamma, S\}$$

and of the observation that no graph γ from $C^{\sim}\{\Gamma, S\}$ can have a proper subgraph $\gamma' \subset \gamma$ such that $\gamma' \in C^{\succ}\{\Gamma, S\}$. In order to prove (4.15) it is sufficient to make use of (4.14) and (4.11).

4.3 Extra notation and definitions for Feynman integrals

Let $\langle \Gamma \rangle (\mathbf{q}, \mathbf{m})$ be a dimensionally regularized FI corresponding to a (connected) graph Γ . Its external momenta may obey some extra linear constraints of the form

$$\sum_{l \in c_i} \varphi_{\Gamma}(l) q_l = 0, \quad c_i \in \mathbf{c}. \quad (4.16)$$

Here $\mathbf{c} = \{c_i\}$ is a collection of subsets of \mathcal{E}_{Γ} which is to comprise the empty set and \mathcal{E}_{Γ} ; this latter constraint is due to the total momentum conservation. Without essential loss of generality, we shall assume that for any pair i, i' such that $c_i \cap c_{i'} = \emptyset$ or $c_i \subset c_{i'}$, one has either $(c_i \cup c_{i'}) \in \mathbf{c}$ or $(c_i' \setminus c_i) \in \mathbf{c}$, respectively. The case of non-exceptional external momenta corresponds to the choice $\mathbf{c} = \{\mathcal{E}, \emptyset\}$.

If γ is a connected graph from $\mathcal{P}\{\Gamma\}$, then the relevant set of external momenta \mathbf{q}^{γ} will be subjected to linear constraints induced by (4.16) and the total momentum conservation i.e.

$$\sum_{l \in c_i} \varphi_{\gamma}(l) q_l^{\gamma} = 0, \quad c_i \in \mathbf{c}^{\gamma}, \quad (4.17)$$

where

$$\mathbf{c}^{\gamma} = \{c_i | c_i \subseteq \mathcal{E}_{\gamma}, c_i \in \mathbf{c}\} \cup \{\mathcal{E}_{\gamma} \setminus c_i | c_i \subseteq \mathcal{E}_{\gamma}, c_i \in \mathbf{c}\}. \quad (4.18)$$

Note that even if the external momenta $q_l \in \mathbf{q}$ are non-exceptional the vectors from \mathbf{q}^{γ} will, in general, form an exceptional set of external momenta. Indeed, if, for instance, $\mathcal{E} \subseteq \mathcal{E}_{\gamma}$, then $\mathbf{c}^{\gamma} = \{\mathcal{E}, \mathcal{E}_{\gamma} \setminus \mathcal{E}, \emptyset\}$ for $\mathbf{c} = \{\mathcal{E}, \emptyset\}$, as it follows from (4.18).

A subset $\mathbf{q}' \subseteq \mathbf{q}^{\gamma} = \{q_l | l \in \mathcal{E}' \subseteq \mathcal{E}_{\gamma}\}$ is said to be a *right* (*r*-)subset of \mathbf{q}^{γ} if the decomposition $\mathbf{q}^{\gamma} = \mathbf{q}' \cup (\mathbf{q}^{\gamma} \setminus \mathbf{q}')$ is well-defined with respect to constraints (4.17). In other words, each of the constraints should remain valid after all the momenta from \mathbf{q}' are multiplied by an auxiliary parameter. The requirement is equivalent to demanding that the intersection $(\mathcal{E}' \cap c_i) \subseteq \mathbf{c}^{\gamma}$ for every c_i from \mathbf{c}^{γ} .

Suppose we are given a partition of external momenta and masses of the FI $\langle \Gamma \rangle$ of the form¹²

$$\begin{aligned} \mathbf{q} &= \mathbf{q}_1 \cup \mathbf{q}_0, & \mathbf{m} &= \mathbf{m}_1 \cup \mathbf{m}_0, \\ \mathbf{q}_i &= \{q_l | l \in \mathcal{E}^i\}, & \mathbf{m}_i &= \{m_l | l \in \mathcal{L}^i\}, & \mathcal{E}^0 \cap \mathcal{E}^1 &= \emptyset = \mathcal{L}^0 \cap \mathcal{L}^1, & i &= 0, 1, \end{aligned} \quad (4.19)$$

¹²It is convenient to assume that all the massless lines from \mathcal{L} belong to \mathcal{L}_0 .

with \mathbf{q}_1 and \mathbf{q}_0 being r - subsets of \mathbf{q} . For every connected graph $\gamma \in \mathcal{P}\{\Gamma\}$ we define

$$\mathbf{q}_i^\gamma = \{q_l^\gamma | l \in \mathcal{E}_\gamma^i\} \quad \text{and} \quad \mathbf{m}_i^\gamma = \{m_l | l \in \mathcal{L}_\gamma^i\}, \quad i = 0, 1,$$

where

$$\mathcal{E}_\gamma^1 = \mathcal{E}^1 \cap \mathcal{E}_\gamma, \quad \mathcal{E}_\gamma^0 = \mathcal{E}_\gamma \setminus \mathcal{E}_\gamma^1 \quad \text{and} \quad \mathcal{L}_\gamma^i = \mathcal{L}^i \cap \mathcal{L}_\gamma, \quad i = 0, 1. \quad (4.20)$$

Let γ be a connected graph from $\mathcal{P}\{\Gamma\}$. If the external momenta \mathbf{q}_1^γ form an r -subset of \mathbf{q}^γ , then γ will be termed as a *1-right* graph. A vertex $v \in \mathcal{V}_\gamma$ is said to be *1-hard*, if

$$\sum_{l \in \mathcal{E}^1 \cap \mathcal{E}(v)} q_l \neq 0$$

and/or at least one massive line from \mathcal{L}_γ^1 has v as its incident vertex; otherwise v will be termed *1-soft*. For example, if $\mathcal{L}^0 = \mathcal{E}^0 = \emptyset$, then for a given vertex of Γ to be 1-hard, the algebraic sum of external momenta flowing into it should be non-zero, or the vertex should be incident at least to one massive internal line. If γ is 1-right, then it will be referred to as

(i) *1-soft*, if \mathcal{V}_γ comprises no 1-hard vertices;

and

(ii) *1-irreducible*, if there is no such line l , $\mathcal{L}^1 \ni l \in \mathcal{L}_\gamma$, that the graph $\gamma - l$ obtained by deleting l from γ remains 1-right and $c(\gamma - l) > c(\gamma)$. In other words for γ to be 1-irreducible it must not involve a cut- line l whose corresponding internal momenta p_l^γ can be identically expressed (via linear constraints on \mathbf{q}^γ) exclusively in terms of the momenta from the set \mathbf{q}_0^γ . Note that a 1-soft graph γ could be 1-irreducible if and only if it is 1PI. Finally, a spinney $S \in \mathcal{W}\{\gamma\}$ consisting of 1-right subgraphs of γ will called

(i) a *1-uniting* one, if every 1-hard vertex from \mathcal{V}_γ belongs to some graph from S ;

and

(ii) a *1-hard* one, if it comprises no 1-soft graphs.

A very useful transformation of a Feynman integral is its expansion in a (formal) Taylor series with respect to some of its external momenta and masses. Suppose one is going to expand the FI $\langle \Gamma \rangle(\mathbf{q}, \mathbf{m})$ in masses and momenta from the collections \mathbf{q}_0 and \mathbf{m}_0 , respectively. The operator t_0 which performs the procedure is defined as

$$t_0 = \sum_{n \geq 0} \xi^n t_0^{(n)}, \quad t_0^{(n)} = \mathcal{T}_{\mathbf{q}_0, \mathbf{m}_0}^{(n)}. \quad (4.21)$$

Thus

$$t_0^{(n)} \langle \Gamma \rangle = \frac{1}{n!} \left(\frac{d}{d\xi} \right)^n \langle \Gamma \rangle(\mathbf{q}_1 \cup \xi \mathbf{q}_0, \mathbf{m}_1 \cup \xi \mathbf{m}_0) |_{\xi=0}, \quad (4.22)$$

and

$$t_0^{(n)} \langle \Gamma \rangle = \sum_i \langle \Gamma \rangle_i(\mathbf{q}_1, \mathbf{m}_1) \mathbf{e}_i(\mathbf{q}_0, \mathbf{m}_0), \quad (4.23)$$

with $\mathbf{e}_i(\mathbf{q}_0, \mathbf{m}_0)$ being basis vectors of a (mass) dimension n from $A(\mathbf{q}, \mathbf{m})$. Here several comments are in order.

(i) The differentiation with respect to ξ in (4.22) may be carried out in three ways. In first place, one could simply differentiate the FI, which is a smooth function of ξ at $\xi \neq 0$. Another way is to differentiate the corresponding *integrand*. Here in turn two options arise. First, we might employ the momentum space representation as described by (2.1).

(It is this option, which is always used in carrying out practical calculations because of flexibility and compactness of this representation.) Second, it is possible to use the α -parametric representation. In what follows we shall follow to the last option since this allows for a direct interpretation of $\langle \Gamma \rangle_i(\mathbf{q}_1, \mathbf{m}_1)$ as a FI corresponding to the *same* graph Γ with the same but somewhat differently treated α -parametric integrand¹³.

(ii) The operation of setting ξ zero is meant to be a formal one, i.e., it should act on the differentiated integrand.

(iii) The infinite series in the rhs of (4.21) is to be understood as a formal one; so no question about its convergence may arise. Moreover, in every application where the operator t_0 can appear, the terms of too high orders in ξ may be dropped for one reason or another, while the remaining (finite) series in ξ is taken at $\xi = 1$. Having this in mind we put $\xi = 1$ in all formulas below.

4.4 R^* -operation in the MS-scheme

Given an arbitrary c -operation, Δ_g , we define another c -operation, $\Delta_{\dot{g}}$, according to the following rule

$$\Delta_{\dot{g}} \langle \gamma \rangle = \begin{cases} \Delta_g \langle \gamma \rangle & \text{if } \mathcal{L}_\Gamma \neq \emptyset; \\ \langle \gamma \rangle & \text{if } \gamma \text{ is an isolated vertex.} \end{cases} \quad (4.24)$$

Further, let $\tilde{C}\{\Gamma\}$ stand for a collection of graphs which could be produced from a graph Γ by reducing some non-empty (possibly disconnected) subgraphs of Γ . In other words,

$$\tilde{C}\{\Gamma\} = \{\gamma | \gamma = \Gamma/S, S \in W\{\Gamma\}\}.$$

It is worth noting that $\Gamma \in \tilde{C}\{\Gamma\}$, since by definition $\Gamma/S_\emptyset = \Gamma$, and if Γ is disconnected, then for every element $\gamma \in \tilde{C}\{\Gamma\}$ the number of its c -component will be equal to that of Γ . Let us also define $\tilde{W}\{\Gamma\}$ as a subset of $W\{\Gamma\}$ comprising all (minimal) spinneys which generate exactly the set $\tilde{C}\{\Gamma\}$ as a result of the operation $S \rightarrow \Gamma/S$.

The \tilde{c} -operation $\tilde{\Delta}_I(\gamma)$ — the "infrared" counterpart of Δ_U — is defined as follows

$$\tilde{\Delta}_I(\gamma) t_0 \langle \Gamma \rangle = \begin{cases} t_0(\langle S_\gamma \rangle) * \Delta_I \langle \Gamma/S_\gamma \rangle, & \text{if } \gamma \in \tilde{C}\{\Gamma\} \text{ and} \\ & S_\gamma \text{ is 1-uniting} \\ 0, & \text{otherwise,} \end{cases} \quad (4.25)$$

where S_γ is the (unique) spinney from $\tilde{W}\{\Gamma\}$ such that $\Gamma/S_\gamma = \gamma$. Here Δ_I such is a c -operation that for any collection of vertex polynomials $\{P_\gamma(\mathbf{q}_0) | \gamma \in S_\gamma\}$ the combination

$$- \left(\prod_{\gamma \in S_\gamma} P_\gamma \right) * \Delta_I \langle G/S_\gamma \rangle \equiv -\Delta_I \left(\left(\prod_{\gamma \in S_\gamma} P_\gamma \right) * \langle G/S_\gamma \rangle \right)$$

is the overall IR divergence of a (tadpole¹⁴) FI:

$$- t_0 \left(\left(\prod_{\gamma \in S_\gamma} P_\gamma \right) * \langle G/S_\gamma \rangle \right)$$

¹³Generally speaking, it is by no means obvious that these ways are equivalent. Fortunately, this is the case within the dimensional regularization [32].

¹⁴This is the case since $\gamma = \Gamma/S_\gamma$ is 1-soft owing to the definition (4.25).

(more details about the possible choices of Δ_I are presented below).
Another useful version of (4.25) reads

$$\tilde{\Delta}_I(\gamma)t_0^{(n)} < \Gamma > = \begin{cases} (t_0^{(n')} < S_\gamma >) * \Delta_j < \Gamma/S_\gamma >, & \text{if } \gamma \in \tilde{C}\{\Gamma\} \text{ and } S_\gamma \\ & \text{is 1-uniting;} \\ 0, & \text{otherwise,} \end{cases} \quad (4.26)$$

with¹⁵ $n' = n - \omega(\Gamma/S_\gamma)$.

The Δ_I -operation has the following easily checked properties:

- (i) $\tilde{\Delta}_I(\Gamma)t_0 < \Gamma > = t_0 < \Gamma >$ if Γ is a trivial graph;
- (ii) $\tilde{\Delta}_I(\Gamma)t_0 < \Gamma > = \Delta_I < \Gamma >$ if Γ is a 1-soft, non-trivial graph;
- (iii) if the spinney S_I comprises at least one 1-reducible element, then $\tilde{\Delta}_I(\gamma)t_0 < \Gamma > = 0$ due to the very FI $t_0 < S_\gamma >$ vanishes.

Finally, we define the combined action of Δ_U and $\tilde{\Delta}_I$ operations in the natural way:

$$\Delta_U(S)\tilde{\Delta}_I(\gamma) \equiv \tilde{\Delta}_I(\gamma)\Delta_U(S), \quad (4.27)$$

$$\Delta_U(S)\tilde{\Delta}_I(\gamma)t_0 < \Gamma > = (t_0\Delta_U(S) < S_\gamma >) * \Delta_j < \Gamma/S_\gamma >. \quad (4.28)$$

It follows from (4.28) that the product $\Delta_U(S)\tilde{\Delta}_I(\gamma)$ can be non-zero if and only if $S \leq S_\gamma$. This condition reflects the simple fact that no propagator can contribute to both UV and IR divergencies simultaneously.

Now we are sufficiently equipped to introduce the R^* -operation which subtracts all kinds of divergences from an (euclidean) dimensionally regularized FI formally expanded in some of its external momenta and masses. We define

$$R^*(\Gamma) = R_U(\Gamma)\tilde{R}_I(\Gamma) \equiv \tilde{R}_I(\Gamma)R_U(\Gamma) \quad (4.29)$$

where we have introduced the "infrared" \tilde{R}_I -operation which subtracts the IR divergences only, and does not care for any UV ones:

$$\tilde{R}_I(\Gamma) = \sum_{\gamma \in \tilde{C}\{\Gamma\}} \tilde{\Delta}_I(\gamma) \equiv \sum_{S \in \tilde{W}\{\Gamma\}} \tilde{\Delta}_I(\Gamma/S) \quad (4.30)$$

Eqs. (4.25,4.30) imply that the R^* -normalized FI $R^*t_0 < \Gamma >$ can be presented in the following convenient form

$$R^*t_0 < \Gamma > = \sum_{S \in {}_uW_1\{\Gamma\}} \sum_{S' \in W\{\Gamma\}}^{S' \leq S} \Delta_U(S')t_0 < S > * \Delta_j < \Gamma/S > \quad (4.31)$$

or, equivalently,

$$R^*t_0 < \Gamma > = \Delta_j(\Gamma) \sum_{S \in {}_uW_1\{\Gamma\}} \sum_{S' \in W\{\Gamma\}}^{S' \leq S} \Delta_U(S')t_0 < S > * < \Gamma/S > \quad (4.32)$$

¹⁵It is understood here that in the evaluation of the UV index of the FI $< \Gamma/S_\gamma >$ the unit vertex polynomial is associated with every vertex $v_\delta \in \mathcal{V}(\Gamma/S_\gamma)$, $\delta \in S_\gamma$.

Here ${}_uW_1\{\Gamma\}$ stands for the collection of 1-uniting spinneys from $\widetilde{W}\{\Gamma\}$, which comprise 1-irreducible elements only.

Note that these definitions coincide with those of ref. [43] in a particular case where $\mathbf{q}_0 = \emptyset$, $\mathbf{c} = \{\mathcal{E}, \emptyset\}$, and the set \mathcal{L}_0 consists of massless lines only (this means that, acting on the very FI $\langle \Gamma \rangle$, the operator t_0 reduces to the unit one, and that the external momenta are non-exceptional ones). We shall not develop the corresponding argument since (4.30,4.31) are much more general than definition (8) of ref. [43] and should be considered in their own right. Moreover, later we shall show that eq.(4.31) provides one with a very convenient regular algorithm to evaluate UV counterterms — the problem which in fact has generated the very idea of the R^* -operation.

Theorem 5. *There exists such a choice of the c-operation Δ_I that the R^* -operation defined by eq.(4.30) makes finite arbitrary dimensionally regularized FI $t_0 \langle \Gamma \rangle$ at $\epsilon \rightarrow 0$.*

The proof of this statement will be presented in the next section. Now we use it to clarify the concept of IR counterterms and to construct an algorithm for their evaluation.

Let us choose $\mathcal{E}^0 = \mathcal{E}$ and $\mathcal{L}^0 = \mathcal{L}$. In this case the operator t_0 converts the (now 1-soft!) FI $\langle \Gamma \rangle$ into a sum of massless tadpoles which are to be put zero within the DR. To put it in another way: in the absence of any dimensional parameters to counterbalance the factor $\mu^{2\epsilon}$ (masses and external momenta do not count due to our choice of t_0), the UV and IR divergences have no choice but to cancel each other. Had we subtracted the UV divergences with the help of the R_U -operation, this fine tuning would surely disappear — the FI $R_U t_0 \langle \Gamma \rangle$ is free of any UV singularities but still suffers from the IR ones, and, thus is, in general, not finite at $\epsilon \rightarrow 0$. On the other hand, Theorem 5 states that $R^* t_0 \langle \Gamma \rangle$ must be well-defined in this limit. It is thus only natural to fix unambiguously the IR Δ_I -operation by demanding

$$R^* t_0 \langle \Gamma \rangle = 0 \quad (4.33)$$

for every 1-soft FI $\langle \Gamma \rangle$. It will be shown soon that this normalization condition leads to great simplifications in all applications of R^* -operation. To begin, we shall demonstrate that this convention immediately leads to an explicit expression for the IR Δ_I -operation in terms of its UV counterpart.

Indeed, if Γ is a 1-soft graph then each of its connected subgraphs is 1-soft, too. This means, in particular, that the combination $t_0(\Delta_U(S') \langle S \rangle)$ is non-zero if and only if $S' = S$. This observation allows one to use (4.31) together with definition (4.24) in order to rewrite the normalization condition (4.33) in the form

$$\Delta_U(\Gamma) \langle \Gamma \rangle + \Delta_I(\Gamma)' R_U \langle \Gamma \rangle \equiv 0. \quad (4.34)$$

The equation holds for *any* FI $\langle \Gamma \rangle$, whence it follows that

$$\Delta_I(\Gamma) = -\Delta_U(\Gamma) R_U^{-1}(\Gamma), \quad (4.35)$$

or, equivalently,

$$\Delta_I = \Delta_U^{-1}. \quad (4.36)$$

Now we are going to derive a more compact representation for the action of the R^* -operation upon $t_0 \langle \Gamma \rangle$. We write

$$R^* t_0 \langle \Gamma \rangle$$

$$= \sum_{S \in {}_u W_1\{\Gamma\}} (R_U t_0 < S >) * \Delta_i < \Gamma/S > \quad (4.37)$$

$$= R_U t_0 < \Gamma > + \sum_{\substack{S \neq \Gamma \\ S \in {}_u W_1\{\Gamma\}}} (R_U t_0 < S >) * \Delta_i < \Gamma/S >. \quad (4.38)$$

Every spinney $S \in {}_u W_1\{\Gamma\}$ can be unambiguously represented in the form

$$S = S' \cup S'', \quad \text{where } S' \sim S'', S' \in {}_u W_1^h\{\Gamma\} \text{ and } S'' \in W_1^s\{\Gamma\}.$$

Here $W_1^h\{\Gamma\}$ ($W_1^s\{\Gamma\}$) stands for the collection of all spinneys from $\widetilde{W}\{\Gamma\}$ that include only 1-hard(soft) elements, while ${}_u W_1^h\{\Gamma\}$ is composed of 1-uniting spinneys from $W_1^h\{\Gamma\}$. As a result, (4.38) could be presented as follows

$$R^* t_0 < \Gamma > = R_U t_0 < \Gamma > + \quad (4.39)$$

$$\sum_{\substack{S' \neq \Gamma \\ S' \in {}_u W_1^h\{\Gamma\}}} R_U t_0 < S' > * \sum_{\substack{S'' \sim S' \\ S'' \in W_1^s\{\Gamma\}}} \Delta_i(\Gamma) \Delta_U(S'') < \Gamma/S' > \quad (4.40)$$

$$= R_U t_0 < \Gamma > +$$

$$\sum_{\substack{S' \neq \Gamma \\ S' \in {}_u W_1^h\{\Gamma\}}} (R_U t_0 < S' >) * \left(\Delta_i(\Gamma) R(W^\sim\{\Gamma, S'\}, \Delta_U) < \Gamma/S' > \right) \quad (4.41)$$

The account of the generic properties of the c -operation allows one to replace the operation $\Delta_i(\Gamma)$ in (4.41) by

$$- \Delta_U(\Gamma) R^{-1}(W^{\widetilde{>}}\{\Gamma, S'\}, \Delta_U).$$

Indeed, according to (4.35)

$$\Delta_i(\Gamma) = -\Delta_U(\Gamma) R(F\{\Gamma\}, -\Delta_U).$$

However, if a forest $F \in F\{\Gamma\}$ does not meet the condition $F \widetilde{>} S'$, then the expression

$$\Delta_U(F) \Delta_U(S'') < \Gamma/S' >$$

vanishes for any spinney S'' . Thus, in the case under consideration, we may use

$$R^{-1}(W^{\widetilde{>}}\{\Gamma, S'\}, \Delta_U) = R(F^{\widetilde{>}}\{\Gamma, S'\}, -\Delta_U)$$

instead of $R(F\{\Gamma\}, -\Delta_U)$. Finally, on performing the substitution, there appears a possibility to use (4.15) and represent $R^* t_0 < \Gamma >$ in a partially summed form

$$R^* t_0 < \Gamma > = R_U t_0 < \Gamma > \quad (4.42)$$

$$+ \sum_{\substack{S \neq \Gamma \\ S \in {}_u W_1^h\{\Gamma\}}} (R_U t_0 < S >) * \left(-\Delta_U(\Gamma) R(F^{\widetilde{>}}\{\Gamma, S\}, -\Delta_U) < \Gamma/S > \right) \quad (4.43)$$

or, equivalently,

$$R^* t_0 < \Gamma > = \sum_{S \in {}_u W_1^h\{\Gamma\}} (R_U t_0 < S >) * \left(-\Delta_U(\Gamma) \sum_{\substack{F \neq \Gamma \\ F \in F^{\widetilde{>}}\{\Gamma, S\}}} (-)^{|F|} \Delta_U(F) < \Gamma/S > \right). \quad (4.44)$$

4.5 R^* -operation and evaluation of UV and IR counterterms

In this subsection we use the above developed machinery of the R^* -operation to give a simple proof to an important theorem [43] which states that an arbitrary UV (or IR) counterterm can be expressed in terms of the divergent and finite parts of some properly constructed massless FI's depending on one external momenta. As a by-product, we also get a new, practically convenient and shorter version of the corresponding algorithm of ref. [43].

Theorem 6. *Let $t_0 \langle \Gamma \rangle$ be an arbitrary dimensionally regularized FI corresponding to a connected graph $\langle \Gamma \rangle$ with $\mathcal{N}_\Gamma = h$ and with its external momenta constrained by (4.16). Then*

- (a) The Laurent expansion in ϵ of the FI $t_0 \langle \Gamma \rangle$ contains only ϵ^i with $i \geq -h$;
- (b) Both polynomials $\Delta_U \langle G \rangle$ and $\Delta_I \langle \Gamma \rangle$ can be identically expressed via the first h -terms¹⁶ of the Laurent expansion in ϵ of some massless propagator-type FI's with the number of loops not exceeding $h - 1$.

Proof.

- (a) This is a well-known fact in the case of FI's without IR divergences which follows naturally way from the transition to the α -parametric representation and a properly chosen change of integration variables (see e.g. [29]). Though the argument is no longer operative in the general case, one can still use it to infer that the polynomial $\Delta_U \langle \Gamma \rangle$ and, thereby, $\Delta_I \langle \Gamma \rangle$ (as a consequence of (4.36)) has poles in ϵ no stronger than $\epsilon^{\mathcal{N}(\Gamma)}$ owing to the argument presented after Theorem 2 of sect. 3. Thus, if a FI $t_0 \langle \Gamma \rangle$ had a pole at $\epsilon \rightarrow 0$ higher than $\epsilon^{-\mathcal{N}(\Gamma)}$, then the R^* -operation would fail to renormalize away *all* the poles from $t_0 \langle \Gamma \rangle$, which is in evident contradiction with Theorem 5.
- (b) The statement is obviously true at $h = 1$. Let us prove it for $h = h_0 + 1$ assuming that it has already been proved for all $h \leq h_0$. Owing to (4.36) it is sufficient to consider only the case of the UV counterterm

$$\Delta_U \langle \Gamma \rangle = \sum_i Z_i(\epsilon) \mathcal{P}_i(\mathbf{q}, \mathbf{m}).$$

Without essential loss of generality, we may also assume that, first, Γ is 1PI and, second, the FI $\langle \Gamma \rangle$ is only log-divergent (that is $\omega(\Gamma) = 0$ and $\Delta_U \langle \Gamma \rangle = Z(\epsilon)$). Indeed it is well-known [35, 37, 38] that if $\omega(\Gamma) > 0$, then every renormalization constant Z_i can be expressed through the UV counterterms of some set of log-divergent FI's obtained from $\langle \Gamma \rangle$ by differentiating the latter with respect to its external momenta and masses.

From Theorem 5 it follows that

$$Z = -K \left('R_U \tilde{R} \langle \Gamma \rangle \right), \quad (4.45)$$

or, in explicit form, (see (4.43))

$$Z = -K \left['R_U \langle \Gamma \rangle + \sum_{\substack{S \neq \Gamma \\ S \in {}_u\mathcal{W}_1\{\Gamma\}}} R_U t_0^{\omega(S)} \langle S \rangle * \Delta_I \langle \Gamma/S \rangle \right] \quad (4.46)$$

¹⁶By definition, the Laurent expansion of an h -loop FI starts from the term $a\epsilon^{-h}$ even if $a \equiv 0$.

where we have used the identity $\omega(\Gamma/S) + \omega(S) = \omega(\Gamma)$ and the definition (4.26). Z is a dimensionless polynomial in ϵ^{-1} , and we hence have the freedom of setting to zero some (or even all) external momenta and masses. Now we put $\mathbf{q} = 0$ and $\mathbf{m} = 0$ and introduce an non-zero auxiliary mass μ_0 into a (arbitrarily chosen) line $l \in \mathcal{L}$ in such a way that the resulting FI $\langle \Gamma \rangle (\mu_0, \mu, \epsilon)$ can be written as¹⁷

$$\langle \Gamma \rangle (\mu_0, \mu, \epsilon) = \int \frac{\langle \Gamma' \rangle (k, \mu, \epsilon)}{(k^2 + \mu_0^2)} d_\mu k. \quad (4.47)$$

Here $\langle \Gamma' \rangle (k, \mu, \epsilon) = \langle \Gamma - l \rangle (k, \mu, \epsilon) P_l(k)$ with $P_l(q_l)/q_l^2$ being the propagator corresponding the line l in the initial FI $\langle \Gamma \rangle$.

After the rearrangement of external momenta and masses has been done, the FI $\langle \Gamma \rangle$ is to be naturally interpreted as having $\mathbf{q} = \emptyset$ and $\mathcal{L}^1 = l$. It is now clear that every spinney from ${}_u W_1^h \{\Gamma\}$ may contain one and only one graph; the graph must have l among its internal lines and must get 1PI after reducing this line. Moreover, if $\gamma \in {}_u W_1^h \{\Gamma\}$ is *not* 1PI and $\omega(\gamma) \geq 0$, then the FI $R_\nu t_0^{\omega(\gamma)} \langle \gamma \rangle$ is, in fact, zero. Indeed, in this case $\gamma = \gamma_1 \cup \gamma_0 \cup \gamma_2$, where γ_1 and γ_2 are two disjoint 1PI subgraphs of Γ attached to the vertices $\pi_-(l)$ and $\pi_+(l)$, respectively, while γ_0 is the (unique) connected subgraph of Γ such that $\mathcal{L}(\gamma_0) = l$. Since $\mathbf{q} = \emptyset$, one has $\mathbf{q}_0^\gamma = \mathbf{q}^\gamma$, and the only possibility to get a non zero contribution to $R_\nu t_0^{\omega(\gamma)} \langle \gamma \rangle$ is due to the term¹⁸

$$t_0^{\omega(\gamma)} ((\Delta_{\dot{\nu}} \langle g_1 \rangle) \langle \gamma_0 \rangle (\Delta_{\dot{\nu}} \langle \gamma_2 \rangle)). \quad (4.48)$$

But this expression is itself equal to zero owing to the fact that (4.48) is evidently a homogeneous polynomial in momenta from \mathbf{q}^γ of degree $\omega(\gamma) + 2 \neq \omega(\gamma)!$ With account of this remark, (4.43) leads to the following simplified form of eq.(4.46)

$$Z = -K \left[R_\nu \langle \Gamma \rangle - K \left[\sum_{\gamma}^{\gamma \neq \Gamma} R_\nu t_0^{\omega(\gamma)} \langle \gamma \rangle * - \Delta_\nu(\Gamma) \sum_{F \in F^\nu \{\Gamma, \gamma\}} (-)^{|F|} \Delta_\nu(F) \langle \Gamma/\gamma \rangle \right] \right], \quad (4.49)$$

where the first sum goes over a 1PI γ such that $\gamma_0 \subset \gamma$ and $\omega(\gamma) \geq 0$. On dimensional grounds we have

$$\langle \Gamma' \rangle (k, \mu, \epsilon) = (\mu/q^2)^{h_0 \epsilon} \frac{P'(k, \epsilon)}{(k^2)^{1+n}},$$

where $P'(k, \epsilon)$ is a homogeneous polynomial in k of degree $2n$ with coefficients being some meromorphic functions of ϵ . This relation means that the pole part of the FI $\langle \Gamma \rangle (\mu_0, \mu, \epsilon)$ itself can be found by performing a trivial one-loop integration over k in (4.47), with the result being expressed through the first $(h_0 + 1)$ terms of the Laura expansion of $P'(k, \epsilon)$ in ϵ .

In order to finish the proof it remains to be checked that the UV c-operation Δ_ν in the rhs of (4.49) acts, in fact, only on FI's with the loop numbers less than or or equal

¹⁷If we did not introduce any auxiliary momenta or masses after the nullification procedure, the rhs of (4.45) would have included the IR counterterm $\Delta_\nu \langle \Gamma \rangle$ whose evaluation is as difficult as that of $\Delta_\nu \langle \Gamma \rangle$.

¹⁸In general, one of two 1PI graphs γ_1 and γ_2 might be trivial. In order to take into account this possibility we shall use the $\Delta_{\dot{\nu}}$ -operation in the expression below.

to h . This is, indeed, the case since a 1PI graph can not contain a proper subgraph with the same number of loops.

Equation (4.49) is a convenient starting point for evaluating the UV (and thereby IR) counterterms. Indeed, the initial algorithm of ref. [43] relies on a relation which is nothing but an unnecessarily complicated form of (4.46). The advantages of (4.49) over the latter are clearly seen. First, everything here is expressed through the usual UV counterterm operation; moreover, the calculational procedure prescribed by (4.49) is technically very similar to the one used in the case where no IR divergences appear. This means that if someone wants to compute some UV renormalization constant then one can hopefully use eq.(4.49) only without any need to understand how it is obtained, not speaking about subtle details of the R^* -operation. Second, the total number of terms in the rhs of (4.49) is, in general, much smaller than in the rhs of (4.46). Finally, eq.(4.49) is applicable without any changes to calculating UV counterterms in two-dimensional field theories, while the definitions of ref. [43] should be somewhat modified in this case.

5 ASYMPTOTIC EXPANSIONS OF FEYNMAN INTEGRALS

This section is mainly devoted to combinatorial problems appearing in studying asymptotic behavior of a generic Feynman integral in the case where some of its momenta and/or masses go to infinity.

Suppose we are given a decomposition of external momenta and masses of a dimensionally regularized FI $\langle \Gamma \rangle(\mathbf{q}, \mathbf{m}, \mu, \epsilon)$, with Γ being a connected graph and \mathbf{q} being \mathbf{q} constrained by eqs.(4.16), namely

$$\mathbf{q} = \mathbf{q}_2 \cup \mathbf{q}_1 \cup \mathbf{q}_0, \quad \mathbf{m} = \mathbf{m}_2 \cup \mathbf{m}_1 \cup \mathbf{m}_0, \quad (5.1)$$

$$\mathbf{q}_i = \{q_l | l \in \mathcal{E}^i\}, \quad \mathbf{m}_i = \{m_l | l \in \mathcal{L}^i\}, \quad \mathcal{E}^i \cap \mathcal{E}^j = \emptyset = \mathcal{L}^i \cap \mathcal{L}^j \text{ for } i \neq j, \quad i, j = 0, 1, 2,$$

where $\mathbf{q}_2, \mathbf{q}_1$, and \mathbf{q}_0 are right subsets of \mathbf{q} . Let us denote also

$$\mathbf{q}_{i\uparrow} = \bigcup_{i'=i}^2 \mathbf{q}_{i'}, \quad \mathbf{q}_{i\downarrow} = \bigcup_0^{i'=i} \mathbf{q}_{i'}, \quad \mathbf{m}_{i\uparrow} = \bigcup_{i'=i}^2 \mathbf{m}_{i'}, \quad \mathbf{m}_{i\downarrow} = \bigcup_0^{i'=i} \mathbf{m}_{i'}, \quad i = 0, 1, 2$$

Note that the sets $\mathbf{q}_{i\downarrow}, \mathbf{q}_{i\uparrow}, i = 0, 1, 2$ are obviously right subsets of \mathbf{q} too. This allows us to use freely the terminology introduced in subsection 4.3 with respect to the partitions

$$\mathbf{q} = \mathbf{q}_i \cup \mathbf{q}_{i'}, \quad \text{and} \quad \mathbf{q} = \mathbf{m}_i \cup \mathbf{m}_{i'} \quad \text{where} \quad (i, i') = (2, 1) \quad \text{or} \quad (1\uparrow, 0).$$

The problem we are interested in is to construct an explicitly finite asymptotic expansion of the FI

$$t_0 \prec \Gamma \rangle(\mathbf{q}, \mathbf{m}, \mu, \epsilon, \rho) \equiv t_0 \langle \Gamma \rangle(\mathbf{q}', \mathbf{m}', \mu, \epsilon)$$

as $\rho \rightarrow 0$. Here, by definition, $\mathbf{q}' = \mathbf{q}_2 \cup \mathbf{q}_1 \cup \mathbf{q}_0$, $\mathbf{m}' = \mathbf{m}_2 \cup \mathbf{m}_1 \cup \mathbf{m}_0$, $\mathbf{q}'_2 = \{q_l/\rho | q_l \in \mathbf{q}_2\}$, and $\mathbf{m}'_2 = \{m_l/\rho | m_l \in \mathbf{m}_2\}$. Without essential loss of generality we shall assume that if a vertex $v \in \mathcal{V}$ is 2-soft, then the corresponding vertex polynomial does not depend on the momenta from the collection \mathbf{q}_2 .

5.1 Dimensionally regularized Feynman integrals

Suppose for a moment that the FI $\langle \Gamma \rangle$ does not suffer on any IR divergences. The general form of the $\rho \rightarrow 0$ expansion of the R -normalized¹⁹ FI $R\langle \Gamma \rangle(\mathbf{q}, \mathbf{m})$ considered in a space-time with an integer dimension is [11, 63–69]

$$R\langle \Gamma \rangle \underset{\rho \rightarrow 0}{=} \sum_{i=i_{\min}}^{\infty} \rho^i \sum_{j=0}^{\mathcal{N}(\Gamma)} (\ln \rho)^j f_{ij}(\mathbf{q}, \mathbf{m}), \quad (5.2)$$

where i runs over the rational values of an increasing arithmetic progression.

Now we describe how the expansion (5.2) is naturally generalized to hold for an unrenormalized FI with all its divergences being regulated by dimensional regularization.

Theorem 7 *For arbitrary dimensionally regularized Feynman integral $t_0\langle \Gamma \rangle(\mathbf{q}, \mathbf{m}, \mu, \epsilon, \rho)$ there holds the asymptotic expansion as $\rho \rightarrow 0$ of the form*

$$t_0\langle \Gamma \rangle \underset{\rho \rightarrow 0}{=} \sum_{i=i_{\min}}^{\infty} \sum_{j=0}^{\mathcal{N}(\Gamma)} \rho^{i-2j\epsilon} F_{ij}(\mathbf{q}, \mathbf{m}, \mu, \epsilon) \quad (5.3)$$

where the functions F_{ij} 's are meromorphic in ϵ and homogeneous with respect to the momenta and masses from the collections \mathbf{q}_2 and \mathbf{m}_2 respectively, to wit:

$$F_{ij}(\mathbf{q}', \mathbf{m}', \mu, \epsilon) = \rho^{i-2j\epsilon} F_{ij}(\mathbf{q}, \mathbf{m}, \mu, \epsilon) \quad (5.4)$$

The expansion (5.3) remains valid after its left and right parts are both subjected to the Laurent expansion in ϵ .

The proof of the theorem generalizes and to some extent repeats the reasoning given in refs. [29–31, 66, 67]. Since it is rather lengthy and not especially instructive, it will be given elsewhere.

Our next task is to construct an explicit representation of the rhs of (5.3) in terms of subgraphs of $\langle \Gamma \rangle$ and the respective reduced graphs. Let us define the *glued*²⁰ FI $t_0\langle \hat{\Gamma} \rangle(\mathbf{q}, \mathbf{m}, \epsilon, \mu, \delta)$ as a Mellin transform of $\theta(\rho-1)t_0\langle \Gamma \rangle(\mathbf{q}, \mathbf{m}, \mu, \epsilon, \rho)$ with respect to ρ , that is

$$t_0\langle \hat{\Gamma} \rangle = \int_0^1 t_0\langle \Gamma \rangle \rho^{-\delta-1} d\rho \quad (5.5)$$

Due to Theorem 7 the integral $\langle \hat{\Gamma} \rangle$ is a meromorphic function of δ with simple poles located at $\delta = \delta_{i,j} \equiv i - 2j\epsilon$; the respective residues are proportional to the functions F_{ij} .

On the other hand, within the α representation technique the glued FI (5.5) is akin to usual dimensionally *and* analytically regularized FI provided the parameter ρ is treated as an extra α -parameter. This key observation allows one to employ the existing powerful methods of studying analytical structure of such integrals [29, 55, 71, 72]. The result is described by the following²¹.

¹⁹By a R -normalized FI we mean the result of subtracting UV divergences from the FI via an R -operation; the similar convention will be also used for the R^* -operation.

²⁰The term comes from works [51, 52, 70] that deal with a particular kinematical regime where only one independent external momentum goes to ∞ .

²¹An elaborated proof of the theorem has been found within the outlined approach by the present author and will be published elsewhere.

Theorem 8. *The expansion (5.3) can be identically rearranged so that it takes the form*

$$t_0 \langle \Gamma \rangle \stackrel{\rho \rightarrow 0}{=} \sum_{S \in {}_u W_2\{\Gamma\}} t_1 \langle S \rangle * \langle \Gamma/S \rangle, \quad (5.6)$$

where $\langle S \rangle \equiv \prod_{\gamma \in S} \langle \gamma \rangle$, with

$$\langle \gamma \rangle \equiv \langle \gamma \rangle (\mathbf{q}^\gamma, \mathbf{m}^\gamma) \quad \text{and} \quad \mathbf{q}^\gamma \equiv (\mathbf{q}_2/\rho) \cup \mathbf{q}_1^\gamma, \quad \mathbf{m}^\gamma \equiv (\mathbf{m}_2^\gamma/\rho) \cup \mathbf{m}_1^\gamma.$$

There are several remarks we would like to make in connection with the theorem.

(i) All the dependence on ρ in the rhs of (5.6) is located in the first factor, with

$$t_1^{(n)} \langle \gamma \rangle = \rho^{-\omega(\gamma)+2\epsilon\mathcal{N}(\gamma)+n} \langle \gamma \rangle. \quad (5.7)$$

(ii) Theorem 7 ensures that the expansion (5.6) does commute with expanding in ϵ . This fact will be of vital importance below in proving Theorem 5.

(iii) The sum in (5.6) would not change if it had gone over $S \in {}_u W_2^h\{\Gamma\}$. Indeed, if S contains a 2-soft element γ , then the factor $t \langle \gamma \rangle$ vanishes since it is a linear combination of the massless tadpoles.

Finally, as a direct consequence of the above theorem we find that the following statement holds.

Theorem 9

(a) *If γ is a 2-right subgraph of Γ , then the $\rho \rightarrow 0$ asymptotic expansion of the respective Feynman subintegral can be written as*

$$t_0 \langle \gamma \rangle \stackrel{\rho \rightarrow 0}{=} \sum_{S \in {}_u W_2\{\gamma\}} t_1 \langle S \rangle * t_0 \langle \gamma/S \rangle. \quad (5.8)$$

(b) *If Φ is an arbitrary spinney from $W_U\{\Gamma\}$, then the asymptotic expansion of the FI $\langle \Gamma/\Phi \rangle$ as $\rho \rightarrow 0$ assumes the form*

$$t_0 \langle \Gamma/S \rangle \stackrel{\rho \rightarrow 0}{=} \sum_{S \in {}_u W_2\{\Gamma\}}^{S \geq \Phi} t_1 \langle S/\Phi \rangle * t_0 \langle \Gamma/S \rangle. \quad (5.9)$$

Proof.

(a) Obvious.

(b) There exists a one-to-one correspondence between two collections of spinneys

$$\{S | S \geq \Phi, S \in {}_u W_2\{\Gamma\}\}$$

and

$$\{S' | S' \in {}_u W_2\{\Gamma/S\},$$

which is established by the mapping (cf. the proof of (3.4))

$$r : S \xrightarrow{r} S' = (S \setminus \Phi) / \Phi. \quad (5.10)$$

Relation (5.9) follows from this correspondence and the observation that

$$\langle (S \setminus \Phi) / \Phi \rangle = \langle S / \Phi \rangle.$$

5.2 R^* -normalized Feynman integrals

We now turn to finding the $\rho \rightarrow 0$ asymptotic expansion of the R^* -normalized FI

$$R^* t_0 \triangleleft \Gamma \triangleright = R^*(\Gamma) t_0 \triangleleft \Gamma \triangleright (\mathbf{q}, \mathbf{m}, \mu, \epsilon) = \sum_{\Phi \in {}_u W_{\mathbb{H}}\{\Gamma\}} R_U t_0 \triangleleft \Phi \triangleright * \Delta_i \triangleleft \Gamma / \Phi \triangleright . \quad (5.11)$$

This is certainly a purely algebraic problem since the FI (5.11) is virtually nothing but a linear combination of unrenormalized FI's multiplied by some UV and IR renormalization constants and its solution will hence rely upon the forest technique developed above.

To begin, we assume for a while that $\mathbf{m}_0 = \mathbf{q}_0 = \emptyset$ and, thus, the operator t_0 in the first line of (5.11) reduces to the unit one. This means in particular that the FI $\triangleleft \Gamma \triangleright$ does not contain any IR divergences²² and the R^* -operation may be safely replaced by the R -one. Under the circumstances the asymptotic expansion we are looking for can be directly read off from (5.9):

$$R \triangleleft \Gamma \triangleright \underset{\rho \rightarrow 0}{=} \sum_{S_a \in {}_u W_2^h\{\Gamma\}} \sum_{S_b \in W_2^s\{\Gamma\}} \sum_{\Phi_a \in W_{UV}\{\Gamma\}} \sum_{\Phi_b \in W_{UV}\{\Gamma\}} \quad (5.12)$$

$$\Delta_U \triangleleft \Phi_a \triangleright * t_1 \triangleleft S_a / \Phi_a \triangleright * \Delta_U \triangleleft \Phi_b \triangleright * t_1 \triangleleft S_b / \Phi_b \triangleright * \triangleleft \Gamma / (S_a \cup S_b) \triangleright ,$$

As explained after Theorem 8 the terms with $S_b \neq \Phi_b$ do not contribute to (5.12). This allows us to rewrite (5.12) in the following compact form

$$R \triangleleft \Gamma \triangleright \underset{\rho \rightarrow 0}{=} \sum_{S \in {}_u W_2^h\{\Gamma\}} R t_1 \triangleleft S \triangleright * R(W^\sim\{\Gamma, S\}, \Delta_U) \triangleleft \Gamma / S \triangleright . \quad (5.13)$$

The asymptotic expansion is *not* explicitly finite: both factors $R t_1 \triangleleft S \triangleright$ and $R(W^\sim\{\Gamma, S\}, \Delta_U) \triangleleft \Gamma / S \triangleright$ do in general suffer from the IR and UV divergences respectively. On the other hand these divergences should cancel out owing to the finiteness of the initial (renormalized!) FI $R \triangleleft \Gamma \triangleright$ at $\epsilon \rightarrow 0$. Hence, there should exist an explicitly finite version of (5.13). The problem can be solved by simply adding the missing subtractions to the both factors in the rhs of (5.13) followed by a proof that the transformation is in fact an identical rearrangement of terms of the expansion.

We begin with two identities (see eqs.(4.13 - 4.15))

$$R_U(W^\sim\{\Gamma, S\}) = R_U(W^{\tilde{>}}\{\Gamma, S\}) R_U^{-1}(W^{\succ}\{\Gamma, S\}) \quad (5.14a)$$

and

$$R_U^{-1}(W^{\succ}\{\Gamma, S\}) = R(W^{\succ}(\Gamma, S), \Delta'_I), \quad (5.14b)$$

where we have denoted

$$\Delta'_I(\gamma) = -\Delta_U(\gamma) \sum_{F \in F^{\succ}\{\Gamma, S\}}^{F < \gamma} (-)^{|F|} \Delta_U(F), \quad (5.14c)$$

and $R_U(W^\sim\{\Gamma, S\}) \equiv R(W^\sim\{\Gamma, S\}, \Delta_U)$ and so on. Note also that due to Theorem 1 for any spinney $S' \succ S$ one has

$$R_U(W^{\tilde{>}}\{\Gamma, S\}) \triangleleft \Gamma / (S' \cup S)_{\max} \triangleright = R_U \triangleleft \Gamma / (S' \cup S)_{\max} \triangleright . \quad (5.15)$$

²²Recall that as in our conventions *all* massless lines are supposed to be assigned to \mathcal{L}^0 .

We now use (5.14,5.15) in order to transform the rhs of (5.13) into

$$\sum_{S \in {}_uW_2^h\{\Gamma\}} \sum_{S' \in W\{\Gamma\}}^{S' \succ S} R_U t_1 \triangleleft S \succ * \Delta'_i \triangleleft S'/S \succ * R_U \triangleleft \Gamma/S'' \succ, \quad (5.16)$$

where $S'' = (S' \cup S)_{\max}$. If S' includes a 2-reducible graph γ' , then the corresponding term in (5.16) can be safely dropped due to appearance of the vanishing factor $\Delta'_i \triangleleft \gamma'/S \succ$. Hence only spinneys $S' \in W_2^h\{\Gamma\}$ do contribute to (5.16). Next, after going to the summation over S'' and employing the identity

$$\Delta'_i \triangleleft S'/S \succ = \Delta'_i \triangleleft S''/S \succ$$

we have

$$R_U \triangleleft \Gamma \succ \stackrel{\rho \rightarrow 0}{=} \sum_{S, S'' \in {}_uW_2^h\{\Gamma\}}^{S \leq S''} R_U t_1 \triangleleft S \succ * \Delta'_i \triangleleft S''/S \succ * R_U \triangleleft \Gamma/S'' \succ \quad (5.17)$$

or, due to the relation (4.44),

$$R_U \triangleleft \Gamma \succ \stackrel{\rho \rightarrow 0}{=} \sum_{S'' \in {}_uW_2^h\{\Gamma\}} R^* t_1 \triangleleft S'' \succ * R_U \triangleleft \Gamma/S'' \succ, \quad (5.18)$$

which is the desired *explicitly finite* writing of (5.13).

In order to cover the general case of the expansion of the R^* -normalized FI (5.11) one should learn to expand products like

$$R_U t_0 \triangleleft \Phi \succ = \prod_{\gamma \in \Phi} R_U t_0 \triangleleft \gamma \succ, \quad (5.19)$$

with $\Phi \in {}_uW_1\{\Gamma\}$. Fortunately, the result of expanding in ρ every factor in the rhs of (5.19) may be directly obtained with the help of (5.18) since its derivation has used neither IR finiteness of $\triangleleft \Gamma \succ$ nor the absence of the operator t_0 . As a result, we get

$$R^* t_0 \triangleleft \Gamma \succ \stackrel{\rho \rightarrow 0}{=} \sum_{\Phi \in {}_uW_1\{\Gamma\}} \sum_{S \in {}_uW_2^h\{\Gamma\}}^{S \leq \Phi} R^* t_1 \triangleleft S \succ * R_U t_0 \triangleleft \Phi/S \succ * \Delta'_i \triangleleft \Gamma/\Phi \succ, \quad (5.20)$$

which, in turn, can be considered as an alternative form of the following remarkable asymptotic expansion

$$R^* t_0 \triangleleft \Gamma \succ \stackrel{\rho \rightarrow 0}{=} \sum_{S \in {}_uW_2^h\{\Gamma\}} R^* t_1 \triangleleft S \succ * R^* t_0 \triangleleft \Gamma/S \succ. \quad (5.21)$$

Indeed, to derive (5.21) from (5.20) it suffices to observe, first, the existence of a natural one-to-one correspondence between two woods

$$\{\Phi | \Phi \in {}_uW_1\{\Gamma\}, \Phi \geq S\} \quad \text{and} \quad \{\Phi' | \Phi \in {}_uW_1\{\Gamma/S\}\}$$

with $\Phi' = (\Phi \setminus S)/S$ and, second, to note that $\triangleleft (\Phi \setminus S)/S \succ = \triangleleft \Phi/S \succ$.

We conclude this subsection by presenting a simple proof of the fact that the R^* -operation does subtracts all kinds of divergences from a generic FI $t_1 < \Gamma >$ (Theorem 5).

Let us introduce an auxiliary positive mass μ_0 into all the propagators from \mathcal{L}^1 and assume that μ_0 is *not* subjected to the Taylor expansion under the action of the operator t_1 . Evidently, the presence of the auxiliary mass prevents any IR divergences from appearing and thus FI $R_U t_1^{(n)} < G >$ is finite at $\epsilon \rightarrow 0$ for every integer $n \geq 0$. It follows from (5.21) that

$$\begin{aligned} R_U t_1^{(n)} < G > \stackrel{\rho \rightarrow 0}{=} & \sum_{k, k' \geq 0}^{k+k' \leq K} \sum_{S \in_u W_2^h \{\Gamma\}}^{S \neq \Gamma} R^* t_1^{(k)} \mathcal{T}_{\mu_0}^{(k')} < S > * R_U t_1^n < \Gamma/S > \\ & + \sum_{k, k' \geq 0}^{k+k' \leq K} R^* t_1^{(k)} \mathcal{T}_{\mu_0}^{(k')} < \Gamma > + \delta(\rho), \end{aligned} \quad (5.22)$$

where, due to Theorems 7 and 8, the reminder²³ $\delta(\rho) = o(\rho^{-N+K+2\mathcal{N}(\Gamma)\epsilon})$ with at $\rho \rightarrow 0$ is an analytical function of ϵ in the vicinity of the point $\epsilon = 0$. Now, reasoning by recurrence with reference to the number of internal lines of Γ one can easily convince oneself that the sum

$$\sum_{k, k' \geq 0}^{k+k' \leq K} R^* t_1^{(k)} \mathcal{T}_{\mu_0}^{(k')} < \Gamma >$$

is finite as $\epsilon \rightarrow 0$, because of the finiteness of the left side of (5.22). This, in turn, means that every term in this sum is finite for the values of ρ and μ_0 can be chosen at will. Finally, choosing $\rho = 1, k = n$ and $k' = 0$ we conclude that the FI $R^* \left[t_1^{(n)} < \Gamma > \Big|_{\mu_0=0} \right]$ is also finite as $\epsilon \rightarrow 0$, which was the thing to be proved.

6 RELATED WORKS: A COMPARATIVE DISCUSSION OF RESULTS

R^{-1} -operation. There exists a deep similarity between the R^{-1} -operation technique and the counterterm formalism [22–25]. To some extent, the former may be considered as another variant of the latter adopted to be able to deals with with separate FI's with the same ease as the counterterm formalism treats the perturbation series as a whole.

Infrared rearrangement and R^ -operation.* After the pioneering work by Vladimirov [37], the trick of infrared rearrangement has been repeatedly rediscovering by many authors (see e.g. refs. [73, 74]). The authors of the second work even claim that they have proved a statement analogous to Theorem but their proof makes no allowance for the IR divergences. Certainly in the absence of any IR subtractions such a statement may be true provided that the FI obtained after the IRR has no IR divergences. This, in turn, means that the procedure suggested in ref. [74] is completely equivalent to the IRR trick as it was described in ref. [37].

²³Here $N = \max_{S \in_u W_2^h \{\Gamma\}} \left(\sum_{\gamma \in S} \omega(\gamma) \right)$ and ϵ is understood to be sufficiently small and negative.

A subtraction procedure to remove the IR divergences which appear if one tries to expand a convergent FI with non-exceptional external momenta when all its masses go to zero, has been discussed in ref. [75]. The procedure is, in fact, a very particular version of the R^* -operation: the assumed UV finiteness of the integral to be expanded makes things much easier and, simultaneously, restricts severely the possible applications. Indeed, both of the most useful applications of the R^* -operation — calculation of the renormalization constants and construction of explicitly finite asymptotic expansions of generic FI's — seem to be lost.

The natural normalization condition (4.33) for the R^* -operation was first suggested in [47] with essentially the same motivation as the one put forward in the present work.

The accurate proof of Theorem 5 — the main statement of the R^* -operation theory — has been presented in [61] for the particular case of a FI with nonexceptional external momenta. The proof is based on the heavy use of the α -parametric representation and technical tools developed early in refs. [29, 71, 72] (for an up-to-date review see a book [55]).

Asymptotic expansions. There are a few series of studies devoted to the problem of constructing a *complete* asymptotic expansion of a generic dimensionally regularized euclidean FI when some of its external momenta and/or masses go to infinity. The authors of refs. [56, 57, 76–78] have investigated the problem within a so-called "extension principle" (EP) which, in fact, was inspired by the ideas of the R^* -operation as formulated in ref. [42]. In particular, in ref. [78] an expansion equivalent to the one (5.13) has been suggested as a natural generalization of the results obtained by applying the EP to a few particular one-loop FI's. As far as we know, no proof of this expansion or of the EP itself has been published yet. Moreover it has been explicitly stated in [78] that the EP is *not* applicable to FI's with IR divergences. This means that the approach fails to expand a generic R^* -normalized FI.

In his two works [79], the late Doctor S.G. Gorishny suggested a method of constructing asymptotic expansions of FI's at large momenta and/or masses. His method is based on a heuristic generalization of the Zimmermann approach [7, 9]. Its starting point is an asymptotic expansion analogous to (5.13) in which, however, the operator t_0 expands in momenta from \mathbf{q}_0 around some fixed *non-zero* point in momentum space. This modification is introduced to avoid the IR divergences. After developing a combinatorial technique similar to ours, the author of ref. [79] found an expansion which should presumably be equivalent (modulo the different definition of the t_0 - operator) to eq.(5.18). He also gave the heuristic argument that within the MS - scheme the choice of the zero momentum expansion point should not lead to IR divergences. Note also work [80] similar in its spirit and results, which, however, deals with a very particular case of the problem under discussion, namely with the short distance expansion of a product of two composite operators.

The fact that it is the R^* -operation which naturally appears when one treats asymptotic expansions of minimally renormalized FI's was first realized in refs. [47, 48]. The expansions (5.13) and (5.18) were derived for the case where only one external momentum is considered to be large in refs. [47, 51, 52]. Here the Wilson short distance expansion of a product of two composite operators was studied in the framework of the gluing method [70].

A heuristic argument in favor of the existence of the expansion (5.18) was suggested in ref. [50].

A practically convenient algorithm to compute the coefficient functions of various

operator expansions in the MS-scheme was presented in refs. [50, 81]. At the level of individual Feynman graphs the algorithm amounts, in fact, to the expansion (5.13), which is taken as granted.

Recently a rigorous proof of the expansion (5.18) has been obtained in [53, 54] along the lines of refs. [7, 9, 25, 82], by constructing a suitable oversubtraction operator which makes use of the \tilde{R} -operation. In addition, in these works the counterterm formalism was effectively employed to derive explicitly finite asymptotic expansions of minimally renormalized Green functions for a variety of physically interesting asymptotic regimes. Note that the generalization of the approach to generic R^* -normalized FI's seems to be nontrivial (see in this connection ref. [49]).

7 CONCLUSIONS

In this work we have elaborated a technique which has allowed us to perform a uniform, self contained and essentially complete study of a variety combinatorial issues involved in the $R-$, $R^{-1}-$ and R^* -operations and their applications. We find it nice that all the three "renormalization" operations are tightly interconnected each other, with the interplay showing up not only in the MS - scheme (what is more-or-less natural, since the inverted and the starred R-operations first appeared in the context of just this scheme) but also in a different renormalization prescription — the momentum subtraction scheme.

There remain few interesting problems that can be solved within the approach. Below we list some of them.

R^{-1} -operation. To construct practically convenient conversion formulas which are to connect the standard MS - scheme with its close relative based on so-called dimensional reduction. By finding a suitable conversion formula to clarify the relationship between different definitions of the γ_5 -matrix in DR .

R^{-1} - and R^ -operations.* It would be interesting to find a generalization of the relations (3.39) between momentum subtractions and the R^{-1} -operation in the case of renormalization with the soft mass. In doing the problem a kind of R^* -operation should presumably appear in a natural way.

The R^* -operation is a natural tool to investigate the IR finiteness of the given theory including massless particles. With its help it is possible e.g. to give a simple and purely combinatorial proof of the IR finiteness of quantum chromodynamics along the way outlined in refs. [29–31]. This has been done by the present author and will be published elsewhere.

Asymptotic expansions. It would be of interest to construct an explicit and purely combinatorial proof the renormalization group equation to which the R^* - normalized FI's should satisfy.

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8 Comments

The work has never been submitted for publication for purely personal reasons. Now, a quarter of century later, I want to make it more accessible as it is the improved and extended formulation of the R^* -operation presented here has been one of the crucial tool (along with the algebraic manipulation language FORM [83,84] and other advances in our

understanding of multiloop Feynman diagrams (see, e.g. [85]) for many record-breaking multiloop calculations in gauge theories performed during last years [86-92].

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