

MULTIVARIATERESIDUES: a Mathematica package for computing multivariate residues

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Abstract

Multivariate residues appear in many different contexts in theoretical physics and algebraic geometry. In theoretical physics, they for example give the proper definition of generalized-unitarity cuts, and they play a central role in the Grassmannian formulation of the S-matrix by Arkani-Hamed et al. In realistic cases their evaluation can be non-trivial. In this paper we provide a Mathematica package for efficient evaluation of multivariate residues based on methods from commutative algebraic geometry.

Keywords: Multivariate residues; computational algebraic geometry; Mathematica

PROGRAM SUMMARY

Program Title: MULTIVARIATERESIDUES

Licensing provisions: GNU General Public License (GPL)

Programming language: Mathematica

Nature of problem: Evaluation of multivariate complex residues

Solution method: Mathematica implementation

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1. Introduction

Multivariate residues appear in many different contexts in theoretical physics and algebraic geometry. In theoretical physics, they for example give the proper definition of generalized-unitarity cuts [1–6], and they play a central role in the Grassmannian formulation of the S-matrix by Arkani-Hamed et al. [7–15]. In algebraic geometry, multivariate residues play an important role in elimination theory in the context of solving systems of multivariate polynomial equations [16].

In practise, the evaluation of multivariate residues can be non-trivial. Nevertheless, implementations of their evaluation have not been made publicly available. In this paper we provide the Mathematica package `MULTIVARIATERESIDUES` for efficient evaluation of multivariate residues based on methods from commutative algebraic geometry.

This paper is organized as follows. In section 2 we give the definition of the multivariate residue along with some of its basic properties, in particular the transformation formula. We explain an algorithm for how the latter can be utilized to compute multivariate residues in general. In section 3 we explain an alternative approach which makes use of powerful methods from modern commutative algebra. Both of these methods are implemented in `MULTIVARIATERESIDUES`. In section 4 we apply the formalism of section 3 to a specific example to illustrate how residues are computed in practise. In section 5 we discuss the application of multivariate residues to the calculation of generalized-unitarity cuts in the context of computations of scattering amplitudes in perturbative quantum field theory. Section 6 provides a manual for `MULTIVARIATERESIDUES` along with benchmarks of the performance, comparisons between the various options and tips for the user to improve performance. In section 7 we give our conclusions. Appendix A provides a topological explanation of why multivariate residues, in contrast to the univariate case, are not uniquely determined by the location of a pole, but have some dependence on the integration cycle.

2. General theory

In this section we give the definition of multivariate complex residues and discuss the transformation formula and how this may be utilized to compute residues in practise.

Our setup is as follows. Let $f(z) = (f_1(z), \dots, f_n(z)) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ and $h : \mathbb{C}^n \rightarrow \mathbb{C}$ be holomorphic functions, and consider the meromorphic n -form,¹

$$\omega = \frac{h(z) dz_1 \wedge \dots \wedge dz_n}{f_1(z) \cdots f_n(z)}. \quad (1)$$

The case where the form has m denominator factors with $m > n$ can be treated as a special case of the above by grouping the m factors into precisely n factors. We will elaborate on the ambiguity of this process and its underlying topological explanation further below. Likewise, the case of $m < n$ denominator factors will be discussed below.

In the multivariate setting, we define a pole as a point $p \in \mathbb{C}^n$ where f has an isolated zero—that is, $f(p) = 0$ and $f^{-1}(0) \cap U = \{p\}$ for a sufficiently small neighborhood U of p . We are interested in computing the residue of ω at its poles. The multivariate residue is defined as a multidimensional generalization of a contour integral: an integral taken over a product of n circles, that is an n -torus,

$$\operatorname{Res}_{\{f_1, \dots, f_n\}, p}(\omega) = \frac{1}{(2\pi i)^n} \oint_{\Gamma_\epsilon} \frac{h(z) dz_1 \wedge \dots \wedge dz_n}{f_1(z) \cdots f_n(z)}, \quad (2)$$

where $\Gamma_\epsilon = \{z \in \mathbb{C}^n : |f_i(z)| = \epsilon_i\}$ and the ϵ_i have infinitesimal real values. Furthermore, the integration cycle is oriented by the condition,

$$d(\arg f_1) \wedge \dots \wedge d(\arg f_n) \geq 0. \quad (3)$$

We note that the definition of the integration cycle differs from the univariate case: rather than being defined directly in terms of the variables z , Γ_ϵ is defined in terms of the denominator factors $f_i(z)$.

The Jacobian determinant evaluated at the pole

$$J(p) \equiv \det_{i,j} \left(\frac{\partial f_i}{\partial z_j} \right) \Big|_{z=p}, \quad (4)$$

plays an important role, since if $J(p) \neq 0$, we can evaluate the residue directly by the coordinate transformation $w = f(z)$,

$$\operatorname{Res}_{\{f_1, \dots, f_n\}, p}(\omega) = \frac{1}{(2\pi i)^n} \oint_{|w_i| \leq \epsilon_i} \frac{h(f^{-1}(w)) dw_1 \wedge \dots \wedge dw_n}{J(p) w_1 \cdots w_n} = \frac{1}{(2\pi i)^n} \frac{h(p)}{J(p)}. \quad (5)$$

¹If one adds a boundary at infinity as needed to apply global residue theorems, we can define the functions on $\mathbb{C}\mathbb{P}^n$ rather than \mathbb{C}^n .

In this case, the residue is termed nondegenerate.

In general, however, a residue may be degenerate, such as is the case for higher-order poles. In this situation, the above coordinate transformation does not suffice to compute it. A central and completely general property of residues is the transformation formula (cf. section 5.1 of ref. [17]). As we will shortly see, this property can be utilized to compute any residue, degenerate or nondegenerate.

Theorem 1. (*Transformation formula*). *Let $I = \langle f_1(z), \dots, f_n(z) \rangle$ be a zero-dimensional ideal² generated by a finite set of holomorphic functions $f_i(z) : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}$ with $f_i(p) = 0$. Furthermore, let $J = \langle g_1(z), \dots, g_n(z) \rangle$ be a zero-dimensional ideal such that $J \subseteq I$; that is, whose generators are related to those of I by $g_i(z) = \sum_{j=1}^n a_{ij}(z)f_j(z)$ with the $a_{ij}(z)$ being polynomials. Letting $A(z) = (a_{ij}(z))_{i,j=1,\dots,n}$ denote the transformation matrix, the residue at p satisfies,*

$$\operatorname{Res}_{\langle f_1, \dots, f_n \rangle, p} \left(\frac{h(z) dz_1 \wedge \dots \wedge dz_n}{f_1(z) \cdots f_n(z)} \right) = \operatorname{Res}_{\langle g_1, \dots, g_n \rangle, p} \left(\frac{h(z) \det A(z) dz_1 \wedge \dots \wedge dz_n}{g_1(z) \cdots g_n(z)} \right). \quad (6)$$

In cases where the form ω has fewer denominator factors than variables, the notion of residue defined in eq. (2) does not apply, and this case is therefore outside the scope of this paper. Nevertheless, we mention that a notion of residue which does apply in this situation is that of *residual forms*. To illustrate the idea, let us consider the following example, taken from section 7.2 of ref. [7],

$$\omega = \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_1(z_1 + z_2 z_3)}. \quad (7)$$

As ω has three variables, but only two denominator factors, the residue in eq. (2) is not well-defined. However, we observe that we can define the 2-form

$$\tilde{\omega} = \operatorname{Res}_{z_1=0} \omega = \frac{dz_2 \wedge dz_3}{z_2 z_3}. \quad (8)$$

This form has two variables and two denominator factors, and hence the notion of residue in eq. (2) applies to $\tilde{\omega}$.

²The ideal I is said to be *zero-dimensional* if and only if the solution to the equation system $f_1(z) = \dots = f_n(z) = 0$ consists of a finite number of points $z \in \mathbb{C}\mathbb{P}^n$.

2.1. Computation of residues via the transformation formula

To apply the transformation formula (6) to the computation of residues, we must first find a useful transformation of the set of ideal generators. Here we restrict attention to the case where the generators $f_i(z)$ are polynomials and follow the approach explained in section 1.5.4 of ref. [16]. The idea is to choose the g_i to be *univariate*—that is, $g_i(z_1, \dots, z_n) = g_i(z_i)$. Then the residue can simply be evaluated as a product of univariate residues.

A set of univariate polynomials g_i can be obtained by generating a Gröbner basis of $\{f_1(z), \dots, f_n(z)\}$ with lexicographic monomial order. Specifying the variable ordering $z_{i+1} \succ z_{i+2} \succ \dots \succ z_n \succ z_1 \succ z_2 \dots \succ z_i$ will produce a Gröbner basis whose first element is a polynomial which depends only on z_i . We let $g_i(z_i)$ denote this polynomial. Now, by considering all n cyclic permutations of the variable ordering $z_1 \succ z_2 \succ \dots \succ z_n$ in this way we generate a set of n univariate polynomials $\{g_1(z_1), \dots, g_n(z_n)\}$.

To illustrate the above method, we consider as an example the following differential form,

$$\omega = \frac{z_1 dz_1 \wedge dz_2}{z_2(a_1 z_1 + a_2 z_2)(b_1 z_1 + b_2 z_2)}, \quad (9)$$

which at the same time will serve to explain how to compute residues in cases with more distinct denominator factors than variables. As eq. (9) depends on two variables and has three distinct denominator factors, we must consider all possible ways of partitioning the denominator into two factors. Denoting the denominator factors of eq. (9) as follows,

$$\begin{aligned} \varphi_1(z_1, z_2) &= z_2 \\ \varphi_2(z_1, z_2) &= a_1 z_1 + a_2 z_2 \\ \varphi_3(z_1, z_2) &= b_1 z_1 + b_2 z_2, \end{aligned} \quad (10)$$

we observe that this can be done in three distinct ways, namely

$$\{\varphi_1, \varphi_2 \varphi_3\}, \quad \{\varphi_2, \varphi_3 \varphi_1\} \quad \text{and} \quad \{\varphi_3, \varphi_1 \varphi_2\}. \quad (11)$$

We are interested in computing the residues of ω at the pole $p = (0, 0)$ corresponding to each of these partitionings. We note that all of these residues are degenerate.

Let us evaluate the residue for the denominator partitioning $\{\varphi_1, \varphi_2 \varphi_3\}$. The lexicographically-ordered Gröbner basis of $\{\varphi_1, \varphi_2 \varphi_3\}$ in the variable

ordering $z_2 \succ z_1$ is $\{a_1 b_1 z_1^2, z_2\}$; in the variable ordering $z_1 \succ z_2$ it is $\{z_2, a_1 b_1 z_1^2\}$. Choosing the first element of each Gröbner basis we obtain,

$$g_1(z_1, z_2) = a_1 b_1 z_1^2 \quad (12)$$

$$g_2(z_1, z_2) = z_2. \quad (13)$$

We can obtain the transformation matrix A as a byproduct of finding the Gröbner basis (or using the approach implemented in ref. [18]). In the simple case considered here, ordinary multivariate polynomial division produces the same result,

$$A = \begin{pmatrix} -(a_1 b_2 + a_2 b_1) z_1 - a_2 b_2 z_2 & 1 \\ 1 & 0 \end{pmatrix}, \quad (14)$$

that relates the two sets of ideal generators,

$$A \cdot \begin{pmatrix} \varphi_1(z_1, z_2) \\ \varphi_2(z_1, z_2) \varphi_3(z_1, z_2) \end{pmatrix} = \begin{pmatrix} g_1(z_1, z_2) \\ g_2(z_1, z_2) \end{pmatrix}. \quad (15)$$

From the transformation law (6) we then find that the residue of ω at $p = (0, 0)$ with respect to the ideal generators $\{\varphi_1, \varphi_2 \varphi_3\}$ is

$$\operatorname{Res}_{\{\varphi_1, \varphi_2 \varphi_3\}, p} \omega = \operatorname{Res}_p \frac{z_1 \det A \, dz_1 \wedge dz_2}{g_1(z_1, z_2) g_2(z_1, z_2)} = - \operatorname{Res}_p \frac{dz_1 \wedge dz_2}{a_1 b_1 z_1 z_2}. \quad (16)$$

As desired, the denominator on the right-hand side of eq. (16) is a product of univariate polynomials. Hence the residue can be computed as a product of univariate residues and yields,

$$R_1 \equiv \operatorname{Res}_{\{\varphi_1, \varphi_2 \varphi_3\}, p} \omega = - \frac{1}{a_1 b_1 (2\pi i)^2} \quad (17)$$

$$R_2 \equiv \operatorname{Res}_{\{\varphi_2, \varphi_3 \varphi_1\}, p} \omega = - \frac{a_2}{a_1 (a_1 b_2 - a_2 b_1) (2\pi i)^2} \quad (18)$$

$$R_3 \equiv \operatorname{Res}_{\{\varphi_3, \varphi_1 \varphi_2\}, p} \omega = \frac{b_2}{b_1 (a_1 b_2 - a_2 b_1) (2\pi i)^2}, \quad (19)$$

where the residues for the two other denominator partitionings $\{\varphi_2, \varphi_3 \varphi_1\}$ and $\{\varphi_3, \varphi_1 \varphi_2\}$ were computed analogously. We remark that in general one must keep in mind that the residue is antisymmetric under interchanges of the denominator factors of ω . This follows from the dependence of the residue on the orientation of the integration cycle, cf. eq. (3).

We observe that only two out of the three residues R_1, R_2, R_3 in eqs. (17)–(19) are independent, as the residues satisfy the identity,

$$R_1 + R_2 + R_3 = 0. \quad (20)$$

Identities of this kind are common for multivariate residues.

In Appendix A we give a topological explanation of why the multivariate residues in eqs. (17)–(19) are not uniquely determined by the pole p , but rather also depend on the choice of partitionings in eq. (11).

3. Evaluation of residues by use of dual structure of quotient ring

The evaluation of residues by use of the transformation formula explained in section 2.1 is completely general. However, in realistic cases the computation of the transformation matrix A can be intensive, and as a result this method is not optimal in all situations.

In this section we explain a more efficient method for residue computations, which we have implemented in `MULTIVARIATERESIDUES`. Our setup is as follows. As in section 2, we restrict ourselves to the case where the denominator factors of ω in eq. (1) are polynomials. We denote these polynomials by $P_1(z), \dots, P_n(z)$ and assume that the ideal $I = \langle P_1(z), \dots, P_n(z) \rangle$ is zero-dimensional; i.e., that the associated variety $V = \{z \in \mathbb{C}^n : P_1(z) = \dots = P_n(z) = 0\}$ consists of a finite number of points,

$$V = \{p_1, \dots, p_m\}. \quad (21)$$

This method exploits that the residue map defines a non-degenerate inner product $\langle \cdot, \cdot \rangle$ on the quotient ring

$$Q \equiv \mathbb{C}[z_1, \dots, z_n]/I \quad (22)$$

of the ring $\mathbb{C}[z_1, \dots, z_n]$ of all polynomials in the variables z_1, \dots, z_n with coefficients in \mathbb{C} modulo the ideal I . As I is zero-dimensional, Q has a finite dimension (cf. section 2.2 of ref. [19]) which we denote by D .

By decomposing the numerator of ω in a canonical (linear) basis of the quotient ring, and the constant 1 in the dual basis wrt. $\langle \cdot, \cdot \rangle$, the *global residue*

$$\operatorname{Res}_I \omega = \sum_{i=1}^m \operatorname{Res}_{I, p_i} \omega \quad (23)$$

can be computed as the dot product of the corresponding coefficient vectors. The corresponding local residue at each pole p_i can then be computed by multiplying the integrand by appropriate polynomials which are unity in the vicinity of p_i and vanish in the vicinity of the remaining poles.

In the following we explain how the canonical and dual (linear) bases are computed and how the above-mentioned polynomials are constructed.

3.1. Computing the canonical basis of the quotient ring

Our first aim is to determine a canonical (linear) basis of the quotient ring Q . To this end we compute a Gröbner basis G of I and consider the ideal $\langle LT(I) \rangle$ generated by the leading term of each element of G . The monomials in the complement of $\langle LT(I) \rangle$ then form a basis of Q .

We can rephrase this statement as the following algorithm.

1. Decide on a monomial order \prec and compute a Gröbner basis $G = \{g_1, \dots, g_s\}$ of I wrt. \prec .
2. Obtain the leading term of each Gröbner basis element,

$$h_i = \text{LT}(g_i). \quad (24)$$

3. Extract the exponent vectors of the leading terms

$$h_i = z_1^{\alpha_{i,1}} \dots z_n^{\alpha_{i,n}} \quad \mapsto \quad (\alpha_{i,1}, \dots, \alpha_{i,n}). \quad (25)$$

4. The elements of

$$E = \mathbb{Z}_{\geq 0}^n \setminus \bigcup_{i=1}^s ((\alpha_{i,1}, \dots, \alpha_{i,n}) + \mathbb{Z}_{\geq 0}^n) \quad (26)$$

then define exponent vectors of the canonical basis elements. That is,

$$\mathcal{C} = \{z_1^{\beta_{1,1}} \dots z_n^{\beta_{1,n}}, \dots, z_1^{\beta_{D,1}} \dots z_n^{\beta_{D,n}}\} \quad \text{where } (\beta_{j,1}, \dots, \beta_{j,n}) \in E, \quad (27)$$

is the desired canonical basis of the quotient ring Q wrt. \prec (cf. proposition 1 of section 5.3 of ref. [20]).

3.2. Computing the dual basis of the quotient ring

Our next aim is to determine the dual (wrt. \mathcal{C}) basis of Q . This basis can be extracted from the determinant of the Bezoutian matrix of the polynomials $P_1(z), \dots, P_n(z)$. Accordingly, we proceed with the following steps.

1. Compute the Bezoutian matrix of the polynomials $P_1(z), \dots, P_n(z)$,

$$\text{Bez}_{ij}(z, y) = \frac{P_i(y_1, \dots, y_{j-1}, z_j, \dots, z_n) - P_i(y_1, \dots, y_j, z_{j+1}, \dots, z_n)}{z_j - y_j}. \quad (28)$$

The entries of the Bezoutian matrix are elements of the direct product $Q \otimes Q$.

2. Take the determinant of the Bezoutian matrix

$$\mathcal{B}(z, y) \equiv \det(\text{Bez}). \quad (29)$$

3. Compute the remainder of $\mathcal{B}(z, y)$ in $Q \otimes Q$. This is carried out in practise by first performing polynomial division of $\mathcal{B}(z, y)$ wrt. the Gröbner basis $G = \{g_1(z), \dots, g_s(z)\}$ where the elements are taken as polynomials in z_1, \dots, z_n , and then performing polynomial division of the result wrt. G whose elements are now taken as polynomials in y_1, \dots, y_n ,

$$\mathcal{B}(z, y) = q_1(z, y)g_1(z) + \dots + q_s(z, y)g_s(z) + \mathcal{B}_Q(z, y), \quad (30)$$

$$\mathcal{B}_Q(z, y) = \hat{q}_1(z, y)g_1(y) + \dots + \hat{q}_s(z, y)g_s(y) + \mathcal{B}_{Q \otimes Q}(z, y). \quad (31)$$

4. Label the elements of the canonical basis as $\mathcal{C} = \{c_1(z), \dots, c_D(z)\}$ and decompose the Bezoutian determinant as,

$$\mathcal{B}_{Q \otimes Q}(z, y) = c_1(z)d_1(y) + \dots + c_D(z)d_D(y). \quad (32)$$

$\mathcal{B}_{Q \otimes Q}$ has a unique such decomposition, and the dual basis of Q (wrt. the canonical basis \mathcal{C}) can now be read off (cf. section 1.5.4 of ref. [16]),

$$\mathcal{D} = \{d_1(z), \dots, d_D(z)\}, \quad (33)$$

where the variables were relabeled into z_1, \dots, z_n . We remark that the elements $d_i(z)$ are in general polynomials rather than monomials (in contrast to the canonical basis elements).

3.3. Constructing partition-of-unity polynomials

One more ingredient is needed to compute residues at all the poles in the variety $V = \{p_1, \dots, p_m\}$ associated with I , namely a set of polynomials $e_1(z), \dots, e_m(z)$ which are unity in the vicinity of a given pole and vanishing in the vicinity of the remaining poles.

To this end, we construct a linear form $\ell(z) = a_1 z_1 + \dots + a_n z_n$ (with $a_i \in \mathbb{C}$) such that $\ell(p_1), \dots, \ell(p_m)$ are all distinct. (In practise, this is done in MULTIVARIATERESIDUES by scanning over a set of coefficient vectors (a_1, \dots, a_n) with integer entries.)

The following set of Lagrange polynomials

$$L_i(z) = \prod_{\substack{j=1, \\ j \neq i}}^m \frac{\ell(z - p_j)}{\ell(p_i - p_j)} \quad (34)$$

then have the desired property of “projecting onto each pole”,

$$L_i(p_k) = \delta_{ik} . \quad (35)$$

However, this set of polynomials will not quite have the desired property of defining a partition of unity,

$$\sum_{i=1}^m e_i = 1 \pmod{I} \quad \text{and} \quad e_i e_j = e_i \delta_{ij} \pmod{I} . \quad (36)$$

Rather (cf. lemma 2.3 of section 4.2 of ref. [19]), a set of polynomials with these additional properties can be obtained as

$$e_i(z) = 1 - (1 - L_i(z)^\delta)^\delta, \quad (37)$$

where δ is a positive integer such that for the intersection of the ideals generated by each pole $J\langle\{p_i\}\rangle = \langle z_1 - p_{i,1}, \dots, z_n - p_{i,n} \rangle$

$$M \equiv \bigcap_{i=1}^m \langle z_1 - p_{i,1}, \dots, z_n - p_{i,n} \rangle, \quad (38)$$

we have that

$$M^\delta \subseteq I. \quad (39)$$

To find an appropriate δ thus requires algorithms to determine the intersection and the product of two ideals and moreover to check if one ideal is contained in another ideal.

To this end, we consider any two ideals in $\mathbb{C}[z_1, \dots, z_n]$,

$$J_1 = \langle h_1, \dots, h_r \rangle \quad \text{and} \quad J_2 = \langle k_1, \dots, k_s \rangle. \quad (40)$$

To compute the intersection $J_1 \cap J_2$, we introduce a parameter t and consider the ideal

$$\langle th_1, \dots, th_r, (1-t)k_1, \dots, (1-t)k_s \rangle. \quad (41)$$

Now compute a Gröbner basis G of the latter ideal wrt. lexicographic order in which t is greater than the z_i . The elements of G which do not contain the parameter t will then form a basis³ of $J_1 \cap J_2$ (cf. section 4.3 of ref. [20]).

The product of J_1 and J_2 is generated by the product of the generators,

$$J_1 J_2 = \langle h_i k_j : 1 \leq i \leq r, 1 \leq j \leq s \rangle, \quad (42)$$

cf. proposition 6 of section 4.3 of ref. [20].

Finally, to check the inclusion of ideals, for example whether $J_1 \subseteq J_2$, compute a Gröbner basis H of J_2 . Then

$$J_1 \subseteq J_2 \iff \forall i = 1, \dots, r : h_i \equiv 0 \pmod{J_2}. \quad (43)$$

That is, the inclusion $J_1 \subseteq J_2$ holds if and only if all the generators of J_1 have a vanishing remainder upon polynomial division wrt. H (cf. exercise 2 of section 1.4 of ref. [20]).

The computation of δ in eq. (37) following the above steps can in some cases be computationally intensive. This is especially true in cases where δ is large and M has many generators, so that a large number of polynomial divisions must be carried out in order to compute the generators of M^j (where $j = 2, \dots, \delta$) in the intermediate stages.

Alternatively, to compute partition-of-unity polynomials, we may compute the maximum pole multiplicity λ_{\max} , and use $\delta = \lambda_{\max}$ in eq. (37). Thus, we turn to explaining how to compute the multiplicities of the poles p_i . To this end we consider a linear form ℓ of the type discussed above eq. (34) with the property of mapping all poles $p_i \in V$ to distinct values. We aim

³In fact, the basis will be a Gröbner basis of $J_1 \cap J_2$.

to find the matrix of the map $P(z) \mapsto \ell(z)P(z)$ acting on polynomials in Q and compute the dimensions of the eigenspaces of the matrix, as these are the desired pole multiplicities (cf. the discussion below Proposition (2.7) of Chapter 4 of ref. [19]).

Let $\mathcal{C} = (c_1(z), \dots, c_D(z))$ denote the canonical basis of Q . To find the matrix M_ℓ of $P(z) \mapsto \ell(z)P(z)$, for a fixed $1 \leq i \leq D$ decompose (the polynomial remainder of) $\ell(z)c_i(z)$ in the basis \mathcal{C} , producing a vector v_i with D entries. Then, cf. section 2.4 of ref. [19], $(M_\ell)_{ij} = (v_i)_j$. The eigenvalues of M_ℓ are $(\lambda_1, \dots, \lambda_m) = (\ell(p_1), \dots, \ell(p_m))$, and the multiplicity of the pole p_i is the dimension of the eigenspace

$$E_{\lambda_i} = \{x \in \mathbb{R}^D : M_\ell x = \lambda_i x\}. \quad (44)$$

In practise, the largest pole multiplicity $\lambda_{\max} = \max\{\lambda_1, \dots, \lambda_m\}$ is strictly greater than the smallest δ satisfying eq. (39). As a result, in cases where the dimension of the quotient ring is large, the subsequent computation of the partition-of-unity polynomials in eq. (37) may prove time-consuming, even though the computation of $\delta = \lambda_{\max}$ itself is typically faster.

3.4. Evaluation of residues

With the construction of the canonical basis $\mathcal{C} = \{c_1(z), \dots, c_D(z)\}$ and of the corresponding dual basis $\mathcal{D} = \{d_1(z), \dots, d_D(z)\}$ of the quotient ring Q , along with the partition-of-unity polynomials $e_i(z)$, all ingredients are now in place to compute the residues of any given rational n -form,

$$\omega = \frac{h(z) dz_1 \wedge \dots \wedge dz_n}{P_1(z) \dots P_n(z)}. \quad (45)$$

The key point is that the residue $\text{Res}_I : Q \rightarrow \mathbb{C}$ defines a symmetric non-degenerate inner product on Q ,

$$\langle h_1, h_2 \rangle \equiv \text{Res}_I(h_1 h_2), \quad (46)$$

and that \mathcal{D} is dual to \mathcal{C} with respect to this inner product,

$$\text{Res}_I(c_a d_b) = \delta_{ab}, \quad (47)$$

cf. ref. [16].

Thus, if we decompose the numerator $h(z)$ of eq. (45) in the canonical basis,

$$h(z) = \lambda_1 c_1(z) + \cdots + \lambda_D c_D(z), \quad (48)$$

and decompose the constant 1 in the dual basis,

$$1 = \mu_1 d_1(z) + \cdots + \mu_D d_D(z), \quad (49)$$

then we can compute the global residue (cf. eq. (23)) of ω wrt. I as the dot product of the coefficient vectors,

$$\operatorname{Res}_I \omega \equiv \operatorname{Res}_I h(z) = \operatorname{Res}_I (h(z) \cdot 1) = \operatorname{Res}_I \left(\sum_{a=1}^D \lambda_a c_a \sum_{b=1}^D \mu_b d_b \right) \quad (50)$$

$$= \sum_{a,b=1}^D \lambda_a \mu_b \operatorname{Res}_I (c_a d_b) = \sum_{a=1}^D \lambda_a \mu_a. \quad (51)$$

This prescription allows us to compute the global residue of ω wrt. I , i.e. the sum of the residues at all poles in the associated variety $V = \{p_1, \dots, p_m\}$. To compute the residue at any given pole p_i , we utilize the corresponding partition-of-unity polynomial e_i ,

$$\operatorname{Res}_{I, z=p_i} \omega = \operatorname{Res}_I (\omega e_i). \quad (52)$$

4. Example of residue computation

In this section we aim to apply the theory explained in section 3 to an example. Thus, let us consider the differential form

$$\omega = \frac{(z_1 - z_2) dz_1 \wedge dz_2}{z_1^2 (\chi z_1 + 1)^2 z_2^3 (z_2 - 1)}, \quad (53)$$

where χ is considered as a parameter. As ω depends on $n = 2$ variables, we must partition the denominator into two distinct factors, cf. the discussion in section 2. We will consider the ideal

$$I = \langle P_1(z), P_2(z) \rangle = \langle z_1^2 (z_2 - 1), (\chi z_1 + 1)^2 z_2^3 \rangle. \quad (54)$$

This has the associated variety

$$V = \{p_1, p_2\} = \left\{ (0, 0), \left(-\frac{1}{\chi}, 1\right) \right\}. \quad (55)$$

As V is finite, I is zero-dimensional, so that ω has well-defined residues at the poles in V .

4.1. Computation of canonical basis of Q

We choose as the monomial order \prec degree lexicographic order. The Gröbner basis is found to be,⁴

$$G = \{z_1^2 z_2 - z_1^2, -2\chi^3 z_1^3 - 3\chi^2 z_1^2 + z_2^3, z_2^4 - z_2^3, 2\chi z_1 z_2^3 + \chi^2 z_1^2 + z_2^3\}. \quad (56)$$

The leading terms of these elements are,

$$\text{LT}(G) = \{z_1^2 z_2, -2\chi^3 z_1^3, z_2^4, 2\chi z_1 z_2^3\}, \quad (57)$$

whose corresponding exponent vectors are

$$\{(2, 1), (3, 0), (0, 4), (1, 3)\}. \quad (58)$$

From these exponent vectors we can now proceed to construct the set E defined in eq. (26). In the case at hand, the construction is made more transparent with the lattice illustration in fig. 1.

Hence we find the following exponent vectors of the canonical basis elements,

$$E = \{(1, 2), (0, 3), (2, 0), (1, 1), (0, 2), (1, 0), (0, 1), (0, 0)\}. \quad (59)$$

Thus, the canonical basis takes the following form,

$$\mathcal{C} = (z_1 z_2^2, z_2^3, z_1^2, z_1 z_2, z_2^2, z_1, z_2, 1). \quad (60)$$

We remark that different choices of monomial order lead to different canonical bases (but of course with the same number of elements). For example, if we choose lexicographic monomial order, we find the following canonical basis,

$$\mathcal{C}_{\text{lex}} = (z_1 z_2^3, z_1 z_2^2, z_1 z_2, z_1, z_2^3, z_2^2, z_2, 1). \quad (61)$$

4.2. Computation of dual basis of Q

To construct the dual basis of Q wrt. \mathcal{C} , we start by computing the Bezoutian matrix of the ideal generators $P_i(z)$ in eq. (54). We find, cf. eq. (28),

$$\text{Bez}(z, y) = \begin{pmatrix} (y_1 + z_1)(z_2 - 1) & y_1^2 \\ \chi(\chi(y_1 + z_1) + 2)z_2^3 & (\chi y_1 + 1)^2 (y_2^2 + z_2 y_2 + z_2^2) \end{pmatrix}. \quad (62)$$

⁴As I has parameters, the Gröbner basis must be computed with the `CoefficientDomain->RationalFunctions` option.

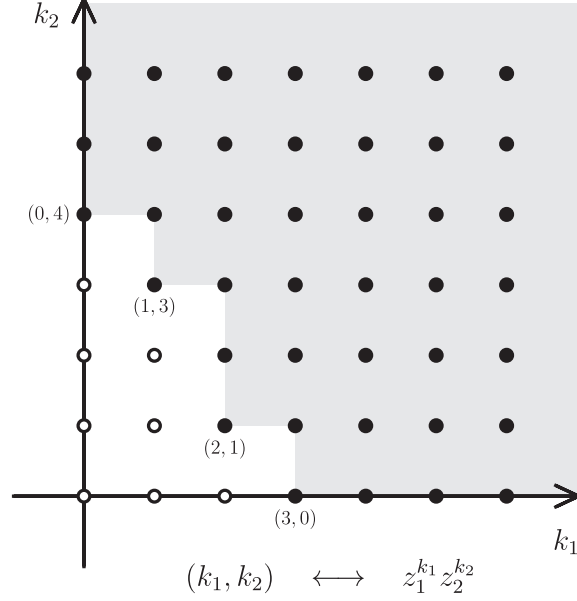


Figure 1: The labeled lattice points are the exponent vectors of the leading term of each element of the Gröbner basis G in eq. (56). The lattice points indicated with white circles are the elements of the set E defined in eq. (26). These elements define exponent vectors of the canonical basis elements.

The Bezoutian determinant is thus,

$$\begin{aligned}
\mathcal{B}(z, y) = & \chi^2 y_1^3 (y_2^2 (z_2 - 1) + y_2 (z_2 - 1) z_2 - z_2^2) \\
& + y_1 (z_2 - 1) (y_2 z_2 + y_2^2 + z_2^2) (2\chi z_1 + 1) + z_1 (z_2 - 1) (y_2 z_2 + y_2^2 + z_2^2) \\
& + \chi y_1^2 (y_2^2 (z_2 - 1) + y_2 (z_2 - 1) z_2 - z_2^2) (\chi z_1 + 2). \tag{63}
\end{aligned}$$

We now perform polynomial division of $\mathcal{B}(z, y)$ with respect to the Gröbner basis G , where the elements are taken as polynomials in the z_i variables.

This produces the following decomposition, cf. eq. (30),

$$(q_1, q_2, q_3, q_4) = (0, 0, 0, \frac{1}{2\chi} + y_1) \quad (64)$$

$$\begin{aligned} \mathcal{B}_Q(z, y) = (2\chi + \chi^2 y_1) & \left[y_1^2 (y_2^2 (z_2 - 1) + y_2 (z_2 - 1) z_2 - z_2^2) \right. \\ & \left. + y_1 z_1 (y_2^2 (z_2 - 1) + y_2 (z_2 - 1) z_2 - z_2^2) - z_1^2 \right] \\ & + (y_1 + z_1) (y_2^2 (z_2 - 1) + y_2 (z_2 - 1) z_2 - z_2^2) - \frac{z_2^3}{2\chi} + \frac{3}{2} \chi z_1^2. \end{aligned} \quad (65)$$

Next we perform polynomial division of the remainder $\mathcal{B}_Q(z, y)$ with respect to G , where the elements are now taken as polynomials in the y_i variables, $P_i(z) \rightarrow P_i(y)$. This produces the decomposition in eq. (31) with

$$(\widehat{q}_1, \widehat{q}_2, \widehat{q}_3, \widehat{q}_4) = \left(\chi(z_2 - 1)(y_2 + z_2 + 1)(\chi(y_1 + z_1) + 2), \frac{1}{2\chi}, 0, 0 \right), \quad (66)$$

where the remainder is $\mathcal{B}_{Q \otimes Q}(z, y) = c_1(z)d_1(y) + \dots + c_D(z)d_D(y)$, with the dual basis elements taking the following form,

$$\begin{aligned} \mathcal{D} = & \left((z_2 - 1)(2\chi z_1 + 1), -\frac{1}{2\chi}, -\chi^2 z_1 - \frac{\chi}{2}, z_2(z_2 - 1)(2\chi z_1 + 1), \right. \\ & \left. z_1(z_2 - 1), -\chi^2 z_1^2 - 2\chi z_1 z_2^2 - z_2^2, z_1 z_2(z_2 - 1), -z_1 z_2^2 - \frac{1}{2\chi} z_2^3 - \frac{\chi}{2} z_1^2 \right), \end{aligned} \quad (67)$$

where we have expressed the elements as functions of the z_i variables.

4.3. Constructing partition-of-unity polynomials

To construct the partition-of-unity polynomials in eq. (37), our first aim is to find a linear form which maps the elements p_i of the variety (55) to distinct values. We observe that the linear form

$$\ell(z_1, z_2) = z_1 + z_2, \quad (68)$$

has this property. From eq. (34) we then obtain the following Lagrange polynomials,

$$(L_1(z), L_2(z)) = \left(\frac{\chi(z_1 + z_2 - 1) + 1}{1 - \chi}, \frac{\chi(z_1 + z_2)}{\chi - 1} \right), \quad (69)$$

and we observe that $L_i(p_j) = \delta_{ij}$, as desired.

Our next aim is to compute δ as defined in eq. (39). To this end, we first compute M , defined in eq. (38) as the intersection of the ideals associated with each pole of the variety. In the case at hand, the ideals associated with each pole of the variety (55) are

$$J\langle\{p_1\}\rangle = \langle z_1, z_2 \rangle \quad (70)$$

$$J\langle\{p_2\}\rangle = \left\langle z_1 + \frac{1}{\chi}, z_2 - 1 \right\rangle. \quad (71)$$

Accordingly, we introduce the parameter t and compute the Gröbner basis of

$$\langle tz_1, tz_2, (1-t)(z_1 + \frac{1}{\chi}), (1-t)(z_2 - 1) \rangle \quad (72)$$

wrt. lexicographic monomial order and the variable order (t, z_1, z_2) . We find $\{z_1 + \frac{z_2}{\chi}, z_2^2 - z_2, z_2 + t - 1\}$. The elements which do not contain the parameter t then form a basis of $J\langle\{p_1\}\rangle \cap J\langle\{p_2\}\rangle$,

$$M = \left\langle z_1 + \frac{z_2}{\chi}, z_2^2 - z_2 \right\rangle. \quad (73)$$

Now, to determine δ , we start by checking whether $M \subseteq I$. Polynomial division of the elements of eq. (73) wrt. the Gröbner basis of I in eq. (56) leaves remainders identical to the original elements.

Thus, we proceed to consider M^2 . Taking the products of the generators in eq. (73) and performing polynomial division we find

$$M^2 = \left\langle (z_1 + \frac{z_2}{\chi})^2, (z_2^2 - z_2)(z_1 + \frac{z_2}{\chi}), z_2^2 - z_2^3 \right\rangle. \quad (74)$$

As the remainders are non-zero, we proceed to consider M^3 . Taking the products of the generators in eqs. (73) and (74) and performing polynomial division we find

$$M^3 = \left\langle \frac{3}{2} \left(\frac{z_2^3}{\chi^3} + \frac{2z_1z_2^2}{\chi^2} + \frac{z_1^2}{\chi} \right), -\frac{z_2^3}{\chi^2} - \frac{2z_1z_2^2}{\chi} - z_1^2, -\frac{z_2^3}{\chi^2} - \frac{2z_1z_2^2}{\chi} - z_1^2, \right. \\ \left. \frac{1}{2} \left(\frac{z_2^3}{\chi} + \chi z_1^2 + 2z_1z_2^2 \right), \frac{1}{2} \left(\frac{z_2^3}{\chi} + \chi z_1^2 + 2z_1z_2^2 \right), 0 \right\rangle. \quad (75)$$

As the remainders are non-zero, we proceed to consider M^4 . Taking the products of the generators in eqs. (73) and (75) and performing polynomial division we find

$$M^4 = \langle 0 \rangle. \quad (76)$$

Hence we conclude that $\delta = 4$.

Alternatively, we may use the maximum pole multiplicity as a value for δ . To this end we compute the matrix M_ℓ of $P(z) \mapsto \ell(z)P(z)$ for the linear form in eq. (68). Using the canonical basis $\mathcal{C} = (c_1(z), \dots, c_8(z))$ in eq. (60), we find

$$M_\ell = \begin{pmatrix} 0 & -\frac{1}{2\chi} & 1-\frac{\chi}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\frac{1}{2\chi} & -\frac{\chi}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2\chi^3} & 1-\frac{3}{2\chi} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad (77)$$

i.e., so that $\ell(z)c_i(z) \equiv \sum_{j=1}^8 (M_\ell)_{ij}c_j(z) \pmod{I}$. From this we find

$$\det(M_\ell - \lambda I) = (\lambda - \ell(p_1))^6 (\lambda - \ell(p_2))^2. \quad (78)$$

We conclude that the poles p_1 and p_2 have the multiplicities 6 and 2, respectively. In particular, $\lambda_{\max} = 6$.

Plugging the value $\delta = 4$ from eq. (76) into eq. (37) with the Lagrange polynomials given in eq. (69) and performing polynomial division wrt. G ,⁵ we find⁶

$$e_1(z) = 1 - z_2^3 \quad \text{and} \quad e_2(z) = z_2^3. \quad (79)$$

It is straightforward to check that these polynomials indeed have the properties stated in eqs. (35)–(36).

Now, to compute the residues at each pole p_i in the variety (55), we utilize the partition-of-unity polynomials computed in eq. (79) and consider the numerator of eq. (53), i.e. $h(z) = z_1 - z_2$, multiplied by these polynomials,

$$\begin{aligned} h(z)e_1(z) &= (z_1 - z_2)(1 - z_2^3) \equiv \frac{(2\chi+1)}{2\chi}z_2^3 + \frac{\chi}{2}z_1^2 + z_1 - z_2 \pmod{I}, \\ h(z)e_2(z) &= (z_1 - z_2)z_2^3 \equiv -\frac{2\chi+1}{2\chi}z_2^3 - \frac{\chi}{2}z_1^2 \pmod{I}. \end{aligned} \quad (80)$$

⁵To calculate a desired power of a polynomial $L_i(z)^\delta$ in the quotient ring Q , MULTIVARIATERESIDUES performs polynomial division wrt. G after taking each product $L_i^k = L_i^{k-1}L_i$. This guarantees that each power computed in the intermediate stages has $D = \dim Q$ terms rather than D^k terms, thereby minimizing intermediate expression swell.

⁶The value $\delta = \lambda_{\max} = 6$ obtained from eq. (78) produces identical results for $e_1(z)$ and $e_2(z)$.

We proceed to decompose these in the canonical basis \mathcal{C} given in eq. (60), finding the following coefficient vectors,

$$\Lambda_1 = \left(0, \frac{2\chi+1}{2\chi}, \frac{\chi}{2}, 0, 0, 1, -1, 0\right), \quad (81)$$

$$\Lambda_2 = \left(0, -\frac{2\chi+1}{2\chi}, -\frac{\chi}{2}, 0, 0, 0, 0, 0\right), \quad (82)$$

i.e., where $h(z)e_i(z) \equiv \Lambda_i \cdot \mathcal{C} \pmod{I}$. Moreover, the constant 1 may be composed in the dual basis \mathcal{D} in eq. (67) as $1 = \mu \cdot \mathcal{D}$ where,

$$\mu = (0, -2\chi, 0, 0, 0, 0, 0, 0). \quad (83)$$

Thus we find for the residues,

$$\begin{aligned} \operatorname{Res}_{I, z=p_1} \omega &= \Lambda_1 \cdot \mu = -2\chi - 1, \\ \operatorname{Res}_{I, z=p_2} \omega &= \Lambda_2 \cdot \mu = 2\chi + 1. \end{aligned} \quad (84)$$

5. Applications of multivariate residues

In this section we give one application of multivariate residues, namely the computation of generalized-unitarity cuts. These are important in several contexts, for example unitarity calculations of loop amplitudes and the Grassmannian formulation of the S-matrix by Arkani-Hamed et al.

Here we focus on unitarity calculations. To provide some context, let us consider a one-loop scattering amplitude in a generic gauge theory. From standard reduction techniques it can be shown that there is a finite basis of one-loop integrals in which the amplitude can be expanded,

$$A_n^{(1)} = \sum_{\text{boxes}} c_{\square} I_{\square} + \sum_{\text{triangles}} c_{\triangle} I_{\triangle} + \sum_{\text{bubbles}} c_{\circ} I_{\circ} + \sum_{\text{tadpoles}} c_{-\circ} I_{-\circ} + \text{rational terms}, \quad (85)$$

where I_{\square} , I_{\triangle} , I_{\circ} and $I_{-\circ}$ represent box, triangle, bubble and tadpole integrals, respectively. As all these integrals are known, this decomposition reduces the computation of $A_n^{(1)}$ to the computation of the expansion coefficients.

The coefficient c_{\square} of the box integral

$$I_{\square} = \int_{\mathbb{R}^D} \frac{d^D \ell}{(2\pi)^D} \frac{1}{\prod_{i=1}^4 p_i^2(\ell)}, \quad (86)$$

can now be computed by replacing the integration contour \mathbb{R}^D in eq. (85) by the contour

$$T_\epsilon^4 = \{\ell \in \mathbb{C}^4 : |p_i^2(\ell)| = \epsilon_i, \quad i = 1, \dots, 4\}. \quad (87)$$

The replacement of contour $\mathbb{R}^D \rightarrow T_\epsilon^4$ has the effect of computing the residue at the poles where all four propagators $p_i^2(\ell)$ of I_\square go on shell. All terms missing any one of the propagators therefore vanish, and eq. (85) becomes [21]

$$c_\square = \frac{1}{2} \sum_{a \in \{L, L^\bullet\}} \sum_{\substack{\text{helicities,} \\ \text{species}}} \prod_{i=1}^4 A_i^{\text{tree}}(p_i(a), p_{i+1}(a)), \quad (88)$$

where L, L^\bullet denote the solutions to $p_i^2(\ell) = 0$ for $i = 1, \dots, 4$, and A_i^{tree} are the tree amplitudes arising from literally cutting the propagators of the box graph.

More to the point, though the integral $\int_{T_\epsilon^4} \frac{d^D \ell}{(2\pi)^D} \frac{1}{\prod_{i=1}^4 p_i^2(\ell)}$ is a multivariate residue, it is non-degenerate and can thus be computed directly from eq. (5). However, starting at two loops, degenerate residues are generic, and the algorithms explained in sections 2.1 and 3 become necessary to evaluate them. To give an explicit example, we consider the generalized-unitarity cut shown in figure 2.

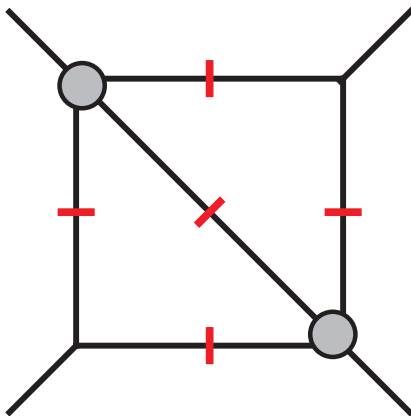


Figure 2: A generalized-unitarity cut where the propagators with superimposed red lines have been put on shell. The gray blobs represent tree amplitudes.

It turns out that the five on-shell constraints can be integrated out as a non-degenerate residue. However, the residues of the resulting integrand

in the remaining $2 \times 4 - 5 = 3$ variables will generically be degenerate: for example, the residue with respect to the ideal I (where χ denotes the ratio t/s of Mandelstam invariants),

$$I = \langle z_1^2, z_2, -\chi z_1 + \chi z_3 z_1 + z_3 z_1 + z_2 z_3 - z_3 \rangle, \quad (89)$$

at $(z_1, z_2, z_3) = (0, 0, 0)$ is degenerate, and the algorithms explained in sections 2.1 and 3 are required to compute the residue.

We remark that multivariate residues also play an important role in elimination theory in the context of solving systems of multivariate polynomial equations [16].

6. Manual

The package `MultivariateResidues.m` can be obtained from ref. [22]. At the beginning of a Mathematica session, the package can be loaded with

```
SetDirectory[NotebookDirectory[]];
<< "MultivariateResidues"
```

where it is assumed that the package and the notebook are located in the same directory. The newly available definitions can be shown by running

```
?MultivariateResidues*
```

The package defines one new function, called `MultivariateResidue`. Below we give a brief introduction to this function and its options.

6.1. New function

The function `MultivariateResidue` computes a multivariate residue, based on the algorithms described in this paper. It has the following syntax:

```
MultivariateResidue[Num, {d[1], ..., d[n]},
                    {z[1] -> z[1,1], ..., z[n] -> z[n,1]}
```

which returns the multivariate residue of $\text{Num}/(d_1 d_2 \cdots d_n)$ at the location given by $(z_1, \dots, z_n) = (z_1^{(1)}, \dots, z_n^{(1)})$. Alternatively,

```
MultivariateResidue[Num, {d[1], ..., d[n]}, {z[1], ..., z[n]},
                    {{z[1,1], ..., z[n,1]}, {z[1,2], ..., z[n,2]}, ...}]
```

returns a list of multivariate residues of $\text{Num}/(d_1 d_2 \cdots d_n)$ at the collection of points $(z_1, \dots, z_n) \in \{(z_1^{(1)}, \dots, z_n^{(1)}), (z_1^{(2)}, \dots, z_n^{(2)}), \dots\}$. This second syntax is better suited for the computation of several residues, because it exploits the fact that part of the computation is common to all poles.

In the univariate case, `MultivariateResidue` is equivalent to the native Mathematica function `Residue`. For instance,

```
MultivariateResidue[f[z], {z}, {z -> 0}]
Residue[f[z]/z, {z, 0}]
Out: f[0]
Out: f[0]
```

As a multivariate example, let us compute the residues considered in section 2.1. Taking $\varphi_1, \varphi_2, \varphi_3$ from eq. (10), we can straightforwardly compute the residues with respect to the ideals given in eq. (11) as follows:

```
f[1] = z[2];
f[2] = a[1] z[1] + a[2] z[2];
f[3] = b[1] z[1] + b[2] z[2];
MultivariateResidue[z[1], {f[1], f[2]*f[3]}, {z[1] -> 0, z[2] -> 0}]
MultivariateResidue[z[1], {f[2], f[3]*f[1]}, {z[1] -> 0, z[2] -> 0}]
MultivariateResidue[z[1], {f[3], f[1]*f[2]}, {z[1] -> 0, z[2] -> 0}]
Out: -(1/(a[1] b[1]))
Out: -(a[2]/(a[1] (-a[2] b[1] + a[1] b[2])))
Out: -(b[2]/(b[1] (a[2] b[1] - a[1] b[2])))
```

When using the second syntax, the input list of poles does not have to contain all points in the variety defined by the ideal (i.e., the set of points where all denominator factors vanish). Moreover, it may contain additional points (although the corresponding residues will be zero). As an example, consider the following ideal,

```
Ideal = {z[1]^2 (1 + z[1] - z[2]), z[3]^3,
         z[2]^3 (-1 - z[1] - a z[1] + z[2] + a z[1] z[3])};
Parameters = {a};
Vars = {z[1], z[2], z[3]};
```

The resulting variety contains three points,

```
Variety = Map[Last, Sort[Solve[Ideal == 0, Vars]], {2}]
Out: {{-1, 0, 0}, {0, 0, 0}, {0, 1, 0}}
```

We may ask for the residue computed at precisely the poles contained in the variety, or alternatively a subset, or alternatively a subset with an added random point,

```
MultivariateResidue[1, Ideal, Vars, Variety]
MultivariateResidue[1, Ideal, Vars, Variety[[1 ;; 2]]]
MultivariateResidue[1, Ideal, Vars, Join[Variety[[1 ;; 2]], {{0, 0,
1}}]]
Out: {-(6/a), 0, 6/a}
Out: {-(6/a), 0}
Out: {-(6/a), 0, 0}
```

In fact, the algorithm based on the dual structure of the quotient ring requires the computation of the residues of all poles in the variety. This particular algorithm will therefore internally compute the variety itself and return the residues at the (sub)set of poles that are specified by the user.

6.2. Options

The following options can be specified in `MultivariateResidue`:

- **Method**: the type of algorithm used for computing multivariate residues. Possible settings for this option are "TransformationFormula" (default) and "QuotientRingDuality", described in sections 2.1 and 3, respectively.
- **CoefficientDomain**: the type of objects assumed to be coefficients of monomials in the computation of Gröbner bases. Possible settings for this option are `InexactNumbers`, `Rationals` and `RationalFunctions` (default).
- **MonomialOrder**: the criterion used for monomial ordering in Gröbner basis computation and polynomial division. This option only has an effect when using `Method -> "QuotientRingDuality"`. Possible values are `Lexicographic` (default), `DegreeLexicographic` and `DegreeReverseLexicographic`.
- **FindMinimumDelta**: compute and use the smallest possible δ entering the partition-of-unity polynomials $e_i = 1 - (1 - L_i^\delta)^\delta$, which are needed to compute residues at all the finite poles of the variety. Possible settings are `True` (default) and `False`. Specifying `True` will compute and use the smallest possible δ ; specifying `False` will drop the calculation of a minimal δ and use a generally larger value (namely the maximum pole multiplicity).

6.3. Global Settings

The following global settings can be specified in a Mathematica session that makes use of the package (the settings can be altered after loading the

package and will impact all subsequent `MultivariateResidue` evaluations):

- `$MultiResInputChecks`: check the input of `MultivariateResidue`. It checks for instance whether the ideal is zero-dimensional. Possible values are `True` (default) and `False`. It can be switched off to improve efficiency.
- `$MultiResInternalChecks`: perform internal cross-checks, for instance on the correctness of the transformation matrix $A(z)$ and the partition of unity $e(i)$. Possible values are `True` (default) and `False`. Similarly, this can also be switched off to improve efficiency.
- `$MultiResUseSingular`: use Singular for computations of Gröbner bases and polynomial division. Possible settings are `True` and `False` (default). When set to `True`, one must specify the path to the Singular executable in the variable `$MultiResSingularPath`. Singular is a computer algebra system dedicated to computational algebraic geometry that can outperform Mathematica for complicated calculations (see section 6.4) and therefore merits an interface between the two programs. Singular is free software under the GNU General Public Licence. It may be obtained from <https://www.singular.uni-kl.de>.
- `$MultiResSingularPath`: the path to Singular. By default the path is set to `"/usr/bin/Singular"`.

6.4. Performance

The speed performance of `MultivariateResidue` depends on the selected options and global settings; in particular the options `Method`, `MonomialOrder` and `$MultiResUseSingular`. This section demonstrates the impact of these options through a few explicit examples.

The default method `"TransformationFormula"` is typically the best choice for simple problems, whereas the sophisticated `"QuotientRingDuality"` can offer speed improvements in more involved computations. For instance, the default method is the fastest method for the residue computation with the simple ideal defined in section 6.1,

```
First[AbsoluteTiming[MultivariateResidue[1, Ideal, Vars, Variety, Method
-> "TransformationFormula"]]]
First[AbsoluteTiming[MultivariateResidue[1, Ideal, Vars, Variety, Method
-> "QuotientRingDuality"]]]
Out: 0.222012
Out: 1.868446
```


On the other hand, if we consider an example with more variables, then the default method becomes less efficient (due to the costly computation of the transformation matrix), and one should opt for "QuotientRingDuality",

```
n = 10;
Ideal = Table[z[i], {i, 1, n}];
Vars = Table[z[i], {i, 1, n}];
Variety = {Table[0, {n}]}];
First[AbsoluteTiming[MultivariateResidue[1, Ideal, Vars, Variety, Method
-> "TransformationFormula"]]]
First[AbsoluteTiming[MultivariateResidue[1, Ideal, Vars, Variety, Method
-> "QuotientRingDuality"]]]
Out: 1.003684
Out: 0.134473
```

Another circumstance under which "QuotientRingDuality" performs better than "TransformationFormula" is when points in the variety are high-degree rational functions of one or more parameters. In such cases, the computation of Gröbner bases in "TransformationFormula" can be rather slow. The following example with two complex variables illustrates this point.

Consider first the following relatively simple variety,

```
Ideal = {1 + c[1] z[1] + c[2] z[2], 1 + c[3] z[1] + c[4] z[2]};
Vars = {z[1], z[2]};
Variety = Map[Last, Sort[Solve[Ideal == 0, Vars]], {2}]
Out: {{-(c[2] - c[4])/(c[2] c[3] - c[1] c[4])},
      -((-c[1] + c[3])/(c[2] c[3] - c[1] c[4]))}}
```

Since this variety is a simple rational polynomial of the coefficients $c(i)$, both methods take roughly the same amount of time,

```
First[AbsoluteTiming[MultivariateResidue[1, Ideal, Vars, Variety, Method
-> "TransformationFormula"]]]
First[AbsoluteTiming[MultivariateResidue[1, Ideal, Vars, Variety, Method
-> "QuotientRingDuality"]]]
Out: 0.143496
Out: 0.043382
```

Polynomial degree p	2	3	4	5	6	7	8
$T_{\text{TransformationFormula}}$ (sec)	0.100	0.149	0.269	0.779	1.669	2.375	8.898
$T_{\text{QuotientRingDuality}}$ (sec)	0.037	0.049	0.061	0.078	0.086	0.087	0.105
Ratio	2.7	3.0	4.4	10.	19.	27.	85.

Table 1: Timing (in seconds) of the computation of multivariate residues with the two methods, as a function of the degree p of the polynomials $c(i)$. The method "TransformationFormula" shows an exponential increase in computation time, whereas "QuotientRingDuality" exhibits a linear trend.

However, once we substitute the $c(i)$ with polynomials of a single parameter a , then the default method becomes much slower,

```

c[1] = 1 + 6 a + 6 a^2 + 9 a^3 + 6 a^4 + 3 a^5 + 9 a^6 + 4 a^7 + 4 a^8;
c[2] = 1 + 7 a + 2 a^2 + 3 a^3 + 7 a^4 + 9 a^5 + 8 a^6 + 8 a^7 + 5 a^8;
c[3] = 1 + 2 a + 4 a^2 + 7 a^3 + 2 a^4 + 2 a^5 + 4 a^6 + 4 a^7 + 8 a^8;
c[4] = 1 + 8 a + 3 a^2 + 8 a^3 + 9 a^4 + 2 a^5 + 3 a^6 + 8 a^7 + 5 a^8;
First[AbsoluteTiming[MultivariateResidue[1, Ideal, Vars, Variety, Method
-> "TransformationFormula"]]]
First[AbsoluteTiming[MultivariateResidue[1, Ideal, Vars, Variety, Method
-> "QuotientRingDuality"]]]
Out: 8.898075
Out: 0.104702

```

Indeed, the more sophisticated method "QuotientRingDuality" shows an excellent behavior with respect to the maximum degree p in the polynomials $c(i) = \sum_{n=1}^p r_{i,n} a^n$, which is evident from table 1. The sharp increase in computation time in "TransformationFormula" is due to the increased complexity of the Gröbner basis computation in the determination of the transformation matrix. In such cases it is beneficial to let the Gröbner basis computation be performed by Singular. For instance, using the same degree 8 polynomials $c(i)$ as above, "TransformationFormula" becomes much faster in conjunction with Singular,

```

$MultiResUseSingular = True;
First[AbsoluteTiming[MultivariateResidue[1, Ideal, Vars, Variety, Method
-> "TransformationFormula"]]]
Out: 0.295190

```

The method "QuotientRingDuality" allows the use of various monomial orderings in the computation of Gröbner bases, which can impact the speed of subsequent polynomial reductions. Considering once again the ideal defined in section 6.1 and computing the multivariate residue using three different monomial orderings,

```

Ideal = {z[1]^2 (1 + z[1] - z[2]), z[3]^3,
         z[2]^3 (-1 - z[1] - a z[1] + z[2] + a z[1] z[3])};
Vars = {z[1], z[2], z[3]};
Variety = Map[Last, Sort[Solve[Ideal == 0, Vars]], {2}];
First[AbsoluteTiming[MultivariateResidue[1, Ideal, Vars, Variety, Method
  -> "QuotientRingDuality", MonomialOrder -> #]]] & /@ {Lexicographic,
DegreeLexicographic, DegreeReverseLexicographic}
Out: {6.017642, 1.938357, 2.157330}

```

shows that the options DegreeLexicographic and DegreeReverseLexicographic are noticeably faster than Lexicographic. The reason for the speed difference (which becomes more pronounced upon raising the powers of factors in the ideal) is that the Lexicographic monomial ordering produces a Gröbner basis for the ideal which contains only four polynomials (with 2, 3, 4 and 16 terms), whereas the other two monomial orderings produce a Gröbner basis for this ideal with eight polynomials (with 2, 2, 3, 3, 4, 5, 16 and 19 terms). The polynomial reduction is faster when using the larger Gröbner basis.

Finally, it is worth mentioning that multivariate residue calculations can be simplified – where possible – by appropriate use of partial fractioning. Suppose one computes the residues of $R(z_1, z_2) = (z_1 f_1 + z_2 f_3)/(f_1 f_2 f_3 f_4)$, where $f_i = a_i z_1 + b_i z_2$, with respect to the ideal $I(z_1, z_2) = \langle f_1 f_2, f_3 f_4 \rangle$. A direct computation of the residue yields

```

f[i_] := a[i] z[1] + b[i] z[2];
Res[1] = MultivariateResidue[z[1] f[1] + z[2] f[3], {f[1] f[2], f[3] f
[4]}, {z[1] -> 0, z[2] -> 0}]
Out: -((a[3] a[4] b[2] + a[4] b[1] b[2] - a[2] a[4] b[3] - a[1] b[2] b
[4])/((a[3] b[2] - a[2] b[3]) (a[4] b[1] - a[1] b[4]) (a[4] b[2] - a[2]
b[4])))

```

Partial fractioning the rational function produces two terms, $R(z_1, z_2) = z_1/(f_2 f_3 f_4) + z_2/(f_1 f_2 f_4)$. The residue of each of these terms should be computed with respect to the corresponding reduced ideals $\langle f_2, f_3 f_4 \rangle$ and $\langle f_1 f_2, f_4 \rangle$. The sum of these two residues reproduces the result of the direct computation,

```

Res[2, 1] = MultivariateResidue[z[1], {f[2], f[3] f[4]}, {z[1] -> 0, z
[2] -> 0}]
Res[2, 2] = MultivariateResidue[z[2], {f[1] f[2], f[4]}, {z[1] -> 0, z
[2] -> 0}]
Simplify[Res[1] == Res[2, 1] + Res[2, 2]]
Out: -(b[2]/((a[3] b[2] - a[2] b[3]) (a[4] b[2] - a[2] b[4])))
Out: -(a[4]/((a[4] b[1] - a[1] b[4]) (a[4] b[2] - a[2] b[4])))
Out: True

```

and is faster than the direct computation (in this case 0.0844 seconds versus 0.4549 seconds).

7. Conclusions

In this paper we have introduced the Mathematica package `MULTIVARIATERESIDUES` for the evaluation of multivariate residues. The implementation can be used to compute any multivariate residue of any rational form, including the case where the denominator ideal involves parameters (corresponding, for example, to Lorentz invariants of external momenta in a scattering process).

We have implemented two different algorithms for the computation of residues, one ("`TransformationFormula`") based on the transformation formula and one ("`QuotientRingDuality`") exploiting that the residue map defines a non-degenerate inner product on the quotient ring.

We have applied our code to 6500 examples arising in the computation of generalized-unitarity cuts of two-loop scattering amplitudes. From doing so, we have observed the following patterns regarding the relative performance of the two algorithms. In cases where the ideals involve only few parameters, and the expressions of the parameters are polynomials of low degree, "`TransformationFormula`" is the faster option. Moreover, when the cumulative computation time of "`TransformationFormula`" is plotted against the range of ideals over which it has been applied, it displays a more linear profile than "`QuotientRingDuality`", showing that the underlying computational procedure is rather insensitive to the precise structure of the ideal.

In contrast, "`QuotientRingDuality`" tends to perform better in more involved computations because this circumvents the need to compute the transformation matrix $A(z)$ in eq. (6). This becomes increasingly apparent in cases

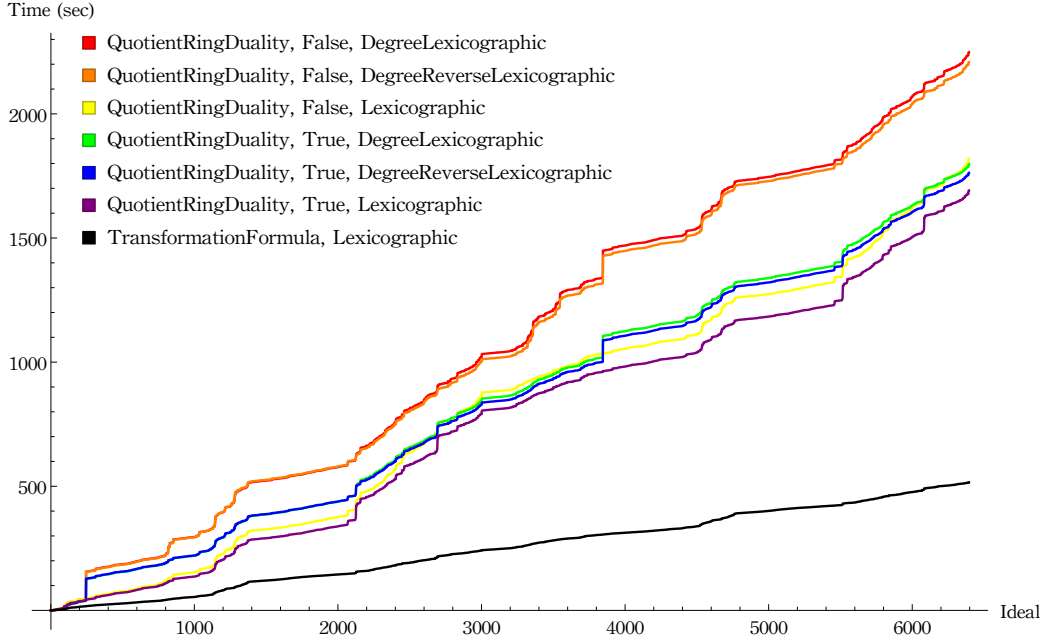


Figure 3: The cumulative evaluation time of residues (vertical axis) over the set of 6395 ideals (horizontal axis) associated with the pentacut of a generic gauge theory two-loop four-point integrand for the different residue computation options in the package MULTIVARIATERESIDUES. The legend items are ordered by descending total computation time. The plateaus present in "QuotientRingDuality" show that the underlying computational procedure is rather sensitive to the precise structure of the ideal. In contrast, "TransformationFormula" exhibits a more linear profile, showing that the underlying computational procedure is rather insensitive to the precise structure of the ideal. (Color online. These timings were obtained on a single laptop with 8 GB RAM and 1.8 GHz Intel Core i5 processor.)

with several variables, or with several parameters, or whenever ideals involve high-degree polynomials in the parameters.

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Appendix A. Topology of multivariate residues

In this appendix we aim to explain the underlying topological reason why the value of a multivariate residue is not uniquely determined by the pole enclosed by the integration cycle, but also depends on the cycle. To gain a concrete understanding, we will consider the example of the differential form ω in eq. (9). The presentation here is largely based on that of section II B of ref. [6] by one of the present authors.

As computed in section 2.1, the differential form (9) has the three distinct residues at $p = (0, 0)$ given in eqs. (17)–(19). As we will see shortly, this is reflected in the fact that there are several distinct integration cycles based at p which yield distinct residues. The higher-dimensional situation is thus quite different from contour integration in one complex variable, where a contour either encloses a pole or doesn’t, and there is a unique value for the residue.

To clarify the situation, we seek to split eq. (9) into terms with two distinct denominator factors. To this end, we make the following change of variables,

$$\begin{aligned} z'_1 &= a_1 z_1 + a_2 z_2 \\ z'_2 &= z_2. \end{aligned} \tag{A.1}$$

After applying this transformation and dropping the primes on z_i , the form then becomes,

$$\omega = \frac{1}{a_1} \left(\frac{1}{z_2(c_1 z_1 + c_2 z_2)} - \frac{a_2}{z_1(c_1 z_1 + c_2 z_2)} \right) dz_1 \wedge dz_2, \tag{A.2}$$

where $c_1 \equiv b_1$ and $c_2 \equiv a_1 b_2 - a_2 b_1$. (This separation is a partial fractioning followed by a change of variables.)

We start by examining the first term of eq. (A.2). The canonical integration contour is a product of two circles. Choosing each circle to be based at

each denominator factor,

$$|z_2| = \delta_2 \tag{A.3}$$

$$|c_1 z_1 + c_2 z_2| = \delta_c, \tag{A.4}$$

where $\delta_2, \delta_c > 0$, the residue of this term is,

$$\frac{1}{a_1 b_1 (2\pi i)^2}, \tag{A.5}$$

independent of the precise values of the radii of the circles. To be a bit more explicit, we can parametrize the integration cycle as

$$\sigma: \begin{cases} z_2 = \delta_2 e^{i\theta_2} \\ c_1 z_1 + c_2 z_2 = \delta_c e^{i\theta_c}, \end{cases} \tag{A.6}$$

so that, as θ_2 and θ_c run over the interval $[0, 2\pi]$, the cycle is traced out. By parametrizing $\int_\sigma \omega$ in this way, the integral can then be evaluated as an ordinary two-fold integral over θ_2 and θ_c .

We now turn to the second term of eq. (A.2). We must now examine whether any choices of the radii δ_2 and δ_c leave the integrand singular on σ , as these would define an *illegitimate* integration cycle. The second denominator factor is of course nonvanishing on the cycle (A.6). For the first factor we obtain from the parametrization (A.6),

$$z_1 = \frac{1}{c_1} (\delta_c e^{i\theta_c} - c_2 \delta_2 e^{i\theta_2}). \tag{A.7}$$

We observe that the first denominator factor will not vanish as long as $\delta_c \neq |c_2| \delta_2$. On the other hand, if $\delta_c = |c_2| \delta_2$, z_1 is guaranteed to vanish for some values of the angles. The illegitimate choice $\delta_c = |c_2| \delta_2$ therefore divides the moduli space (δ_c, δ_2) into two regions,

$$\begin{aligned} (1): & \quad \delta_c > |c_2| \delta_2 \\ (2): & \quad \delta_c < |c_2| \delta_2, \end{aligned} \tag{A.8}$$

which we will consider in turn.

We start by observing from the second equation of eq. (A.6) that the parameter θ_c traces out a circle around the zero of the second denominator factor. For the cycle σ to enclose the pole at $(z_1, c_1 z_1 + c_2 z_2) = (0, 0)$, the

issue is therefore whether the zero of the first denominator factor is encircled by the *other* independent parameter θ_2 . That is, whether for a fixed value of θ_c , the contour (A.7) traced out by θ_2 encloses $z_1 = 0$ or not.

Now, in region (1), for a fixed value of θ_c , the contour (A.7) traced out by θ_2 is a circle centered at $\delta_c e^{i\theta_c}$. As the radius $|c_2|\delta_2$ is less than δ_c , the circle fails to enclose $z_1 = 0$. For the torus σ , this translates into saying that the pole $(z_1, z_2) = (0, 0)$ is sitting at the center of the symmetry plane of σ , but not inside the “tube”. We conclude that in region (1), the second term in eq. (A.2) integrated over the cycle (A.6) produces a vanishing residue.

In contrast, in region (2), the θ_2 -parametrized contour does encircle $z_1 = 0$, and hence the second term in eq. (A.2) integrated over σ produces a nonvanishing residue. In particular, we observe that the residue of eq. (A.2) differs in the two regions (A.8) and thus depends on the relative radii δ_2 and δ_c of the integration cycle.

More generally, let us consider a generic torus,

$$\begin{aligned} z_1 &= \delta_{1,1}e^{i\theta_1} + \delta_{1,2}e^{i\theta_2} \\ z_2 &= \delta_{2,1}e^{i\theta_1} + \delta_{2,2}e^{i\theta_2}, \end{aligned} \tag{A.9}$$

where the $\delta_{i,j}$ are real positive constants which determine the shape of the cycle. For the two-form at hand, we can rescale all the $\delta_{i,j}$ uniformly without loss of generality, so that we only have three independent real parameters.

The integration cycle is legitimate for the first term in eq. (A.2) if and only if $\delta_{2,1} \neq \delta_{2,2}$ and $r_1 \neq r_2$, where

$$r_1 = |c_1\delta_{1,1} + c_2\delta_{2,1}| \quad \text{and} \quad r_2 = |c_1\delta_{1,2} + c_2\delta_{2,2}|. \tag{A.10}$$

The cycle is legitimate for the second term if and only if $\delta_{1,1} \neq \delta_{1,2}$ and $r_1 \neq r_2$. Thus, we must consider eight regions, corresponding to choosing the upper or lower inequality in each of the three relations,

$$\delta_{2,1} \gtrless \delta_{2,2}, \quad \delta_{1,1} \gtrless \delta_{1,2}, \quad r_1 \gtrless r_2. \tag{A.11}$$

We denote the upper choice by ‘+’ and the lower choice by ‘-’. Each region is then labeled by a string of signs. We see that in the region M^{+++} , corresponding to $\delta_{2,1} > \delta_{2,2}$, $\delta_{1,1} > \delta_{1,2}$ and $r_1 > r_2$, the zeros of all denominator factors of the two terms in eq. (A.2) are all encircled by the parameter θ_1 , so that the torus fails to enclose the pole of either term and hence produces a vanishing residue. In M^{++-} , the torus will enclose both terms, and the

residue will be the sum of the two terms' residues. In M^{+-+} , the torus only encloses the second term, and in M^{+--} , the torus only encloses the first term. The remaining four regions are related to these four by flipping all inequalities which leaves the results invariant (up to a sign).

The above analysis shows that multivariate residues are in general not fully characterized by the location of the pole. Rather, the value of the residue depends also on the shape of the cycle enclosing the pole. In the present example we found that the moduli space of allowed integration cycles is divided into several regions. These regions correspond to distinct homology classes of the (z_1, z_2) space

$$\mathbb{C}^2 \setminus \bigcup_{i=1}^3 D_i, \quad (\text{A.12})$$

where each $D_i \equiv \{(z_1, z_2) \in \mathbb{C}^2 : \varphi_i(z_1, z_2) = 0\}$ is the surface where the i th denominator factor of ω vanishes (cf. eq. (10)) and ω hence is not well-defined. The surfaces D_i are called the *divisors* of ω .

Integration cycles with moduli taken from distinct regions M^{+++} , M^{++-} etc. are *non-homologous*, and as a result are not guaranteed to produce identical residues. Figure A.4 gives a schematic representation of the divisors of ω and two non-homologous integration contours.

We can apply the residue evaluation algorithm of section 2.1 to each of the two terms in eq. (A.2) separately, yielding R_1 and R_2 for the first and second term, respectively. (Recall eqs. (17)–(18) for the expressions for R_1 and R_2 .) Combining this with the observations made in the discussion below eq. (A.11), we see that in the region M^{+++} of the moduli space, the residue evaluates to 0; in M^{+--} to R_1 ; in M^{+-+} to R_2 ; and in M^{+--} to $R_1 + R_2 = -R_3$ (cf. eq. (20)). From these observations we conclude that we have the following one-to-one map between the partitionings of the denominator of ω in eq. (11) and the regions of the moduli space of integration cycles,

$$\begin{aligned} \{\varphi_1, \varphi_2\varphi_3\} &\longleftrightarrow M^{+--} \\ \{\varphi_2, \varphi_3\varphi_1\} &\longleftrightarrow M^{+-+} \\ \{\varphi_3, \varphi_1\varphi_2\} &\longleftrightarrow M^{+--}. \end{aligned} \quad (\text{A.13})$$

This map provides a dictionary between the algebraic and geometric pictures of the distinct residues defined at the given pole.

We remark that the relation (20) shows that only two of the regions M^{+--} , M^{+-+} , \dots define linearly independent integration cycles.

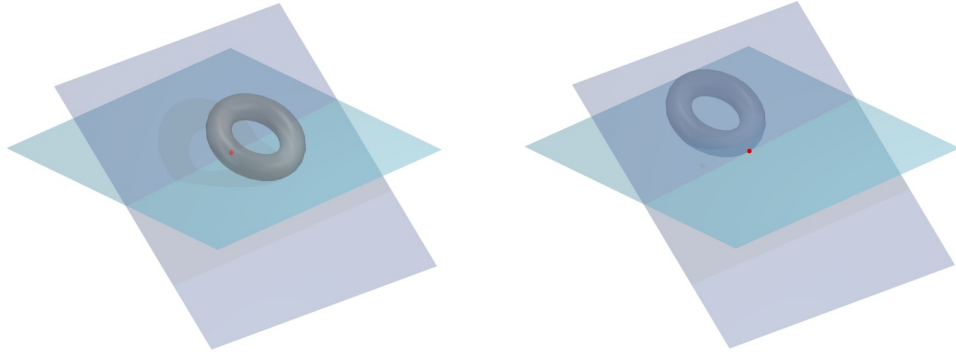


Figure A.4: A schematic illustration of the topological reason why the value of a multivariate residue is not uniquely determined by the pole enclosed by the integration cycle, but also depends on the cycle. The figure is a projection of the two-complex-dimensional (i.e., four-real-dimensional) situation for the differential form ω in eq. (9). The planes represent the divisors of ω . For visual clarity only two planes are shown. The red dot represents the pole $p = (0,0)$, and the tori two distinct integration cycles enclosing the pole. Because of the presence of the divisors, the cycle in the left figure cannot be continuously deformed into the one in the right figure: the cycles are non-homologous. As a result, the contours are not guaranteed to produce equal residues, and in general they will not. It is an artifact of the two-dimensional projection of the four-dimensional situation that the torus in the right figure appears to intersect the divisors and not to enclose the pole. (Color online.)

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