Prepared for submission to JHEP TU–1037, TTP16-056

Subtracting IR Renormalons from Wilson Coefficients: Uniqueness and power dependences on $\Lambda_{\rm QCD}$

Go Mishima[a,b,c], **Yukinari Sumino**[a], **Hiromasa Takaura**[a,][1]

[a] *Department of Physics, Tohoku University, Sendai, 980-8578 Japan*

[b] *Institute for Theoretical Particle Physics (TTP), Karlsruhe Institute of Technology, Engesserstraße 7, D-76128 Karlsruhe, Germany*

[c] *Institute for Nuclear Physics (IKP), Karlsruhe Institute of Technology, Hermann-von-Helmholtz-Platz 1, D-76344 Eggenstein-Leopoldshafen, Germany*

E-mail: go.mishima@kit.edu, sumino@tuhep.phys.tohoku.ac.jp, t_hiromasa@tuhep.phys.tohoku.ac.jp

Abstract: In the context of OPE and using the large-β_0 approximation, we propose a method to define Wilson coefficients free from uncertainties due to IR renormalons. We first introduce a general observable $X(Q^2)$ with an explicit IR cutoff, and then we extract a genuine UV contribution $X_{\rm UV}$ as a cutoff-independent part. $X_{\rm UV}$ includes power corrections $\sim (\Lambda_{\rm QCD}^2/Q^2)^n$ which are independent of renormalons. Using the integration-by-regions method, we observe that $X_{\rm UV}$ coincides with the leading Wilson coefficient in OPE and also clarify that the power corrections originate from UV region. We examine scheme dependence of $X_{\rm UV}$ and single out a specific scheme favorable in terms of analytical properties. Our method would be optimal with respect to systematicity, analyticity and stability. We test our formulation with the examples of the Adler function, QCD force between $Q\bar{Q}$, and R-ratio in e^+e^- collision.

[1] Corresponding author.

Contents

1 Introduction

In perturbative quantum field theory, perturbative series are considered to be asymptotic and divergent. It suggests that we have to truncate the series at finite order, and thus perturbative calculation cannot reach arbitrary precision. The idea of renormalon is a powerful tool to discuss an inevitable uncertainty of perturbative calculation [1]. It is related to divergent behaviors of perturbative series, and it provides an estimate of the size of uncertainty in an optimal prediction. In perturbative QCD, infrared (IR) renormalons give essential uncertainties of order $(\Lambda_{\rm QCD}/Q)^n$ in the prediction, where Q is a typical energy scale of an observable X. IR renormalons stem from low-energy region of loop momenta in Feynman integrals. Such uncertainties cannot be removed even by a resummation or Borel summation. This indicates that another framework is needed to overcome perturbative uncertainties induced by IR renormalons.

Operator product expansion (OPE) is a framework, in which the perturbative uncertainties can be eliminated systematically. An OPE of an observable $X(Q^2)$ consists of two components: Wilson coefficients and non-perturbative matrix elements. In the Wilsonian picture, Wilson coefficients are calculated from ultraviolet (UV) modes, which are higher than a factorization scale μ_f, whereas non-perturbative matrix elements are described by a low-energy effective theory valid below the scale μ_f. As a result, Wilson coefficients are free from uncertainties induced by IR renormalons and can be calculated unambiguously in perturbation theory (in principle). Non-perturbative matrix elements are determined from IR dynamics and show the same power dependence on $\Lambda_{\rm QCD}/Q$ as the uncertainties due to IR renormalons in the original perturbative series of X. Note, however, that each non-perturbative matrix element is no longer an uncertainty but a definite quantity, at least conceptually. Therefore, one can go beyond perturbation theory in the OPE framework.

In OPE an observable $X(Q^2)$ is evaluated by expansion in $1/Q^2$. To realize the concept of the Wilsonian picture, it is natural to introduce a hard cutoff (μ_f) in momentum space for factorizing UV and IR dynamics.[1] Then the IR renormalons are clearly eliminated from perturbative calculation of Wilson coefficients, and the $1/Q^2$-expansion (derivative expansion) in the low-energy effective theory is well justified since the active modes satisfy $k/Q \leq \mu_f/Q \ll 1$. It is, however, disadvantageous in practical computations to introduce a hard cutoff due to the following reasons: (1) One should include an additional scale μ_f in computations, which complicates the computations considerably. (2) Generally it generates apparent power-like strong dependences on μ_f of Wilson coefficients. Although they should eventually cancel in physical predictions, they can be sources of strong instability of the predictions in

[1] In conventional analyses of renormalons, a UV scale is assumed to be much larger than any scale involved in the calculation. In this paper, however, we use the terminology "UV" for scales above the factorization scale μ_f in the context of OPE. In particular Q is regarded as a UV scale.

practice [2]. (3) If we adopt a too naive cutoff regularization scheme, it may violate gauge invariance. For these reasons today it is customary to compute perturbative series of Wilson coefficients in dimensional regularization. This regularization circumvents the above difficulties. Nevertheless, as a trade-off, the perturbative series contain IR renormalon uncertainties since each integral region extends from $k \sim 0$ to infinity. Hence, several ways to subtract the contributions of IR renormalons have been explored [3–7].

In this paper we investigate the Wilson coefficient of the leading operator in OPE (equals to the identity operator in our explicit examples) and aim at removing a factorization scale dependent part, which destabilizes the prediction. Our basic tool is perturbation theory in the so-called large-β_0 approximation [8–10]. We proceed in the following steps: (i) We consider an observable $X(Q^2)$ with an IR cutoff μ_f. (ii) We extract a μ_f-independent part $X_{\rm UV}$ systematically, which can be regarded as a genuine UV contribution. (iii) We examine the scheme dependence of $X_{\rm UV}$. (iv) We single out a favorable scheme in terms of analyticity of $X_{\rm UV}$.

It turns out that $X_{\rm UV}$ includes power corrections $\sim (\Lambda_{\rm QCD}^2/Q^2)^n$ which stem from UV physics. We will see that (1) the power corrections are consistent with the framework of OPE, and (2) the power corrections are crucial for understanding the short-distance behavior of $X(Q^2)$. This is one of the main focuses of our discussion. Our method would also be useful in extracting non-perturbative matrix elements numerically, since the leading Wilson coefficient which we construct does not contain intrinsic uncertainties of the order of the matrix elements.

An analytical evaluation of a resummed perturbative series in the large-β_0 approximation was first performed in Ref. [11]. In fact many building blocks in our method are taken from their analysis. Their analysis starts from a regularized Borel integral, which removes IR renormalons by contour deformation. In their method a physical quantity can be separated into the real part and imaginary part. The real part is predicted reliably within perturbation theory, whereas the imaginary part is regarded as a perturbative uncertainty. The real part in their method and the cutoff independent part in our method have the same expanded form in $1/Q$. We also use an idea in their analysis related with the pseudo gluon mass to extract a cutoff independent part in our method.

Characteristic features of our method can be stated as follows. By starting from a well-defined integral with an explicit cutoff, we give a solid basis to our method, thereby the relation to OPE in the Wilsonian picture is made clear. We also reinforce our argument using the integration-by-regions (expansion-by-regions) method or comparison with the perturbative series up to large orders. Furthermore, we compare the perturbative series in the large-β_0 approximation with the known exact perturbative series and confirm consistency or validity of the approximation we use. These analyses utilize theoretical developments which took place after the analysis [11], and it is worthwhile to examine their impact.

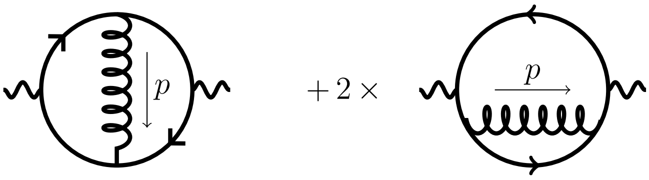


Figure 1. Leading-order diagrams which contribute to the reduced Adler function. The spiral line represents a gluon (with momentum p), and the solid line represents a massless quark. The external wavy line represents an insertion of the electromagnetic current.

Related subjects have also been studied in [8, 9, 12] (see also [13, 14]). In particular, existence of power corrections in the UV contribution to observables has been discussed, e.g., using a resummation of the perturbative series [11], and in certain model calculations [13, 14]. Our work can be regarded as an extension of the analyses in refs. [8, 9, 11] and more directly of the formulation used in the analysis of the static QCD potential [15–17]. Part of the analysis presented in this paper, in particular its application to the Adler function, have been reported in the letter article [18].

The outline of this paper is as follows. In Sec. 2, we explain our method to extract a cutoff independent part from a general observable defined with an IR cutoff. We also test our method with the Adler function and the force between $Q\bar{Q}$. In Sec. 3, we investigate the relation between our method and OPE using the method of integration by regions and also clarify which region gives each power correction. In Sec. 4, we show that the power corrections in $X_{\rm UV}$ is included in the large-order perturbative series. We also compare our results with known exact perturbative series. Through Secs. 2–4 only Euclidean quantities are examined. In Sec. 5, we study the R-ratio in e^+e^- collision as an example of a timelike quantity, and how our method can be applied. Conclusions and discussion are given in Sec. 6. Details of our analyses are collected in Appendixes.

2 Extraction of cutoff-independent part from UV contributions

In this section we present a method to extract a cutoff-independent part from UV contributions to physical quantities. In Sec. 2.1, basic notions are reviewed. In Sec. 2.2, the method to extract a cutoff-independent part is explained. As examples, we investigate the Adler function in Sec. 2.3 and the force between static quark and antiquark in Sec. 2.4. In Sec. 2.5, we examine a scheme dependence inherent in our method. In Sec. 2.6, we show that a specific scheme is favored from analytical properties of the extracted UV part. In Sec. 2.7, some detailed features of this specific scheme are analyzed.

2.1 Definitions and basics (review)

We consider a dimensionless spacelike observable $X(Q^2)$ whose leading order contribution is given by one-gluon-exchange diagrams, such as the ones shown in Fig. 1. For simplicity we focus on a quantity which depends on a single scale $Q^2 > 0$ in perturbative QCD. All the external and loop momenta are taken to be in the Euclidean region and we use the Euclidean metric through Secs. 2–4, except where stated otherwise. Explicitly we consider the case where the leading order (LO) contribution to $X(Q^2)$ in perturbation theory can be written in the form

$$X_{\rm LO}(Q^2) = \alpha_s(\mu) \int_0^\infty \frac{d\tau}{2\pi\tau}\, w_X\!\left(\frac{\tau}{Q^2}\right) . \tag{2.1}$$

τ represents the modulus-squared of the Euclidean gluon momentum p ($\tau = p^2$), and integrations over all the other loop momentum variables are included in w_X. We call w_X as "weight function," or simply "weight." In this form w_X reduces to a function of the single variable τ/Q^2. We assume that the integral is finite both in IR ($\tau \to 0$) and UV ($\tau \to \infty$) regions. The strong coupling constant $\alpha_s(\mu)$ is factored out, where μ is the renormalization scale. We adopt the modified minimal-subtraction ($\overline{\rm MS}$) renormalization scheme, in which $\alpha_s(\mu)$ at the one-loop level is given by

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0} \frac{1}{\log\left(\mu^2/\Lambda_{\rm QCD}^2\right)} . \tag{2.2}$$

$\beta_0 = 11 - 2n_f/3$ denotes the leading-order coefficient of the beta function for n_f active quark flavors.

We evaluate $X(Q^2)$ in the large-β_0 approximation, which can be obtained as follows. We consider insertions of a chain of fermion bubbles into the gluon propagator of $X_{\rm LO}$. Each bubble diagram produces a factor proportional to $\alpha_s(\mu) n_f \log(\mu^2 e^{-C}/p^2)$, where C is a scheme dependent constant and $C = -5/3$ in the $\overline{\rm MS}$ scheme. Taking the infinite sum of the chains and replacing $n_f \to n_f - 33/2 = -3\beta_0/2$, we obtain the all-order perturbative series in the large-β_0 approximation [8, 9]

$$X_{\beta_0}(Q^2) = \alpha_s(\mu) \sum_{n=0}^{\infty} \int_0^\infty \frac{d\tau}{2\pi\tau}\, w_X\!\left(\frac{\tau}{Q^2}\right) \left[\frac{\beta_0 \alpha_s(\mu)}{4\pi} \log\left(\frac{\mu^2 e^{5/3}}{\tau}\right)\right]^n . \tag{2.3}$$

After resummation of the infinite series in Eq. (2.3), the expression reduces to the same form as Eq. (2.1) with the strong coupling replaced by an effective coupling $\alpha_{\beta_0}(\tau)$:

$$X_{\beta_0}^{\rm resum}(Q^2) = \int_0^\infty \frac{d\tau}{2\pi\tau}\, w_X\!\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau) , \tag{2.4}$$

where

$$\alpha_{\beta_0}(\tau) = \frac{4\pi}{\beta_0} \frac{1}{\log(\tau e^{-5/3}/\Lambda_{\rm QCD}^2)} . \tag{2.5}$$

The effective coupling $\alpha_{\beta_0}(\tau)$ has a pole at $\tau = e^{5/3}\Lambda_{\rm QCD}^2$, and the existence of this pole on the integral path makes the integral ill-defined. The uncertainty which arises from this pole in this approach is attributed to IR renormalons.

We can make use of the Borel transformation to understand properties of the series in Eq. (2.3). The Borel transform of $X_{\beta_0}(Q^2)$ is defined as

$$\begin{aligned}\hat{B}_X(u) &\equiv \sum_{n=0}^{\infty} \frac{u^n}{n!} \int_0^\infty \frac{d\tau}{2\pi\tau}\, w_X\!\left(\frac{\tau}{Q^2}\right) \left[\log\left(\frac{\mu^2 e^{5/3}}{\tau}\right)\right]^n \\ &= \int_0^\infty \frac{d\tau}{2\pi\tau}\, w_X\!\left(\frac{\tau}{Q^2}\right)\left(\frac{\mu^2 e^{5/3}}{\tau}\right)^u . \end{aligned} \tag{2.6}$$

$\hat{B}_X(u)$ plays the role of a generating function for the coefficients of the original series d_n after accelerating convergence by $1/n!$:

$$\hat{B}_X(u) = \sum_{n=0}^{\infty} \frac{d_n}{n!} u^n \,, \tag{2.7}$$

$$X_{\beta_0}(Q^2) = \alpha_s(\mu) \sum_{n=0}^{\infty} d_n \left[\frac{\beta_0 \alpha_s(\mu)}{4\pi}\right]^n . \tag{2.8}$$

In general, singularities of $\hat{B}_X(u)$ characterize diverging behaviors of the original series. Singularities of $\hat{B}_X(u)$ located on the positive real axis are called IR renormalons and those on the negative real axis are called UV renormalons. Due to assumed finiteness of the integral Eq. (2.1), $\hat{B}_X(u)$ is regular at $u = 0$ [since Eq. (2.6) reduces to Eq. (2.1)]. The first IR renormalon at $u = u_{\rm IR} > 0$, closest to the origin, is known to give an inevitable uncertainty of $\mathcal{O}((\Lambda_{\rm QCD}^2/Q^2)^{u_{\rm IR}})$ in perturbative prediction.

Since the renormalization scale dependence of $\hat{B}_X(u)$ is factorized in Eq. (2.6), we further define

$$B_X(u) \equiv \left(\frac{Q^2 e^{-5/3}}{\mu^2}\right)^u \hat{B}_X(u) \,. \tag{2.9}$$

The weight $w_X(x)$ and the Borel transform $B_X(u)$ are related by [9]

$$B_X(u) = \int_0^\infty \frac{dx}{2\pi}\, w_X(x)\, x^{-u-1} \,, \tag{2.10}$$

$$w_X(x) = \frac{1}{i} \int_{u_0 - i\infty}^{u_0 + i\infty} du\, B_X(u)\, x^u \,, \tag{2.11}$$

where u_0 is located between the first IR renormalon and the first UV renormalon. In particular, the small-x behavior of the weight $w_X(x)$ is detected from the singularities of $B_X(u)$ explicitly as[2]

$$w_X(x) = \sum_{n \in U_{\rm IR}} b_n x^n = -2\pi \sum_{n \in U_{\rm IR}} {\rm Res}_{u=n}[B_X(u) x^u] \,, \tag{2.12}$$

[2] In the case that $B_X(u)$ has a multiple pole in u, the corresponding residue includes a polynomial of $\log x$. For simplicity we neglect such terms in the small-x expansion of w_X.

where $U_{\rm IR}$ denotes the set of IR renormalons ($U_{\rm IR} = \{u_{\rm IR}, \dots\}$).

As mentioned below Eq. (2.5), the expression Eq. (2.4) has an ambiguity because of the pole of $\alpha_{\beta_0}(\tau)$. In order to avoid this ambiguity we introduce an IR cutoff scale μ_f to the gluon momentum and eliminate contributions whose momentum scales are smaller than μ_f [9]:

$$X_{\beta_0}(Q^2;\mu_f) \equiv \int_{\mu_f^2}^{\infty} \frac{d\tau}{2\pi\tau}\, w_X\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau)\,. \tag{2.13}$$

The factorization scale is chosen to satisfy $e^{5/3}\Lambda_{\rm QCD}^2 \ll \mu_f^2 \ll Q^2$. Now that the integral path does not contain the pole, the integral is well defined. We choose this well-defined quantity as the starting point of our discussion. We will see in explicit examples that $X_{\beta_0}(Q^2;\mu_f)$ corresponds to the Wilson coefficient of the leading operator in OPE (see Sec. 3).

The subtraction of IR contributions also removes the IR renormalons of $X_{\beta_0}(Q^2)$ since they stem from the divergence of the integral (2.10) around $x = 0$ for some positive u. One can verify this by restarting from Eq. (2.3) with the IR cutoff μ_f and tracing the above discussion.

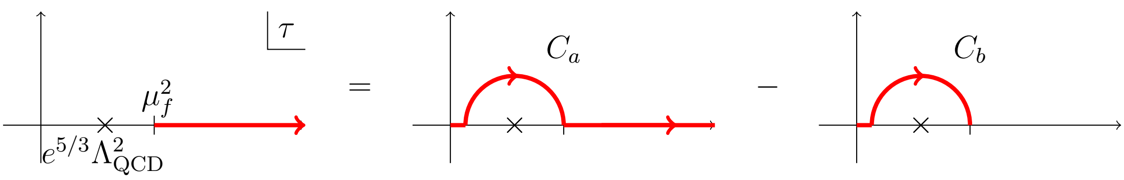
2.2 Extraction of cutoff-independent part: General case

The IR-subtracted quantity (2.13) is free from the ambiguity caused by IR renormalons. However, it has a cutoff dependence. This dependence makes the prediction of Eq. (2.13) unstable under the change of the artificial cutoff scale μ_f (which should eventually be canceled in a physical prediction). In this subsection we explain a method to extract a cutoff-independent part from this quantity.

Our method consists of two steps: (i) Rewrite the weight $w_X(x)$ by a new function $W_X(z)$ which is analytic in the upper half-plane and is related to $w_X(x)$ by

$$2\,{\rm Im}\,W_X(x) = w_X(x) \qquad (x \in \mathbb{R} \text{ and } x > 0)\,. \tag{2.14}$$

We call W_X as "pre-weight." (We will shortly present a construction of W_X.) (ii) Deform the integral path in the complex τ-plane. The original integral path is decomposed as follows:



Then Eq. (2.13) is rewritten as

$$X_{\beta_0}(Q^2;\mu_f) = {\rm Im}\int_{\mu_f^2}^{\infty} \frac{d\tau}{\pi\tau}\, W_X\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau) \tag{2.15}$$

$$= {\rm Im}\int_{C_a} \frac{d\tau}{\pi\tau}\, W_X\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau) - {\rm Im}\int_{C_b} \frac{d\tau}{\pi\tau}\, W_X\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau)\,. \tag{2.16}$$

The first term of Eq. (2.16) (integral along C_a) is clearly independent of μ_f. Although the second term (integral along C_b) is apparently μ_f-dependent, we can show that it also includes μ_f-independent part.

Since $\mu_f^2 \ll Q^2$ it would be justified to expand $W_X(\tau/Q^2)$ about $\tau = 0$ along C_b. In this way the second term of Eq. (2.16) is expressed in the large-Q^2 expansion:

$$\mathrm{Im} \int_{C_b} \frac{d\tau}{\pi\tau} W_X\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau) = \mathrm{Im} \sum_{n\geq 0} c_n \int_{C_b} \frac{d\tau}{\pi\tau} \left(\frac{\tau}{Q^2}\right)^n \alpha_{\beta_0}(\tau) \,, \tag{2.17}$$

with[3]

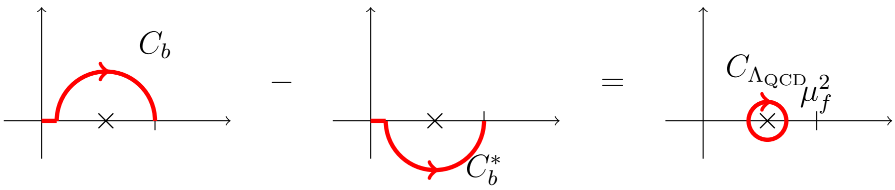
$$W_X(z) = \sum_{n\geq 0} c_n z^n \,. \tag{2.18}$$

The μ_f-dependence of the integral of each term of Eq. (2.17) can be classified into two cases.

Case (I): If the coefficient c_n is real, the complex conjugate of the integral along C_b becomes the integral along C_b^* since the integrand satisfies the relation $\{f(z)\}^* = f(z^*)$. Hence, we obtain

$$\begin{aligned} \mathrm{Im} \int_{C_b} \frac{d\tau}{\pi\tau} c_n \left(\frac{\tau}{Q^2}\right)^n \alpha_{\beta_0}(\tau) &= \frac{1}{2i} \left(\int_{C_b} - \int_{C_b^*} \right) \frac{d\tau}{\pi\tau} c_n \left(\frac{\tau}{Q^2}\right)^n \alpha_{\beta_0}(\tau) \\ &= \frac{1}{2\pi i} \int_{C_{\Lambda_{\mathrm{QCD}}}} \frac{d\tau}{\tau} c_n \left(\frac{\tau}{Q^2}\right)^n \alpha_{\beta_0}(\tau) \\ &= -\frac{4\pi c_n}{\beta_0} \left(\frac{e^{5/3}\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)^n \,, \end{aligned} \tag{2.19}$$

where the integration contours C_b^* and $C_{\Lambda_{\mathrm{QCD}}}$ are defined as below.



Here we use the fact that $C_b - C_b^*$ becomes a closed contour surrounding the pole at $e^{5/3}\Lambda_{\mathrm{QCD}}^2$. Therefore the result is μ_f-independent and can be calculated analytically by the Cauchy theorem. We see that positive powers of Λ_{QCD} appear.

Case (II): If the coefficient c_n has a non-zero imaginary part, the above argument

[3] We assume that the small-z expansion of $W_X(z)$ exists, where the expansion can include half-integer powers of z or powers of $\log z$. For simplicity we explain in the case where W_X is expanded as Taylor series in z. In other cases, it only matters whether the integrand satisfies the relation $\{f(z)\}^* = f(z^*)$ or not in classifying the Cases (I) and (II) in the following discussion.

does not hold since the integrand does not satisfy the relation $\{f(z)\}^* = f(z^*)$. In this case μ_f-dependence remains in the result:

$$\mathrm{Im} \int_{C_b} \frac{d\tau}{\pi\tau} c_n \left(\frac{\tau}{Q^2}\right)^n \alpha_{\beta_0}(\tau) = \mathcal{O}((\mu_f^2/Q^2)^n)\,. \tag{2.20}$$

Thus, μ_f-independent part appears not only from the integral along C_a but also from the integral along C_b depending on whether the expansion coefficient c_n is real or complex.

We can find whether the coefficient c_n in Eq. (2.18) is real or complex without knowing the concrete form of W_X. The insight is obtained using the expansions of w_X [Eq. (2.12)] and W_X [Eq. (2.18)] and the relation between them [Eq. (2.14)]. Schematically the relation can be understood as follows:

$$\begin{array}{lllllll} n \notin U_{\mathrm{IR}} & \longleftrightarrow & 2\,\mathrm{Im}\, c_n = b_n = 0 & \longleftrightarrow & c_n \in \mathbb{R} & \longleftrightarrow & \text{case (I)} \\ n \in U_{\mathrm{IR}} & \longleftrightarrow & 2\,\mathrm{Im}\, c_n = b_n \neq 0 & \longleftrightarrow & c_n \notin \mathbb{R} & \longleftrightarrow & \text{case (II)} \end{array} \tag{2.21}$$

Namely, the knowledge on the IR renormalons of $X_{\beta_0}(Q^2)$ is sufficient to judge μ_f-independence of each term of Eq. (2.17).

From the above discussion, by taking the terms for $0 \leq n < u_{\mathrm{IR}}$ of Eq. (2.17) and the first term of Eq. (2.16), we obtain the general result for $X_{\beta_0}(Q^2;\mu_f)$, where μ_f-independent part is separated:

$$X_{\beta_0}(Q^2;\mu_f) = X_{\mathrm{UV}}(Q^2) + \mathcal{O}\left((\mu_f^2/Q^2)^{u_{\mathrm{IR}}}\right)\,. \tag{2.22}$$

We have extracted the μ_f-independent part X_{UV} given by

$$X_{\mathrm{UV}}(Q^2) = \mathrm{Im} \int_{C_a} \frac{d\tau}{\pi\tau} W_X\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau) + \sum_{0\leq n<u_{\mathrm{IR}}} \frac{4\pi c_n}{\beta_0} \left(\frac{e^{5/3}\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)^n\,. \tag{2.23}$$

This is one of the main results in this paper. $X_{\mathrm{UV}}(Q^2)$ is insensitive to IR physics and can be regarded as a genuine UV contribution.

We rewrite X_{UV} as

$$X_{\mathrm{UV}}(Q^2) = X_0(Q^2) + \sum_{0<n<u_{\mathrm{IR}}} \frac{4\pi c_n}{\beta_0} \left(\frac{e^{5/3}\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)^n\,, \tag{2.24}$$

with

$$X_0(Q^2) = \mathrm{Im} \int_{C_a} \frac{d\tau}{\pi\tau} W_X\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau) + \frac{4\pi c_0}{\beta_0}\,. \tag{2.25}$$

The asymptotic form of X_0 as $Q^2 \to \infty$ is given by

$$X_0(Q^2) \to d_0\alpha_s(Q) = \frac{4\pi d_0}{\beta_0} \frac{1}{\log(Q^2/\Lambda_{\mathrm{QCD}}^2)}\,. \tag{2.26}$$

This is the leading term of the asymptotic expansion of X_0 that will be derived in Eq. (2.70) below; it is also a consequence of the renormalization-group (RG) equation.[4] This gives a more dominant contribution than power behaviors for large Q^2. Therefore $X_{\rm UV}(Q^2)$ indeed gives a leading behavior of $X_{\beta_0}(Q^2;\mu_f)$ for large Q^2. In explicit examples in Secs. 2.3 and 2.4, we will see that Eq. (2.24) represents a separation of $X_{\rm UV}(Q^2)$ into a logarithmic term[5] (non-power correction term) X_0 and power correction terms $\sim (\Lambda_{\rm QCD}^2/Q^2)^n$.

Up to this point we have considered a general pre-weight $W_X(z)$, which is analytic in the upper half-plane and satisfies the relation (2.14). A pre-weight which satisfies these conditions can be constructed systematically as

$$W_X(z) = \int_0^\infty \frac{dx}{2\pi}\frac{w_X(x)}{x-z-i0}\,, \tag{2.27}$$

due to the relation $\mathrm{Im}\{(x-z-i0)^{-1}\} = \pi\delta(x-z)$ for $z\in\mathbb{R}$. The integral in Eq. (2.27) always converges according to our assumption on the convergence of $X_{\rm LO}$. Note that there are potentially an infinite number of candidates for the pre-weight W_X since Eq. (2.14) does not restrict its real part on the positive real axis. Thus, W_X defined by Eq. (2.27) represents just one possibility and we refer to the choice Eq. (2.27) as "massive gluon scheme." This is because this construction is equivalent to replacing the gluon propagator to that with a tachyonic mass $m^2=-\tau$ in the leading order contribution Eq. (2.1): [6]

$$\int_0^\infty \frac{d(p^2)}{2\pi}\frac{w_X(p^2/Q^2)}{p^2} \quad\to\quad \int_0^\infty \frac{d(p^2)}{2\pi}\frac{w_X(p^2/Q^2)}{p^2-\tau-i0} = W_X^{(m)}(\tau/Q^2)\,, \tag{2.28}$$

where $W_X^{(m)}$ denotes the pre-weight in the massive gluon scheme.

We note that one does not have to start from w_X to obtain W_X in the massive gluon scheme. It is sufficient to use the gluon propagator with a tachyonic mass in the usual loop calculation, i.e., starting from the expression retaining all the loop momentum integrals, since it coincides with the right-hand side of Eq. (2.28). If we take this route, we rather obtain the weight w_X via the relation (2.14) *after* calculating the pre-weight W_X.

---

[4] Since the leading logarithmic terms are proportional to $\alpha_s(\mu)[\beta_0\alpha_s(\mu)\log(Q/\mu)]^n$, they are incorporated correctly by the large-β_0 approximation. The modification of the perturbative series by the IR cutoff is power-suppressed $\sim(\mu_f^2/Q^2)^k$, hence the leading large-Q^2 behavior is determined by the one-loop RG equation.

[5] By a "logarithmic term" we mean a term which is closest to $(Q^2/\Lambda_{\rm QCD}^2)^P$ with $P=0$ in the entire range $0<Q^2<\infty$, if it is compared with a single power dependence on Q^2 (for an integer P); see Figs. 2, 3 and Sec. 2.7.

[6] There exist many studies on low-energy QCD phenomena (especially chiral symmetry breaking and confinement) in terms of massive gluons [19–21]. We stress, however, that we study perturbative (UV) contributions using W_X.

For later convenience, we introduce W_{X+} from the pre-weight in the massive gluon scheme as

$$W_{X+}^{(m)}(z) \equiv W_X^{(m)}(-z) = \int_0^\infty \frac{dx}{2\pi} \frac{w_X(x)}{x+z-i0} \,. \tag{2.29}$$

This function is real for $z > 0$ since $w_X(x)$ is real and $x+z > 0$. Using this function, Eq. (2.25) can be expressed as[7]

$$X_0(Q^2) = \int_0^\infty \frac{d\tau}{\pi\tau} W_{X+}^{(m)}\left(\frac{\tau}{Q^2}\right) \mathrm{Im}\, \alpha_{\beta_0}(-\tau + i0) + \frac{4\pi c_0}{\beta_0} \,, \tag{2.30}$$

$$\mathrm{Im}\, \alpha_{\beta_0}(-\tau + i0) = \frac{4\pi}{\beta_0} \frac{-\pi}{\log^2\left(\tau e^{-5/3}/\Lambda_{\mathrm{QCD}}^2\right) + \pi^2} \,, \tag{2.31}$$

in the case that it is justified to deform the integral path C_a to the straight line connecting $\tau = 0$ to $-\infty$. This expression has a good analytical property as we will see later (end of Sec. 2.3 and Sec. 2.6). In calculating the asymptotic form of $X_0(Q^2)$ as $Q^2 \to \infty$ or $Q^2 \to 0$, the following expression, obtained by partial integration, is useful:

$$\begin{aligned} X_0(Q^2) = &- \int_0^\infty \frac{dx}{\pi} W_{X+}^{(m)\prime}(x) \frac{4\pi}{\beta_0} \mathrm{Im} \log\log\left(Q^2/(e^{5/3}\Lambda_{\mathrm{QCD}}^2)\right) \\ &- \int_0^\infty \frac{dx}{\pi} W_{X+}^{(m)\prime}(x) \frac{4\pi}{\beta_0} \tan^{-1}\left[\frac{\pi}{\log\left(Q^2/(e^{5/3}\Lambda_{\mathrm{QCD}}^2)\right) + \log x}\right] \,. \end{aligned} \tag{2.32}$$

2.3 Example 1: Adler function

As an application of the general framework presented in the previous subsection, we examine large-Q^2 behavior of the Adler function [18]. This observable is suited to test our method, in particular since OPE can be performed. The first IR renormalon is located at $u_{\mathrm{IR}} = 2$ [12], and thus the renormalon uncertainty is fairly suppressed.

We study the reduced Adler function $D(Q^2)$ with one massless quark, defined as

$$D(Q^2) = 4\pi^2 Q^2 \frac{d\Pi(Q^2)}{dQ^2} - 1 \,, \tag{2.33}$$

where $\Pi(Q^2)$ is a correlator[8] of the quark current $J^\mu(x) = \bar{q}(x)\gamma^\mu q(x)$,

$$(q^\mu q^\nu - g^{\mu\nu} q^2)\Pi(Q^2) = -i \int d^4x\, e^{-iq\cdot x} \langle 0|\mathrm{T} J^\mu(x) J^\nu(x)|0\rangle \,, \quad Q^2 = -q^2 > 0 \,. \tag{2.34}$$

[7] A quantity similar to X_{UV} with this X_0 is derived in Ref. [11] using a regularized Borel integral. Our derivation is different from theirs in that our result does not contain renormalon uncertainties since we subtract IR modes in Eq. (2.13).

[8] Eq. (2.34) uses the Minkowski metric, where q denotes the four-momentum of the vacuum polarization. In our letter [18] the sign of the corresponding equation [Eq. (2)] was incorrect and should be reversed.

We define the reduced Adler function in the large-β_0 approximation with an IR cutoff as

$$D_{\beta_0}(Q^2;\mu_f) = \int_{\mu_f^2}^{\infty} \frac{d\tau}{2\pi\tau}\, w_D\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau)\,. \tag{2.35}$$

The weight $w_D(x)$ is given by [9]

$$w_D(x) = \frac{N_C C_F}{3}\times \\ \begin{cases} (7-4\log x)x^2 + 4x(1+x)\{\mathrm{Li}_2(-x) + \log x \log(1+x)\} & ; x<1 \\ 3+2\log x + 4x(1+\log x) + 4x(1+x)\{\mathrm{Li}_2(-x^{-1}) - \log x \log(1+x^{-1})\} & ; x>1 \end{cases}, \tag{2.36}$$

where $N_C = 3$ is the number of colors and $C_F = 4/3$ is the Casimir operator of the fundamental representation. The first IR renormalon is located at $u_{\mathrm{IR}} = 2$, as can be seen from the expansion of $w_D(x)$ and Eq. (2.12):

$$w_D(x) = N_C C_F x^2 + \dots\,. \tag{2.37}$$

The pre-weight $W_D^{(m)}$ and $W_{D+}^{(m)}$ in the massive gluon scheme, obtained via Eq. (2.27) or by calculating the two-loop integral, are given by

$$\begin{aligned} W_D^{(m)}(z) = \frac{N_C C_F}{12\pi}\Big[&3 + 16z(z+1)H(z) - 14z^2\log(-z) \\ &+ 8z(z+1)\{-\log(-z)\mathrm{Li}_2(-z) + \mathrm{Li}_3(z) + \mathrm{Li}_3(-z)\} \\ &+ 4\{2z^2+2z+1-4z(z+1)\log(1+z)\}\mathrm{Li}_2(z) \\ &+ 2(7z^2-4z-3)\log(1-z) - 8\zeta_2 z(z+1)\log(1+z) \\ &+ 4\{z^2 - z(z+1)\log(1+z)\}\log^2(-z) \\ &+ 2(4\zeta_2 - 7\zeta_3)z^2 + 2(11-7\zeta_3)z\Big] \end{aligned} \tag{2.38}$$

and

$$W_{D+}^{(m)}(z) \equiv W_D^{(m)}(-z)\,. \tag{2.39}$$

Here, we define $H(z) = \int_z^1 dx\, x^{-1}\log(1+x)\log(1-x)$; $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n}$ denotes the polylogarithm; $\zeta_k = \zeta(k)$ denotes the Riemann zeta function.[9] We present another expression of $W_D^{(m)}$ in App. B, which is lengthier but exhibits the structure of the singularities more clearly. The first few terms of the small-z expansion of $W_D^{(m)}$ is given by[10]

$$W_D^{(m)}(z) = N_C C_F\left[\frac{1}{4\pi} + \frac{2(4-3\zeta_3)}{3\pi}z + \frac{10-12\zeta_3 - 3\log z + 3i\pi}{6\pi}z^2 + \dots\right]\,. \tag{2.40}$$

[9] $H(z)$ can be expressed using the harmonic polylogarithms.

[10] This series expansion was obtained in Ref.[11].

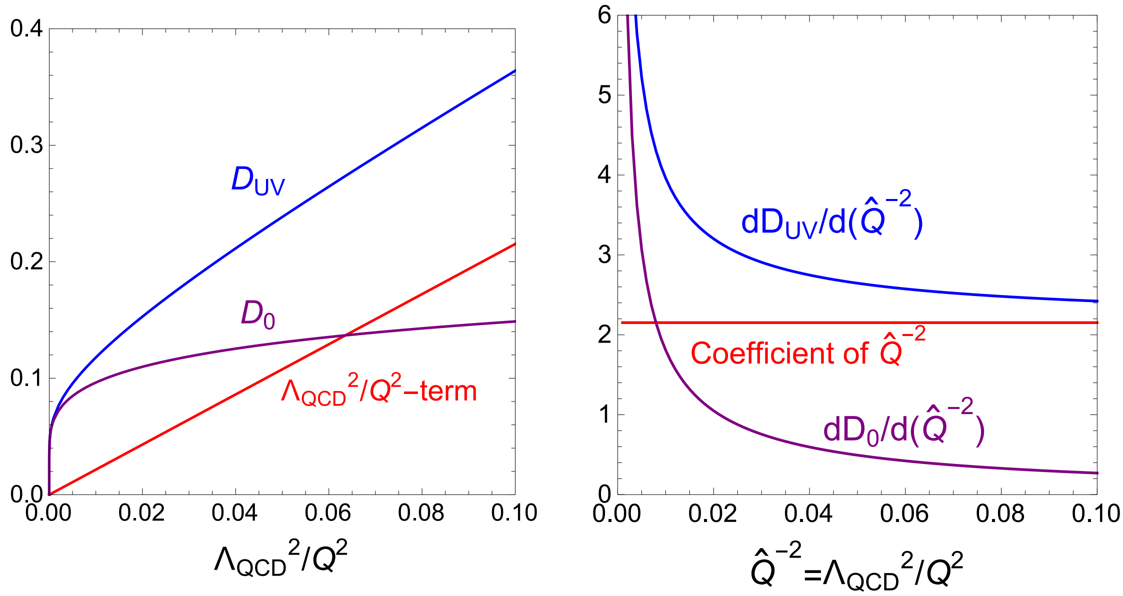


Figure 2. [Left] D_{UV} [Eq. (2.42)], D_0 [Eq. (2.43)] and the $\Lambda^2_{\mathrm{QCD}}/Q^2$-term [Eq.(2.42)] as functions of $\Lambda^2_{\mathrm{QCD}}/Q^2$. [Right] Derivatives of D_{UV}, D_0 and the $\Lambda^2_{\mathrm{QCD}}/Q^2$-term with respect to $\hat{Q}^{-2} \equiv \Lambda^2_{\mathrm{QCD}}/Q^2$.

Following the disucussion in the general case, we can extract the μ_f-independent part D_{UV}:

$$D_{\beta_0}(Q^2;\mu_f) = D_{\mathrm{UV}}(Q^2) + \mathcal{O}(\mu_f^4/Q^4) \tag{2.41}$$

with

$$D_{\mathrm{UV}}(Q^2) = D_0(Q^2) + \frac{8(4-3\zeta_3)e^{5/3}N_C C_F}{3\beta_0}\frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\,, \tag{2.42}$$

$$D_0(Q^2) = \int_0^\infty \frac{d\tau}{\pi\tau} W^{(m)}_{D+}\left(\frac{\tau}{Q^2}\right) \mathrm{Im}\,\alpha_{\beta_0}(-\tau+i0) + \frac{N_C C_F}{\beta_0}\,. \tag{2.43}$$

The $\Lambda^2_{\mathrm{QCD}}/Q^2$-term arises from the z^1-term of the pre-weight $W_D(z)$; see Eq. (2.40). The large-z behavior of $W_D(z)$ allows rotation of the integration contour and we write D_0 as in Eq. (2.30). The asymptotic behaviors of $D_0(Q^2)$ are obtained as

$$D_0(Q^2) \to \begin{cases} \frac{N_C C_F}{\beta_0}\frac{1}{\log(Q^2/\Lambda^2_{\mathrm{QCD}})} & \text{as } Q^2 \to \infty \\ \frac{N_C C_F}{\beta_0} & \text{as } Q^2 \to 0 \end{cases}\,, \tag{2.44}$$

and these asymptotic forms are interpolated smoothly in the intermediate region. Hence, qualitatively D_0 behaves as a constant term with a logarithmic correction at large Q^2.

In Fig. 2, D_{UV}, D_0 and the $\Lambda^2_{\mathrm{QCD}}/Q^2$-term of Eq. (2.42) are plotted as functions of $\Lambda^2_{\mathrm{QCD}}/Q^2$. The $\Lambda^2_{\mathrm{QCD}}/Q^2$-term naturally explains the power-like behavior of D_{UV}, which looks linear in this figure. In fact, the derivative of $D_{\rm UV}$ is given by the $\Lambda_{\rm QCD}^2/Q^2$-term dominantly in the range $\Lambda_{\rm QCD}^2/Q^2 \gtrsim 0.01$. In Sec. 4 we will compare $D_{\rm UV}$ with the large-order perturbative prediction in the large-β_0 approximation as well as with the known exact perturbative series, where we will find good agreement.

The μ_f-dependence of the $1/Q^4$-term in Eq. (2.41) shows a sensitivity to IR dynamics and can be interpreted in the context of OPE. In OPE, the reduced Adler function is expressed in terms of vacuum expectation values (VEVs) of operators which are invariant under Lorentz and gauge symmetries:

$$D(Q^2) = C_1 + C_{GG}\frac{\langle 0|G^{a\mu\nu}G^a_{\mu\nu}|0\rangle}{Q^4} + \ldots\,, \tag{2.45}$$

where C_1 and C_{GG} represent the Wilson coefficients of the operators $\mathbf{1}$ and $G^{a\mu\nu}G^a_{\mu\nu}$, respectively. The VEV of $G^{a\mu\nu}G^a_{\mu\nu}$, known as the local gluon condensate, has mass-dimension four and hence it is accompanied by the factor $1/Q^4$. The gluon condensate is determined by IR dynamics and it would have a dependence on the UV cutoff scale μ_f of the low energy effective theory. We can interpret that the IR cutoff dependence of $D_{\beta_0}(Q^2;\mu_f)$ at the order $1/Q^4$ in Eq. (2.41) is a counterpart of the UV cutoff dependence of the gluon condensate. In other words, if we include the gluon condensate as determined by IR dynamics, the leading μ_f-dependence of $D_{\beta_0}(Q^2;\mu_f)$ would be canceled and the $1/Q^4$-term is expected to be reduced to order $\Lambda_{\rm QCD}^4/Q^4$.

In the OPE framework, $D_{\rm UV}$ including the $\Lambda_{\rm QCD}^2/Q^2$-term is identified with C_1 in Eq. (2.45) as we will clarify in Sec. 3. In this sense, the μ_f-independent $\Lambda_{\rm QCD}^2/Q^2$-term does not conflict with the structure of OPE, and what we have found in this subsection is a non-trivial behavior of the Wilson coefficient C_1 of the reduced Adler function. Due to this power correction, we conclude that the Adler function has the leading power dependence as $\Lambda_{\rm QCD}^2/Q^2$ rather than $\Lambda_{\rm QCD}^4/Q^4$ at large Q^2 as long as the large-β_0 approximation is valid.

Finally we comment on the analytic structure of the Adler function. It is known that the Adler function in perturbative QCD is an analytic function in the complex Q^2-plane, with a cut along the negative axis from $Q^2 = 0$ corresponding to the threshold of massless partons, and with the $1/(\beta_0 \log Q^2)$ singularity at $Q^2 = \infty$ dictated by the RG equation. One can see that the expression of $D_{\rm UV}$ of Eq. (2.42) with Eq. (2.43) indeed satisfies these requirements. The cut arises from the property of $W_{D+}^{(m)}$ that it has an imaginary part when the argument becomes negative due to the relation (2.14). However, if we represent D_0 as in Eq. (2.25), it cannot be regarded as an analytic function of Q^2 since it is given by the imaginary part of a function. The representation (2.25) is defined only for real positive Q^2, whereas the representation (2.30) is defined in the entire complex Q^2 plane. They are equivalent only if we limit Q^2 to a real positive parameter. Thus, from the viewpoint of analyticity, the latter representation turns out to be superior to the former.

2.4 Example 2: Force between static quark-antiquark pair

As another application of the method presented in Sec. 2.2, we consider the short-distance behavior of the force between a static quark-antiquark pair, which is obtained from the derivative of the static QCD potential. The static QCD potential has been studied extensively to understand the nature of the force between the quark and antiquark. At short distances perturbative QCD prediction is accurate, whereas at large distances lattice QCD predictions are accurate. There is a significant overlap region at intermediate distances, where both predictions agree well. Presently the exact perturbative series are known up to NNNLO [22–24]. In addition, the low energy effective theory "potential non-relativistic QCD (pNRQCD)" is known, in which OPE can be performed, and there is a good theoretical understanding of the connection between UV and IR contributions. Therefore the QCD potential (or the force) is an optimal observable to examine our formulation.

The potential energy between the static quark Q and antiquark $\bar{Q}$ (QCD potential) in the large-β_0 approximation and with an IR cutoff is given by

$$V_{\beta_0}(r;\mu_f) = -\int_{p>\mu_f} \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{r}}\, \frac{4\pi C_F}{p^2}\, \alpha_{\beta_0}(p^2) = -\frac{1}{r}\int_{\mu_f^2}^{\infty} \frac{d\tau}{2\pi\tau} 2C_F \sin(\sqrt{\tau}r)\alpha_{\beta_0}(\tau)\,. \tag{2.46}$$

Here, the typical (energy) scale is r^{-1}, the inverse of the distance between $Q\bar{Q}$. Comparing Eq. (2.46) with Eq. (2.13), the weight of the (dimensionless) QCD potential $rV_{\beta_0}(r)$ is given by

$$w_V(x) = -2C_F \sin(\sqrt{x})\,. \tag{2.47}$$

Comparing its expansion and Eq. (2.12) we find that the first IR renormalon of the QCD potential is located at $u = 1/2$. However, this renormalon is not serious since it only gives an uncertainty to the constant (r-independent) part of the potential. Several prescriptions to eliminate the $u = 1/2$ renormalon are known. Here we adopt the prescription to consider the force between $Q\bar{Q}$, $F_{\beta_0}(r^2) = -dV_{\beta_0}(r)/dr$ [25].

The force between $Q\bar{Q}$ with an IR cutoff is obtained by differentiating Eq. (2.46) with respect to r as

$$F_{\beta_0}(r^2;\mu_f) = -C_F\frac{\alpha_{F,\beta_0}(1/r^2;\mu_f)}{r^2} = -\frac{C_F}{r^2}\int_{\mu_f^2}^{\infty}\frac{d\tau}{2\pi\tau}\, w_F(\tau r^2)\,\alpha_{\beta_0}(\tau)\,, \tag{2.48}$$

where the weight $w_F(x)$ is given by

$$w_F(x) = 2(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x})\,. \tag{2.49}$$

In the following we deal with the dimensionless force (or the F-scheme coupling) α_{F,β_0}, defined by Eq. (2.48). The expansion of the weight reads

$$w_F(x) = \frac{2}{3}x^{3/2} - \frac{1}{15}x^{5/2} + \cdots\,, \tag{2.50}$$

hence, the first IR renormalon of α_{F,β_0} is indeed shifted to $u_{\rm IR}=3/2$. The pre-weight $W_F^{(m)}$ and $W_{F+}^{(m)}$ in the massive gluon scheme can be obtained using Eq. (2.27) as

$$W_F^{(m)}(z)=e^{i\sqrt{z}}(1-i\sqrt{z}) \tag{2.51}$$

and

$$W_{F+}^{(m)}(z)=e^{-\sqrt{z}}(1+\sqrt{z})\,. \tag{2.52}$$

The expansion of the pre-weight is given by

$$W_F^{(m)}(z)=1+\frac{z}{2}+\frac{i}{3}z^{3/2}+\cdots\,. \tag{2.53}$$

From the general discussion we can extract the μ_f-independent part $\alpha_{F,\rm UV}(1/r^2)$ from $\alpha_{F,\beta_0}(1/r^2;\mu_f)$ as

$$\alpha_{F,\beta_0}(1/r^2;\mu_f)=\alpha_{F,\rm UV}(1/r^2)+\mathcal{O}(\mu_f^3 r^3) \tag{2.54}$$

with

$$\alpha_{F,\rm UV}(1/r^2)=\alpha_{F,0}(1/r^2)+\frac{2\pi}{\beta_0}\Lambda_{\rm QCD}^2 e^{5/3}r^2\,, \tag{2.55}$$

$$\alpha_{F,0}(1/r^2)=\int_0^\infty \frac{d\tau}{\pi\tau}\,W_{F+}^{(m)}(\tau r^2)\,{\rm Im}\,\alpha_{\beta_0}(-\tau+i0)+\frac{4\pi}{\beta_0}\,. \tag{2.56}$$

The $\Lambda_{\rm QCD}^2 r^2$-term arises from the z^1-term of the pre-weight $W_F^{(m)}$ [Eq. (2.53)]. This power behavior corresponds to a linear potential in the QCD potential. The asymptotic behaviors of $\alpha_{F,0}$ are obtained via Eq. (2.32) as

$$\alpha_{F,0}(1/r^2)\to\begin{cases}\frac{4\pi}{\beta_0}\frac{1}{|\log(r^2\Lambda_{\rm QCD}^2)|} & \text{as } r^2\to 0\\ \frac{4\pi}{\beta_0} & \text{as } r^2\to\infty\end{cases}\,. \tag{2.57}$$

In Fig. 3, $\alpha_{F,\rm UV}$, $\alpha_{F,0}$, and the $\Lambda_{\rm QCD}^2 r^2$-term of Eq. (2.55) are plotted as functions of $\Lambda_{\rm QCD}^2 r^2$. Qualitatively they show similar behaviors to those of the reduced Adler function (Fig. 2), and the derivative of $\alpha_{F,\rm UV}$ is dominated by the $\Lambda_{\rm QCD}^2 r^2$-term especially in the range $\Lambda_{\rm QCD}^2 r^2\gtrsim 0.02$. Comparisons with the large-order predictions in the large-β_0 approximation and with the known exact perturbative series will be presented in Sec. 4.

In Eq. (2.54), the μ_f-dependent term starts from order r^3. Let us discuss this μ_f-dependence in the context of OPE. The relevant low-energy effective theory is known as pNRQCD, in which the QCD potential is expressed in expansion in $\vec{r}$ (multipole expansion) as [26]

$$V_{\rm QCD}(r)\approx V_S(r)-\frac{2\pi i\alpha_s}{N_C}\int_0^\infty e^{-it\Delta V(r)}\,\langle \vec{r}\cdot\vec{E}^a(t)\,\vec{r}\cdot\vec{E}^a(0)\rangle+\mathcal{O}(r^3)\,. \tag{2.58}$$

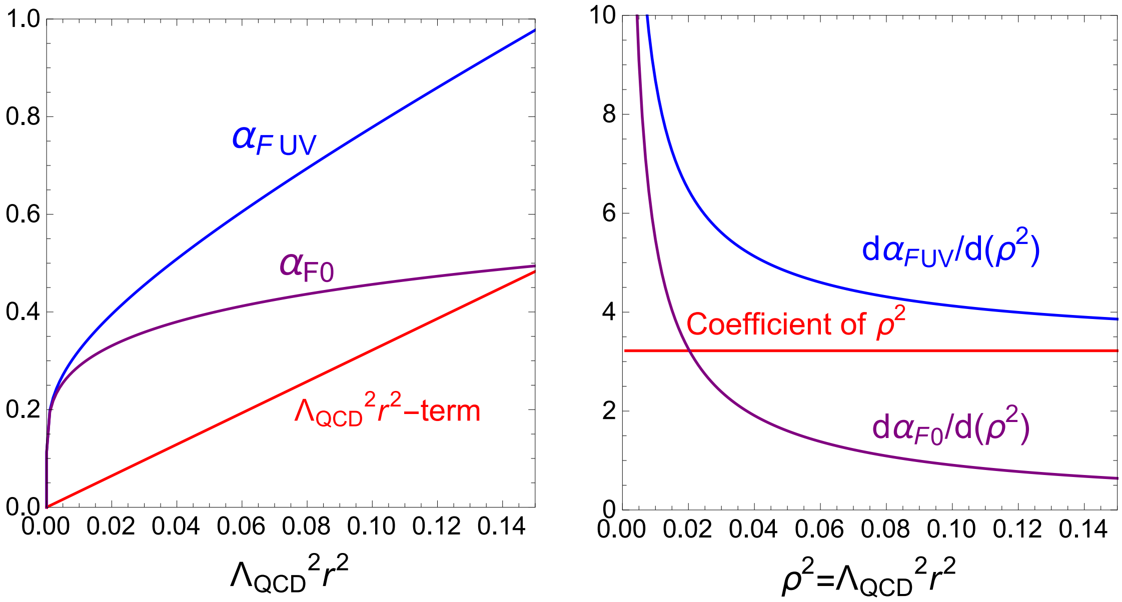


Figure 3. [Left] $\alpha_{F,\mathrm{UV}}$ [Eq. (2.55)], $\alpha_{F,0}$ [Eq. (2.56)], and the $\Lambda_{\mathrm{QCD}}^2 r^2$-term [Eq. (2.55)] as functions of $\Lambda_{\mathrm{QCD}}^2 r^2$. The number of flavors is set to $n_f = 1$. [Right] Derivatives of $\alpha_{F,\mathrm{UV}}$, $\alpha_{F,0}$ and the $r^2\Lambda_{\mathrm{QCD}}^2$-term with respect to $\rho^2 \equiv \Lambda_{\mathrm{QCD}}^2 r^2$.

Here, $V_S(r)$ represents the Wilson coefficient for the (leading) identity operator and has the meaning of the energy of the $Q\bar{Q}$ singlet state; ΔV denotes the energy difference between the octet and singlet states; $\vec{E}^a$ denotes the color electric field. If we compute $V_S(r)$ in the large-β_0 approximation and with an explicit cutoff in the gluon momentum, it is identified with $V_{\beta_0}(r;\mu_f)$.[11] It has been confirmed that the μ_f-dependence of $V_{\beta_0}(r;\mu_f)$ at order $\mu_f^3 r^2$, originating from the $u = 3/2$ renormalon, is canceled against the μ_f-dependence of the non-perturbative matrix element in the second term of Eq. (2.58) [17, 26]. Differentiating with respect to r, the leading μ_f-dependence of $\alpha_{F,\beta_0}(1/r^2;\mu_f)$ at order $\mu_f^3 r^3$ is also canceled by that of the non-perturbative matrix element. We expect that a similar cancellation between $D_{\beta_0}(Q^2;\mu_f)$ and the local gluon condensate would hold for the Adler function, although the relevant low energy effective theory is as yet unknown.

2.5 Scheme dependence by choice of pre-weight W_X

As we already pointed out, the pre-weight W_X introduced in Sec. 2.2 is not unique, and we clarify its effect in this subsection. We first show that the dependences of X_0 and the power corrections in X_{UV} on the choice of W_X almost cancel in the sum (X_{UV}). We then discuss its relevance in determination of a non-perturbative matrix element in OPE. Finally we discuss why the power corrections in X_{UV} can vary from

[11] See also the discussion at the end of Sec. 3.3.

the viewpoint of the asymptotic property of the perturbative series and how it is related to variation of W_X.

The pre-weight W_X which satisfies Eq. (2.14) is not unique since its real part on the positive real axis is not restricted. Although the original μ_f-dependent integral (2.15) is independent of the choice of W_X, the μ_f-independent part $X_{\rm UV}$ generally depends on the choice of W_X. Namely, $X_{\rm UV}$ is a functional of W_X. We can regard that $X_{\rm UV}$ determined by different W_X correspond to different scheme choices. We first discuss the scheme dependence of $X_{\rm UV}$.

Consider two different pre-weights $W_X^{(i)}$ (i=1,2) both satisfying the relation (2.14). Correspondingly we obtain $X_{\rm UV}$ in different schemes via Eqs. (2.24) and (2.25):

$$X_{\rm UV}^{(i)} = X_0^{(i)}(Q^2) + \sum_{0<n<u_{\rm IR}} \frac{4\pi c_n^{(i)}}{\beta_0} \left(\frac{e^{5/3}\Lambda_{\rm QCD}^2}{Q^2} \right)^n \tag{2.59}$$

with

$$X_0^{(i)}(Q^2) = {\rm Im} \int_{C_a} \frac{d\tau}{\pi\tau} W_X^{(i)}\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau) + \frac{4\pi c_0^{(i)}}{\beta_0}\,, \tag{2.60}$$

where $W_X^{(i)}(z) = \sum_{n\geq 0} c_n^{(i)} z^n$. The difference between $X_0^{(1)}$ and $X_0^{(2)}$ is given by

$$X_0^{(2)} - X_0^{(1)} = {\rm Im} \int_{C_a} \frac{d\tau}{\pi\tau} \left\{ W_X^{(2)}\left(\frac{\tau}{Q^2}\right) - W_X^{(1)}\left(\frac{\tau}{Q^2}\right) \right\} \alpha_{\beta_0}(\tau) + \frac{4\pi(c_0^{(2)} - c_0^{(1)})}{\beta_0}\,. \tag{2.61}$$

In the integral along C_b we assumed that it is justified to expand $W_X(z)$ for sufficiently small $|z|$. Accordingly, we assume that $\delta W(z) \equiv W_X^{(2)}(z) - W_X^{(1)}(z)$ is regular at any point $z_0 \in \mathbb{R}$ and $0 < z_0 < \epsilon$ for $\exists \epsilon > 0$ (sufficiently close to the origin).[12] Namely, $\delta W(z)$ can be expanded in Taylor series about $z = z_0$ with a non-zero radius of convergence:

$$\delta W(z) = \sum_{n\geq 0} A_n(z_0)\,(z - z_0)^n\,, \qquad z_0 \in \mathbb{R} \ \text{ and } \ 0 < z_0 < \epsilon\,. \tag{2.62}$$

Since ${\rm Im}\,\delta W = 0$ on the positive real axis, (i) the integral along C_a in Eq. (2.61) is equal to that along C_b, and (ii) $A_n(z_0) \in \mathbb{R}$, hence $\{\delta W(z)\}^* = \delta W(z^*)$ is satisfied along the path C_b if $Q^2 \gg \mu_f^2$. Then, by exploiting the same procedure as in

[12] The reason to exclude $z_0 = 0$ is to cope with possible existence of $\log z$ or $\sqrt{z}$. (See footnote 3.) Note that even if the small-z expansion of $W_X(z)$ includes $\log z$ or $\sqrt{z}$ we expect that the expansion has a domain of convergence close to the origin; see the examples in Secs. 2.3 and 2.4.

Eq. (2.19), the first term of Eq. (2.61) can be reduced to

$$\begin{aligned}
&\mathrm{Im}\int_{C_a}\frac{d\tau}{\pi\tau}\,\delta W\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau)=\mathrm{Im}\int_{C_b}\frac{d\tau}{\pi\tau}\,\delta W\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau)\\
&=-\frac{4\pi}{\beta_0}\,\delta W\left(\frac{e^{5/3}\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)=-\frac{4\pi}{\beta_0}\sum_{n\geq 0}(c_n^{(2)}-c_n^{(1)})\left(\frac{e^{5/3}\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)^n. \qquad (2.63)
\end{aligned}$$

It means that the difference of $X_0^{(i)}(Q^2)$ is given by[13]

$$X_0^{(2)}(Q^2)-X_0^{(1)}(Q^2)=-\frac{4\pi}{\beta_0}\sum_{n>0}(c_n^{(2)}-c_n^{(1)})\left(\frac{e^{5/3}\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)^n. \qquad (2.64)$$

Furthermore, according to Eq. (2.59) we obtain the difference of $X_{\mathrm{UV}}^{(i)}$ as

$$\begin{aligned}
X_{\mathrm{UV}}^{(2)}(Q^2)-X_{\mathrm{UV}}^{(1)}(Q^2)&=-\sum_{n\geq u_{\mathrm{IR}}}\frac{4\pi(c_n^{(2)}-c_n^{(1)})}{\beta_0}\left(\frac{e^{5/3}\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)^n\\
&=\mathcal{O}\left((\Lambda_{\mathrm{QCD}}^2/Q^2)^{u_{\mathrm{IR}}}\right). \qquad (2.65)
\end{aligned}$$

Thus, the differences of the power corrections $(1/Q^2)^n$ with $0<n<u_{\mathrm{IR}}$ in Eq. (2.59) are canceled by the change of $X_0(Q^2)$. As a result, the difference of X_{UV} in different schemes is smaller than the last included term of the $(\Lambda_{\mathrm{QCD}}^2/Q^2)^n$-terms in $X_{\mathrm{UV}}(Q^2)$. Namely, the μ_f-independent part X_{UV} has a minor dependence on the scheme, which is the same order as an uncertainty induced by the first IR renormalon, and we confirm validity of our result of X_{UV} taking into account the scheme dependence.

It is worth emphasizing that the scheme dependence discussed above is *not* a renormalon uncertainty. In fact the scheme dependence can be removed by including higher orders of the $1/Q^2$ expansion. Let us clarify this point. Suppose we consider $X_{\beta_0}(Q^2;\mu_f)$ up to $1/(Q^2)^n$ in different schemes:

$$X_{\beta_0}^{(i)}(Q^2;\mu_f)\Big|_{1/(Q^2)^n}=X_0^{(i)}(Q^2)-\sum_{k=1}^{n}\mathrm{Im}\int_{C_b}\frac{d\tau}{\pi\tau}\,c_k^{(i)}\left(\frac{\tau}{Q^2}\right)^k\alpha_{\beta_0}(\tau).\quad(i=1,2) \qquad (2.66)$$

We show that $X_{\beta_0}^{(2)}-X_{\beta_0}^{(1)}\Big|_{1/(Q^2)^n}$ is order $(\Lambda_{\mathrm{QCD}}^2/Q^2)^{n+1}$. (The previous argument already proves this for the case $n=u_{\mathrm{IR}}-1$.)

Note that since $\mathrm{Im}\,c_k^{(i)}$ is fixed by Eq. (2.14), there is no scheme dependence, hence $c_k^{(2)}-c_k^{(1)}\in\mathbb{R}$. This enables reducing the difference of the second term of Eq. (2.66) as

$$\sum_{k=1}^{n}\mathrm{Im}\int_{C_b}\frac{d\tau}{\pi\tau}\,(c_k^{(2)}-c_k^{(1)})\left(\frac{\tau}{Q^2}\right)^k\alpha_{\beta_0}(\tau)=\frac{4\pi}{\beta_0}\sum_{k=1}^{n}(c_k^{(2)}-c_k^{(1)})\left(\frac{e^{5/3}\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)^k. \qquad (2.67)$$

[13]Note that the right-hand side of Eq. (2.64) is $\mathcal{O}(\Lambda_{\mathrm{QCD}}^2/Q^2)$ and the asymptotic form of $X_0(Q^2)$ at $Q^2\to\infty$ shown in Eq. (2.26) is not modified.

Combining with Eq. (2.64), we see that $X_{\beta_0}^{(2)} - X_{\beta_0}^{(1)}\Big|_{1/(Q^2)^n} = \mathcal{O}((\Lambda_{\rm QCD}^2/Q^2)^{n+1})$. Such a property follows from the fact that the original μ_f-dependent integral (2.15) is independent of the choice of W_X. Therefore the scheme dependence is gradually eliminated by including higher order terms in $1/Q^2$.

In the case of the Adler function, this fact is important if we want to determine the local gluon condensate using our formulation, for instance, by comparing with an evaluation of $D(Q^2)$ by a lattice calculation. The OPE up to the $\mathcal{O}(1/Q^4)$ terms (in the large-β_0 approximation) is written as

$$D(Q^2) = D_{\beta_0}(Q^2;\mu_f)\Big|_{1/Q^4} + C_{GG}(\mu_f)\frac{\langle 0|G^{a\mu\nu}G^a_{\mu\nu}|0\rangle\,(\mu_f)}{Q^4} + \mathcal{O}(\Lambda_{\rm QCD}^6/Q^6)\,. \quad (2.68)$$

We expect that μ_f-dependences up to $1/Q^4$-terms are canceled. According to the above discussion, the variation due to the scheme difference (choice of W_X) satisfies

$$\Delta_{\rm scheme}\left(C_{GG}(\mu_f)\frac{\langle 0|G^{a\mu\nu}G^a_{\mu\nu}|0\rangle\,(\mu_f)}{Q^4}\right) = \mathcal{O}(\Lambda_{\rm QCD}^6/Q^6)\,. \quad (2.69)$$

Thus, the error becomes higher order than the term which we want to determine. [Note that C_{GG} would also include power corrections $\sim (\Lambda_{\rm QCD}^2/Q^2)^n$.]

Although we have shown that δW changes $X_{\rm UV}$ only at subleading order, it alters X_0 and the power corrections $(\Lambda_{\rm QCD}^2/Q^2)^n$ with $n < u_{\rm IR}$ individually; see Eqs. (2.59) and (2.64). In the rest of this subsection, we discuss the reason why the coefficients of the $(\Lambda_{\rm QCD}^2/Q^2)^n$-terms can be altered.

We can show that $X_0(Q^2)$ has the same asymptotic expansion in α_s as the perturbative series of $X_{\beta_0}(Q^2)$:

$$X_0(Q^2) - \sum_{k=0}^{n-1} d_k(\mu = Q)\left(\frac{\beta_0}{4\pi}\right)^k \alpha_s^{k+1}(Q) = \mathcal{O}(\alpha_s(Q)^{n+1})\,, \quad (2.70)$$

as $\alpha_s(Q) \to 0$. (We sketch the proof in App. D.) This shows that, although $X_0(Q^2)$ is expansible with respect to $\alpha_s(Q)$, it is *not* expansible with respect to $\Lambda_{\rm QCD}^2/Q^2$ since $\alpha_s(Q) \sim 1/\log(Q^2/\Lambda_{\rm QCD}^2)$. Reflecting this fact, $X_{\beta_0}(Q^2;\mu_f)$, which is related to $X_0(Q^2)$ by Eqs. (2.22) and (2.24), is also not expansible with respect to $1/Q^2$. This is a short answer to the question why the $(\Lambda_{\rm QCD}^2/Q^2)^n$-terms in $X_{\beta_0}(Q^2;\mu_f)$ is not uniquely determined.

Note that $X_{\beta_0}(Q^2;\mu_f) - X_0(Q^2)$ is expansible in $1/Q^2$ and the $(\Lambda_{\rm QCD}^2/Q^2)^n$-terms are regarded as a part of this series expansion. In this respect Eq. (2.70) is essential since it ensures that the singularities of $X_{\beta_0}(Q^2;\mu_f)$ caused by $\alpha_s(Q)^k$ cancel with those of $-X_0(Q^2)$. Considering the fact that $X_{\beta_0}(Q^2;\mu_f)$ is a uniquely-defined quantity, it is deduced that the non-uniqueness of the $(\Lambda_{\rm QCD}^2/Q^2)^n$-terms in $X_{\beta_0}(Q^2;\mu_f) - X_0(Q^2)$ is caused by the non-uniqueness of $X_0(Q^2)$. In fact there are potentially many candidates of $X_0(Q^2)$ satisfying the property (2.70). A new X_0

constructed by adding $(\Lambda_{\rm QCD}^2/Q^2)^n$ to the old one also satisfies Eq. (2.70), since all the series coefficients of $\Lambda_{\rm QCD}^2/Q^2 = e^{-4\pi/(\beta_0\alpha_s(Q^2))}$ in $\alpha_s(Q^2)$ are zero.

The non-uniqueness of X_0 stems from the non-uniqueness of W_X in our method. The variation of W_X indeed changes X_0 by powers of $\Lambda_{\rm QCD}^2/Q^2$ as shown in Eq. (2.64) while keeping the asymptotic expansion (2.70). This change of X_0 is compensated by the change of the $(\Lambda_{\rm QCD}^2/Q^2)^n$-terms as shown below Eq. (2.67). Thus, the non-uniqueness of the power corrections is also attributed to the non-uniqueness of W_X.

At this stage, it suggests that it would be meaningless to focus on the power corrections $(\Lambda_{\rm QCD}^2/Q^2)^n$ alone in $X_{\rm UV}$ since it becomes definite only after we specify X_0, and only the sum of them ($X_{\rm UV}$) is a meaningful quantity. Nevertheless, it turns out that if we limit schemes to a reasonable class, the separation of $X_{\rm UV}$ into X_0 and $(\Lambda_{\rm QCD}^2/Q^2)^n$-terms becomes unique by a uniqueness of W_X. We will elaborate on this point in the next subsection.

2.6 Massive gluon scheme as the optimal scheme

We discuss which scheme is favored from the analytical properties of $X_{\rm UV}(Q^2)$ when we extend it to a function of the complex variable $\Lambda_{\rm QCD}^2/Q^2$. Since the power-correction terms in $X_{\rm UV}$ are obviously analytic in the whole $\Lambda_{\rm QCD}^2/Q^2$-plane, we mainly focus on the analytic structure of $X_0(Q^2)$.

$X_0(Q^2)$ in the massive gluon scheme can be expressed as an analytic function of $\Lambda_{\rm QCD}^2/Q^2$ by Eq. (2.30), provided that the integral path can be rotated.[14] Using this expression we can show that X_0 has a cut along the negative real axis starting from the origin and is regular everywhere else in the $\Lambda_{\rm QCD}^2/Q^2$-plane. It follows from the fact that $W_{X+}^{(m)}(z)$ in this scheme can have cuts along the negative real axis starting only from $z=0$ and $z=-1$ and is regular everywhere else.[15] Thus, $X_0(Q^2)$ in this scheme (hence, $X_{\rm UV}(Q^2)$) satisfies the required analyticity in the complex plane in terms of perturbative QCD, where the form of the singularity at $\Lambda_{\rm QCD}^2/Q^2=0$ is dictated by the renormalization-group equation. We have already seen this favorable feature of the massive gluon scheme for the Adler function in Sec. 2.3.

In a general scheme, i.e., for a general pre-weight, $X_0(Q^2)$ can be expressed as an analytic function in the following manner. We rewrite the pre-weight as the sum of $W_X^{(m)}(z)$, which is the pre-weight in the massive gluon scheme, and the rest as $W_X(z) = W_X^{(m)}(z) + \delta W_X(z)$. We can follow the same steps which led to Eq. (2.63) in the previous section, assuming regularity of $\delta W_X(z)$ close to the origin, and obtain,

[14] We can show that the rotation of integral path is possible if $|w_X(z)| = \mathcal{O}(|z|^a)$ for $\exists a<0$ for sufficiently large $|z|$ in the lower half-plane.

[15] This can be shown using the property that $W_X^{(m)}(z)$ can have singularities only at $z=0$, 1, ∞, where $z=\tau/Q^2=(-\tau)/q^2=1$ corresponds to the threshold of the massive gluon plus massless partons. In passing, since $2\,{\rm Im}\,W_X^{(m)}(x) = w_X(x)$ holds for $x>0$, $w_X(x)$ can have singularities only at $x=0$, 1, ∞ along the integral path of Eq. (2.29); c.f., Eqs. (2.36) and (2.49).

for sufficiently small $|\Lambda_{\rm QCD}^2/Q^2|$,

$$X_0(Q^2) = X_0^{(m)}(Q^2) + \frac{4\pi}{\beta_0}\left[\delta W_X(0) - \delta W_X\left(\frac{e^{5/3}\Lambda_{\rm QCD}^2}{Q^2}\right)\right], \tag{2.71}$$

where $X_0^{(m)}(Q^2)$ represents X_0 in the massive gluon scheme. Then we can enlarge the domain of this function by analytic continuation to the entire $\Lambda_{\rm QCD}^2/Q^2$-plane, except at singular points of $\delta W_X(e^{5/3}\Lambda_{\rm QCD}^2/Q^2)$ and the origin.

In this construction we can regard that the essential part is determined by $W_X^{(m)}(z)$, which already gives the required analyticity of $X_0(Q^2)$. $\delta W_X(z)$ is subsidiary in the sense that it is not necessary in an essential way and should not have singularities (except at $\Lambda_{\rm QCD}^2/Q^2 = \infty$) in order not to violate the required analyticity of $X_0(Q^2)$ or $X_{\rm UV}(Q^2)$. Thus, we may say that the massive gluon scheme is an optimal (or minimal) scheme in terms of the analyticity, according to this construction of $X_0(Q^2)$.

We would like to know how many pre-weights are allowed as a reference scheme in the above construction of $X_0(Q^2)$, or in other words, how many minimal schemes exist. The integral expression (2.30) is used to define the reference scheme, and this expression is realized naturally by the following conditions on the pre-weight:[16]

(0) $W_X(z)$ is analytic in the upper half-plane, and

$$2\,{\rm Im}\,W_X(x) = w_X(x) \quad \text{for} \quad x \geq 0\,. \tag{2.72}$$

(1) $${\rm Im}\,W_X(x) = 0 \quad \text{for} \quad x \leq 0\,. \tag{2.73}$$

(2) $\displaystyle\int_{C_R} \frac{dz}{\pi z}\, W_X(z)$ is absolutely convergent to 0 as $R \to \infty$,

$$\text{where } C_R = \{Re^{i\theta} | 0 \leq \theta \leq \pi\}. \tag{2.74}$$

The pre-weight in the massive gluon scheme $W_X^{(m)}(z)$ satisfies the conditions (0) and (1). If it also satisfies the condition (2) (see footnote 14), we can rotate the integration path to the negative axis, and the expression (2.30) is obtained, namely, $X_{\rm UV}$ satisfies the required analyticity.

We now prove that the above conditions (0)–(2) are sufficient to determine the pre-weight uniquely. Let us examine the difference of the pre-weights satisfying the above conditions:

$$\delta W_X(z) = W_X^{(2)}(z) - W_X^{(1)}(z). \tag{2.75}$$

We can translate the conditions (0) – (2) into conditions for δW_X as

$${\rm Im}\,\delta W_X(x) = 0 \text{ for } x \in \mathbb{R}\,, \tag{2.76}$$

[16] The condition (0) is already included in the definition of a general $W_X(z)$. Note also that $w_X(0) = 0$ due to our assumption that $X_{\rm LO}$ is IR finite [see Eq. (2.1)], hence the conditions (0) and (1) are mutually consistent at $x = 0$.

$$\int_{C_R} \frac{dz}{\pi z} \delta W_X(z) \text{ is absolutely convergent to 0 as } R \to \infty. \tag{2.77}$$

Using Eq.(2.77), we can show

$$\text{Pr.} \int_{-\infty}^{\infty} \frac{dx}{\pi} \frac{\delta W_X(x)}{x - x'} = i\, \delta W_X(x'), \tag{2.78}$$

where Pr. denotes the principal value integral and x' is assumed to be a real parameter. Taking the imaginary part of this equation and using Eq.(2.76), we obtain

$$\mathrm{Re}\, \delta W_X(x) \equiv 0 \text{ for } x \in \mathbb{R}. \tag{2.79}$$

One can see from Eq.(2.76) and Eq.(2.79) that δW_X is identically zero in the upper half-plane including the real axis (by the identity theorem). Since we do not expect any physical singularity to disturb enlargement of this analyticity domain,[17] we can conclude that $\delta W_X = 0$ in the entire complex plane:

$$\delta W_X(z) \equiv 0 \text{ for } z \in \mathbb{C}. \tag{2.80}$$

Hence, if $W_X^{(m)}(z)$ in the massive gluon scheme satisfies the condition (2), the allowed scheme is uniquely determined to this one. This is the case in the explicit examples considered in Secs. 2.3 and 2.4.

This argument shows that, instead of requiring the analyticity of X_0, through the above conditions (0)–(2) we can satisfy the analyticity requirement and single out the pre-weight uniquely simultaneously. Note that once the pre-weight is uniquely fixed, the separation of X_{UV} into X_0 and power corrections $\sim (\Lambda_{\mathrm{QCD}}^2/Q^2)^n$ becomes unambiguous.

2.7 Behaviors of X_{UV} in massive gluon scheme

In the previous subsection, we pointed out that the massive gluon scheme can be regarded as special among all the schemes. We examine some details of the behaviors of $X_0(Q^2)$ and the power corrections in X_{UV}, respectively, in this scheme. [18]

Behavior of $X_0(Q^2)$

As discussed below Eq. (2.70), the behavior of $X_0(Q^2)$ close to $1/Q^2 = 0$ is determined by the fact that $X_0(Q^2)$ has the same asymptotic expansion as the perturbative series of $X(Q^2)$; see Eq. (2.26). Namely the behavior of $X_0(Q^2)$ at large Q^2 is almost insensitive to the scheme of $W_X^{(m)}$. In contrast, the global behavior of $X_0(Q^2)$ generally depends on the scheme of W_X.

[17] Note that a singularity in $\delta W_X(z)$ except for a cut along the negative real axis generates an additional singularity in $X_0^{(2)}(Q^2)$ compared with $X_0^{(1)}(Q^2)$ as one can see from Eq.(2.71).

[18] In this subsection, we assume that $W_X^{(m)}(z)$ has a good convergence for large $|z|$ in the upper half-plane including the real axis.

Let us examine some details about the massive gluon scheme. The limit of X_0 in this scheme at $1/Q^2 \to \infty$ is calculated from Eq. (2.32) as

$$X_0(Q^2) \to \frac{4\pi}{\beta_0} W_{X+}^{(m)}(0) = \frac{4\pi}{\beta_0} d_0 \,. \tag{2.81}$$

Namely $X_0(Q^2)$ approaches a constant for sufficiently large $1/Q^2$.

In addition, if we regard $X_0(Q^2)$ as a function of $\hat{Q}^{-2} = \Lambda_{\rm QCD}^2/Q^2$, we can see that X_0 and its derivatives have definite signs at least for the two examples which we studied:

$$X_0 \geq 0 \quad ; \quad \frac{dX_0}{d(\hat{Q}^{-2})} \geq 0 \quad ; \quad \frac{d^2 X_0}{d(\hat{Q}^{-2})^2} \leq 0 \quad \text{for } X = D, \alpha_F \,. \tag{2.82}$$

This property follows from $W'_{X+}(x) \leq 0$, $W''_{X+}(x) \geq 0$ for $x \geq 0$, and

$$\frac{d^n X_0}{d(\hat{Q}^{-2})^n} = \int_0^\infty \frac{d\tau}{\pi\tau} \left(\frac{\tau}{\Lambda_{\rm QCD}^2} \right)^n \left. \frac{d^n W_{X+}^{(m)}(x)}{dx^n} \right|_{x=\frac{\tau}{Q^2}} \frac{4\pi}{\beta_0} \frac{-\pi}{\log^2\left(\tau e^{-5/3}\Lambda_{\rm QCD}^2\right) + \pi^2} \,. \tag{2.83}$$

As a result, combined with the asymptotic forms at $1/Q^2 = 0, \infty$, the behavior of each X_0 is determined globally and the form is simple (and similar), as seen from Figs. 2 or 3.

Power corrections in $X_{\rm UV}$

We show that the power corrections in $X_{\rm UV}$ can be detected generally from the Borel transformation. Consider an integral

$$C_X(v) \equiv \int_0^\infty \frac{dz}{2\pi} W_X^{(m)}(z)\, z^{-v-1} \,. \tag{2.84}$$

The expansion of $W_X^{(m)}(z)$ for small-z is determined by the singularities of $C_X(v)$ as [c.f., Eqs. (2.10) and (2.12)]

$$W_X^{(m)}(z) = -2\pi \sum_{n \in V_{\rm IR}} {\rm Res}_{v=n}[C_X(v) z^v] = \sum c_n z^n \,, \tag{2.85}$$

where $V_{\rm IR}$ denotes a set of non-negative poles of $C_X(v)$. Using Eq. (2.27), $C_X(v)$ is explicitly calculated in the massive gluon scheme as

$$\begin{aligned} C_X(v) &= \int_0^\infty \frac{dx}{2\pi} w_X(x) \int_0^\infty \frac{dz}{2\pi} \frac{z^{-v-1}}{x - z - i0} \\ &= -\frac{1}{2} \frac{e^{-i\pi v}}{\sin(\pi v)} \int_0^\infty \frac{dx}{2\pi} w_X(x) x^{-u-1} \\ &= -\frac{1}{2} \frac{e^{-i\pi v}}{\sin(\pi v)} B_X(v) = -\frac{1}{2} \frac{\cos(\pi v)}{\sin(\pi v)} B_X(v) + \frac{i}{2} B_X(v) \,, \end{aligned} \tag{2.86}$$

where we used Eq. (2.10). (The same equation was derived in Ref.[11] in a different context.) By taking the imaginary part of $C_X(v)$, we can check that the usual Borel transformation is obtained consistently with Eq. (2.14). The factor $\{\sin(\pi v)\}^{-1}$ in the real part of Eq. (2.86) generates additional integer poles, that is, $U_{\rm IR} \subset V_{\rm IR}$. In particular, the first few terms of the expansion of $W_X^{(m)}(z)$ stem from this factor and reduce to real coefficients:

$$W_X^{(m)}(z) = \sum_{0 \le n < u_{\rm IR}} B_X(n) z^n + \dots\,, \tag{2.87}$$

where we use Eq. (2.85). Therefore, from Eqs. (2.18), (2.23) and (2.87), the coefficient of the $(e^{5/3}\Lambda_{\rm QCD}^2/Q^2)^n$-term of $X_{\rm UV}$ is revealed to be $4\pi B_X(n)/\beta_0$.

Incidentally, we have a similar relation for $W_{X+}^{(m)}$ in the massive gluon scheme as

$$\begin{aligned} C_{X+}(u) &\equiv \int_0^\infty \frac{dz}{2\pi} W_{X+}^{(m)}(z) z^{-u-1} \\ &= \int_0^\infty \frac{dx}{2\pi} w_X(x) \int_0^\infty \frac{dz}{2\pi} \frac{z^{-u-1}}{x+z-i0} \\ &= -\frac{1}{2}\frac{1}{\sin(\pi u)} \int_0^\infty \frac{dx}{2\pi} w_X(x) x^{-u-1} \\ &= -\frac{1}{2}\frac{1}{\sin(\pi u)} B_X(u), \end{aligned} \tag{2.88}$$

where we used Eqs. (2.29) and (2.10).

Note that Eq. (2.87) does *not* mean that the power corrections included in $X_{\rm UV}$ are related to perturbative ambiguity, but it is purely a mathematical relation. We explore the origin of the expansion of $W_X^{(m)}(z)$ and clarify the meaning in terms of the method of expansion by regions in the next section.

3 Power corrections and OPE in light of expansion by regions

In this section we investigate (i) the origin of the power corrections in $X_{\rm UV}$, and (ii) the relation of $X_{\rm UV}$ to Wilson coefficients in OPE, by means of the method of expansion by regions, or asymptotic expansion in limits of large momentum [27–29]. With this method, we can identify which momentum region contributes to each power correction. We show that the power corrections in $X_{\rm UV}$ for the Adler function Eq. (2.42) and the interquark force Eq. (2.55), respectively, are genuine UV contributions. We also provide an effective field theoretical point of view of our framework presented in the previous section.

We first discuss some general aspects (Sec. 3.1) and subsequently clarify detailed features explicitly in the examples of the Adler function (Sec. 3.2) and the interquark force or QCD potential (Sec. 3.3).

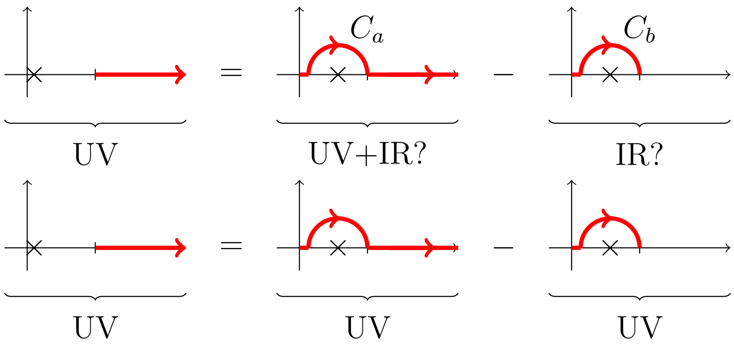


Figure 4. Deformed integral path introduced in Sec. 2.2 and different interpretations on relevant kinematical regions. [Upper figure] The pole contribution is interpreted to be an IR effect. [Bottom figure] The pole contribution is interpreted to be a UV effect, which is shown to be legitimate for the Adler function and the force between $Q\bar{Q}$.

3.1 General aspects

We discuss two issues using the method of expansion by regions. First we answer the question: "Which kinematical regions do the power corrections $(\Lambda_{\rm QCD}^2/Q^2)^n$ originate from?" This question can be addressed accurately in the massive gluon scheme. The motivation to ask this question is as follows. Since the power corrections stem from the contour C_b in Eq. (2.16) close to the IR pole at $\tau = e^{5/3}\Lambda_{\rm QCD}^2$, one may suspect that the power corrections originate from IR contributions, although we claim that $X_{\rm UV}$ consists of UV contributions. (See Figs. 4.)

We can use the method of expansion by regions in the following manner. The $1/Q^2$ expansion of $X_{\beta_0}(Q^2;\mu_f)$ is determined by the small-z expansion of $W_X^{(m)}(z)$ in the integral along C_b [Eq. (2.17)], and the $(\Lambda_{\rm QCD}^2/Q^2)^n$-terms are included as a part of it. Since the pre-weight in the massive gluon scheme $W_X^{(m)}(\tau/Q^2)$ is expressed as a usual Feynman integral with a massive gluon propagator, we can use the expansion-by-regions technique to decompose the small-(τ/Q^2) expansion of $W_X^{(m)}(\tau/Q^2)$ into contributions from different kinematical regions. In this computation Q^2 plays the role of a hard scale, whereas τ plays the role of a soft scale, since $|\tau| \leq \mu_f^2 \ll Q^2$ along C_b.

If the first few terms of the power corrections are found to originate from UV region, we can further deduce that the integral along C_a [$= X_0(Q^2) - 4\pi c_0/\beta_0$] also originates (dominantly) from UV region for large Q^2. This is because, $X_{\beta_0}(Q^2;\mu_f)$ consists of UV contributions (given by an integral over $\tau \geq \mu_f^2$), and the C_a-integral is given by the difference of X_{β_0} and the C_b-integral.

Secondly the correspondence between $X_{\beta_0}(Q^2;\mu_f)$ in our formulation and OPE in a low-energy effective field theory can be examined using expansion-by-regions of Feynman diagrams [27].[19] Since an early stage of the development of the expansion-

[19] This part of the analysis deals with X_{β_0} at each order of perturbation and has only minor

by-regions method, its relation to effective field theory and OPE has been explored [27–29]. The hard contributions in the context of expansion by regions are interpreted as Wilson coefficients in the effective field theory, and the soft contributions are interpreted as perturbative quantum corrections due to low-energy degrees of freedom. In other words, the low-energy effective Lagrangian is constructed by including hard contributions in terms of effective vertices whereas the soft contributions are left to be evaluated. This procedure is what is usually called "integrating out hard modes." In OPE, the correspondence between hard contributions and Wilson coefficients are unchanged, while quantum corrections due to low-energy degrees of freedom are evaluated as non-perturbative matrix elements.

$X_{\beta_0}(Q^2;\mu_f)$ introduced in Eq. (2.13) can also be interpreted as a Wilson coefficient in OPE, as we will see in explicit examples below. However, the way to separate UV and IR effects is different from that of expansion by regions in the following sense: (i) The Wilson coefficient of our method is regularized by a cutoff, whereas the one in the expansion-by-regions method is formulated in dimensional regularization. (ii) We separate the UV contribution from the IR contribution by only one measure, i.e., scale of the gluon momentum. On the other hand, the method of expansion by regions distinguishes momentum regions with a finer resolution in general.

In the case that the relevant low-energy effective field theory is known, the expansion-by-regions technique is a standard tool to systematically compute Wilson coefficients to high orders. Detailed connection between the full theory and the effective field theory can be made, including correspondence of relevant kinematical regions. Since $X_{\beta_0}(Q^2;\mu_f)$ in our formulation is well defined in the full theory, using this information it is possible to establish a firm connection between $X_{\beta_0}(Q^2;\mu_f)$ and Wilson coefficients, as we have briefly reviewed in Sec. 2.4.

On the other hand, in the case that the relevant effective field theory is unknown, we can still infer its structure using the expansion by regions, as well as factorize UV and IR contributions to the Wilson coefficients and non-perturbative matrix elements, respectively, in dimensional regularization. Changing to a cutoff regularization would be less founded, since consistent treatment is not guaranteed by an effective field theory framework. Furthermore, since the analysis necessarily becomes diagram-based, correspondence with operators, gauge symmetry, or the equation of motion is not transparent.

We can clarify these two issues in explicit computations for the observables which we studied already, the Adler function and the QCD force (or the QCD potential).

3.2 Example 1: Adler function

Using the expansion-by-region method we first compute the small-z expansion of the pre-weight $W_D^{(m)}(z)$ for the Adler function in the massive gluon scheme. In this way

connection with the separation of X_{β_0} into X_0 and power corrections or with the scheme dependence.

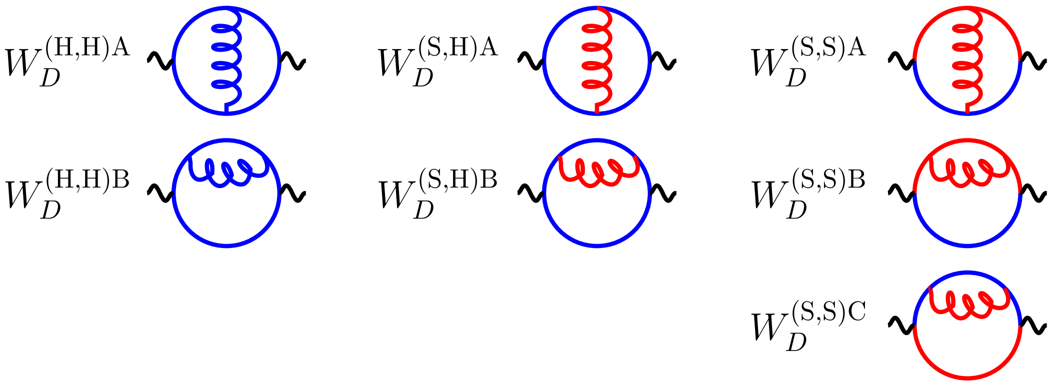


Figure 5. Different kinematical regions contributing to the Adler function in light of expansion by regions. A blue (red) line represents that a hard (soft) momentum $\sim Q\,(\sim \sqrt{\tau})$ is flowing through the line.

we can identify which kinematical region the power correction in Eq. (2.42) originate from.

The kinematical regions contributing to the expansion of $W_D^{(m)}(z)$ are shown in Fig. 5, in which blue (red) lines carry hard (soft) momenta. Note that, since the external momentum is hard, a hard momentum should flow through between the two external vertices. There is no contribution from the kinematical region, in which the gluon is hard and some of the quark lines are soft.[20] Hence, we divide the kinematical regions into three regions, (H,H), (S,H) and (S,S), as shown in the figure:

$$W_D^{(m)}(z) = W_D^{(\mathrm{H,H})}(z) + W_D^{(\mathrm{S,H})}(z) + W_D^{(\mathrm{S,S})}(z)\,, \tag{3.1}$$

where $W_D^{(\mathrm{H,H})} = W_D^{(\mathrm{H,H})\mathrm{A}} + W_D^{(\mathrm{H,H})\mathrm{B}}$, etc.

For instance, the "all-hard" contribution $W_D^{(\mathrm{H,H})}$ is computed as follows. Recall that $W_D^{(m)}$ in the massive gluon scheme is given as [c.f., Eq. (2.28)]

$$W_D^{(m)}(\tau/Q^2) = \int_0^\infty \frac{d(p^2)}{2\pi}\,\frac{1}{p^2-\tau}\,w_D(p^2/Q^2)\,. \tag{3.2}$$

$W_D^{(\mathrm{H,H})}$ is obtained by expanding the gluon propagator in τ as

$$W_D^{(\mathrm{H,H})}(\tau/Q^2) = \sum_{n=0}^{\infty}\int_0^\infty \frac{d(p^2)}{2\pi}\,\frac{\tau^n}{(p^2)^{n+1}}\,w_D(p^2/Q^2)\,, \tag{3.3}$$

where it is understood that w_D is regularized by dimensional regularization.[21] Apart from the gluon propagator, the integrand does not receive any modification since the

[20] This is because in such a region the soft scale integral becomes scaleless, since the soft scale τ is included only in the gluon propagator, and if gluon is hard, after expansion in τ no soft scale remains in the denominator.

[21] It means that one should not use Eq. (2.36) for w_D. One expands the integrand before performing any momentum integral while keeping $\varepsilon \neq 0$.

soft-scale parameter τ is contained only in the factor $1/(p^2-\tau)$. The result of the computation reads

$$W_D^{(\mathrm{H,H})}(z) = N_C C_F \left[\frac{1}{4\pi} + \frac{8-6\zeta_3}{3\pi} z + \left(-\frac{[\epsilon^{-1}]}{2\pi} + \frac{4-12\zeta_3}{6\pi} \right) z^2 + \left(-\frac{[\epsilon^{-2}]}{6\pi} + \frac{5[\epsilon^{-1}]}{12\pi} + \frac{265}{216\pi} + \frac{\pi}{36} \right) z^3 + O\left(z^4\right) \right]. \tag{3.4}$$

Divergent terms are denoted as

$$[\epsilon^{-1}] = \frac{1}{\varepsilon} - \gamma_{\mathrm{E}} + \log\left(\frac{4\pi Q^2}{\mu^2}\right) + \mathcal{O}(\varepsilon)\,, \tag{3.5}$$

$$[\epsilon^{-2}] = \frac{1}{\varepsilon^2} - \frac{2}{\varepsilon}\left[\gamma_{\mathrm{E}} - \log\left(\frac{4\pi Q^2}{\mu^2}\right)\right] + 2\left[\gamma_{\mathrm{E}} - \log\left(\frac{4\pi Q^2}{\mu^2}\right)\right]^2 + \mathcal{O}(\varepsilon), \tag{3.6}$$

where the space-time dimension is denoted as $d = 4-2\varepsilon$; $\gamma_{\mathrm{E}} = 0.57721...$ is the Euler constant and μ is the renormalization scale. If we neglect $\log(Q^2/\mu^2)$, these terms correspond to those which are subtracted in the usual $\overline{\mathrm{MS}}$ renormalization.

The results of the other two contributions are given by

$$W_D^{(\mathrm{S,H})}(z) = N_C C_F \left[\left(\frac{[\epsilon^{-1}]}{2\pi} + \frac{6-3\log z + 3i\pi}{6\pi} \right) z^2 + \left(\frac{[\epsilon^{-2}]}{3\pi} - \frac{1+\log(-z)}{3\pi}[\epsilon^{-1}] + \frac{\log^2(-z)}{6\pi} + \frac{\log(-z)}{3\pi} - \frac{91}{54\pi} \right) z^3 + O\left(z^4\right) \right], \tag{3.7}$$

$$W_D^{(\mathrm{S,S})}(z) = N_C C_F \left[\left(-\frac{[\epsilon^{-2}]}{6\pi} + \frac{4\log(-z)-1}{12\pi}[\epsilon^{-1}] - \frac{\log^2(-z)}{3\pi} + \frac{\log(-z)}{6\pi} - \frac{\pi}{12} + \frac{35}{216\pi} \right) z^3 + O\left(z^4\right) \right], \tag{3.8}$$

where we use a short-hand notation $\log(-z) \equiv \log z - i\pi$. Although $W_D^{(\mathrm{H,H})}$, $W_D^{(\mathrm{S,H})}$ and $W_D^{(\mathrm{S,S})}$ individually contain the divergent terms (3.5), (3.6), which are μ-dependent, these terms cancel altogether in the sum (3.1).[22]

The first two terms of the all-hard contribution [Eq. (3.4)] are exactly equal to the first two terms (c_0 and c_1) of the expansion of $W_D^{(m)}(z)$ [Eq. (2.40)], while the order z^0 and z^1 terms are absent in $W_D^{(\mathrm{S,H})}(z)$ and $W_D^{(\mathrm{S,S})}(z)$. Therefore we conclude that the μ_f-independent $\Lambda_{\mathrm{QCD}}^2/Q^2$-term of $D_{\mathrm{UV}}(Q^2)$ belongs to the hard contribution. Consequently the dominant part of $D_0(Q^2)$ is also UV origin, according to the argument in the previous subsection. Thus, the lower figure of Figs. 4 corresponds to the proper interpretation up to order $1/Q^2$.

In Sec. 2.2 we found that the imaginary part of the expansion coefficients of $W_X^{(m)}(z)$ results in μ_f-dependent terms, and the μ_f-dependent terms are related to

[22] Cancellation of divergent terms is a common feature in the method of expansion by regions and signifies that the result is independent of the factorization scale separating the soft and hard regions.

	c_0	c_1	c_2	c_3
$W_D^{(H,H)}(z)$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
$W_D^{(S,H)}(z)$			$\mathbb{C}$	$\mathbb{C}$
$W_D^{(S,S)}(z)$				$\mathbb{C}$

Table 1. Expansion coefficients c_n of the contribution from each kinematical region to $W_D^{(m)}(z)$ up to order z^3 [Eqs. (3.4), (3.7), (3.8)]. The symbol "$\mathbb{R}$" stands for a non-zero real value, while "$\mathbb{C}$" represents a complex value with non-zero imaginary part. A blank represents that the coefficient is zero.

IR contributions. The expansion-by-regions analysis shows that the imaginary part of the expansion coefficients stems only from the region where the gluon has a soft-scale momentum. This is because the only source of imaginary part is the integral of $1/(p^2-\tau)$. Indeed $W_D^{(\mathrm{S,H})}(z)$ and $W_D^{(\mathrm{S,S})}(z)$ include imaginary part. Oppositely, the all-hard contribution $W_D^{(\mathrm{H,H})}(z)$ in Eq. (3.4) is explicitly real. In fact, Eq. (3.3) shows that $W_D^{(\mathrm{H,H})}(z)$ is Euclidean and real to all orders in $1/Q^2$ expansion.

In Ref. [11], using the method of massive gluon, terms which are non-analytic in the gluon mass λ are identified as IR contributions, while terms which are power-like in λ as UV contributions. Written in the form of Eq. (3.3), the source of the imaginary part can be attributed to the same origin. For example, a non-analytic term $\log \lambda^2$ generates an imaginary part when we substitute $\lambda^2 = -\tau$ with $\tau > 0$.

In Tab. 1 we summarize the contribution from each momentum region to the expansion coefficients of $W_D^{(m)}(z)$ up to $\mathcal{O}(z^3)$ [Eqs. (3.4), (3.7), (3.8)]. The first two coefficients of $W_D^{(m)}(z)$ originate only from the all-hard region [(H,H) region], and there is no divergence up to this order. From the order z^2, the contribution from each region diverges and only the sum is finite. In the case that each contribution is divergent, it is μ-dependent and the separation between different regions becomes somewhat vague. The contribution from the soft-gluon and hard-quark region [(S,H) region] starts at order z^2, and the region where the gluon is soft and some of the quarks are soft [(S,S) region] contributes from order z^3. As already mentioned above, (S,H) and (S,S) contributions have non-zero imaginary part. Notably these regions also contribute to the real part, although the values are divergent and become definite only after they are added to the (H,H) contributions. Namely, from the order z^2, the real part of the expansion coefficients receive mixed contributions from the hard and soft momentum regions of the gluon. This is in contrast to the imaginary part, to which only the soft-gluon-momentum regions can contribute.

It is worth emphasizing that the soft contributions are not always pure imaginary, i.e., not all of the real part of the expansion coefficients originate from the hard-gluon

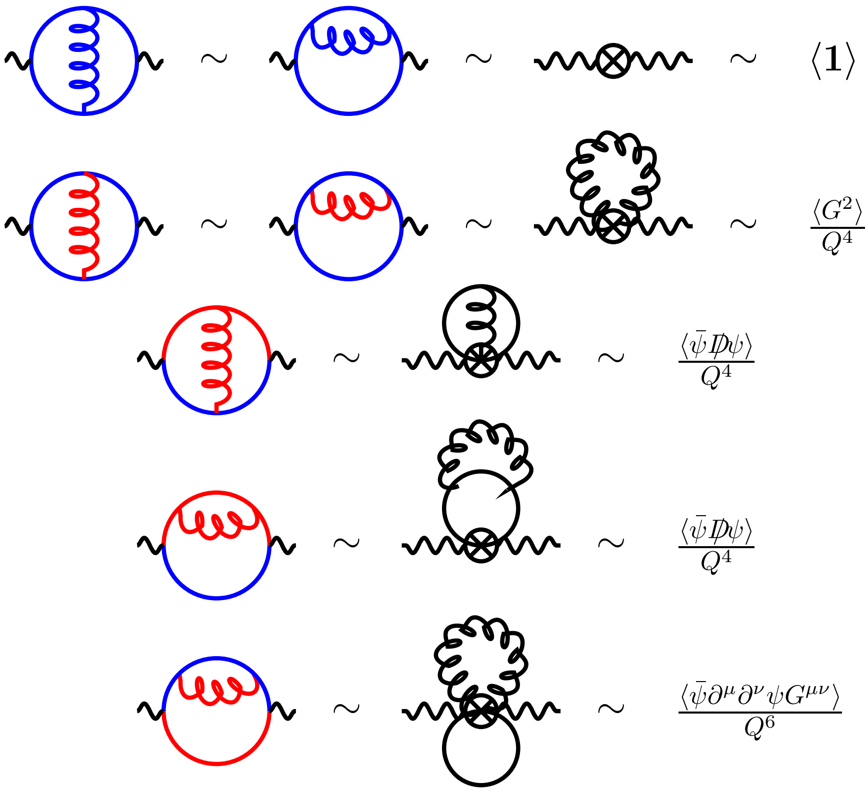


Figure 6. Relations between contributions to the Adler function evaluated with the expansion-by-regions method (colored graphs; c.f. Fig. 5) and those with a (would-be) low-energy effective field theory (black graphs). The corresponding terms in OPE are also shown.

region. Thus, the method of expansion by regions has a finer resolution than our analysis given in Sec. 2.2 and detects soft contributions even in the real part of c_n for $n \geq u_{\rm IR}$. From this detailed examination, we confirm consistency[23] of our treatment of $X_{\rm UV}$ in Sec. 2.2, where we classify as the genuine UV contribution the $(\Lambda_{\rm QCD}^2/Q^2)^n$-terms for $0 \leq n < u_{\rm IR}$.

Let us turn to examine OPE of the Adler function using the method of expansion by regions. The relevant low-energy effective field theory is not known. Applying the expansion-by-regions method to the diagrams for the reduced Adler function, they are decomposed into contributions from different kinematical regions as shown in Fig. 6. For instance, the all-hard region can be identified with the Wilson coefficient for the identity operator, as illustrated in the figure. Similarly, a contribution involving soft gluons/quarks can be identified with the matrix element of a higher-dimensional local operator times its Wilson coefficient. This includes the local gluon condensate at order $1/Q^4$.

[23] There may occur a contradiction to the result of Sec. 2.2 in exceptional cases where the leading soft-gluon contribution happens to be pure real. This may happen if the leading IR renormalon vanishes accidentally.

Using this correspondence, we argue that $D_{\rm UV}(Q^2)$ is almost identified with the Wilson coefficient of the identity operator $C_1(Q^2;\mu_f)$. Recall that $D_{\rm UV}(Q^2)$ is a μ_f-independent part of $D_{\beta_0}(Q^2;\mu_f)$, which is diagrammatically given by Fig. 1 with an effective coupling $\alpha_{\beta_0}(\tau)$ and an IR cutoff μ_f of the gluon momentum. As inferred from the above correspondence, in general the all-hard region of a diagram, where all the momenta are larger than the cutoff scale μ_f, contributes to $C_1(Q^2;\mu_f)$, since the entire loop integral shrinks to a local vertex. In contrast a contribution which includes soft modes becomes a non-perturbative matrix element times its Wilson coefficient.

$D_{\beta_0}(Q^2;\mu_f)$ is slightly different from C_1 in that it includes both hard and soft quarks. The leading contribution involving soft quarks reduces to the matrix element of the dimension-six operator $(\bar{q}\gamma^\mu q)(\bar{q}\gamma_\mu q)$ made only of the quark field.[24] (Note that the dimension-three operator $\bar{q}q$ is absent since it appears together with the quark mass.) Thus, we obtain[25]

$$D_{\beta_0}(Q^2;\mu_f) = C_1(Q^2;\mu_f) + \mathcal{O}(\mu_f^6/Q^6)\,. \tag{3.9}$$

As a result, the μ_f-independent part $D_{\rm UV}$ is identified with $C_1(Q^2;\mu_f)$ up to $\mathcal{O}(\mu_f^4/Q^4)$ via Eq. (2.41):

$$C_1(Q^2;\mu_f) - D_{\rm UV}(Q^2) = \mathcal{O}(\mu_f^4/Q^4)\,. \tag{3.10}$$

In particular, the power correction $\Lambda_{\rm QCD}^2/Q^2$ in $D_{\rm UV}$ is a part of the Wilson coefficient C_1.

We note that the matrix element of the dimension-four operator $\bar{q}\not{D}q$ vanishes by the equation of motion $\not{D}q = 0$. Essentially the same effect is observed in the computation of the expansion coefficient of $W_D^{(m)}$ in the first part of this subsection. $W_D^{(S,S)}$ represents contributions in which the gluon and some of the quarks have soft momenta (see Fig. 5). By explicit computation we confirm that each of $W_D^{(S,S)A}$ and $W_D^{(S,S)B}$ is nonzero at $\mathcal{O}(z^2)$, while they cancel in the sum, resulting in the z^3 term as the leading term in their sum.[26] This property is considered to be a consequence of gauge invariance and the equation of motion. Since each diagram does not respect gauge invariance, the soft contribution from each diagram at order $1/Q^4$ is non-vanishing, but the sum of all the diagrams should vanish at this order. The first gauge invariant operator involving soft quarks is dimension six, as already noted.

[24] Although one may be worried that the cutoff regularization would break gauge invariance and generate gauge non-invariant operators, in fact our regularization method preserves gauge invariance.

[25] Contributions from the soft region of the fermion bubble subgraphs are also suppressed.

[26] The same cancellation mechanism cannot be seen explicitly in the computation of the massless diagrams in Fig. 6, since the soft-scale integrals are scaleless and vanish for all the diagrams. On the other hand, in Fig. 5 the gluon mass τ acts as the soft scale.

3.3 Example 2: QCD potential

The UV contribution to the force between the static quark and antiquark $\alpha_{F,\mathrm{UV}}(1/r^2)$ and its power behavior are analyzed in Sec. 3.3, where it is shown that the μ_f-independent r^2-term exists in $\alpha_{F,\mathrm{UV}}(1/r^2)$. This result is obtained from the one-dimensional integral representation of the QCD potential (2.46), and we investigate the QCD potential in this subsection.

The pre-weight $W_V^{(m)}(z)$ of the QCD potential in the massive gluon scheme, where $z = \tau r^2$, is given by

$$W_V^{(m)}(z) = -\frac{2C_F}{\pi}\int_0^\infty \frac{\sin(pr)p}{p^2-\tau-i0}dp \tag{3.11}$$

$$= -C_F e^{i\sqrt{z}}\,. \tag{3.12}$$

We investigate the kinematical regions which contribute to the small-z expansion of $W_V^{(m)}(z)$. To apply the method of expansion by regions, we introduce a variant of the dimensional regularization by replacing $dp \to p^{-2\varepsilon}dp$ in Eq. (3.11). While in the conventional expansion-by-regions method [29] only Feynman integrals in momentum space are considered, Eq. (3.11) contains both coordinate-space variable (r) and momentum-space variable (p). To our knowledge, there is no systematic argument concerning validity of the expansion-by-regions method in such cases. Nevertheless in the current specific example, we can show validity of the method using the argument in Ref.[30].

Similarly to Eq. (3.3), the contribution from the hard region $p \sim 1/r$ is given as

$$W_V^{(H)}(z) = -\frac{2C_F}{\pi}\sum_{n=0}^{\infty}\int_0^\infty \frac{\tau^n}{(p^2)^{n+1}}\sin(pr)p^{1-2\varepsilon}dp \tag{3.13}$$

$$= -\frac{2C_F}{\pi}r^{2\varepsilon}z^n\frac{\pi\Gamma(-2n-2\varepsilon)}{\Gamma(n+\varepsilon+1)\Gamma(-n-\varepsilon)} \tag{3.14}$$

$$= -C_F\cos(\sqrt{z}) + \mathcal{O}(\varepsilon)\,, \tag{3.15}$$

while the contribution from the soft region $p \sim \sqrt{\tau}$ is given as

$$W_V^{(S)}(z) = -\frac{2C_F}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^\infty \frac{(pr)^{2n+1}}{p^2-\tau-i0}p^{1-2\varepsilon}dp \tag{3.16}$$

$$= -\frac{C_F}{\pi}\frac{(\sqrt{z})^{2n+1}}{(2n+1)!}\tau^{-\varepsilon}\Gamma(\varepsilon-\frac{1}{2})\Gamma(\frac{3}{2}-\varepsilon)e^{i\pi(n-\varepsilon+1/2)} \tag{3.17}$$

$$= -iC_F\sin(\sqrt{z}) + \mathcal{O}(\varepsilon)\,. \tag{3.18}$$

Both hard and soft contributions are finite as $\varepsilon \to 0$ to all orders in the small-z expansion.

The hard and soft contributions separate into the real and imaginary part, respectively. There is no mixed contribution from both regions to each expansion coefficient, so that the correspondence is simpler than the Adler function. Namely, each coefficient is either real or pure imaginary, where the former originates from the hard region and the latter from the soft region. The real and imaginary coefficients appear alternately. The order z^1 term of $\cos(\sqrt{z})$ gives the $\Lambda_{\rm QCD}^2 r^2$ term of $\alpha_{F,\rm UV}$, which indeed stems from the hard region.

As already explained in Sec. 2.4, the effective field theory for the QCD potential is known as pNRQCD, and its construction can be understood using the integration-by-regions method. According to this understanding, pNRQCD for the static QCD potential is constructed by integrating out the so-called "hard" and "soft" scales. The remaining active dynamical degrees of freedom are those in the "ultra-soft" scale and the $\Lambda_{\rm QCD}$ scale.[27]

Computations in the framework of pNRQCD is systematically organized using the multipole expansion, which gives an OPE in this effective field theory. A number of Wilson coefficients in pNRQCD have been computed using the method of expansion by regions. Wilson coefficients are usually regularized by dimensional regularization and they contain divergences in general. It is possible to change to another regularization scheme within pNRQCD framework, and the physical predictions should not depend on the regularization scheme. Hence, through such a route, computation of the QCD potential in our framework can be related to that of pNRQCD or full QCD without any ambiguity.

In dimensional regularization and in strict expansion in α_s, the leading Wilson coefficient $V_S(r)$ in Eq. (2.58) coincides with $V_{\rm QCD}(r)$ to all orders in α_s, since contributions from the ultra-soft and $\Lambda_{\rm QCD}$ scales (e.g., the second term) evaluate to scaleless integrals at each order of α_s in the expansion-by-regions method. In OPE the ultra-soft and $\Lambda_{\rm QCD}$ contributions turn into non-perturbative matrix elements. If we adopt the large-β_0 approximation and the cutoff in the gluon momentum, $V_S(r)$ coincides with $V_{\beta_0}(r;\mu_f)$ in our formulation. At the same time, the leading non-perturbative matrix element is estimated as order $\mu_f^3 r^2$ in this regularization scheme. By examining the matching between full QCD and pNRQCD in detail, we can check that with this regularization Eq. (2.58) also achieves a consistent separation of the UV (hard+soft) and IR (ultrasoft+$\Lambda_{\rm QCD}$) contributions to the whole static QCD potential $V_{\rm QCD}(r)$. Details of the computation can be found, e.g., in [17].

4 Relation between $X_{\rm UV}$ and perturbative series at large orders

In this section we show that $X_{\rm UV}(Q^2)$ derived in Sec. 2.2 is reproduced from the perturbative series in the large-β_0 approximation at large orders. Based on this

[27] There is no contribution from the "potential region" in the computation of the static QCD potential due to the fact that the static propagator originating from the Wilson line does not include the kinetic energy term $\sim \vec{p}^{\,2}/(2m)$.

observation we confirm that our result $X_{\rm UV}$ is consistent with the renormalon uncertainty and the OPE framework.

The smallest term of the perturbative series of Eq. (2.3) is given at around $n_* = 4\pi u_{\rm IR}/(\beta_0\alpha_s)$, hence, it is natural to regard the truncated series at this order $X_{n_*}(Q^2)$ as an optimal prediction within perturbation theory. The uncertainty of the prediction $X_{n_*}(Q^2)$ is of the order of $(\Lambda_{\rm QCD}^2/Q^2)^{u_{\rm IR}}$. By taking a small $\alpha_s(\mu)$, we can examine the large order perturbative series keeping the perturbative series finite, since n_* becomes large in this case. The truncated series X_{n_*} is written as

$$\begin{aligned}
X_{n_*}(Q^2) &= \sum_{n=0}^{n_*-1}\int_0^\infty \frac{d\tau}{2\pi\tau}\, w_X\left(\frac{\tau}{Q^2}\right)\alpha_s(\mu)\left[\frac{\beta_0\alpha_s(\mu)}{4\pi}\log\left(\frac{\mu^2 e^{5/3}}{\tau}\right)\right]^n \\
&= \int_0^\infty \frac{d\tau}{2\pi\tau}\, w_X\left(\frac{\tau}{Q^2}\right)\alpha_s(\mu)\frac{1-L^{n_*}}{1-L} \qquad (4.1)\\
&= \mathrm{Im}\int_0^\infty \frac{d\tau}{\pi\tau}\, W_X\left(\frac{\tau}{Q^2}\right)\alpha_s(\mu)\frac{1-L^{n_*}}{1-L}\,, \qquad (4.2)
\end{aligned}$$

where we define

$$L = \frac{\beta_0\alpha_s(\mu)}{4\pi}\log\left(\frac{\mu^2 e^{5/3}}{\tau}\right). \qquad (4.3)$$

Since the integrand of Eq. (4.2) is regular along the integral path (positive real axis), we can deform the path into C_a:

$$\begin{aligned}
X_{n_*}(Q^2) &= \mathrm{Im}\int_{C_a}\frac{d\tau}{\pi\tau}\, W_X\left(\frac{\tau}{Q^2}\right)\alpha_s(\mu)\frac{1-L^{n_*}}{1-L}\\
&= \mathrm{Im}\int_{C_a}\frac{d\tau}{\pi\tau}\, W_X\left(\frac{\tau}{Q^2}\right)\alpha_s(\mu)\frac{1}{1-L} + \mathrm{Im}\int_{C_a}\frac{d\tau}{\pi\tau}\, W_X\left(\frac{\tau}{Q^2}\right)\alpha_s(\mu)\frac{-L^{n_*}}{1-L}\,.
\end{aligned} \qquad (4.4)$$

The first term is a part of $X_0(Q^2)$ since

$$\alpha_s(\mu)\frac{1}{1-L} = \alpha_{\beta_0}(\tau)\,. \qquad (4.5)$$

In the second term of Eq. (4.4), the integrand has a pole at $\tau = e^{5/3}\Lambda_{\rm QCD}^2$. We decompose the integral into the principle value part and the delta function part, after taking the integral path again on the positive axis:

$$\alpha_{\beta_0}(\tau) = \mathrm{Pr.}\,\alpha_{\beta_0}(\tau) - \frac{4\pi}{\beta_0}\pi\tau i\,\delta(\tau - e^{5/3}\Lambda_{\rm QCD}^2)\,. \qquad (4.6)$$

Thus, we obtain

$$\begin{aligned}
X_{n_*}(Q^2) =& \mathrm{Im}\int_{C_a}\frac{d\tau}{\pi\tau}\, W_X\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau) + \frac{4\pi}{\beta_0}\mathrm{Re}\, W_X\left(\frac{e^{5/3}\Lambda_{\rm QCD}^2}{Q^2}\right)\\
&+ \mathrm{Pr.}\int_0^\infty \frac{d\tau}{2\pi\tau}\, w_X\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau)(-L^{n_*})\,,
\end{aligned} \qquad (4.7)$$

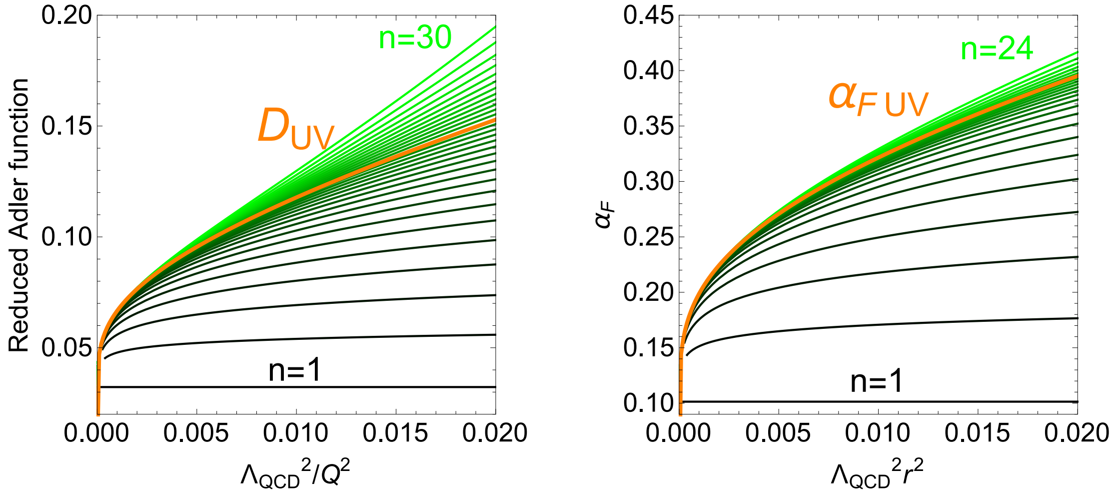


Figure 7. Truncated perturbative series up to $\mathcal{O}(\alpha_s^n(\mu))$ in the large-β_0 approximation and $X_{\rm UV}$, for the reduced Adler function $X = D$ (left) and the F-scheme coupling $X = \alpha_F$ (right). We choose $\alpha_s(\mu) = 0.0243$ corresponding to $n_* = 24$ ($n_* = 18$) for $X = D$ ($X = \alpha_F$) and $u_{\rm IR} = 2$ ($u_{\rm IR} = 3/2$). Optimal perturbative prediction X_{n_*} lies close to $X_{\rm UV}$ in each figure. We set $n_f = 1$.

where we used Eq. (2.14) for the third term. By expanding $\mathrm{Re}\, W_X$ in $\Lambda_{\rm QCD}^2/Q^2$, we can see that $X_{n_*}(Q^2)$ indeed includes $X_{\rm UV}(Q^2)$; see Eqs. (2.21) and (2.23).

In Fig. 7, we show $X_{\rm UV}$ and perturbative series truncated at various orders for $X = D$ and α_F. (The truncated order is denoted as n.) One can see that the truncated perturbative series gradually approaches to $X_{\rm UV}$ for $n \lesssim n_*$ as we raise n. For $n \gtrsim n_*$ it starts to deviate from $X_{\rm UV}$.

The difference between X_{n_*} and $X_{\rm UV}$ is given by

$$X_{n_*}(Q^2) - X_{\rm UV}(Q^2) = \frac{4\pi}{\beta_0}\left[\mathrm{Re}\, W_X\left(\frac{e^{5/3}\Lambda_{\rm QCD}^2}{Q^2}\right) - \sum_{0\le n<u_{\rm IR}} c_n \left(\frac{e^{5/3}\Lambda_{\rm QCD}}{Q^2}\right)^n\right] + \mathrm{Pr.}\int_0^\infty \frac{d\tau}{2\pi\tau}\, w_X\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau)(-L^{n_*})\,. \tag{4.8}$$

For large n_*, this difference has a power behavior $\Lambda_{\rm QCD}^2/Q^2$ whose order is the same as the renormalon uncertainty:[28]

$$X_{n_*}(Q^2) - X_{\rm UV}(Q^2) \sim \mathcal{O}((\Lambda_{\rm QCD}^2/Q^2)^{u_{\rm IR}})\,. \tag{4.9}$$

More precisely, we can detect the n_*-dependence of Eq. (4.8) analytically as

$$X_{n_*}(Q^2) - X_{\rm UV}(Q^2) \sim \log n_* \times \frac{b_{u_{\rm IR}}}{\beta_0}\left(\frac{e^{5/3}\Lambda_{\rm QCD}^2}{Q^2}\right)^{u_{\rm IR}}, \tag{4.10}$$

[28] The difference Eq. (4.8) can contain a polynomial of $\log(Q^2/\Lambda_{\rm QCD}^2)$, $\log\log(Q^2/\Lambda_{\rm QCD}^2)$, $\cdots$, as a factor in front of the $(\Lambda_{\rm QCD}^2/Q^2)^{u_{\rm IR}}$-term. Therefore, strictly speaking, the difference is $o(\Lambda_{\rm QCD}^2/Q^2)^{u_{\rm IR}-\delta}$ with $0 < \forall\delta \le 1$.

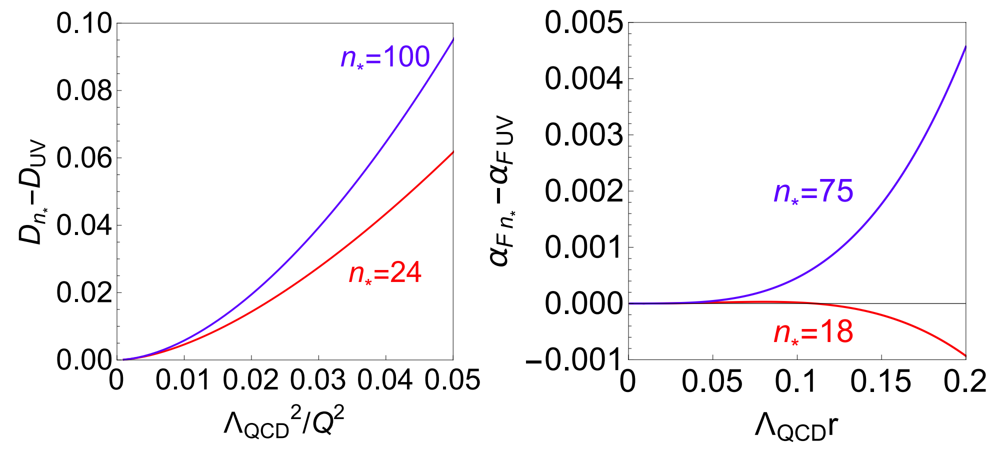


Figure 8. Difference between truncated perturbative series X_{n_*} and $X_{\rm UV}$ for the reduced Adler function (left) and α_F (right). The red (blue) lines correspond to the input $\alpha_s(\mu) = 0.1013\,(0.0243)$. The truncation orders n_* are shown in the plots. We set $n_f = 1$.

where $b_{u_{\rm IR}}$ is an expansion coefficient of w_X; c.f., Eq. (2.12). (We give a derivation in Appendix C.) In Fig. 8 we check Eqs. (4.9) and (4.10) numerically for $X = D$ and α_F. We confirm the predicted behavior, although $n_* = 18$ for α_F is not large enough to reach the asymptotic forms.

We can draw some conclusions from Eq. (4.9). First, it shows that the power behaviors in $X_{\rm UV}$ are not a new contribution which should be added to the perturbative prediction, rather they are already contained in the perturbative series. Secondly, using Eq. (4.9) and the assumption of the renormalon uncertainty, we can extract the following relation between $X_{\rm UV}$ and the true value $X(Q^2)$:

$$\begin{aligned} X(Q^2) - X_{\rm UV}(Q^2) &= \left[X(Q^2) - X_{n_*}(Q^2)\right] + \left[X_{n_*}(Q^2) - X_{\rm UV}(Q^2)\right] \\ &\sim \mathcal{O}((\Lambda_{\rm QCD}^2/Q^2)^{u_{\rm IR}})\,. \end{aligned} \tag{4.11}$$

This result is consistent with the interpretation that $X_{\rm UV}(Q^2)$ is the leading order contribution to $X(Q^2)$ in the OPE framework and the deviation from $X(Q^2)$ starts from the next-to-leading order in OPE and has the same order of magnitude as the non-perturbative matrix element of the order of $(\Lambda_{\rm QCD}^2/Q^2)^{u_{\rm IR}}$.

We end this section with comparisons between the known exact perturbative series and those obtained under the large-β_0 approximation to make sure how far we can trust the result based on the large-β_0 approximation. Figs. 9 and 10 show that the large-β_0 approximation reproduces qualitatively the same behavior of the exact series of $D(Q^2)$ [31, 32] and $\alpha_F(1/r^2)$ [22–24] [29]. Therefore we expect that the results in this paper (especially Fig. 7) grasp an essential feature of QCD.

[29] There is an IR divergence in the exact series of α_F from the three-loop order, and the divergence cancels with contributions from the ultra-soft scale. We do not include the contribution of the ultra-soft scale because this contribution cannot be regarded as a part of the Wilson coefficient. Instead,

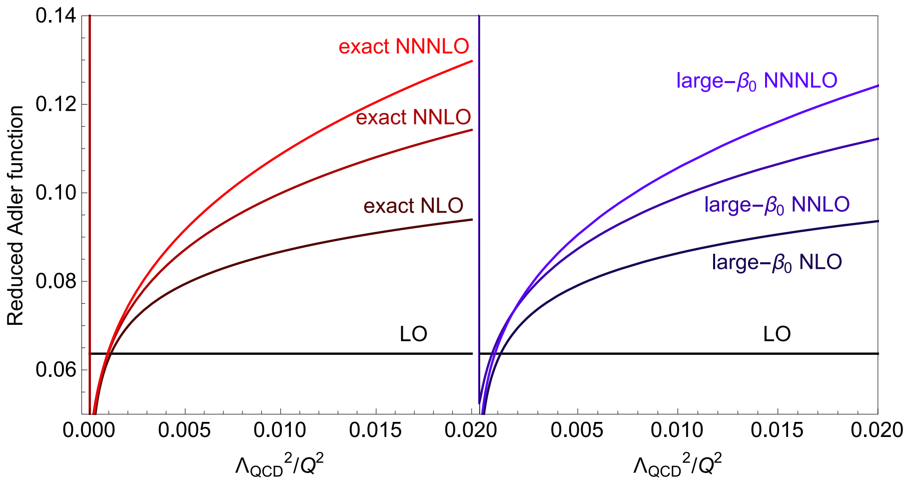


Figure 9. Perturbative series of $D(Q^2)$: exact result for the non-singlet component (left) and large-β_0 approximation (right). N^{k}LO line represents the sum of the series up to $\mathcal{O}(\alpha_s^{k+1})$. The input is taken as $\alpha_s(\mu) = 0.2$, and we set $n_f = 1$.

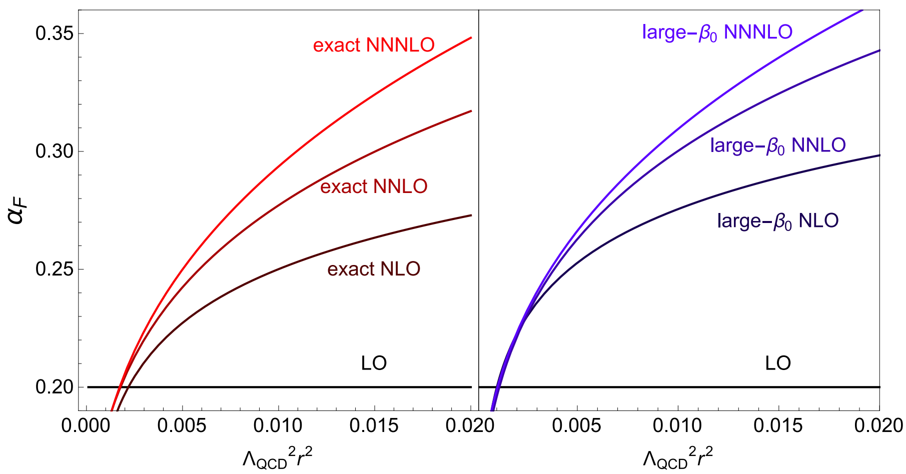


Figure 10. Perturbative series of $\alpha_F(1/r^2)$. The parameters are the same as those of Fig.9.

5 Example of timelike quantity: R-ratio in e^+e^- collision

So far we have considered Euclidean quantities. In this section we investigate how our method can be extended to the case of the R-ratio in e^+e^- collision as an example of a timelike quantity. We obtain a result which can be regarded as an extension of the massive gluon scheme.

In calculating the R-ratio, we set $Q^2 < 0$ (i.e. $q^2 > 0$) and take the imaginary part of $\Pi(Q^2)$ according to the optical theorem. The difference from Euclidean

we simply subtract the term proportional to $(1/\epsilon + 4(2\log(\mu/p) + \log(4\pi) - \gamma_{\rm E}))$ in momentum space in dimensional regularization.

quantities is that we do not have a one-dimensional integral representation of the R-ratio in the form of Eq. (2.4). Thus, we cannot directly apply the method developed for Euclidean quantities in Sec. 2, and a reconsideration is needed. Our strategy is to start from a Euclidean quantity and to use the analytic continuation to derive the result for the R-ratio.

We consider the reduced vacuum polarization and reduced R-ratio, in which α_s-independent terms are subtracted. Let us start from the reduced vacuum polarization[30] in the Euclid region $Q^2 > 0$ with an IR cutoff scale,

$$\Pi_{\beta_0}(Q^2;\mu_f) = \int_{\mu_f^2}^{\infty} \left[\frac{d\tau}{2\pi\tau}\right]_r w_\Pi\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau)\,, \tag{5.1}$$

where we denote by $[d\tau/(2\pi\tau)]_r$ a regularized integral measure which makes the integral UV finite. We do not need to specify a way of regularization since the R-ratio, which we are interested in, is finite, and thus the final result should be independent of the regularization method. The weight is given as [9]

$$w_\Pi(x) = w_1(x)\theta(1-x) + w_2(x)\theta(x-1) \tag{5.2}$$

with

$$w_1(x) = A\left[2(1-\log x)x + (5-3\log x)x^2 + 2(1+x)^2\{\mathrm{Li}_2(-x) + \log x \log(1+x)\}\right]\,, \tag{5.3}$$

$$w_2(x) = A\left[5 + 3\log x + 2(1+\log x)x + 2(1+x)^2\{\mathrm{Li}_2(-x^{-1}) - \log x \log(1+x^{-1})\}\right]\,, \tag{5.4}$$

where $A = -N_C C_F/(12\pi^2)$. The small-x and large-x behaviors of $w_\Pi(x)$ are given, respectively, by

$$w_\Pi(x) = A\left[\frac{3}{2}x^2 - \frac{11-6\log x}{9}x^3 + \dots\right] \qquad (x \ll 1)\,, \tag{5.5}$$

$$w_\Pi(x) = A\left[\frac{3}{2} - \frac{11+6\log x}{9}\frac{1}{x} + \dots\right] \qquad (x \gg 1). \tag{5.6}$$

We can see that the first IR renormalon is located at $u = 2$ for the reduced vacuum polarization. The constant $c_\infty \equiv 3A/2$ in Eq. (5.6) stems from the UV renormalon at $u = 0$, which is the source of the UV divergence of the integral (5.1). As shown in Eqs. (5.3) and (5.4), w_Π has different analytic forms for $x < 1$ and $x > 1$, hence, we separate the integral path at $\tau = Q^2$ in order to represent Eq. (5.1) as an analytic

[30] Note that we define the reduced vacuum polarization (5.1) such that its perturbative expansion does not contain the α_s^0-part $\frac{N_C}{12\pi^2}(\log(Q^2/\mu^2) + C)$, which is included in the renormalized $\Pi(Q^2)$ of Eq. (2.34). Correspondingly, the reduced R-ratio is different by $N_C e_q^2$ for each quark flavor compared with the R-ratio.

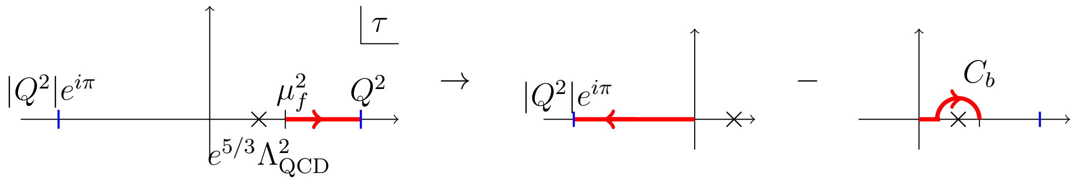
function of Q^2:

$$\Pi_{\beta_0}(Q^2;\mu_f) = \int_{\mu_f^2}^{Q^2} \frac{d\tau}{2\pi\tau} w_1\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau) + \int_{Q^2}^{\infty} \frac{d\tau}{2\pi\tau}\left[w_2\left(\frac{\tau}{Q^2}\right) - c_\infty\right]\alpha_{\beta_0}(\tau)$$
$$+ \int_{Q^2}^{\infty}\left[\frac{d\tau}{2\pi\tau}\right]_r c_\infty\,\alpha_{\beta_0}(\tau)\,. \tag{5.7}$$

We also separate the divergent part which needs a regularization.

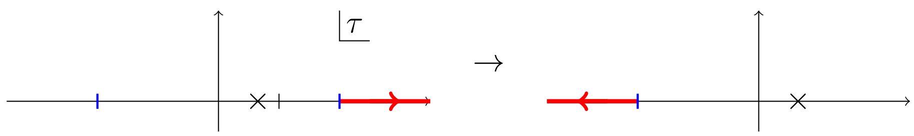
Now we replace $Q^2 \to |Q^2|e^{i\pi}$ in Eq. (5.7) and derive an expression for the timelike region. The integral path of the first term in Eq. (5.7) can be deformed after replacing $Q^2 \to |Q^2|e^{i\pi}$ as

$$\int_{\mu_f^2}^{Q^2} \frac{d\tau}{2\pi\tau}\, w_1\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau) \to \int_0^{-|Q^2|} \frac{d\tau}{2\pi\tau}\, w_1\left(\frac{\tau}{|Q^2|}e^{-i\pi}\right)\alpha_{\beta_0}(\tau)$$
$$- \int_{C_b} \frac{d\tau}{2\pi\tau}\, w_1\left(\frac{\tau}{|Q^2|}e^{-i\pi}\right)\alpha_{\beta_0}(\tau). \tag{5.8}$$



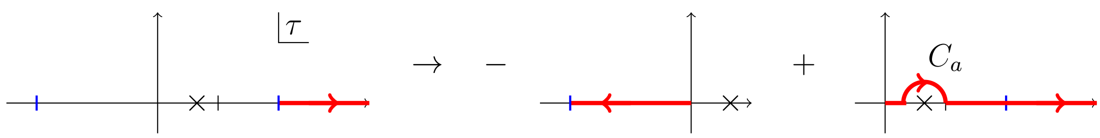
The second term in Eq. (5.7) changes as

$$\int_{Q^2}^{\infty} \frac{d\tau}{2\pi\tau}\left[w_2\left(\frac{\tau}{Q^2}\right) - c_\infty\right]\alpha_{\beta_0}(\tau) \to \int_{-|Q^2|}^{-\infty} \frac{d\tau}{2\pi\tau}\left[w_2\left(\frac{\tau}{|Q^2|}e^{-i\pi}\right) - c_\infty\right]\alpha_{\beta_0}(\tau), \tag{5.9}$$



where the end point of the integral path is changed from ∞ to $-\infty$ using the fact that the contribution from C_R [defined in Eq. (2.74)] vanishes as $R \to \infty$. The third term becomes

$$\int_{Q^2}^{\infty}\left[\frac{d\tau}{2\pi\tau}\right]_r c_\infty\,\alpha_{\beta_0}(\tau) \to -\int_0^{-|Q^2|}\left[\frac{d\tau}{2\pi\tau}\right]_r c_\infty\alpha_{\beta_0}(\tau) + \int_{C_a}\left[\frac{d\tau}{2\pi\tau}\right]_r c_\infty\alpha_{\beta_0}(\tau). \tag{5.10}$$



Collecting these terms, we obtain an expression for the reduced vacuum polarization in the timelike region:

$$\Pi_{\beta_0}(|Q^2|e^{i\pi};\mu_f) = \int_0^\infty \frac{d\tau}{2\pi\tau}\left[w_\Pi\left(\frac{\tau}{q^2}\right) - c_\infty\right]\alpha_{\beta_0}(-\tau+i0)$$
$$+\int_{C_a}\left[\frac{d\tau}{2\pi\tau}\right]_r c_\infty\alpha_{\beta_0}(\tau) - \int_{C_b}\frac{d\tau}{2\pi\tau}w_1\left(\frac{\tau}{q^2}e^{-i\pi}\right)\alpha_{\beta_0}(\tau). \quad (5.11)$$

By taking the imaginary part, we obtain the reduced R-ratio:

$$R_{\beta_0}(q^2;\mu_f) = 12\pi\left(\sum_q e_q^2\right)\mathrm{Im}\,\Pi_{\beta_0}(|Q^2|e^{i\pi};\mu_f). \quad (5.12)$$

Setting $\sum_q e_q^2 = 1$ for simplicity, we have

$$R_{\beta_0}(q^2;\mu_f) = \int_0^\infty \frac{d\tau}{\pi\tau}W_{R+}\left(\frac{\tau}{q^2}\right)\mathrm{Im}\,\alpha_{\beta_0}(-\tau+i0) - \mathrm{Im}\int_{C_b}\frac{d\tau}{\pi\tau}W_R\left(\frac{\tau}{q^2}\right)\alpha_{\beta_0}(\tau). \quad (5.13)$$

We regard W_R and W_{R+} as pre-weights (although we do not have a weight), which are defined as

$$W_R(z) = 6\pi\{w_1(ze^{-i\pi}) - c_\infty\} \quad (|z|<1)\,, \quad (5.14)$$
$$W_{R+}(z) = 6\pi\{w_\Pi(z) - c_\infty\}\,. \quad (5.15)$$

In taking the imaginary part of the second term of Eq. (5.11) to obtain Eq. (5.13), we used

$$2\,\mathrm{Im}\int_{C_a}\left[\frac{d\tau}{2\pi\tau}\right]_r c_\infty\alpha_{\beta_0}(\tau) = \mathrm{Im}\int_{C_b}\frac{d\tau}{\pi\tau}c_\infty\alpha_{\beta_0}(\tau)\,, \quad (5.16)$$

since the imaginary part of the integrand is zero on the positive real axis. The way to evaluate the second term of Eq. (5.13), i.e., the integral along C_b, is no longer different from the case of Euclidean quantities (see the discussion in Sec. 2.2). The expansion of $W_R(z)$ in z reads

$$W_R(z) = N_C C_F\left[\frac{3}{4\pi} - \frac{3}{4\pi}z^2 + \frac{11-6\log z+6i\pi}{18\pi}z^3 + \dots\right]. \quad (5.17)$$

As a result, we can separate the μ_f-dependence of Eq. (5.13) and obtain a μ_f-independent part R_{UV} as

$$R_{\beta_0}(q^2;\mu_f) = R_{\mathrm{UV}}(q^2) + \mathcal{O}(\mu_f^6/q^6)\,, \quad (5.18)$$

where

$$R_{\mathrm{UV}}(q^2) = R_0(q^2) - \frac{3N_C C_F}{\beta_0}\frac{e^{10/3}\Lambda_{\mathrm{QCD}}^4}{q^4}\,, \quad (5.19)$$

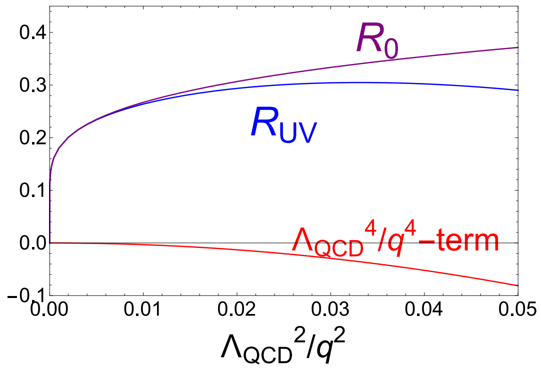


Figure 11. $R_{\rm UV}$ [Eq. (5.19)], R_0 [Eq. (5.20)] and the $\Lambda_{\rm QCD}^4/q^4$-term [Eq. (5.19)] as functions of $\Lambda_{\rm QCD}^2/q^2$.

$$R_0(q^2)=\int_0^\infty \frac{d\tau}{\pi\tau}W_{R+}\left(\frac{\tau}{q^2}\right)\mathrm{Im}\,\alpha_{\beta_0}(-\tau+i0)+\frac{3N_C C_F}{\beta_0}. \tag{5.20}$$

The μ_f-dependence appears first at order $1/q^6$. In fact, the first IR renormalon of the reduced R-ratio is known to be located at $u_{\rm IR}=3$. Therefore the result is consistent with Eq. (2.22). However, note that the absence of $u=2$ renormalon is considered to be an artifact of the large-β_0 approximation, and there is a possibility that the result [Eqs. (5.18)–(5.20)] is not based on a good approximation of the exact perturbative series. Hence, we are cautious in applying our formulation to serious studies of the R-ratio at the current stage. Even in such a case, nevertheless, we can still learn some lessons from the above result.

First, the $1/q^2$-term is absent in Eq. (5.19) due to the vanishing z^1-term in Eq. (5.17).[31] As a result, we obtain a very different behavior of the reduced R-ratio from those of the reduced Adler function and α_F, as seen in Figs. 2, 3 and 11. This fact serves as an evidence that the power corrections indeed play an important role in the determination of the behavior of a physical quantity and understanding of it.

Secondly, our formulation in this section has common features to those of the Euclidean observables in the massive gluon scheme. Let us clarify this point. $R_{\rm UV}$ and R_0 [Eq. (5.19) and (5.20)] have the same expressions as those of a Euclidean observable obtained in the massive gluon scheme [Eqs. (2.24) and (2.30)]. In addition, W_{R+} defined in Eq. (5.15) can be regarded to be "constructed by massive gluon scheme." To justify this statement, we can use Eq. (2.88), which is satisfied by W_{X+} in the massive gluon scheme. We regard it as an abstract property of the massive gluon scheme, since this relation can be checked as long as the observable has Borel transformation. The Borel transformation of the (reduced) R-ratio is known and

[31] As discussed below Eq. (5.22), the absence of z^1-term stems from $B_R(1)=0$, which stems from the absence of a $u=1$ renormalon in the reduced Adler function.

given in appendix A. We can show that W_{R+} satisfies the same relation as Eq. (2.88):

$$\begin{aligned}\int_0^\infty \frac{dz}{2\pi} W_{R+}(z) z^{-u-1} &= 6\pi \int_0^\infty \frac{dz}{2\pi}(w_\Pi(z) - c_\infty) z^{-u-1} \\ &= 6\pi \int_0^\infty \frac{dz}{2\pi} w_\Pi(z) z^{-u-1} \\ &= 6\pi B_\Pi(u)|_{Q^2>0} \\ &= -6\pi \frac{1}{\sin(\pi u)} B_{\mathrm{Im}\Pi}(u)|_{q^2>0} \\ &= -\frac{1}{2\sin(\pi u)} B_R(u),\end{aligned} \tag{5.21}$$

where[32] we use the relation between the Borel transformations with opposite signs of Q^2 (or q^2) [1, 33] and Eq. (5.12). Similarly, we confirm that W_R defined in Eq. (5.14) is consistent with the massive gluon scheme, since the expansion of W_R is correctly reproduced from the relation

$$C_R(v) = -\frac{e^{-i\pi v}}{2\sin(\pi v)} B_R(v) \tag{5.22}$$

and the inverted formula (2.87), which are also obtained in the case of the massive gluon scheme.[33] Namely, our formulation used here can be regarded as a natural extension of the massive gluon scheme developed in Sec. 2 to the timelike quantity.

Thus, our formulation for the R-ratio derived by analytic continuation is an extension of the massive gluon scheme. This is natural if one recalls the discussion in Sec. 2.6 that the massive gluon scheme is unique with respect to the analyticity of an observable. Namely, if we adopt a formulation which has a good property in terms of analyticity, the same result is likely to be obtained.

6 Conclusions and discussion

In this paper we proposed a method to extract a cutoff-independent UV contribution X_{UV} from a general observable $X(Q^2)$ with an explicit IR cutoff, which is free from IR renormalon ambiguities. Our method can be applied in the deep Euclidean region ($Q^2 \gg \Lambda_{\mathrm{QCD}}^2$) and in the large-$\beta_0$ approximation of perturbative series to all orders. The UV contribution X_{UV} consists of the non-powerlike (logarithmic) term X_0 and the power correction terms $\sim (\Lambda_{\mathrm{QCD}}^2/Q^2)^n$.

In our method we introduce an analytic function W_X, which we call "pre-weight," for the systematic treatment of various observables. General properties of the pre-weight, such as its scheme dependence, were investigated. Separation of X_{UV} into

[32] The integral of $(w_\Pi(z) - c_\infty)z^{-u-1}$ has the same form as that of $w_\Pi(z)z^{-u-1}$ as a function of u by analytic continuation.

[33] In Ref.[11], the functions W_R and W_{R+} were obtained by the massive gluon method directly. Our method can be used to circumvent complicated calculations.

X_0 and the power corrections (in particular the coefficients of the power corrections) depends on the scheme choice. Among various schemes, the "massive gluon scheme," in which W_X is given by a dispersive integral, has particularly good analytical properties: (1) The analyticity of $X_{\rm UV}$ satisfies physical requirements within perturbative QCD; (2) Origins of the power corrections can be analyzed accurately using the integration-by-regions method. We showed that the feature (1) is satisfied optimally in the massive gluon scheme. We also find that the analyticity of $X_{\rm UV}$ and a unique scheme choice follow simultaneously if the pre-weight satisfies certain good analytical properties in the upper half-plane.

We can use the integration-by-regions method to elucidate the relation between our formulation and OPE. Using this relation we showed that $X_{\rm UV}$ coincides with the leading Wilson coefficient in the explicit examples considered. Thus, we can systematically subtract IR renormalons from the leading Wilson coefficient in a cutoff-independent way. Furthermore, we used the integration-by-regions method to clarify that the leading power corrections in $X_{\rm UV}$ indeed originate from UV regions.

As applications of our method, we investigated the Adler function and the force between a static quark-antiquark pair. For each observable, there is a nontrivial power correction in $X_{\rm UV}$, which originates from UV region. In the context of OPE, this power correction is a part of the Wilson coefficient of the leading identity operator, and it is consistent with the structure of OPE. Comparison with the exact perturbative series indicates that the large-β_0 approximation is fairly good, hence, it is natural to regard that the power correction (in the massive gluon scheme) is inherent in the perturbative series or the UV contribution.

By now there exist ample numerical evidences for validity of the large-β_0 approximation and IR renormalon dominance hypothesis. Apart from these assumptions, we tried to avoid including ad hoc assumptions into our method. Thus, we believe that we provide a firm connection between the OPE framework and our method for subtracting IR renormalons from Wilson coefficients. Moreover, we consider that our method (in particular in the massive gluon scheme) would be an optimal one within the OPE framework, with respect to systematicity, analyticity, and insensitivity to the factorization scale (IR cutoff scale).

There remain two directions toward generalization of our method: one is to extend it to timelike quantities and the other is to go beyond the large-β_0 approximation. For the former, we presented an example (R-ratio) but the generalization is left to be done. We do not have a clear guide to the latter, since the improvement of the large-β_0 approximation in the ordinary perturbation theory is still incomplete and we need a control up to any order in α_s. We speculate that the method of integration by regions may play a key role to achieve the generalization since the method enables more complicated scale separation than the single scale separation which we adopted in this paper. In addition, we note that a systematic improvement beyond the large-β_0 approximation has been achieved for the static QCD potential and better consistency with OPE has been observed [15, 34], using the fact that the pre-weight takes a simple form to all orders in α_s.

Acknowledgments

The authors are grateful to Y. Kiyo for fruitful discussion. The works of G.M. and Y.S. were supported in part by JSPS KAKENHI Grant Number14J10887 and by Grant-in-Aid for scientific research (No. 26400238) from MEXT, Japan, respectively.

Appendices

A Borel tranformations

We list formulas for the Borel transformations in the large-β_0 approximation of the dimensionless observables analyzed in this paper (reduced Adler function, F-scheme coupling defined from the static QCD force, and R-ratio).

$$B_D(u) = \frac{8N_C C_F}{3\pi}\frac{1}{2-u}\sum_{k=2}^{\infty}\frac{(-1)^k k}{(k^2-(1-u)^2)^2}\,, \qquad [35] \tag{A.1}$$

$$B_{\alpha_F}(u) = \frac{\sin(\pi u)}{\pi u}\Gamma(2-2u)\,, \tag{A.2}$$

$$B_R(u) = \frac{\sin(\pi u)}{\pi u}B_D(u)\,. \qquad [1] \tag{A.3}$$

B Pre-weight of Adler function

Another expression of the pre-weight of the reduced Adler function in the massive gluon scheme is given by

$$\begin{aligned}
&W_{D+}^{(m)}(z) = W_D^{(m)}(-z)\\
&= \frac{N_C C_F}{36\pi(z+1)}\Bigg[-48z^3\mathrm{Li}_2(1-z) + 48z^3\mathrm{Li}_2(-z) + 24z^3\mathrm{Li}_2\left(\frac{1}{z+1}\right) - 24z^3\mathrm{Li}_3\left(1-\frac{1}{z}\right)\\
&- 72z^3\mathrm{Li}_3(1-z) + 24z^3\mathrm{Li}_3(-z) - 48z^3\mathrm{Li}_3\left(\frac{1}{z+1}\right) + 24z^3\mathrm{Li}_2(1-z)\log(z)\\
&- 48z^3\mathrm{Li}_2(1-z)\log(z+1) - 12z^2\mathrm{Li}_2\left(1-\frac{1}{z}\right) + 36z^2\mathrm{Li}_2(1-z) - 48z^2\mathrm{Li}_2\left(\frac{1}{z+1}\right)\\
&- 24z^2\mathrm{Li}_2\left(1-z^2\right) - 24z\mathrm{Li}_3\left(1-z^2\right) + 24z^3\mathrm{Li}_2\left(1-z^2\right) + 24z^3\mathrm{Li}_3\left(1-z^2\right)\\
&- 36z\mathrm{Li}_2(-z) + 24z\mathrm{Li}_2\left(\frac{1}{z+1}\right) + 24z\mathrm{Li}_3\left(1-\frac{1}{z}\right) + 72z\mathrm{Li}_3(1-z) - 24z\mathrm{Li}_3(-z)\\
&+ 48z\mathrm{Li}_3\left(\frac{1}{z+1}\right) + 12\mathrm{Li}_2\left(1-\frac{1}{z}\right) + 12\mathrm{Li}_2(1-z) + 12\mathrm{Li}_2(-z) - 24z\mathrm{Li}_2(1-z)\log(z)\\
&+ 48z\mathrm{Li}_2(1-z)\log(z+1) + 24z^3\zeta(3) + 4\pi^2z^3 + 4z^3\log^3(z) + 8z^3\log^3(z+1)\\
&+ 12z^3\log^2(z) + 12z^3\log^2(z+1) - 42z^3\log(z) + 24z^3\log(z)\log(z+1) - 4\pi^2z^3\log(z+1)\\
&+ 42z^3\log(z+1) + 8\pi^2z^2 - 66z^2 + 6z^2\log^2(z) - 24z^2\log^2(z+1) - 42z^2\log(z)\\
&+ 66z^2\log(z+1) - 24z\zeta(3) - 4\pi^2 z - 57z - 4z\log^3(z) - 8z\log^3(z+1) + 12z\log^2(z+1)\\
&+ 6\log^2(z) - 24z\log(z)\log(z+1) + 4\pi^2z\log(z+1) + 6z\log(z+1) - 18\log(z+1) + 9\Bigg]\,.
\end{aligned} \tag{B.1}$$

This expression is suited for verifying its analytical properties, such as, that $W_D^{(m)}(z)$ has a branch cut along the positive real axis from $z=0$, and that $W_{D+}^{(m)}(z)$ takes a real value for $z>0$. (Note that the polylogarithm $\mathrm{Li}_n(z)$ for $n\geq 2$ has a branch cut along the positive real axis from $z=1$. In the above expression the arguments of Li_n are less than or equal to one for $z\geq 0$.)

C Evaluation of $X_{n_*}(Q^2)-X_{\rm UV}(Q^2)$

We examine the principal value integral appearing in $X_{n_*}(Q^2)-X_{\rm UV}(Q^2)$ (the second term of Eq. (4.8)) for large-n_*.

$$\mathrm{Pr.}\int_0^\infty \frac{d\tau}{2\pi\tau} w_X\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau)L^{n_*} \tag{C.1}$$

Write L of Eq. (4.3) as a function of n_*

$$L(\tilde{\Lambda}^2/\tau)=\frac{\beta_0\alpha_s}{4\pi}(\log(\mu^2/\Lambda_{\rm QCD}^2)+\log(e^{5/3}\Lambda_{\rm QCD}^2/\tau))=1+\frac{u_{\rm IR}}{n_*}\log(\tilde{\Lambda}^2/\tau)\,, \tag{C.2}$$

where $\tilde{\Lambda}^2\equiv e^{5/3}\Lambda_{\rm QCD}^2$, then we get

$$L(\tilde{\Lambda}^2/\tau)^{n_*}\to\left(\frac{\tilde{\Lambda}^2}{\tau}\right)^{u_{\rm IR}}\quad \text{as}\quad n_*\to\infty\,. \tag{C.3}$$

If we use this form, the integral (C.1) does not converge around the region $\tau\sim 0$ due to the behavior $w_X(\tau/Q^2)=b_{u_{\rm IR}}(\tau/Q^2)^{u_{\rm IR}}+\dots$. Therefore we should calculate keeping n_* finite for this part. It is useful to factorize the integral as follows:

$$\begin{aligned}
&\mathrm{Pr.}\int_0^\infty \frac{d\tau}{2\pi\tau} w_X\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau)L(\tilde{\Lambda}^2/\tau)^{n_*}\\
&=\mathrm{Pr.}\int_0^\infty\frac{dx}{2\pi x}\left[w_X\left(\frac{\tilde{\Lambda}^2}{Q^2}x\right)-b_{u_{\rm IR}}\left(\frac{\tilde{\Lambda}^2}{Q^2}x\right)^{u_{\rm IR}}\theta(1-x)\right]\frac{4\pi}{\beta_0}\frac{1}{\log x}L(1/x)^{n_*}\\
&\quad+\mathrm{Pr.}\int_0^\infty\frac{dx}{2\pi x}b_{u_{\rm IR}}\left(\frac{\tilde{\Lambda}^2}{Q^2}x\right)^{u_{\rm IR}}\theta(1-x)\frac{4\pi}{\beta_0}\frac{1}{\log x}L(1/x)^{n_*}
\end{aligned}\tag{C.4}$$

The second term of Eq. (C.4) is separated as

$$\begin{aligned}
&\mathrm{Pr.}\int_0^\infty\frac{dx}{2\pi x}b_{u_{\rm IR}}\left(\frac{\tilde{\Lambda}^2}{Q^2}x\right)^{u_{\rm IR}}\theta(1-x)\frac{4\pi}{\beta_0}\frac{1}{\log x}\left\{L(1/x)^{n_*}-1\right\}\\
&+\mathrm{Pr.}\int_0^\infty\frac{dx}{2\pi x}b_{u_{\rm IR}}\left(\frac{\tilde{\Lambda}^2}{Q^2}x\right)^{u_{\rm IR}}\theta(1-x)\frac{4\pi}{\beta_0}\frac{1}{\log x}\\
&\equiv\frac{4\pi}{\beta_0}b_{u_{\rm IR}}\left(\frac{\tilde{\Lambda}^2}{Q^2}\right)^{u_{\rm IR}}F(n_*)+\mathrm{Pr.}\int_0^\infty\frac{dx}{2\pi x}b_{u_{\rm IR}}\left(\frac{\tilde{\Lambda}^2}{Q^2}x\right)^{u_{\rm IR}}\theta(1-x)\frac{4\pi}{\beta_0}\frac{1}{\log x}
\end{aligned}\tag{C.5}$$

Substituting this into Eq. (C.4), we obtain

$$
\begin{aligned}
&\mathrm{Pr.}\int_0^\infty \frac{d\tau}{2\pi\tau} w_X\left(\frac{\tau}{Q^2}\right)\alpha_{\beta_0}(\tau)L^{n_*}\\
&\simeq \frac{4\pi b_{u_{\mathrm{IR}}}}{\beta_0}\left(\frac{\tilde{\Lambda}^2}{Q^2}\right)^{u_{\mathrm{IR}}} F(n_*)\\
&+\mathrm{Pr.}\int_0^\infty \frac{dx}{2\pi x}\left\{w_X\left(\frac{\tilde{\Lambda}^2}{Q^2}x\right)-b_{u_{\mathrm{IR}}}\left(\frac{\tilde{\Lambda}^2}{Q^2}x\right)^{u_{\mathrm{IR}}}(1-x^{u_{\mathrm{IR}}})\theta(1-x)\right\}\frac{4\pi}{\beta_0}\frac{1}{\log x}\left(\frac{1}{x}\right)^{u_{\mathrm{IR}}},
\end{aligned}
\tag{C.6}
$$

where we used the limit (C.3) for the second term. One can show that the second term of Eq. (C.6) is $o((\Lambda_{\mathrm{QCD}}^2/Q^2)^{u_{\mathrm{IR}}-\delta})$ (see the footnote of § 4) although it is fairly complicated. $F(n_*)$ behaves for large-n_* as

$$
\begin{aligned}
F(n_*) &= \int_0^1 \frac{dx}{2\pi x} x^{u_{\mathrm{IR}}}\frac{1}{\log x}\left[\left\{1+\frac{u_{\mathrm{IR}}}{n_*}\log\left(\frac{1}{x}\right)\right\}^{n_*}-1\right]\\
&= -\int_0^\infty \frac{dt}{2\pi}\frac{e^{-t}}{t}\left[\left(1+\frac{t}{n_*}\right)^{n_*}-1\right]\\
&= -\frac{1}{4\pi}(\log n_* + \log 2 + \gamma_E) + \mathcal{O}\left(\frac{1}{\sqrt{n_*}}\right),
\end{aligned}
\tag{C.7}
$$

which gives the result Eq. (4.10).

D Asymptotic expansion of $X_0(Q^2)$

We sketch how to derive the relation (2.70). Similarly to Eqs. (4.2)–(4.7), we can rewrite the truncated series X_n as follows.

$$
\begin{aligned}
X_n(Q^2) &= \sum_{k=0}^{n-1}\int_0^\infty \frac{d\tau}{2\pi\tau} w_X\left(\frac{\tau}{Q^2}\right)\alpha_s^{k+1}\ell^k\\
&= \int_0^\infty \frac{d\tau}{2\pi\tau} w_X\left(\frac{\tau}{Q^2}\right)\alpha_s\frac{1-(\alpha_s\ell)^n}{1-\alpha_s\ell}\\
&= \mathrm{Im}\int_0^\infty \frac{d\tau}{\pi\tau} W_X\left(\frac{\tau}{Q^2}\right)\alpha_s\frac{1-(\alpha_s\ell)^n}{1-\alpha_s\ell}\\
&= \mathrm{Im}\int_{C_a}\frac{d\tau}{\pi\tau} W_X\left(\frac{\tau}{Q^2}\right)\frac{\alpha_s}{1-\alpha_s\ell} - \mathrm{Im}\int_{C_a}\frac{d\tau}{\pi\tau} W_X\left(\frac{\tau}{Q^2}\right)\frac{\alpha_s^{n+1}\ell^n}{1-\alpha_s\ell},
\end{aligned}
\tag{D.1}
$$

where $\alpha_s = \alpha_s(\mu)$, and

$$
\ell = \frac{\beta_0}{4\pi}\log(e^{5/3}\mu^2/\tau)\,. \tag{D.2}
$$

In the first term of Eq. (D.1), we can rewrite $\alpha_s/(1-\alpha_s\ell) = \alpha_{\beta_0}(\tau)$. In the second term, we can deform the integral path back to the positive real axis and rewrite

$$\frac{\alpha_s^{n+1}\ell^n}{1-\alpha_s\ell} \quad\rightarrow\quad \mathrm{Pr.}\,\frac{\alpha_s^{n+1}\ell^n}{1-\alpha_s\ell} + \frac{4\pi i}{\beta_0}\pi\tau\delta(\tau - e^{5/3}\Lambda_{\mathrm{QCD}}^2)\,, \tag{D.3}$$

where Pr. denotes the principal value.

Therefore, the difference between $X_0(Q^2)$, given by Eq. (2.25), and $X_n(Q^2)$ can be written as

$$X_0(Q^2)-X_n(Q^2) = \frac{4\pi}{\beta_0}\left[c_0 - \mathrm{Re}\,W_X\left(\frac{e^{5/3}\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)\right] + \mathrm{Pr.}\int_0^\infty \frac{d\tau}{2\pi\tau}\,w_X\left(\frac{\tau}{Q^2}\right)\frac{\alpha_s^{n+1}\ell^n}{1-\alpha_s\ell}\,. \tag{D.4}$$

The first term is $\mathcal{O}(\Lambda_{\mathrm{QCD}}^2/Q^2)$. This follows from $W_X(z) = \sum_n c_n z^n$ and $\mathrm{Im}\,c_0 = w_X(0) = 0$. Hence, the first term is smaller than $\mathcal{O}(\alpha_s(\mu)^k)$ for an arbitrary positive integer k (or it is zero in expansion in $\alpha_s(\mu)$). It remains to show that

$$\mathrm{Pr.}\int_0^\infty \frac{d\tau}{2\pi\tau}\,w_X\left(\frac{\tau}{Q^2}\right)\frac{\alpha_s^{n+1}\ell^n}{1-\alpha_s\ell} = \mathcal{O}(\alpha_s(\mu)^{n+1})\,. \tag{D.5}$$

It can be shown that the left-hand side is $\mathcal{O}(\alpha_s(\mu)^{n+1})$ in the case that $\int^t \frac{dx}{x}\,w_X(x)\times$ [Polynomial of $\log x$] is absolutely convergent as $t\to\infty$, and that the first IR renormalon is a single pole. (Although the QCD potential does not satisfy the first condition, we can show Eq. (D.5) in another way.) It is valid for general μ, and in particular if we set $\mu = Q$, we obtain the relation (2.70).

References

[1] M. Beneke, "Renormalons," *Phys. Rept.* **317** (1999) 1–142, arXiv:hep-ph/9807443 [hep-ph].

[2] Y. Kiyo, G. Mishima, and Y. Sumino, "Strong IR Cancellation in Heavy Quarkonium and Precise Top Mass Determination," *JHEP* **11** (2015) 084, arXiv:1506.06542 [hep-ph].

[3] M. Beneke, "A Quark Mass Definition Adequate for Threshold Problems," *Phys. Lett.* **B434** (1998) 115–125, arXiv:hep-ph/9804241 [hep-ph].

[4] A. Pineda, "Determination of the Bottom Quark Mass from the Υ(1S) System," *JHEP* **06** (2001) 022, arXiv:hep-ph/0105008 [hep-ph].

[5] A. H. Hoang, A. Jain, I. Scimemi, and I. W. Stewart, "R-evolution: Improving Perturbative QCD," *Phys. Rev.* **D82** (2010) 011501, arXiv:0908.3189 [hep-ph].

[6] A. H. Hoang, D. W. Kolodrubetz, V. Mateu, and I. W. Stewart, "C-parameter Distribution at N^{3}LL Including Power Corrections," *Phys. Rev.* **D91** no. 9, (2015) 094017, arXiv:1411.6633 [hep-ph].

[7] I. Scimemi and A. Vladimirov, "Power Corrections and Renormalons in Transverse Momentum Distributions," arXiv:1609.06047 [hep-ph].

[8] M. Beneke and V. M. Braun, "Naive Nonabelianization and Resummation of Fermion Bubble Chains," *Phys. Lett.* **B348** (1995) 513–520, arXiv:hep-ph/9411229 [hep-ph].

[9] M. Neubert, "Scale Setting in QCD and the Momentum Flow in Feynman Diagrams," *Phys. Rev.* **D51** (1995) 5924–5941, arXiv:hep-ph/9412265 [hep-ph].

[10] D. J. Broadhurst and A. L. Kataev, "Connections Between Deep Inelastic and Annihilation Processes at Next-to-Next-to-Leading Order and Beyond," *Phys. Lett.* **B315** (1993) 179–187, arXiv:hep-ph/9308274 [hep-ph].

[11] P. Ball, M. Beneke, and V. M. Braun, "Resummation of $(\beta_0\alpha_s)^N$ Corrections in QCD: Techniques and Applications to the Tau Hadronic Width and the Heavy Quark Pole Mass," *Nucl. Phys.* **B452** (1995) 563–625, arXiv:hep-ph/9502300 [hep-ph].

[12] A. H. Mueller, "On the Structure of Infrared Renormalons in Physical Processes at High-Energies," *Nucl. Phys.* **B250** (1985) 327–350.

[13] S. Narison and V. I. Zakharov, "Hints on the Power Corrections from Current Correlators in x-Space," *Phys. Lett.* **B522** (2001) 266–272, arXiv:hep-ph/0110141 [hep-ph].

[14] K. G. Chetyrkin, S. Narison, and V. I. Zakharov, "Short Distance Tachyonic Gluon Mass and $1/Q^2$ Corrections," *Nucl. Phys.* **B550** (1999) 353–374, arXiv:hep-ph/9811275 [hep-ph].

[15] Y. Sumino, "QCD potential as a Coulomb-plus-linear potential," *Phys. Lett.* **B571** (2003) 173–183, arXiv:hep-ph/0303120 [hep-ph].

[16] Y. Sumino, "'Coulomb+Linear' Form of the Static QCD Potential in Operator Product Expansion," *Phys. Lett.* **B595** (2004) 387–392, arXiv:hep-ph/0403242 [hep-ph].

[17] Y. Sumino, "Understanding Interquark Force and Quark Masses in Perturbative QCD," 2014. arXiv:1411.7853 [hep-ph]. http://inspirehep.net/record/1331450/files/arXiv:1411.7853.pdf.

[18] G. Mishima, Y. Sumino, and H. Takaura, "UV Contribution and Power Dependence on Λ_{QCD} of Adler Function," *Phys. Lett.* **B759** (2016) 550–554, arXiv:1602.02790 [hep-ph].

[19] T. Maskawa and H. Nakajima, "Spontaneous Symmetry Breaking in Vector-Gluon Model," *Prog. Theor. Phys.* **52** (1974) 1326–1354.

[20] J. M. Cornwall, "Entropy, Confinement, and Chiral Symmetry Breaking," *Phys. Rev.* **D83** (2011) 076001, arXiv:1011.3524 [hep-ph].

[21] J. Rodriguez-Quintero, "On the Massive Gluon Propagator, the PT-BFM Scheme

and the Low-Momentum Behaviour of Decoupling and Scaling DSE Solutions," *JHEP* **01** (2011) 105, arXiv:1005.4598 [hep-ph].

[22] C. Anzai, Y. Kiyo, and Y. Sumino, "Static QCD Potential at Three-Loop Order," *Phys. Rev. Lett.* **104** (2010) 112003, arXiv:0911.4335 [hep-ph].

[23] A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, "Three-Loop Static Potential," *Phys. Rev. Lett.* **104** (2010) 112002, arXiv:0911.4742 [hep-ph].

[24] R. N. Lee, A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, "Analytic Three-Loop Static Potential," *Phys. Rev.* **D94** no. 5, (2016) 054029, arXiv:1608.02603 [hep-ph].

[25] Y. Sumino, "A Connection Between the Perturbative QCD Potential and Phenomenological Potentials," *Phys. Rev.* **D65** (2002) 054003, arXiv:hep-ph/0104259 [hep-ph].

[26] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, "Potential NRQCD: an Effective Theory for Heavy Quarkonium," *Nucl. Phys.* **B566** (2000) 275, arXiv:hep-ph/9907240 [hep-ph].

[27] M. Beneke and V. A. Smirnov, "Asymptotic Expansion of Feynman Integrals Near Threshold," *Nucl. Phys.* **B522** (1998) 321–344, arXiv:hep-ph/9711391 [hep-ph].

[28] V. A. Smirnov, "Problems of the Strategy of Regions," *Phys. Lett.* **B465** (1999) 226–234, arXiv:hep-ph/9907471 [hep-ph].

[29] V. A. Smirnov, "Applied Asymptotic Expansions in Momenta and Masses," *Springer Tracts Mod. Phys.* **177** (2002) 1–262.

[30] B. Jantzen, "Foundation and Generalization of the Expansion by Regions," *JHEP* **12** (2011) 076, arXiv:1111.2589 [hep-ph].

[31] P. A. Baikov, K. G. Chetyrkin, and J. H. Kuhn, "Adler Function, Bjorken Sum Rule, and the Crewther Relation to Order α_s^4 in a General Gauge Theory," *Phys. Rev. Lett.* **104** (2010) 132004, arXiv:1001.3606 [hep-ph].

[32] P. A. Baikov, K. G. Chetyrkin, J. H. Kuhn, and J. Rittinger, "Adler Function, Sum Rules and Crewther Relation of Order $\mathrm{O}(\alpha_s^4)$: the Singlet Case," *Phys. Lett.* **B714** (2012) 62–65, arXiv:1206.1288 [hep-ph].

[33] M. Beneke, "Large Order Perturbation Theory for a Physical Quantity," *Nucl. Phys.* **B405** (1993) 424–450.

[34] Y. Sumino, "Static QCD Potential at $r < \Lambda_{\mathrm{QCD}}^{-1}$: Perturbative Expansion and Operator-Product Expansion," *Phys. Rev.* **D76** (2007) 114009, arXiv:hep-ph/0505034 [hep-ph].

[35] D. J. Broadhurst, "Large$N$ Expansion of QED: Asymptotic Photon Propagator and Contributions to the Muon Anomaly, for Any Number of Loops," *Z. Phys.* **C58** (1993) 339–346.